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Distance Two Labeling of Generalized Cacti

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Abstract: A distance two labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$. The $L(2,1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-labeling with $\max\{f(v): v \in V(G)\}=k$. Here we introduce a new graph family called generalized cactus and investigate the λ-number for the same.

Key Words: Channel assignment, interference, distance two labeling, block, cactus.

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§1. Introduction

In a communication network, the main task is to assign a channel (non negative integer) to each TV or radio transmitters located at different places such that communication do not interfere. This problem is known as channel assignment problem which was introduced by Hale [4]. Usually, the interference between two transmitters is closely related with the geographic location of the transmitters. If we consider two level interference namely major and minor then two transmitters are *very close* if the interference is major while *close* if the interference is minor. Robert [7] proposed a variation of the channel assignment problem in which *close* transmitters must receive different channels and *very close* transmitters must receive channels that are at two apart.

In a graph model of this problem, the transmitters are represented by the vertices of a graph; two vertices are *very close* if they are adjacent and *close* if they are at distance two apart in the graph. Motivated through this problem Griggs and Yeh [3] introduced $L(2,1)$-labeling which is defined as follows:

Definition 1.1 *A distance two labeling (or $L(2,1)$-labeling) of a graph $G=(V(G),E(G))$ is a function f from vertex set $V(G)$ to the set of all nonnegative integers such that the following conditions are satisfied:*

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(1) $|f(x) - f(y)| \geq 2$ *if* $d(x, y) = 1$;

(2) $|f(x) - f(y)| \geq 1$ *if* $d(x, y) = 2$.

The span of f is defined as $\max\{| f(u) - f(v) |: u, v \in V(G)\}$. The λ-number for a graph G, denoted by $\lambda(G)$ which is the minimum span of a distance two labeling for G. The $L(2,1)$-labeling is studied in the past two decades by many researchers like Yeh [14]-[15], Georges and Mauro [2], Sakai [8], Chang and Kuo [1], Kuo and Yan [5], Lu et al. [6], Shao and Yeh [9], Wang [12], Vaidya et al. [10] and by Vaidya and Bantva [11].

We begin with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For the graph G, Δ denotes the maximum degree of the graph and $N(v)$ denotes the neighborhood of v. Also in the discussion of distance two labeling [0, k] denotes the set of integers $\{0, 1, \cdots, k\}$. For all other standard terminology and notations we refer to West [13]. Now we will state some existing results for ready reference.

Proposition 1.2([1]) *$\lambda(H) \leq \lambda(G)$, for any subgraph H of a graph G.*

Proposition 1.3([14]) *The λ-number of a star $K_{1,\Delta}$ is $\Delta+1$, where Δ is the maximum degree.*

Proposition 1.4([1]) *If $\lambda(G) = \Delta + 1$ then $f(v) = 0$ or $\Delta + 1$ for any $\lambda(G)$-$L(2,1)$-labeling f and any vertex v of maximum degree Δ. In this case, $N[v]$ contains at most two vertices of degree Δ, for any vertex $v \in V(G)$.*

§2. Main Results

The problem of labeling of trees with a condition at distance two remained the focus of many research papers as its λ-number depends upon the maximum degree of a vertex. In [3], Griggs and Yeh proved that the λ-number of any tree T with maximum degree Δ is either $\Delta+1$ or $\Delta+2$. They obtained the λ-number by first-fit greedy algorithm. Later, trees are classified according to their λ-numbers. The trees with λ-number $\Delta + 1$ are classified as *class one* otherwise they are of *class two.* Earlier it was conjectured that the classification problem is NP-complete but Chang and Kuo [1] presented a polynomial time classification algorithm. But even today the classification of trees of class two is an open problem. Motivated through this problem, we present here a graph family which is not a tree but its λ-number is either $\Delta + 1$ or $\Delta + 2$ and it is a super graph of tree.

A block of a graph G is a maximal connected subgraph of G that has no cut-vertex. An n-complete cactus is a simple graph whose all the blocks are isomorphic to K_n. We denote it by $C(K_n)$. An n-complete k-regular cactus is an n-complete cactus in which each cut vertex is exactly in k blocks. We denote it by $C(K_n(k))$. The block which contains only one cut vertex is called leaf block and that cut vertex is known as leaf block cut vertex. We illustrate the definition by means of following example.

