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Degree Splitting Graph on Graceful, Felicitous and Elegant Labeling

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Abstract: We show that the degree splitting graphs of $B_{n,n}; P_n; K_{m,n}; n(k_4 - 3e)I; n(k_4 - 3e)II(b); n(k_4 - e)II$ and $n(k_4 - 2e)II(a)$ are graceful [3]. We prove $C_3\hat{O}K_{1,n}$ is graceful, felicitous and elegant [2], Also we prove $K_{2,n}$ is felicitous and elegant.

Key Words: Degree splitting graph, graceful graph, elegant graph, felicitous graph, star and path.

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§1. Introduction

Graph labeling methods were introduced by Rosa in 1967 or one given by Graham and Sloane in 1980. For a graph G, the splitting graph $S(G)$ is obtained from G by adding for each vertex v of G, a new vertex v' so that v' is adjacent to every vertex in G.

Let G be a graph with q edges. A graceful labeling of G is an injection from the set of its vertices into the set $\{0, 1, 2, \cdots q\}$ such that the values of the edges are all the numbers from 1 to q, the value of an edge being the absolute value of the difference between the numbers attributed to their end vertices.

In 1981 Chang, Hiu and Rogers defined an elegant labeling of a graph G, with p vertices and q edges as an injective function from the vertices of G to the set $\{0, 1, 2, \cdots q\}$ such that when each edge xy is assigned by the label $f(x) + f(y)(mod\ (q + 1))$, the resulting edge labels are distinct and non zero. Note that the elegant labeling is in contrast to the definition of a harmonious labeling [1].

Another generalization of harmonious labeling is felicitous labeling. An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, 2, \cdots q\}$ is called felicitous labeling if the edge label induced by $f(x) + f(y)(mod\, q)$ for each edge xy is distinct.

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§2. Degree Splitting Graph $DS(G)$

Definition 2.1 *Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \cdots S_i \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree and $T = V \setminus \cup S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, w_3, \cdots, w_t$ and joining to each vertex of S_i for $1 \leq i \leq t$.*

§3. Main Theorems

Theorem 3.1 *The $DS(B_{n,n})$ is graceful for $n \geq 2$.*

Proof Let $G = B_{n,n}$ be a graph.Let $V(G) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{uv, uw_2, vw_2\} \cup \{uu_i, vv_i, w_1u_i, w_1v_i : 1 \leq i \leq n\}$ and $| E(DS(G)) |= 4n + 3$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 4n + 3\}$ is as follows:

$f(u) = 1; f(v) = 3; f(w_1) = 0; f(w_2) = 2n + 4; f(u_i) = 4n - 2i + 5$ and $f(v_i) = 2i + 2$ for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The edge label of uv is 2; uu_i is $4n - 2i + 4$ for $1 \leq i \leq n; vv_i$ is $2i - 1$ for $1 \leq i \leq n; w_1u_i$ is $4n - 2i + 5$ for $1 \leq i \leq n; w_1v_i$ is $2i + 2$ for $1 \leq i \leq n; uw_2$ is $2n + 3$ and vw_2 is $2n + 1$. Hence the induced edge labels of $DS(G)$ are $4n + 3$ distinct integers. Hence $DS(G)$ is graceful for $n \geq 2$. □

Theorem 3.2 *The $DS(P_n)$ is graceful for $n \geq 4$.*

Proof Let $G = P_n$ be a graph. Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{w_1v_1, w_1v_n\} \cup \{w_2v_i : 2 \leq i \leq n - 1\}$ and $| E(DS(G)) |= 2n - 1$.

The required vertex labeling $f : V(DS(G) \to \{0, 1, 2, \cdots, 2n - 1\}$ is as follows:

Case 1 n is odd.

Then $f(w_1) = n + 1; f(w_2) = 0; f(v_i) = i$ for $1 \leq i \leq n$, i is odd and $f(v_i) = 2n - i + 1$ for $1 \leq i \leq n$, i is even.

The corresponding edge labels are as follows:

The edge label of w_2v_i is i for $3 \leq i \leq n - 1$ and i is odd; w_2v_i is $2n - i + 1$ for $1 \leq i \leq n$ and i is even; w_1v_1 is n; w_1v_n is 1 and v_iv_{i+1} is $2n - 2i$ for $1 \leq i \leq n - 1$. Hence the induced edge labels of $DS(G)$ are $2n - 1$ distinct integers. Hence $DS(G)$ for $n \geq 4$ is graceful.

Case 2 n is even.

The required vertex labeling is as follows:

$f(w_1) = n + 2; f(w_2) = 0; f(v_i) = i$ for $1 \leq i \leq n, i$ is odd and $f(v_i) = 2n - i + 1$ for

$1 \leq i \leq n, i$ is even.

