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Laplacian Energy of Certain Graphs

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Abstract: Let G be a graph with n vertices and m edges. Let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of the Laplacian matrix of G. The Laplacian energy $LE(G) = \sum\limits_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$. In this paper, we calculate the exact Laplacian energy of complete graph, complete bipartite graph, path, cycle and friendship graph.

Key Words: Complete graph, complete bipartite graph, path, cycle, friendship graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph G with n vertices and m edges. Let d_i be the degree of the i^{th} vertex of $G, i = 1, 2, \cdots, n$.

Definition 1.1([3]) *Let $A(G) = [a_{ij}]$ be the $(0, 1)$ adjacency matrix, $D(G) = diag(d_1, d_2, \cdots, d_n)$, the diagonal matrix with vertex degrees $d_1, d_2, \cdots, d_n$ of its vertices $v_1, v_2, \cdots, v_n$ of a graph G. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of the graph G.*

It is symmetric, singular and positive semi - definite. All its eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ are real and nonnegative and form the Laplacian spectrum. It is well known that one of the eigen values is zero.

Definition 1.2([3]) *If G is a graph with n vertices and m edges, and its Laplacian eigen values are $\mu_1, \mu_2, \cdots, \mu_n$ then the Laplacian energy of G , denoted by LE(G), is $\sum\limits_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$. i.e.,* $LE(G) = \sum\limits_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

This quantity has a long known chemical application for details see the surveys [1,4,5]. If the graph G has one vertex then the Laplacian energy is zero.

Property 1.3([3])

(1) $LE(G) \leq \sqrt{2Mn}$;

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(2) $LE(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2M - (\frac{2m}{n})^2\right]}$;

(3) $2\sqrt{M} \leq LE(G) \leq 2M$, *where* $M = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2$.

§2. The Laplacian Energy of Complete Graphs

Definition 2.1([2]) *A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph.*

Theorem 2.2 *The Laplacian energy of the complete graph K_n on n vertices is $2(n-1)$.*

Proof The eigenvalues of the Laplacian matrix of the complete graph K_n on n vertices and $\frac{n(n-1)}{2}$ edges are $\mu_1 = 0$ and multiplicity of the eigen values n as $n-1$, i.e., $\mu_1 = 0, \mu_2 = \mu_3 = \cdots = \mu_n = n$. Thus

$$LE(K_n) = \sum_{i=1}^{n} |(\mu_i - (n-1)| = |0-(n-1)| + (n-1)|n-(n-1)| = 2(n-1).$$

□

§3. The Laplacian Energy of Complete Bipartite Graphs

Definition 3.1([2]) *A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition(X,Y) is called a bipartition of the graph.*

Definition 3.2([2]) *A complete bipartite graph is a simple bipartite graph with bipartition (X,Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.*

Definition 3.3([6]) *The Star graph $K_{1,n}$ is a tree on n+1 vertices with one vertex having degree n and the other n vertices having degree 1.*

Theorem 3.4 *The Laplacian energy of the complete bipartite graph $K_{m,n}$ with $m+n$ vertices and mn edges is*

$$\frac{(m+n)^2 + |m-n|\,(2mn - (m+n))}{(m+n)}.$$

Proof In this graph, the Laplacian spectrum is $\mu_1 = 0$, the multiplicity of the eigen values m as $n-1$, the multiplicity of the eigen values n as $m-1$ and $\mu_{m+n} = m+n$.

The Laplacian energy

$$\begin{aligned} LE(K_{m,n}) &= \sum_{i=1}^{n+m} \left|\mu_i - \frac{2mn}{m+n}\right| \\ &= |0 - \frac{2mn}{m+n}| + (n-1)|m - \frac{2mn}{m+n}| \\ &\quad +(m-1)|n - \frac{2mn}{m+n}| + (m+n) - \frac{2mn}{m+n} \\ &= \frac{2mn}{m+n} + \frac{m(n-1)}{m+n}|m-n| + \frac{n(m-1)}{m+n}|n-m| \\ &= \frac{(m+n)^2 + |m-n|(2mn-(m+n))}{m+n}. \end{aligned}$$

□

Corollary 3.5 *The Laplacian energy of a star graph $K_{1,n}$ is* $\dfrac{2(n^2+1)}{n+1}$.

Proof Let m be replaced by one in Theorem 3.4. We get the following

$$LE(K_{1,n}) = \frac{(1+n)^2 + |1-n|\,(2n-(1+n))}{1+n} = \frac{2(n^2+1)}{n+1}.$$

□

§4. The Laplacian Energy of Paths P_n and Cycles C_n

Definition 4.1 *A path P_n with n vertices has $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ for its vertex set and $E(P_n) = \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n\}$ is its edge set. This path P_n is said to have length $n-1$.*

Definition 4.2 *A cycle C_n with n points is a graph with vertex set $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(C_n) = \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n, v_nv_1\}$.*

Theorem 4.3 *The Laplacian energy of the path P_n with n vertices is* $\sum\limits_{i=0}^{n-1}\left|2\left[\frac{1}{n} - cos\left(\frac{\pi i}{n}\right)\right]\right|$.

