

Non-Solvable Spaces of Linear Equation Systems

Linfan Mao

(Chinese Academy of Mathematics and System Science, Beijing 100190, P.R.China)

E-mail: maolinfan@163.com

Abstract: A *Smarandache system* $(\Sigma; \mathcal{R})$ is such a mathematical system that has at least one *Smarandachely denied rule* in $\mathcal{R}$, i.e., there is a rule in $(\Sigma; \mathcal{R})$ that behaves in at least two different ways within the same set Σ , i.e., validated and invalidated, or only invalidated but in multiple distinct ways. For such systems, the linear equation systems without solutions, i.e., non-solvable linear equation systems are the most simple one. We characterize such non-solvable linear equation systems with their homeomorphisms, particularly, the non-solvable linear equation systems with 2 or 3 variables by combinatorics. It is very interesting that every planar graph with each edge a straight segment is homologous to such a non-solvable linear equation with 2 variables.

Key Words: Smarandachely denied axiom, Smarandache system, non-solvable linear equations, $\vee$ -solution, $\wedge$ -solution.

AMS(2010): 15A06, 68R10

§1. Introduction

Finding the exact solution of equation system is a main but a difficult objective unless the case of linear equations in classical mathematics. Contrary to this fact, *what is about the non-solvable case?* In fact, such an equation system is nothing but a contradictory system, and characterized only by non-solvable equations for conclusion. But our world is overlap and hybrid. The number of non-solvable equations is more than that of the solvable. The main purpose of this paper is to characterize the behavior of such linear equation systems.

Let $\mathbb{R}^m, \mathbb{R}^n$ be Euclidean spaces with dimensional $m, n \geq 1$ and $T: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a $\mathbb{C}^k, 1 \leq k \leq \infty$ function such that $T(\bar{x}_0, \bar{y}_0) = \bar{0}$ for $\bar{x}_0 \in \mathbb{R}^n, \bar{y}_0 \in \mathbb{R}^m$ and the $m \times m$ matrix $\partial T^j / \partial y^i(\bar{x}_0, \bar{y}_0)$ is non-singular, i.e.,

$$\det\left(\frac{\partial T^j}{\partial y^i}\right)\Big|_{(\bar{x}_0, \bar{y}_0)} \neq 0, \text{ where } 1 \leq i, j \leq m.$$

Then the implicit function theorem ([1]) implies that there exist opened neighborhoods $V \subset \mathbb{R}^n$ of $\bar{x}_0, W \subset \mathbb{R}^m$ of $\bar{y}_0$ and a $\mathbb{C}^k$ function $\phi: V \rightarrow W$ such that

$$T(\bar{x}, \phi(\bar{x})) = \bar{0}.$$

Thus there always exists solutions for the equation $T(\bar{x}, \overline{\phi(\bar{x})}) = \bar{0}$ if T is $\mathbb{C}^k, 1 \leq k \leq \infty$. Now let $T_1, T_2, \dots, T_m, m \geq 1$ be different $\mathbb{C}^k$ functions $\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ for an integer $k \geq 1$. An

¹Received March 6, 2012. Accepted June 5, 2012.

equation system discussed in this paper is with the form following

$$T_i(\bar{x}, \bar{y}) = \bar{0}, \quad 1 \leq i \leq m. \quad (Eq)$$

A point $(\bar{x}_0, \bar{y}_0)$ is a $\vee$ -solution of the equation system (Eq) if

$$T_i(\bar{x}_0, \bar{y}_0) = \bar{0}$$

for some integers i , $1 \leq i \leq m$, and a $\wedge$ -solution of (Eq) if

$$T_i(\bar{x}_0, \bar{y}_0) = \bar{0}$$

for all integers $1 \leq i \leq m$. Denoted by S_i^0 the solutions of equation $T_i(\bar{x}, \bar{y}) = \bar{0}$ for integers $1 \leq i \leq m$. Then $\bigcup_{i=1}^m S_i^0$ and $\bigcap_{i=1}^m S_i^0$ are respectively the $\vee$ -solutions and $\wedge$ -solutions of equations (Eq). By definition, we are easily knowing that the $\wedge$ -solution is nothing but the same as the classical solution.