Example 2.1 *A 3-complete cactus $C(K_3)$ and 3-complete 3-regular cactus are shown in Fig.1 and Fig.2 respectively.*

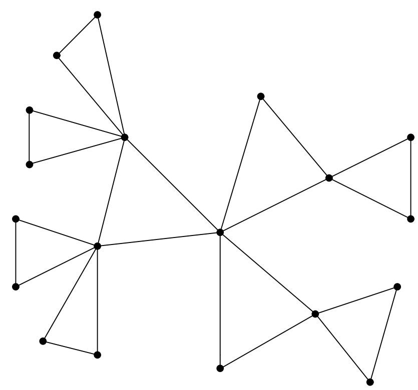

Fig.1

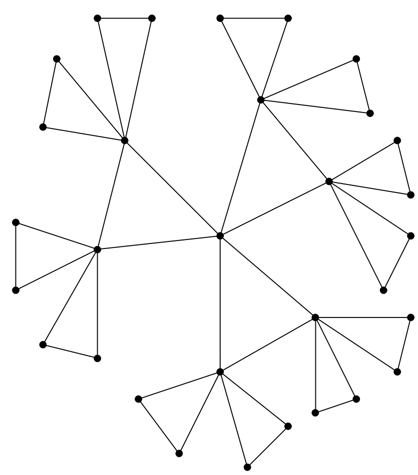

Fig.2

Theorem 2.2 *Let $C(K_n(k))$ be an n-complete k-regular cactus with maximum degree Δ and $k \geq 3$. Then $\lambda(C(K_n(k)))$ is either $\Delta + 1$ or $\Delta + 2$.*

Proof Let $C(K_n(k))$ be an n-complete k-regular cactus with maximum degree Δ. The star $K_{1,\Delta}$ is a subgraph of $C(K_n(k))$ and hence $\lambda(C(K_n(k))) \geq \Delta + 1$.

For upper bound, we apply the following Algorithm:

Algorithm 2.3 The L(2,1)-labeling of given n-complete k-regular cactus.

Input An n-complete k-regular cactus graph with maximum degree Δ.

Idea Identify the vertices which are at distance one and two apart.

Initialization Let v_0 be the vertex of degree Δ. Label the vertex v_0 by 0 and take S = $\{v_0\}$.

Iteration Define $f : V(G) \rightarrow \{0,1,2,...\}$ as follows.

Step 1 Find N(v_0). If N(v_0) = $\{v_1, v_2, ..., v_\Delta\}$ then partition N(v_0) into k sets V_1, V_2, ... ,V_k such that for each i = 1,2,...,k the graph induced by $V_i \cup \{v_0\}$ forms a complete subgraph of $C(K_n(k))$. The definition of $C(K_n(k))$ itself confirms the existence of such partition with the characteristic that for $i \neq j$, $u \in V_i$, $v \in V_j$, $d(u, v) = 2$.

Step 2 Choose a vertex $v_1 \in N(v_0)$ and define $f(v_1) = 2$. Find a vertex $v_2 \in N(v_0)$ such that $d(v_1, v_2) = 2$ and define $f(v_2) = 3$. Continue this process until all the vertices of $N(v_0)$ are labeled. Take $S = \{v_0\} \cup \{v \in V(G)/f(v)$ is a label of $v\}$.

Step 3 For $f(v_i) = i$. Find $N(v_i)$ and define $f(v)$ = the smallest number from the set $\{0, 1, 2, \cdots\} - \{i-1, i, i+1\}$, where $v \in N(v_i) - S$ such that $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$. Denote $S \cup \{v \in V(G)/f(v)$ is a label of $v\} = S^1$.

Step 4 Continue this recursive process till S^n = V(G), where $S^n = S^{n-1} \cup \{v \in V(G)/f(v)$ is a label of $v\}$.

Output $\max\{f(v)/v \in V(G)\} = \Delta + 2$.

Hence, $\lambda(C(K_n(k))) \leq \Delta + 2$.Thus, $\lambda(C(K_n(k)))$ is either $\Delta + 1$ or $\Delta + 2$. □

Example 2.4 In Fig.3, the $L(2, 1)$-labeling of 3-complete 3-regular cactus $C(K_3(3))$ is shown for which $\lambda(C(K_3(3))) = \Delta + 2 = 8$.

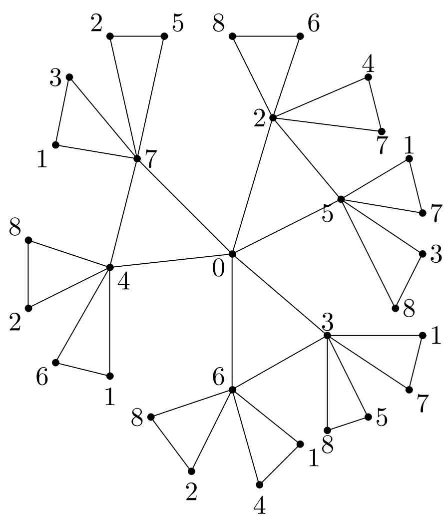

Fig.3

Griggs and Yeh [3] have proved that:

(1) $\lambda(P_2) = 2$;

(2) $\lambda(P_3) = \lambda(P_4) = 3$, and (3) $\lambda(P_n) = 4$, for $n \geq 5$. This can be verified by Proposition 1.4 and using our Algorithm 2.3. In fact, any path P_n is 2-complete 2-regular cactus $C(K_2(2))$. Thus a single Algorithm will work to determine the λ-number of path P_n. Using Algorithm 2.3 the $L(2, 1)$-labeling of P_2, P_3, P_4 and P_5 is demonstrated in Fig.4.