Proof The eigen values of the Laplacian matrix of P_n are $2\left[1 - cos\left(\frac{\pi i}{n}\right)\right]$, $i = 0, 1, \cdots, n-1$. Then,

$$LE(P_n) = \sum_{i=0}^{n-1}\left|2\left[1 - cos\left(\frac{\pi i}{n}\right)\right] - \frac{2(n-1)}{n}\right| = \sum_{i=0}^{n-1}\left|2\left[\frac{1}{n} - cos\left(\frac{\pi i}{n}\right)\right]\right|.$$

□

Theorem 4.4 *The Laplacian energy of the cycle C_n with n vertices is* $2\sum_{i=0}^{n-1}\left|cos\left(\frac{2\pi i}{n}\right)\right|$.

Proof The Laplacian spectrum of the cycle C_n is $2\left[1 - cos\left(\frac{2\pi i}{n}\right)\right]$, $i = 0, 1, \cdots, (n-1)$. Then

$$LE(C_n) = \sum_{i=0}^{n-1}\left|2\left[1 - cos\left(\frac{2\pi i}{n}\right)\right] - 2\right| = 2\sum_{i=0}^{n-1}\left|cos\left(\frac{2\pi i}{n}\right)\right|.$$

□

§5. The Laplacian Energy of Friendship Graphs

Definition 5.1([6]) *The friendship graph $F_r(r \geq 1)$ consists of r triangles with a common vertex.*

Illustration. The Friendship graph F_4 consists of 4 triangles with a common vertex is as shown in Fig.1.

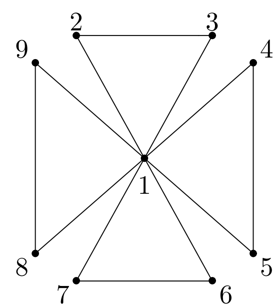

Fig.1 Friendship graph F_4

The Laplacian matrix of F_2 is

$$\begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Theorem 5.2 *The Laplacian energy of the friendship graph F_r is $\dfrac{8r^2+2r+2}{2r+1}$, where $r \geq 1$.*

Proof The friendship graph F_r has $2r+1$ vertices and $3r$ edges. Its Laplacian matrix has $2r+1$ eigen values. These eigen values are $\mu_1 = 2r+1$, the multiplicity of the eigen value 3 as r, the multiplicity of the eigen value 1 as $r-1$ and $\mu_{2r+1} = 0$.

By definition, the Laplacian energy

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

Thus,

$$\begin{aligned} LE(F_r) &= \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| = \sum_{i=1}^{n} \left| \mu_i - \frac{6r}{2r+1} \right| \\ &= \left| 2r+1-\frac{6r}{2r+1} \right| + r \left| 3-\frac{6r}{2r+1} \right| + (r-1) \left| 1-\frac{6r}{2r+1} \right| + \left| 0-\frac{6r}{2r+1} \right| \\ &= \left| \frac{4r^2-2r+1}{2r+1} \right| + r\frac{3}{2r+1} + (r-1) \left| \frac{4r-1}{2r+1} \right| + \frac{6r}{2r+1} = \frac{8r^2+2r+2}{2r+1} \end{aligned}$$

since $4r^2 + 1 > 2r$ and $1 - 4r < 0$. □

Corollary 5.1 *If G is the friendship graph of n vertices then* $LE(G) = \dfrac{2n^2 - 3n + 3}{n}$.

Proof Replacing r by $\dfrac{n-1}{2}$ in Theorem 5.2, we get the result. □

Corollary 5.2 *If G is the friendship graph of m edges then* $LE(G) = \dfrac{2}{3}\left[\dfrac{4m^2 + 3m + 9}{2m + 3}\right]$.

Proof Let r be replaced by $\dfrac{m}{3}$ in Theorem 5.2, we get the result. From [3], $M = m + \dfrac{1}{2}\sum_{i=1}^{n}\left(d_i - \dfrac{2m}{n}\right)^2$. In a friendship graph $M = \dfrac{r}{2r+1}\left(4r^2 - 2r + 7\right)$. Therefore, $2Mn = 2r\left(4r^2 - 2r + 7\right)$. Hence, using Property 1.3, we get the following

$$2\sqrt{\frac{r}{2r+1}\left(4r^2 - 2r + 7\right)} \le LE(G) \le \frac{2r}{2r+1}\left(4r^2 - 2r + 7\right).$$

□

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