Definition 1.1 The $\vee$ -solvable, $\wedge$ -solvable and non-solvable spaces of equations (Eq) are respectively defined by

$$\bigcup_{i=1}^m S_i^0, \quad \bigcap_{i=1}^m S_i^0 \quad \text{and} \quad \bigcup_{i=1}^m S_i^0 - \bigcap_{i=1}^m S_i^0.$$

Now we construct a finite graph $G[Eq]$ of equations (Eq) following:

$$V(G[Eq]) = \{v_i | 1 \leq i \leq m\},$$

$$E(G[Eq]) = \{(v_i, v_j) | \exists (\bar{x}_0, \bar{y}_0) \Rightarrow T_i(\bar{x}_0, \bar{y}_0) = \bar{0} \wedge T_j(\bar{x}_0, \bar{y}_0) = \bar{0}, \quad 1 \leq i, j \leq m\}.$$

Such a graph $G[Eq]$ can be also represented by a vertex-edge labeled graph $G^L[Eq]$ following:

$$V(G^L[Eq]) = \{S_i^0 | 1 \leq i \leq m\},$$

$$E(G[Eq]) = \{(S_i^0, S_j^0) \text{ labeled with } S_i^0 \cap S_j^0 | S_i^0 \cap S_j^0 \neq \emptyset, 1 \leq i, j \leq m\}.$$

For example, let $S_1^0 = \{a, b, c\}$, $S_2^0 = \{c, d, e\}$, $S_3^0 = \{a, c, e\}$ and $S_4^0 = \{d, e, f\}$. Then its edge-labeled graph $G[Eq]$ is shown in Fig.1 following.

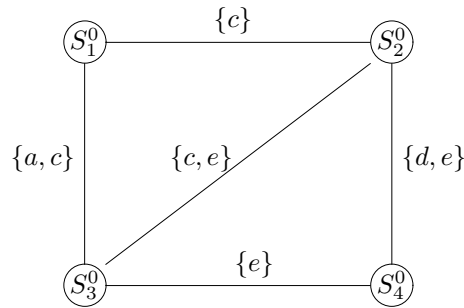


Fig.1

Notice that $\bigcup_{i=1}^m S_i^0 = \bigcup_{i=1}^m S_i^0$, i.e., the non-solvable space is empty only if $m = 1$ in (Eq). Generally, let $(\Sigma_1; \mathcal{R}_1), (\Sigma_2; \mathcal{R}_2), \dots, (\Sigma_m; \mathcal{R}_m)$ be mathematical systems, where $\mathcal{R}_i$ is a rule on Σ_i for integers $1 \leq i \leq m$. If for two integers i, j , $1 \leq i, j \leq m$, $\Sigma_i \neq \Sigma_j$ or $\Sigma_i = \Sigma_j$ but $\mathcal{R}_i \neq \mathcal{R}_j$, then they are said to be *different*, otherwise, *identical*.

Definition 1.2([12]-[13]) *A rule in $\mathcal{R}$ a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ , i.e., validated and invalidated, or only invalidated but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule in $\mathcal{R}$.

Thus, such a Smarandache system is a contradictory system. Generally, we know the conception of Smarandache multi-space with its underlying combinatorial structure defined following.

Definition 1.3([8]-[10]) *Let $(\Sigma_1; \mathcal{R}_1), (\Sigma_2; \mathcal{R}_2), \dots, (\Sigma_m; \mathcal{R}_m)$ be $m \geq 2$ mathematical spaces, different two by two. A Smarandache multi-space $\tilde{\Sigma}$ is a union $\bigcup_{i=1}^m \Sigma_i$ with rules $\tilde{\mathcal{R}} = \bigcup_{i=1}^m \mathcal{R}_i$ on $\tilde{\Sigma}$, i.e., the rule $\mathcal{R}_i$ on Σ_i for integers $1 \leq i \leq m$, denoted by $(\tilde{\Sigma}; \tilde{\mathcal{R}})$.*

Similarly, the underlying graph of a Smarandache multi-space $(\tilde{\Sigma}; \tilde{\mathcal{R}})$ is an edge-labeled graph defined following.

Definition 1.4([8]-[10]) *Let $(\tilde{\Sigma}; \tilde{\mathcal{R}})$ be a Smarandache multi-space with $\tilde{\Sigma} = \bigcup_{i=1}^m \Sigma_i$ and $\tilde{\mathcal{R}} = \bigcup_{i=1}^m \mathcal{R}_i$. Its underlying graph $G[\tilde{\Sigma}, \tilde{\mathcal{R}}]$ is defined by*

$$\begin{aligned} V(G[\tilde{\Sigma}, \tilde{\mathcal{R}}]) &= \{\Sigma_1, \Sigma_2, \dots, \Sigma_m\}, \\ E(G[\tilde{\Sigma}, \tilde{\mathcal{R}}]) &= \{(\Sigma_i, \Sigma_j) \mid \Sigma_i \cap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\} \end{aligned}$$

with an edge labeling

$$l^E : (\Sigma_i, \Sigma_j) \in E(G[\tilde{\Sigma}, \tilde{\mathcal{R}}]) \rightarrow l^E(\Sigma_i, \Sigma_j) = \varpi(\Sigma_i \cap \Sigma_j),$$

where ϖ is a characteristic on $\Sigma_i \cap \Sigma_j$ such that $\Sigma_i \cap \Sigma_j$ is isomorphic to $\Sigma_k \cap \Sigma_l$ if and only if $\varpi(\Sigma_i \cap \Sigma_j) = \varpi(\Sigma_k \cap \Sigma_l)$ for integers $1 \leq i, j, k, l \leq m$.