The corresponding edge labels are as follows:

The edge label of w_2v_i is i for $3 \leq i \leq n$ and i is odd; w_2v_i is $2n-i+1$ for $1 \leq i \leq n-1$ and i is even; w_1v_1 is $n+1$; w_1v_n is 1 and v_iv_{i+1} is $2n-2$ for $1 \leq i \leq n-1$. Hence the induced edge labels of G are $2n-1$ distinct integers. From case (i) and (ii) the $DS(P_n)$ for $n \geq 4$ is graceful. □

Theorem 3.3 *The graph $DS(K_{m,n})$ is graceful.*

Proof The proof is divided into two cases following.

Case 1 $m > n$.

Let $G = K_{m,n}$ be a graph. Let $V(G) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{w_1u_i : 1 \leq i \leq m\} \cup \{w_2v_j : 1 \leq j \leq n\} \cup \{u_iv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $| E(DS(G)) |= mn + m + n$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \cdots, mn + m + n\}$ is as follows:

$f(u_i) = m(n+2-i)+n$ for $1 \leq i \leq m; f(v_j) = j$ for $1 \leq j \leq n; f(w_1) = 0$ and $f(w_2) = n+1$.

The corresponding edge labels are as follows:

The edge label of w_1u_i is $m(n+2-i)+n$ for $1 \leq i \leq m; u_iv_j$ is $m(n+2-i)+n-j$ for $1 \leq i \leq m, 1 \leq j \leq n$ and w_2v_j is $n+1-j$ for $1 \leq j \leq n$. Hence the induced edge labels of G are $mn+m+n$ distinct integers. Hence the graph $DS(K_{m,n})$ is graceful.

Case 2 $m = n$.

Let $G = K_{m,n}$ be a graph. Let $V(G) = \{u_i, v_i : 1 \leq i \leq m\}$ and $V(DS(G)) \setminus V(G) = \{w_1\}$. Let $E(DS(G)) = \{w_1u_i, w_1v_i : 1 \leq i \leq m\} \cup \{u_iv_j : 1 \leq i, j \leq m\}$ and $| E(DS(G)) |= m(m+2)$.

The required vertex labeling $f : V(DS(G)) \rightarrow \{0, 1, 2, \cdots, m(m+2)\}$ is as follows:

$f(w_1) = 0; f(u_i) = m(m+3) - mi$ for $1 \leq i \leq m$ and $f(v_i) = i$ for $1 \leq i \leq m$.

The corresponding edge labels are as follows:

The edge label of w_1v_i is i for $1 \leq i \leq m; u_iv_j$ is $m(m+3) - mi - j$ for $1 \leq i, j \leq m$ and w_1u_i is $m(m+3) - mi$ for $1 \leq i \leq m$. Hence the induced edge labels of G are $m(m+2)$ distinct integers. Hence the graph $DS(K_{m,n})$ is graceful. □

Corollary 3.4 *The $DS(K_n)$ is K_{n+1}.*

Theorem 3.5 *The $DS(n(K_4 - 3e)I))$ is graceful.*

Proof Let $G = n(K_4 - 3e)I$ be a graph. Let $V(G) = \{x, y\} \cup \{z_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_2, yw_2, xy\} \cup \{w_1z_i, xz_i, yz_i : 1 \leq i \leq n\}$

and $| E(DS(G)) |= 3n + 3$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 3n + 3\}$ is as follows:

$f(x) = 1; f(y) = 2; f(w_1) = 0; f(w_2) = 4$ and $f(z_i) = 3n + 6 - 3i$ for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The edge label of xy is $1; xw_2$ is $3; yw_2$ is $2; xz_i$ is $3n+5-3i$ for $1 \leq i \leq n; yz_i$ is $3n+4-3i$ for $1 \leq i \leq n$ and w_1z_i is $3n + 6 - 3i$ for $1 \leq i \leq n$. Hence the induced edge labels of G are $3n + 3$ distinct integers. Hence the $DS(G)$ is graceful. □

Theorem 3.6 *The $DS((n(K_4 - 3e)II(b))$ is graceful.*

Proof Let $G = n(K_4 - 3e)II(b)$ be a graph. Let $V(G) = \{x, y\} \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$ .Let $E(DS(G)) = \{xw_1, xy\} \cup \{w_1v_i, u_iv_i, yu_i, w_2u_i : 1 \leq i \leq n\}$ and $| E(DS(G)) |= 4n + 2$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 4n + 2\}$is as follows:

$f(x) = 3n + 2; f(y) = 1; f(w_1) = 0; f(w_2) = 2; f(v_i) = 3n + 2 + i$ for $1 \leq i \leq n$ and $f(u_i) = 2i + 1$ for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The edge label of w_1v_i is $3n + 2 + i$ for $1 \leq i \leq n; u_iv_i$ is $3n - i + 1$ for $1 \leq i \leq n; yu_i$ is $2i$ for $1 \leq i \leq n; w_2u_i$ is $2i - 1$ for $1 \leq i \leq n; xw_1$ is $3n + 2$ and xy is $3n + 1$. Hence the induced edge labels of G are $4n + 2$ distinct integers. Hence the graph $DS(G)$ is graceful. □

