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Enumeration of Rooted Nearly 2-Regualr Simple Planar Maps

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Abstract: This paper discusses the enumeration of rooted nearly 2-regular simple planar maps and presents some formulae for such maps with the valency of the root-face, the numbers of nonrooted vertices and inner faces as three parameters.

Key Words: Smarandachely map, simple map, nearly 2-regular map, enumerating function, functional equation, Lagrangian inversion, Lagrangian inversion.

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§1. Introduction

Let S be a surface. For an integer $k \geq 0$, a *Smarandachely k-map* on S is such a pseudo-map on S just with k faces that not being 2-cell. If $k = 0$, such a Smarandachely map is called *map*. In the field of enumerating planar maps, many functional equations for a variety of sets of planar maps have been found and some solutions of the equations are obtained. Some nice skills are applied in this area and they have set up the foundation of enumerative theory [2], [5], [6] and [9–13]. But the discussion on enumerating function of simple planar maps is very few. All the results obtained so far are almost concentrated in general simple planar maps [3], [4], [7] and [8]. In 1997, Cai [1] investigated for the first time the enumeration of simple Eulerian planar maps with the valency of root-vertex, the number of inner edges and the valency of root-face as parameters and a functional equation satisfied by its enumerating function was obtained, but it is very complicated and its solution has not been found up to now.

In this paper we treat the enumeration of rooted nearly 2-regular simple planar maps with the valency of the root-face, the numbers of nonrooted vertices and inner faces as three parameters. Several explicit expressions of its enumerating functions are obtained and one of them is summation-free.

Now, we define some basic concepts and terms. In general, rooting a map means distinguishing one edge on the boundary of the outer face as the root-edge, and one end of that edge as the root-vertex. In diagrams we usually represent the root-edge as an edge with an arrow on

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the outer face, the arrow being drawn from the root-vertex to the other end. So the outer face is also called the root-face. A planar map with a rooting is said to be a rooted planar map. We say that two rooted planar maps are combinatorially equivalent or up to root-preserving isomorphism if they are related by 1-1 correspondence of their elements, which maps vertices onto vertices, edges onto edges and faces onto faces, which preserves incidence relations and which preserves the root-vertex, root-edge and root-face. Otherwise, combinatorially inequivalent or nonisomorphic here.

A nearly 2-regular map is a rooted map such that all vertices probably except the root-vertex are of valency 2. A map is said to be simple, if there is neither loop nor parallel edge.

For a set of some maps $\mathscr{M}$, the enumerating function discussed in this paper is defined as

$$f_{\mathscr{M}}(x,y,z)=\sum_{M\in\mathscr{M}} x^{l(M)}y^{p(M)}z^{q(M)}, \tag{1}$$

where $l(M), p(M)$ and $q(M)$ are the root-face valency, the number of nonrooted vertices and the number of inner faces of M, respectively.

Furthermore, we introduce some other enumerating functions for $\mathscr{M}$ as follows:

$$\begin{aligned} g_{\mathscr{M}}(x,y) &= \sum_{M\in\mathscr{M}} x^{l(M)}y^{n(M)},\\ h_{\mathscr{M}}(y,z) &= \sum_{M\in\mathscr{M}} y^{p(M)}z^{q(M)},\\ H_{\mathscr{M}}(y) &= \sum_{M\in\mathscr{M}} y^{n(M)}, \end{aligned} \tag{2}$$

where $l(M), p(M)$ and $q(M)$ are the same in (1) and $n(M)$ is the number of edges of M, that is

$$\begin{aligned} g_{\mathscr{M}}(x,y) =& f_{\mathscr{U}}(x,y,y), \quad h_{\mathscr{M}}(y,z) = f_{\mathscr{M}}(1,y,z),\\ H_{\mathscr{M}}(y) =& g_{\mathscr{M}}(1,y) = h_{\mathscr{M}}(y,y) = f_{\mathscr{M}}(1,y,y). \end{aligned} \tag{3}$$

For the power series $f(x),\ f(x,y)$ and $f(x,y,z)$, we employ the following notations:

$$\partial_x^m f(x), \quad \partial_{(x,y)}^{(m,p)} f(x,y) \quad and \quad \partial_{(x,y,z)}^{(m,p,q)} f(x,y,z)$$

to represent the coefficients of x^m in $f(x)$, x^my^p in $f(x,y)$ and $x^my^pz^q$ in $f(x,y,z)$, respectively. Terminologies and notations not explained here can be found in [11].

§2. **Functional Equations**

In this section we will set up the functional equations satisfied by the enumerating functions for rooted nearly 2-regular simple planar maps.