We consider the simplest case, i.e., all equations in (Eq) are linear with integers $m \geq n$ and $m, n \geq 1$ in this paper because we are easily know the necessary and sufficient condition of a linear equation system is solvable or not in linear algebra. For terminologies and notations not mentioned here, we follow [2]-[3] for linear algebra, [8] and [10] for graphs and topology.

Let

$$AX = (b_1, b_2, \dots, b_m)^T \quad (LEq)$$

be a linear equation system with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

for integers $m, n \geq 1$. Define an augmented matrix A^+ of A by $(b_1, b_2, \dots, b_m)^T$ following:

$$A^+ = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

We assume that all equations in (LEq) are non-trivial, i.e., there are no numbers λ such that

$$(a_{i1}, a_{i2}, \dots, a_{in}, b_i) = \lambda(a_{j1}, a_{j2}, \dots, a_{jn}, b_j)$$

for any integers $1 \leq i, j \leq m$. Such a linear equation system (LEq) is *non-solvable* if there are no solutions $x_i, 1 \leq i \leq n$ satisfying (LEq) .

§2. A Necessary and Sufficient Condition for Non-Solvable Linear Equations

The following result on non-solvable linear equations is well-known in linear algebra([2]-[3]).

Theorem 2.1 *The linear equation system (LEq) is solvable if and only if $\text{rank}(A) = \text{rank}(A^+)$. Thus, the equation system (LEq) is non-solvable if and only if $\text{rank}(A) \neq \text{rank}(A^+)$.*

We introduce the conception of parallel linear equations following.

Definition 2.2 *For any integers $1 \leq i, j \leq m, i \neq j$, the linear equations*

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n &= b_j \end{aligned}$$

are called parallel if there exists a constant c such that

$$c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in} \neq b_j/b_i.$$

Then we know the following conclusion by Theorem 2.1.

Corollary 2.3 *For any integers $i, j, i \neq j$, the linear equation system*

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \end{cases}$$

is non-solvable if and only if they are parallel.

Proof By Theorem 2.1, we know that the linear equations

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n &= b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n &= b_j \end{aligned}$$

is non-solvable if and only if $\text{rank}A' \neq \text{rank}B'$, where

$$A' = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix}, \quad B' = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} & b_1 \\ a_{j1} & a_{j2} & \cdots & a_{jn} & b_2 \end{bmatrix}.$$

It is clear that $1 \leq \text{rank}A' \leq \text{rank}B' \leq 2$ by the definition of matrixes A' and B' . Consequently, $\text{rank}A' = 1$ and $\text{rank}B' = 2$. Thus the matrix A' , B' are respectively elementary equivalent to matrixes

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix}.$$

i.e., there exists a constant c such that $c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in}$ but $c \neq b_j/b_i$. Whence, the linear equations

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n &= b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n &= b_j \end{aligned}$$

is parallel by definition. □

We are easily getting another necessary and sufficient condition for non-solvable linear equations (LEq) by three elementary transformations on a $m \times (n+1)$ matrix A^+ defined following:

- (1) *Multiplying one row of A^+ by a non-zero scalar c ;*
- (2) *Replacing the i th row of A^+ by row i plus a non-zero scalar c times row j ;*
- (3) *Interchange of two row of A^+ .*

Such a transformation naturally induces a transformation of linear equation system (LEq), denoted by $T(LEq)$. By applying Theorem 2.1, we get a generalization of Corollary 2.3 for non-solvable linear equation system (LEq) following.

Theorem 2.4 *A linear equation system (LEq) is non-solvable if and only if there exists a composition T of series elementary transformations on A^+ with $T(A^+)$ the forms following*

$$T(A^+) = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ a'_{21} & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a'_{m1} & a'_{m2} & \cdots & a'_{mn} & b'_m \end{bmatrix}$$

Proof Let $T(A^+)$ be

$$T(A^+) = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ a'_{21} & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a'_{m1} & a'_{m2} & \cdots & a'_{mn} & b'_m \end{bmatrix}.$$

If there are integers $1 \leq i, j \leq m$ such that the linear equations

$$\begin{aligned} a'_{i1}x_1 + a'_{i2}x_2 + \cdots a'_{in}x_n &= b'_i, \\ a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n &= b'_j \end{aligned}$$

are parallel, then there must be $S_i^0 \cap S_j^0 = \emptyset$, where S_i^0, S_j^0 are respectively the solutions of linear equations $a'_{i1}x_1 + a'_{i2}x_2 + \cdots a'_{in}x_n = b'_i$ and $a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n = b'_j$. Whence, there are no edges (S_i^0, S_j^0) in $G[LEq]$ by definition. Thus $G[LEq] \not\simeq K_m$. $\square$