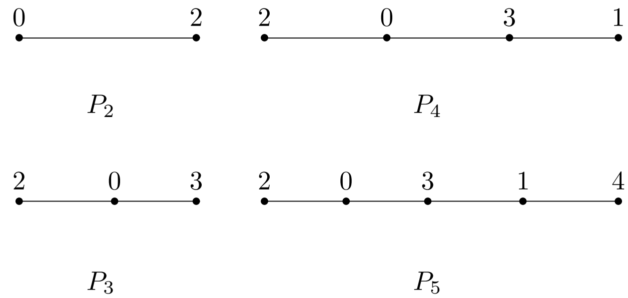

Fig.4

Theorem 2.5 *Let $C(K_n)$ be an n-complete cactus with at least one cut vertex which belongs to at least three blocks. Then $\lambda(C(K_n))$ is either $\Delta+1$ or $\Delta+2$.*

Proof Let $C(K_n)$ be the arbitrary an n-complete cactus with maximum degree Δ. The graph $K_{1,\Delta}$ is a subgraph of $C(K_n)$ and hence by Propositions 1.2 and 1.3 $\lambda(C(K_n)) \geq \Delta+1$. Moreover $C(K_n)$ is a subgraph of $C(K_n(k))$ (where k is $\frac{\Delta}{n-1}$) and hence by Proposition 1.2 and Theorem 2.2, $\lambda(C(K_n)) \leq \Delta+2$. Thus, we proved that $\lambda(C(K_n))$ is either $\Delta+1$ or $\Delta+2$. $\square$

Now as a corollary of above result it is easy to show that the λ-number of any tree with maximum degree Δ is either $\Delta+1$ or $\Delta+2$. We also present some other graph families as a particular case of above graph families whose λ-number is either $\Delta+1$ or $\Delta+2$.

Corollary 2.6 *Let T be a tree with maximum degree $\Delta \geq 2$. Then $\lambda(T)$ is either $\Delta+1$ or $\Delta+2$.*

Proof Let T be a tree with maximum degree $\Delta \geq 2$. If $\Delta = 2$ then T is a path and problem is settled. But if $\Delta > 2$ then $\lambda(T) \geq \Delta+1$ as $K_{1,\Delta}$ is a subgraph of T. The upper bound of λ-number is $\Delta+2$ according to Theorem 2.5 as any tree T is a 2-complete cactus $C(K_2)$. Thus, we proved that $\lambda(T)$ is either $\Delta+1$ or $\Delta+2$. $\square$

Example 2.7 In Fig.5, the $L(2,1)$-labeling of tree T is shown which is 2-complete cactus $C(K_2)$ with maximum degree $\Delta = 3$ for which $\lambda(T) = \lambda(C(K_2)) = \Delta+2 = 5$.

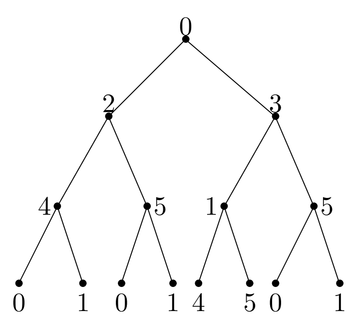

Fig.5

Corollary 2.8 $\lambda(K_{1,n}) = n+1$.

Proof The star $K_{1,n}$ is a 2-complete n-regular cactus. Then by Theorem 2.2, $\lambda(K_{1,n}) = n+1$. □

Corollary 2.9 *For the Friendship graph F_n, $\lambda(F_n) = 2n+1$.*

Proof The Friendship graph F_n is a 3-complete $2n$-regular cactus. Then by Theorem 2.2, $\lambda(F_n) = 2n+1$. □

Example 2.10 In Fig.6 and Fig.7, the $L(2,1)$-labeling of star $K_{1,4}$ and Friendship graph F_4 are shown for which λ-number is 5 and 9 respectively.

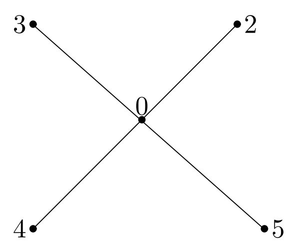

Fig.6

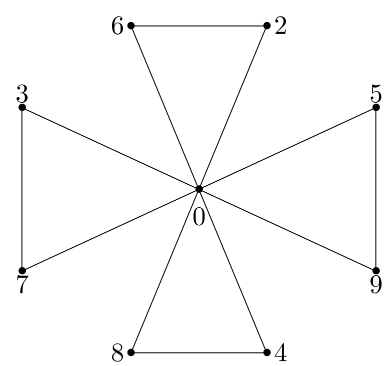

Fig.7

§3. Concluding Remarks

We have achieved the λ-number of an n-complete k-regular cactus. The λ-numbers of some standard graphs determined earlier by Griggs and Yeh [3] can be obtained as particular cases of our results which is the salient feature of our investigations.

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