Let $\mathscr{E}$ be the set of all rooted nearly 2-regular simple planar maps with convention that the vertex map ϑ is in $\mathscr{E}$ for convenience. From the definition of a nearly 2-regular simple map, for any $M\in\mathscr{E}-\vartheta$, each edge of M is contained in only one circuit. The circuit containing the root-edge is called the root circuit of M, and denoted by $C(M)$.

It is clear that the length of the root circuit is no more than the root-face valency, and

$$\mathscr{E} = \mathscr{E}_0 + \bigcup_{i\geq 3} \mathscr{E}_i, \tag{4}$$

where

$$\mathscr{E}_i = \{M \mid M \in \mathscr{E}, the\ length\ of\ C(M)\ is\ i\} \tag{5}$$

and $\mathscr{E}_0$ is only consist of the vertex map ϑ.

It is easy to see that the enumerating function of $\mathscr{E}_0$ is

$$f_{\mathscr{E}_0}(x, y, z) = 1. \tag{6}$$

For any $M \in \mathscr{E}_i (i \geq 3)$, the root circuit divides $M - C(M)$ into two domains, the inner domain and outer domain. The submap of M in the outer domain is a general map in $\mathscr{E}$, while the submap of M in the inner domain does not contribute the valency of the root-face of M. Therefore, the enumerating function of $\mathscr{E}_i$ is

$$f_{\mathscr{E}_i}(x, y, z) = x^i y^{i-1} z h f, \tag{7}$$

where $h = h_{\mathscr{E}}(y, z) = f_{\mathscr{E}}(1, y, z)$.

Theorem 2.1 *The enumerating function $f = f_{\mathscr{E}}(x, y, z)$ satisfies the following equation:*

$$f = \left[1 - \frac{x^3 y^2 z h}{1 - xy}\right]^{-1}, \tag{8}$$

where $h = h_{\mathscr{E}}(y, z) = f_{\mathscr{E}}(1, y, z)$.

Proof By (4), (6) and (7), we have

$$\begin{aligned} f =& 1 + \sum_{i\geq 3} x^i y^{i-1} z h f \\ =& 1 + \frac{x^3 y^2 z h f}{1 - xy}, \end{aligned}$$

which is equivalent to the theorem. □

Let $y = z$ in (8). Then we have

Corollary 2.1 *The enumerating function $g = g_{\mathscr{E}}(x, y)$ satisfies the following equation:*

$$g = \left[1 - \frac{x^3 y^3 H}{1 - xy}\right]^{-1}, \tag{9}$$

where $H = H_{\mathscr{E}}(y) = g_{\mathscr{E}}(1, y)$.

Let $x = 1$ in (8). Then we obtain

Corollary 2.2 *The enumerating function $h = h_{\mathscr{E}}(y, z)$ satisfies the following equation:*

$$y^2 z h^2 - (1 - y)h - y + 1 = 0. \tag{10}$$

Further, let $y = z$ in (10). Then we have

Corollary 2.3 *The enumerating function $H = H_{\mathscr{E}}(y)$ satisfies the following equation:*

$$y^3H^2 - (1-y)H - y + 1 = 0. \tag{11}$$

§3. Enumeration

In this section we will find the explicit formulae for enumerating functions $f = f_{\mathscr{E}}(x,y,z), g = g_{\mathscr{E}}(x,y), h = h_{\mathscr{E}}(y,z)$ and $H = H_{\mathscr{E}}(y)$ by using Lagrangian inversion.

By (10) we have

$$h = \frac{(1-y)\left(1 - \sqrt{1 - \frac{4y^2z}{1-y}}\right)}{2y^2z}. \tag{12}$$

Let

$$y = \frac{\theta}{1+\theta}, \quad yz = \eta(1-\theta\eta). \tag{13}$$

By substituting (13) into (12), one may find that

$$h = \frac{1}{1-\theta\eta}. \tag{14}$$

By (13) and (14), we have the following parametric expression of $h = h_{\mathscr{E}}(y,z)$:

$$\begin{aligned} y =& \frac{\theta}{1+\theta}, \quad yz = \eta(1-\theta\eta), \\ h =& \frac{1}{1-\theta\eta} \end{aligned} \tag{15}$$

and from which we get

$$\Delta_{(\theta,\eta)} = \begin{vmatrix} \frac{1}{1+\theta} & 0 \\ * & \frac{1-2\theta\eta}{1-\theta\eta} \end{vmatrix} = \frac{1-2\theta\eta}{(1+\theta)(1-\theta\eta)}. \tag{16}$$