We wish to find conditions for non-solvable linear equation systems (LEq) without elementary transformations. In fact, we are easily determining $G[LEq]$ of a linear equation system (LEq) by Corollary 2.3. Let L_i be the i th linear equation. By Corollary 2.3, we divide these equations $L_i, 1 \leq i \leq m$ into *parallel families*

$$\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_s$$

by the property that all equations in a family $\mathcal{C}_i$ are parallel and there are no other equations parallel to lines in $\mathcal{C}_i$ for integers $1 \leq i \leq s$. Denoted by $|\mathcal{C}_i| = n_i, 1 \leq i \leq s$. Then the following conclusion is clear by definition.

Theorem 2.6 *Let (LEq) be a linear equation system for integers $m, n \geq 1$. Then*

$$G[LEq] \simeq K_{n_1, n_2, \dots, n_s}$$

with $n_1 + n_2 + \cdots + n_s = m$, where $\mathcal{C}_i$ is the parallel family with $n_i = |\mathcal{C}_i|$ for integers $1 \leq i \leq s$ in (LEq) and (LEq) is non-solvable if $s \geq 2$.

Proof Notice that equations in a family $\mathcal{C}_i$ is parallel for an integer $1 \leq i \leq m$ and each of them is not parallel with all equations in $\bigcup_{1 \leq l \leq m, l \neq i} \mathcal{C}_l$. Let $n_i = |\mathcal{C}_i|$ for integers $1 \leq i \leq s$ in (LEq) . By definition, we know

$$G[LEq] \simeq K_{n_1, n_2, \dots, n_s}$$

with $n_1 + n_2 + \cdots + n_s = m$.

Notice that the linear equation system (LEq) is solvable only if $G[LEq] \simeq K_m$ by definition. Thus the linear equation system (LEq) is non-solvable if $s \geq 2$. $\square$

Notice that the conditions in Theorem 2.6 is not sufficient, i.e., if $G[LEq] \simeq K_{n_1, n_2, \dots, n_s}$, we can not claim that (LEq) is non-solvable or not. For example, let (LEq^*) and (LEq^{**}) be

two linear equations systems with

$$A_1^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad A_2^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix} .$$

Then $G[LEq^*] \simeq G[LEq^{**}] \simeq K_4$. Clearly, the linear equation system (LEq^*) is solvable with $x_1 = 0, x_2 = 0$ but (LEq^{**}) is non-solvable. We will find necessary and sufficient conditions for linear equation systems with two or three variables just by their combinatorial structures in the following sections.

§3. Linear Equation System with 2 Variables

Let

$$AX = (b_1, b_2, \dots, b_m)^T \quad (LEq2)$$

be a linear equation system in 2 variables with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \dots & \dots \\ a_{m1} & a_{m2} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for an integer $m \geq 2$. Then Theorem 2.4 is refined in the following.

Theorem 3.1 *A linear equation system (LEq2) is non-solvable if and only if one of the following conditions hold:*

- (1) *there are integers $1 \leq i, j \leq m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} \neq b_i/b_j$;*
- (2) *there are integers $1 \leq i, j, k \leq m$ such that*

$$\frac{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{vmatrix}}{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{k1} & a_{k2} \end{vmatrix}} \neq \frac{\begin{vmatrix} a_{i1} & b_i \\ a_{j1} & b_j \end{vmatrix}}{\begin{vmatrix} a_{i1} & b_i \\ a_{k1} & b_k \end{vmatrix}} .$$

Proof The condition (1) is nothing but the conclusion in Corollary 2.3, i.e., the i th equation is parallel to the j th equation. Now if there no such parallel equations in $(LEq2)$, let T be the elementary transformation replacing all other j th equations by the j th equation plus $(-a_{j1}/a_{i1})$

times the i th equation for integers $1 \leq j \leq m$. We get a transformation $T(A^+)$ of A^+ following

$$T(A^+) = \left[\begin{array}{c|cc|cc} 0 & a_{i1} & a_{i2} & a_{i1} & b_i \\ & a_{11} & a_{12} & a_{11} & b_1 \\ \dots & & \dots & & \dots \\ 0 & a_{i1} & a_{i2} & a_{i1} & b_i \\ & a_{s1} & a_{s2} & a_{s1} & b_s \\ a_{i1} & a_{i2} & & b_i & \\ 0 & a_{i1} & a_{i2} & a_{i1} & b_i \\ & a_{t1} & a_{t2} & a_{t1} & b_t \\ \dots & & \dots & & \dots \\ 0 & a_{i1} & a_{i2} & a_{i1} & b_i \\ & a_{m1} & a_{m2} & a_{m1} & b_m \end{array} \right],$$