Theorem 3.7 *The $DS(n(K_4 - e)II)$ is graceful.*

Proof Let $G = n(K_4 - e)II)$ be a graph. Let $V(G) = \{x, y\} \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $(DS(G)) \setminus V(G) = \{w_1, w_2\}$ Let $E(DS(G)) = \{xw_2, yw_2\} \cup \{w_1u_i, w_1v_i$ for $1 \leq i \leq n\}$ and $| E(DS(G)) |= 6n + 3$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 6n + 3\}$ is as follows:

$f(x) = 0; f(y) = 4n + 2; f(w_1) = 2n + 2; f(w_2) = 4n + 3; f(v_i) = 5n + 4 - i$ and $f(u_i) = 6n + 4 - i$ for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The edge label of w_1u_i is $4n + 2 - i$ for $1 \leq i \leq n; w_1v_i$ is $4n - 1 - i$ for $1 \leq i \leq n; xw_2$ is $4n + 3; xy$ is $4n + 2$ and yw_2 is $2n$. Hence the induced edge labels of G are $6n + 3$ distinct integers. The $DS(n(K_4 - e)II)$ is graceful. □

Theorem 3.8 *The $DS(n(K_4 - 2e)II(a))$ is graceful.*

Proof Let $G = n(K_4 - 2e)II(a)$ be a graph. Let $V(G) = \{x, y, u_i, v_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xu_i, xy, yu_i, xv_i, v_iw_1, u_iw_2 : 1 \leq i \leq n\}$ and $| E(DS(G)) |= 5n + 1$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 5n + 1\}$ is as follows:

$f(x) = 0; f(y) = 2n+1; f(w_1) = 1; f(w_2) = n+1; f(v_i) = 5n+3-2i$ and $f(u_i) = 3n+2-i$ for $1 \le i \le n$.

The corresponding edge labels are as follows:

The edge label of xu_i is $3n-i+2$ for $1 \le i \le n$; xy is $2n+1$; yu_i is $n-i+1$ for $1 \le i \le n$; xv_i is $5n+3-2i$ for $1 \le i \le n$; v_iw_1 is $5n+2-2i$ for $1 \le i \le n$ and u_iw_2 is $2n-i+1$ for $1 \le i \le n$. Hence the induced edge labels of G are $5n+1$ distinct integers. The $DS(n(K_4-2e)II(a)))$ is graceful. □

Theorem 3.9 *The $DS(C_3\hat{O}K_{1,n})$ is graceful for $n \ge 3$.*

Proof Let graph $G = C_3\hat{O}K_{1,n}$ be a graph. Let $V(K_{1,n}) = \{z\} \cup \{u_i : 1 \le i \le n\}$ and C_3 be the cycle $xyzx$. Let $V(DS(G))\setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_2, yw_2, xy, yz, zx\} \cup \{w_1u_i, zu_i : 1 \le i \le n\}$and $|\ E(DS(G))\ | = 2n + 5$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 2n + 5\}$ is as follows:

$f(w_1) = 1; f(w_2) = 2; f(x) = 4; f(y) = 5; f(z) = 0$ and $f(u_i) = 2i + 5$ for $1 \le i \le n$.

The corresponding edge labels are as follows:

The edge label of xy is 1; xw_2 is 2; yw_2 is 3; xz is 4; yz is 5; zu_i is $2i + 5$ for $1 \le i \le n$ and w_1u_i is $2i + 4$ for $1 \le i \le n$. Hence the induced edge labels of G are $2n + 5$ distinct integers. Hence the $DS(G)$ is graceful for $n \ge 3$. □

Theorem 3.10 *The $DS(C_3\hat{O}K_{1,n})$ is felicitous when $n \ge 3$.*

Proof Let $G = C_3\hat{O}K_{1,n}$ be a graph. Let $V(K_{1,n}) = \{z\} \cup \{u_i : 1 \le i \le n\}$ and C_3 be the cycle $xyzx$. Let $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_2, yw_2, xy, yz, zx\} \cup \{w_1u_i, zu_i : 1 \le i \le n\}$ and $|\ E(DS(G))\ |= 2n + 5$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 2n + 5\}$ is as follows:

$f(w_1) = n; f(w_2) = 2n + 2; f(x) = 2n + 3; f(y) = 2n + 4; f(z) = 2n + 5$ and $f(u_i) = i - 1$ for $1 \le i \le n$.