Theorem 3.1 *The enumerating function $h = h_{\mathscr{E}}(y,z)$ has the following explicit expression:*

$$h_{\mathscr{E}}(y,z) = 1 + \sum_{p\geq 2}\sum_{q=1}^{\lfloor \frac{p}{2} \rfloor} \frac{(2q)!(p-q-1)!}{q!(q+1)!(p-2q)!(q-1)!} y^p z^q. \tag{17}$$

Proof By employing Lagrangian inversion with two parameters, from (15) and (16) one may find that

$$\begin{aligned}h_{\mathscr{E}}(y,z)=&\sum_{p,q\geq 0}\partial_{(\theta,\eta)}^{(p,q)}\frac{(1+\theta)^{p-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+2}}y^{p+q}z^{q}\\ =&\sum_{p\geq 0}\sum_{q=0}^{p}\partial_{(\theta,\eta)}^{(p-q,q)}\frac{(1+\theta)^{p-q-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+2}}y^{p}z^{q}\\ =&1+\sum_{p\geq 1}\sum_{q=1}^{p}\left[\partial_{(\theta,\eta)}^{(p-q,q)}\frac{(1+\theta)^{p-q-1}}{(1-\theta\eta)^{q+2}}-2\partial_{(\theta,\eta)}^{(p-q-1,q-1)}\frac{(1+\theta)^{p-q-1}}{(1-\theta\eta)^{q+2}}\right]y^{p}z^{q}\\ =&1+\sum_{p\geq 2}\sum_{q=1}^{\lfloor\frac{p}{2}\rfloor}\frac{(2q)!}{q!(q+1)!}\partial_{\theta}^{p-2q}(1+\theta)^{p-q-1}y^{p}z^{q}\\ =&1+\sum_{p\geq 2}\sum_{q=1}^{\lfloor\frac{p}{2}\rfloor}\frac{(2q)!(p-q-1)!}{q!(q+1)!(p-2q)!(q-1)!}y^{p}z^{q},\end{aligned}$$

which is just the theorem. □

In what follows we present a corollary of Theorem 3.1.

Corollary 3.1 *The enumerating function $H=H_{\mathscr{E}}(y)$ has the following explicit expression:*

$$H_{\mathscr{E}}(y)=1+\sum_{n\geq 3}\sum_{q=1}^{\lfloor\frac{n}{3}\rfloor}\frac{(2q)!(n-2q-1)!}{q!(q+1)!(n-3q)!(q-1)!}y^{n}. \tag{18}$$

Proof It follows immediately from (17) by putting $x=y$ and $n=p+q$. □

Now, let

$$x=\frac{\xi(1+\theta)}{1+\xi\theta}. \tag{19}$$

By substituting (15) and (19) into Equ. (8), one may find that

$$f=\frac{1}{1-\frac{\xi^{3}\theta\eta(1+\theta)^{2}}{(1+\xi\theta)^{2}}}. \tag{20}$$

By (15), (19) and (20), we have the parametric expression of the function $f=f_{\mathscr{E}}(x,y,z)$ as follows:

$$\begin{aligned}x=&\frac{\xi(1+\theta)}{1+\xi\theta},\quad y=\frac{\theta}{1+\theta},\\ yz=&\eta(1-\theta\eta),\quad f=\frac{1}{1-\frac{\xi^{3}\theta\eta(1+\theta)^{2}}{(1+\xi\theta)^{2}}}.\end{aligned} \tag{21}$$

According to (21), we have

$$\Delta_{(\xi,\theta,\eta)}=\begin{vmatrix}\frac{1}{1+\xi\theta} & * & 0\\ 0 & \frac{1}{1+\theta} & 0\\ 0 & * & \frac{1-2\theta\eta}{1-\theta\eta}\end{vmatrix}=\frac{1-2\theta\eta}{(1+\xi\theta)(1+\theta)(1-\theta\eta)}. \tag{22}$$

Theorem 3.2 *The enumerating function $f = f_{\mathscr{E}}(x,y,z)$ has the following explicit expression:*

$$f_{\mathscr{E}}(x,y,z) = 1 + \sum_{p\geq 2}\sum_{q=1}^{\lfloor \frac{p}{2}\rfloor}\sum_{l=3}^{p+q}\sum_{k=\max\{1,\lceil\frac{l+2q-p}{3}\rceil\}}^{\min\{\lfloor\frac{l}{3}\rceil,q\}} \frac{(2q-k-1)!k}{(q-k)!q!}\binom{l-2k-1}{l-3k} \times \binom{p-q-l+2k-1}{p-2q-l+3k} x^l y^p z^q. \tag{23}$$