where $s = i - 1$, $t = i + 1$. Applying Corollary 2.3 again, we know that there are integers $1 \leq i, j, k \leq m$ such that

$$\frac{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{vmatrix}}{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{k1} & a_{k2} \end{vmatrix}} \neq \frac{\begin{vmatrix} a_{i1} & b_i \\ a_{j1} & b_j \end{vmatrix}}{\begin{vmatrix} a_{i1} & b_i \\ a_{k1} & b_k \end{vmatrix}}.$$

if the linear equation system (LEQ2) is non-solvable. $\square$

Notice that a linear equation $ax_1 + bx_2 = c$ with $a \neq 0$ or $b \neq 0$ is a straight line on $\mathbb{R}^2$. We get the following result.

Theorem 3.2 *A linear equation system (LEq2) is non-solvable if and only if one of conditions following hold:*

- (1) *there are integers $1 \leq i, j \leq m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} \neq b_i/b_j$;*
- (2) *let $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ and*

$$x_1^0 = \frac{\begin{vmatrix} b_1 & a_{21} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2^0 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Then there is an integer i , $1 \leq i \leq m$ such that

$$a_{i1}(x_1 - x_1^0) + a_{i2}(x_2 - x_2^0) \neq 0.$$

Proof If the linear equation system ($LEq2$) has a solution (x_1^0, x_2^0) , then

$$x_1^0 = \frac{\begin{vmatrix} b_1 & a_{21} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2^0 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

and $a_{i1}x_1^0 + a_{i2}x_2^0 = b_i$, i.e., $a_{i1}(x_1 - x_1^0) + a_{i2}(x_2 - x_2^0) = 0$ for any integers $1 \leq i \leq m$. Thus, if the linear equation system ($LEq2$) is non-solvable, there must be integers $1 \leq i, j \leq m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} \neq b_i/b_j$, or there is an integer $1 \leq i \leq m$ such that

$$a_{i1}(x_1 - x_1^0) + a_{i2}(x_2 - x_2^0) \neq 0.$$

This completes the proof. $\square$

For a non-solvable linear equation system ($LEq2$), there is a naturally induced *intersection-free graph* $I[LEq2]$ by ($LEq2$) on the plane $\mathbb{R}^2$ defined following:

$$V(I[LEq2]) = \{(x_1^{ij}, x_2^{ij}) | a_{i1}x_1^{ij} + a_{i2}x_2^{ij} = b_i, a_{j1}x_1^{ij} + a_{j2}x_2^{ij} = b_j, 1 \leq i, j \leq m\}.$$

$E(I[LEq2]) = \{(v_{ij}, v_{il}) | \text{the segment between points } (x_1^{ij}, x_2^{ij}) \text{ and } (x_1^{il}, x_2^{il}) \text{ in } \mathbb{R}^2\}$. (where $v_{ij} = (x_1^{ij}, x_2^{ij})$ for $1 \leq i, j \leq m$).

Such an intersection-free graph is clearly a planar graph with each edge a straight segment since all intersection of edges appear at vertices. For example, let the linear equation system be ($LEq2$) with

$$A^+ = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}.$$

Then its intersection-free graph $I[LEq2]$ is shown in Fig.2.

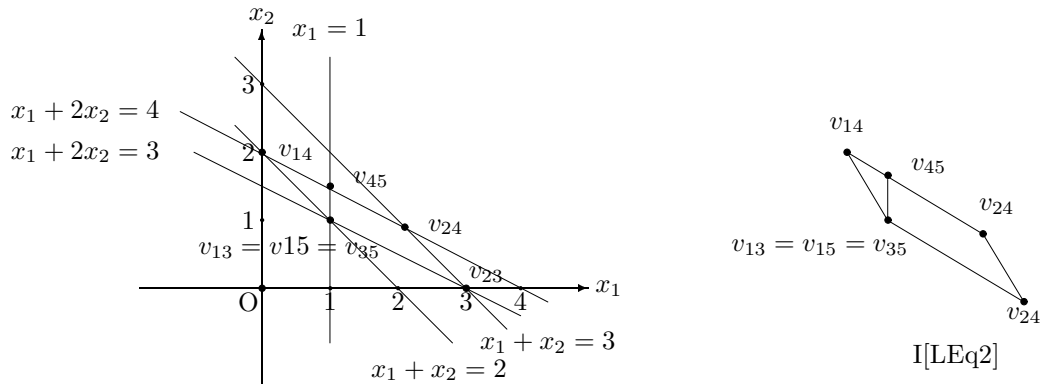


Fig.2

Let H be a planar graph with each edge a straight segment on $\mathbb{R}^2$. Its c -line graph $L_C(H)$ is defined by

$$\begin{aligned} V(L_C(H)) &= \{\text{straight lines } L = e_1 e_2 \cdots e_l, l \geq 1 \text{ in } H\}; \\ E(L_C(H)) &= \{(L_1, L_2) \mid \text{if } e_i^1 \text{ and } e_j^2 \text{ are adjacent in } H \text{ for } L_1 = e_1^1 e_2^1 \cdots e_l^1, L_2 = e_1^2 e_2^2 \cdots e_s^2, l, s \geq 1\}. \end{aligned}$$