The corresponding edge labels are as follows:

The labels of the edges xy is $(4n + 7)(mod\ 2n + 5)$; xw_2 is $(4n + 5)(mod\ \ 2n + 5)$; yw_2 is $(4n + 6)(mod\ 2n + 5)$; xz is $(4n + 8)(mod\ 2n + 5)$; yz is $(4n + 9)(mod\ 2n + 5)$; zu_i is $i - 1$ for $1 \le i \le n$ and w_1u_i is $n + i - 1$ for $1 \le i \le n$. Hence the induced edge labels of G are $2n + 5$ distinct integers. Hence the $DS(C_3\hat{O}K_{1,n})$ is a felicitous for $n \ge 3$. □

Theorem 3.11 *The $DS(C_3\hat{O}K_{1,n}$ is elegant for $n \ge 3$.*

Proof Let $G = C_3\hat{O}K_{1,n}$ be a graph. Let $V(K_{1,n}) = \{z\} \cup \{u_i : 1 \le i \le n\}$ and C_3 be the cycle $xyzx$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{xw_2, yw_2, xy, yz, zx\} \cup$

$\{w_1u_i, zu_i : 1 \leq i \leq n\}$ and $| E(DS(G) |= 2n+5$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 2n+5\}$ is as follows:

$f(w_1) = n+2; f(w_2) = 2n+4; f(x) = n+3; f(y) = 0; f(z) = 2n+5$ and $f(u_i) = i+1$; for $1 \leq i \leq n$.

The corresponding edge labels are as follows:

The edge labels of xy is $(n+3)$; xw_2 is $(3n+7)(mod\, 2n+6)$; yw_2 is $(2n+4)(mod\, 2n+6)$; xz is $(3n+8)(mod\ 2n+6)$; yz is $(2n+5)(mod\ 2n+6)$; zu_i is $(2n+6+i)(mod\ 2n+6)$ for $1 \leq i \leq n$; w_1u_i is $(n+3+i)(mod\, 2n+6)$ $for\ 1 \leq i \leq n$. Hence the induced edge labels of G are $2n+5$ distinct integers. Hence the $DS(G)$ is elegant for $n \geq 3$. □

Theorem 3.12 *The* $DS(K_{2,n})$ *is felicitous for* $n \geq 3$.

Proof Let $G = (K_{2,n})$ be a graph. Let $V(G) = \{v_1, v_2\} \cup \{u_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{w_2v_1, w_2v_2\} \cup \{w_1u_i : 1 \leq i \leq n\}$ and $| E(DS(G)) |= 3n+2$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 3n+2\}$ is as follows:

$f(u_i) = 3n+5-3i$ for $1 \leq i \leq n; f(v_1) = 0; f(v_2) = 3n+1; f(w_1) = 1$ and $f(w_2) = 3$.

The corresponding edge labels are as follows:

The edge label of w_1u_i is $(3n+6-3i)(mod\ 3n+2)$ for$1 \leq i \leq n; u_iv_1$ is $(3n+5-3i)(mod\ 3n+2)$ for $1 \leq i \leq n; u_iv_2$ is $(6n+6-3i)(mod\ 3n+2)$ for $1 \leq i \leq n; w_2v_1$ is 3 and w_2v_2 is $3n+4(mod\ 3n+2)$. Hence the induced edge labels of G are $3n+2$ distinct integers. Hence $DS(K_{2,n})$ is felicitous for $n \geq 3$. □

Theorem 3.13 *The* $DS(K_{2,n})$ *is elegant for* $n \geq 3$.

Proof Let $G = (K_{2,n})$ be a graph, Let$V(G) = \{v_1, v_2\} \cup \{u_i : 1 \leq i \leq n\}$ and $V(DS(G)) \setminus V(G) = \{w_1, w_2\}$. Let $E(DS(G)) = \{w_2v_1, w_2v_2\} \cup \{w_1u_i : 1 \leq i \leq n\}$ and $| E(DS(G)) |= 3n+2$.

The required vertex labeling $f : V(DS(G)) \to \{0, 1, 2, \cdots, 3n+2\}$ is as follows:

$f(u_i) = 3n+5-3i$ for $1 \leq i \leq n; f(v_1) = 0; f(v_2) = 3n+1; f(w_1) = 2$ and $f(w_2) = 4$.

The corresponding edge labels are as follows:

The edge label of w_1u_i is $(3n+7-3i)(mod\ 3n+3)$ for $1 \leq i \leq n; u_iv_1$ is $(3n+5-3i)(mod\ 3n+3)$ for $1 \leq i \leq n$; u_iv_2 is $(6n+6-3i)(mod\ 3n+3)$ for $1 \leq i \leq n; w_2v_1$ is 4 and w_2v_2 is $3n+5\,(mod\ 3n+3)$. Hence the induced edge labels of G are $3n+2$ distinct integers .The $DS(G)$ is elegant for $n \geq 3$. □

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