Proof By using Lagrangian inversion with three variables, from (21) and (22) one may find that

$$\begin{aligned}
f_{\mathscr{E}}(x,y,z) &= \sum_{l,p,q\geq 0} \partial_{(\xi,\theta,\eta)}^{(l,p,q)} \frac{(1+\xi\theta)^{l-1}(1+\theta)^{p-l-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+1}\left[1-\frac{\xi^3\theta\eta(1+\theta)^2}{(1+\xi\theta)^2}\right]} x^l y^{p+q} z^q \\
&= \sum_{l,p\geq 0}\sum_{q=0}^{p} \partial_{(\xi,\theta,\eta)}^{(l,p-q,q)} \frac{(1+\xi\theta)^{l-1}(1+\theta)^{p-q-l-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+1}\left[1-\frac{\xi^3\theta\eta(1+\theta)^2}{(1+\xi\theta)^2}\right]} x^l y^p z^q \\
&= 1 + \sum_{l,p\geq 1}\sum_{q=1}^{p}\sum_{k=0}^{\min\{\lfloor\frac{l}{3}\rfloor,p-q,q\}} \partial_{(\xi,\theta,\eta)}^{(l-3k,p-q-k,q-k)} \frac{(1+\xi\theta)^{l-2k-1}}{(1-\theta\eta)^{q+1}} \\
&\quad \times (1+\theta)^{p-q-l+2k-1}(1-2\theta\eta) x^l y^p z^q \\
&= 1 + \sum_{l,p\geq 1}\sum_{q=1}^{p}\sum_{k=\max\{0,\lceil\frac{l+q-p}{2}\rceil\}}^{\min\{\lfloor\frac{l}{3}\rfloor,q\}} \binom{l-2k-1}{l-3k} \\
&\quad \times \partial_{(\theta,\eta)}^{(p-q-l+2k,q-k)} \frac{(1+\theta)^{p-q-l+2k-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+1}} x^l y^p z^q \\
&= 1 + \sum_{l,p\geq 1}\sum_{q=1}^{p}\sum_{k=\max\{0,\lceil\frac{l+q-p}{2}\rceil\}}^{\min\{\lfloor\frac{l}{3}\rfloor,q\}} \binom{l-2k-1}{l-3k} \\
&\quad \times \left[\partial_{(\theta,\eta)}^{(p-q-l+2k,q-k)} \frac{(1+\theta)^{p-q-l+2k-1}}{(1-\theta\eta)^{q+1}}\right. \\
&\quad \left. - 2\partial_{(\theta,\eta)}^{(p-q-l+2k-1,q-k-1)} \frac{(1+\theta)^{p-q-l+2k-1}}{(1-\theta\eta)^{q+1}}\right] x^l y^p z^q \\
&= 1 + \sum_{p\geq 2}\sum_{q=1}^{\lfloor \frac{p}{2}\rfloor}\sum_{l=3}^{p+q}\sum_{k=\max\{1,\lceil\frac{l+2q-p}{3}\rceil\}}^{\min\{\lfloor\frac{l}{3}\rceil,q\}} \frac{(2q-k-1)!k}{(q-k)!q!}\binom{l-2k-1}{l-3k} \\
&\quad \times \partial_{\theta}^{p-2q-l+3k}(1+\theta)^{p-q-l+2k-1} x^l y^p z^q \\
&= 1 + \sum_{p\geq 2}\sum_{q=1}^{\lfloor \frac{p}{2}\rfloor}\sum_{l=3}^{p+q}\sum_{k=\max\{1,\lceil\frac{l+2q-p}{3}\rceil\}}^{\min\{\lfloor\frac{l}{3}\rceil,q\}} \frac{(2q-k-1)!k}{(q-k)!q!}\binom{l-2k-1}{l-3k} \\
&\quad \times \binom{p-q-l+2k-1}{p-2q-l+3k} x^l y^p z^q,
\end{aligned}$$

which is what we wanted. □

Finally, we give a corollary of Theorem 3.2.

Corollary 3.2 *The enumerating function $g = g_{\mathscr{E}}(x,y)$ has the following explicit expression:*

$$g_{\mathscr{E}}(x,y) = 1 + \sum_{n\geq 3}\sum_{l=3}^{n}\sum_{q=1}^{\lfloor \frac{n}{3}\rfloor}\sum_{k=\max\{1,\lceil\frac{l+3q-n}{3}\rceil\}}^{\min\{\lfloor\frac{l}{3}\rceil,q\}} \frac{(2q-k-1)!k}{(q-k)!q!}\binom{l-2k-1}{l-3k} \times \binom{n-2q-l+2k-1}{n-3q-l+3k} x^l y^n. \tag{24}$$

Proof It follows soon from (23) by putting $x = y$ and $n = p + q$. □

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