The following result characterizes the combinatorial structure of non-solvable linear equation systems with two variables by intersection-free graphs $I[LEq2]$.

Theorem 3.3 *A linear equation system (LEq2) is non-solvable if and only if*

$$G[LEq2] \simeq L_C(H),$$

where H is a planar graph of order $|H| \geq 2$ on $\mathbb{R}^2$ with each edge a straight segment

Proof Notice that there is naturally a one to one mapping $\phi : V(G[LEq2]) \rightarrow V(L_C(I[LEq2]))$ determined by $\phi(S_i^0) = S_i^1$ for integers $1 \leq i \leq m$, where S_i^0 and S_i^1 denote respectively the solutions of equation $a_{i1}x_1 + a_{i2}x_2 = b_i$ on the plane $\mathbb{R}^2$ or the union of points between (x_1^{ij}, x_2^{ij}) and (x_1^{il}, x_2^{il}) with

$$\begin{cases} a_{i1}x_1^{ij} + a_{i2}x_2^{ij} = b_i \\ a_{j1}x_1^{ij} + a_{j2}x_2^{ij} = b_j \end{cases}$$

and

$$\begin{cases} a_{i1}x_1^{il} + a_{i2}x_2^{il} = b_i \\ a_{l1}x_1^{il} + a_{l2}x_2^{il} = b_l \end{cases}$$

for integers $1 \leq i, j, l \leq m$. Now if $(S_i^0, S_j^0) \in E(G[LEq2])$, then $S_i^0 \cap S_j^0 \neq \emptyset$. Whence,

$$S_i^1 \cap S_j^1 = \phi(S_i^0) \cap \phi(S_j^0) = \phi(S_i^0 \cap S_j^0) \neq \emptyset$$

by definition. Thus $(S_i^1, S_j^1) \in L_C(I[LEq2])$. By definition, ϕ is an isomorphism between $G[LEq2]$ and $L_C(I[LEq2])$, a line graph of planar graph $I[LEq2]$ with each edge a straight segment.

Conversely, let H be a planar graph with each edge a straight segment on the plane $\mathbb{R}^2$. Not loss of generality, we assume that edges $e_{1,2}, \dots, e_l \in E(H)$ is on a straight line L with equation $a_{L1}x_1 + a_{L2}x_2 = b_L$. Denote all straight lines in H by $\mathcal{C}$. Then H is the intersection-free graph of linear equation system

$$a_{L1}x_1 + a_{L2}x_2 = b_L, \quad L \in \mathcal{C}. \quad (LEq2^*)$$

Thus,

$$G[LEq2^*] \simeq H.$$

This completes the proof. $\square$

Similarly, we can also consider the liner equation system $(LEq2)$ with condition on x_1 or x_2 such as

$$AX = (b_1, b_2, \dots, b_m)^T \quad (L^-Eq2)$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \dots & \dots \\ a_{m1} & a_{m2} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and $x_1 \geq x^0$ for a real number x^0 and an integer $m \geq 2$. In geometry, each of there equation is a ray on the plane $\mathbb{R}^2$, seeing also references [5]-[6]. Then the following conclusion can be obtained like with Theorems 3.2 and 3.3.

Theorem 3.4 *A linear equation system (L-Eq2) is non-solvable if and only if*

$$G[LEq2] \simeq L_C(H),$$

where H is a planar graph of order $|H| \geq 2$ on $\mathbb{R}^2$ with each edge a straight segment.

§4. Linear Equation Systems with 3 Variables

Let

$$AX = (b_1, b_2, \dots, b_m)^T \quad (LEq3)$$

be a linear equation system in 3 variables with

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

for an integer $m \geq 3$. Then Theorem 2.4 is refined in the following.

Theorem 4.1 *A linear equation system (LEq3) is non-solvable if and only if one of the following conditions hold:*

- (1) *there are integers $1 \leq i, j \leq m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} = a_{i3}/a_{j3} \neq b_i/b_j$;*
- (2) *if (a_{i1}, a_{i2}, a_{i3}) and (a_{j1}, a_{j2}, a_{j3}) are independent, then there are numbers λ, μ and an integer l , $1 \leq l \leq m$ such that*

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3})$$

but $b_l \neq \lambda b_i + \mu b_j$;

- (3) *if (a_{i1}, a_{i2}, a_{i3}) , (a_{j1}, a_{j2}, a_{j3}) and (a_{k1}, a_{k2}, a_{k3}) are independent, then there are numbers λ, μ, ν and an integer l , $1 \leq l \leq m$ such that*

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3}) + \nu(a_{k1}, a_{k2}, a_{k3})$$

but $b_l \neq \lambda b_i + \mu b_j + \nu b_k$.

Proof By Theorem 2.1, the linear equation system (*LEq3*) is non-solvable if and only if $1 \leq \text{rank}A \neq \text{rank}A^+ \leq 4$. Thus the non-solvable possibilities of (*LEq3*) are respectively $\text{rank}A = 1, 2 \leq \text{rank}A^+ \leq 4, \text{rank}A = 2, 3 \leq \text{rank}A^+ \leq 4$ and $\text{rank}A = 3, \text{rank}A^+ = 4$. We discuss each of these cases following.

Case 1 $\text{rank}A = 1$ but $2 \leq \text{rank}A^+ \leq 4$

In this case, all row vectors in A are dependent. Thus there exists a number λ such that $\lambda = a_{i1}/a_{j1} = a_{i2}/a_{j2} = a_{i3}/a_{j3}$ but $\lambda \neq b_i/b_j$.

Case 2 $\text{rank}A = 2, 3 \leq \text{rank}A^+ \leq 4$

In this case, there are two independent row vectors. Without loss of generality, let (a_{i1}, a_{i2}, a_{i3}) and (a_{j1}, a_{j2}, a_{j3}) be such row vectors. Then there must be an integer $l, 1 \leq l \leq m$ such that the l th row can not be the linear combination of the i th row and j th row. Whence, there are numbers λ, μ such that

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3})$$

but $b_l \neq \lambda b_i + \mu b_j$.

Case 3 $\text{rank}A = 3, \text{rank}A^+ = 4$

In this case, there are three independent row vectors. Without loss of generality, let $(a_{i1}, a_{i2}, a_{i3}), (a_{j1}, a_{j2}, a_{j3})$ and (a_{k1}, a_{k2}, a_{k3}) be such row vectors. Then there must be an integer $l, 1 \leq l \leq m$ such that the l th row can not be the linear combination of the i th row, j th row and k th row. Thus there are numbers λ, μ, ν such that

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3}) + \nu(a_{k1}, a_{k2}, a_{k3})$$

but $b_l \neq \lambda b_i + \mu b_j + \nu b_k$. Combining the discussion of Case 1-Case 3, the proof is complete. $\square$

Notice that the linear equation system (*LEq3*) can be transformed to the following (*LEq3**) by elementary transformation, i.e., each j th row plus $-a_{j3}/a_{i3}$ times the i th row in (*LEq3*) for an integer $i, 1 \leq i \leq m$ with $a_{i3} \neq 0$,

$$A'X = (b'_1, b'_2, \dots, b'_m)^T \quad (\text{LEq3}^*)$$

with

$$A'^+ = \begin{bmatrix} a'_{11} & a'_{12} & 0 & b'_1 \\ \dots & \dots & \dots & \dots \\ a'_{(i-1)1} & a'_{(i-1)2} & 0 & b'_{i-1} \\ a_{i1} & a_{i2} & a_{i3} & b_i \\ a'_{(i+1)1} & a'_{(i+1)2} & 0 & b'_{i+1} \\ a'_{m1} & a'_{m2} & 0 & b'_m \end{bmatrix},$$

where $a'_{j1} = a_{j1} - a_{j3}a_{i1}/a_{i3}$, $a'_{j2} = a_{j2} - a_{j3}a_{i2}/a_{i3}$ and $b'_j = b_j - a_{j3}b_i/a_{i3}$ for integers $1 \leq j \leq m$. Applying Theorem 3.3, we get the a combinatorial characterizing on non-solvable linear systems (*LEq3*) following.

Theorem 4.2 *A linear equation system (LEq3) is non-solvable if and only if $G[LEq3] \not\simeq K_m$ or $G[LEq3^*] \simeq u + L_C(H)$, where H denotes a planar graph with order $|H| \geq 2$, size $m - 1$ and each edge a straight segment, $u + G$ the join of vertex u with G .*

Proof By Theorem 2.4, the linear equation system (LEq3) is non-solvable if and only if $G[LEq3] \not\simeq K_m$ or the linear equation system (LEq3*) is non-solvable, which implies that the linear equation subsystem following

$$BX' = (b'_1, \dots, b'_{i-1}, b'_{i+1}, \dots, b'_m)^T \quad (LEq2^*)$$

with

$$B = \begin{bmatrix} a'_{11} & a'_{12} \\ \dots & \dots \\ a'_{(i-1)1} & a'_{(i-1)2} \\ a'_{(i+1)1} & a'_{(i+1)2} \\ a'_{m1} & a'_{m2} \end{bmatrix} \quad \text{and} \quad X' = (x_1, x_2)^T$$

is non-solvable. Applying Theorem 3.3, we know that the linear equation subsystem (LEq2*) is non-solvable if and only if $G[LEq2^*] \simeq L_C(H)$, where H is a planar graph H of size $m - 1$ with each edge a straight segment. Thus the linear equation system (LEq3*) is non-solvable if and only if $G[LEq3^*] \simeq u + L_C(H)$. $\square$

§5. Linear Homeomorphisms Equations

A homeomorphism on $\mathbb{R}^n$ is a continuous 1 - 1 mapping $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that its inverse h^{-1} is also continuous for an integer $n \geq 1$. There are indeed many such homeomorphisms on $\mathbb{R}^n$. For example, the linear transformations T on $\mathbb{R}^n$. A *linear homeomorphisms equation system* is such an equation system

$$AX = (b_1, b_2, \dots, b_m)^T \quad (L^hEq)$$

with $X = (h(x_1), h(x_2), \dots, h(x_n))^T$, where h is a homeomorphism and

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

for integers $m, n \geq 1$. Notice that the linear homeomorphism equation system

$$\begin{cases} a_{i1}h(x_1) + a_{i2}h(x_2) + \dots + a_{in}h(x_n) = b_i, \\ a_{j1}h(x_1) + a_{j2}h(x_2) + \dots + a_{jn}h(x_n) = b_j \end{cases}$$

is solvable if and only if the linear equation system

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = b_j \end{cases}$$

is solvable. Similarly, two linear homeomorphism equations are said *parallel* if they are non-solvable. Applying Theorems 2.6, 3.3, 4.2, we know the following result for linear homeomorphism equation systems (L^hEq).

Theorem 5.1 *Let (L^hEq) be a linear homeomorphism equation system for integers $m, n \geq 1$. Then*

- (1) $G[LEq] \simeq K_{n_1, n_2, \dots, n_s}$ with $n_1 + n_2 + \dots + n_s = m$, where $\mathcal{C}_i^h$ is the parallel family with $n_i = |\mathcal{C}_i^h|$ for integers $1 \leq i \leq s$ in (L^hEq) and (L^hEq) is non-solvable if $s \geq 2$;
- (2) If $n = 2$, (L^hEq) is non-solvable if and only if $G[L^hEq] \simeq L_C(H)$, where H is a planar graph of order $|H| \geq 2$ on $\mathbb{R}^2$ with each edge a homeomorphism of straight segment, and if $n = 3$, (L^hEq) is non-solvable if and only if $G[L^hEq] \not\simeq K_m$ or $G[LEq3^*] \simeq u + L_C(H)$, where H denotes a planar graph with order $|H| \geq 2$, size $m - 1$ and each edge a homeomorphism of straight segment.

References

- [1] R.Abraham, J.E.Marsden and T.Ratiu, *Manifolds, Tensors Analysis and Applications*, Addison-Wesley Publishing Company, Inc., 1983.
- [2] G.Birkhoff and S.MacLane, *A Survey of Modern Algebra* (4th edition), Macmillan Publishing Co., Inc, 1977.
- [3] K.Hoffman and R.Kunze, *Linear Algebra* (2th edition), Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.
- [4] L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, *JP Journal of Geometry and Topology*, 5(1), 2005,77-81.
- [5] Linfan Mao, Parallel bundles in planar map geometries, *Scientia Magna*, Vol.1(2005), No.2, 120-133.
- [6] Linfan Mao, On multi-metric spaces, *Scientia Magna*, Vol.2,No.1(2006), 87-94.
- [7] Linfan Mao, Euclidean pseudo-geometry on $\mathbf{R}^n$, *International J.Math. Combin.*, Vol.1 (2009), 90-95.
- [8] Linfan Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries* (Second edition), Graduate Textbook in Mathematics, The Education Publisher Inc. 2011.
- [9] Linfan Mao, *Smarandache Multi-Space Theory* (Second edition), Graduate Textbook in Mathematics, The Education Publisher Inc. 2011.
- [10] Linfan Mao, *Combinatorial Geometry with Applications to Field Theory* (Second edition), Graduate Textbook in Mathematics, The Education Publisher Inc. 2011.
- [11] W.S.Massey, *Algebraic Topology: An Introduction*, Springer-Verlag, New York, etc., 1977.
- [12] F.Smarandache, Mixed noneuclidean geometries, *Eprint arXiv: math/0010119*, 10/2000.
- [13] F.Smarandache, *A Unifying Field in Logics-Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.