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Surface Embeddability of Graphs via Tree-travels

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Abstract: This paper provides a characterization for surface embeddability of a graph with any given orientable and nonorientable genus not zero via a method discovered by the author thirty years ago.

Key Words: Surface, graph,Smarandache λ^S-drawing, embeddability, tree-travel.

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§1. Introduction

A drawing of a graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache λ^S-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S, if existing, is called an embedding of G on S. Along the Kurotowski research line for determining the embeddability of a graph on a surface of genus not zero, the number of forbidden minors is greater than a hundred even for the projective plane, a nonorientable surface of genus 1 in [1].

However, this paper extends the results in [3] which is on the basis of the method established in [3-4] by the author himself for dealing with the problem on the maximum genus of a graph in 1979. Although the principle idea looks like from the joint trees, a main difference of a tree used here is not corresponding to an embedding of the graph considered.

Given a graph $G = (V, E)$, let T be a spanning tree of G. If each cotree edge is added to T as an articulate edge, what obtained is called a *protracted tree* of G, denoted by $\breve{T}$. An protracted tree $\breve{T}$ is oriented via an orientation of T or its fundamental circuits. In order to guarantee the well-definedness of the orientation for given rotation at all vertices on G and a selected vertex of T, the direction of a cotree edge is always chosen in coincidence with its direction firstly appeared along the the face boundary of $\breve{T}$. For convenience, vertices on the boundary are marked by the ordinary natural numbers as the root vertex, the starting vertex, by 0. Of course, the boundary is a travel on G, called a *tree-travel*.

In Fig.1, (a) A spanning tree T of K_5(i.e., the complete graph of order 5), as shown by bold lines; (b) the protracted tree $\breve{T}$ of T.

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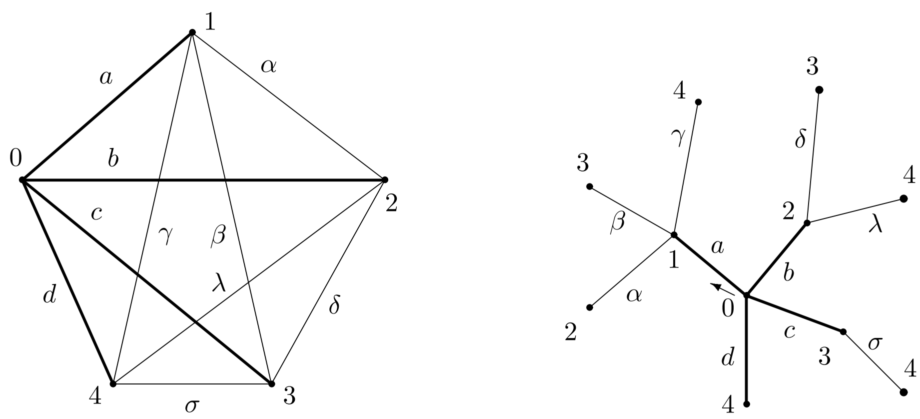


(a) T (b) $\breve{T}$

Fig.1

§2. Tree-Travels

Let $C = C(V;e)$ be the tree travel obtained from the boundary of $\breve{T}$ with 0 as the starting vertex. Apparently, the travel as a edge sequence $C = C(e)$ provides a double covering of $G = (V, E)$, denoted by

$$C(V;e) = 0P_{0,i_1}v_{i_1}P_{i_1,i_2}v_{i_2}P_{i_2,i'_1}v_{i'_1}P_{i'_1,i'_2}v_{i'_2}P_{i'_2,2\epsilon}0 \tag{1}$$

where $\epsilon = |E|$.

For a vertex-edge sequence Q as a tree-travel, denote by $[Q]_{\rm eg}$ the edge sequence induced from Q missing vertices, then $C_{\rm eg} = [C(V;e)]_{\rm eg}$ is a polyhegon(*i.e.*, a polyhedron with only one face).

Example 1 From $\breve{T}$ in Fig.1(b), obtain the thee-travel

$$C(V;e) = 0P_{0,8}0P_{8,14}0P_{14,18}0P_{18,20}0$$

where $v_0 = v_8 = v_{14} = v_{18} = v_{20} = 0$ and

$$P_{0,8} = a1\alpha2\alpha^{-1}1\beta3\beta^{-1}1\gamma4\gamma^{-1}1a^{-1};$$
$$P_{8,14} = b2\delta3\delta^{-1}2\lambda4\lambda^{-1}2b^{-1};$$
$$P_{14,18} = c3\sigma4\sigma^{-1}3c^{-1};$$
$$P_{18,20} = d4d^{-1}.$$

For natural number i, if av_ia^{-1} is a segment in C, then a is called a *reflective edge* and then v_i, the *reflective vertex* of a.

Because of nothing important for articulate vertices(1-valent vertices) and 2-valent vertices in an embedding, we are allowed to restrict ourselves only discussing graphs with neither 1-valent nor 2-valent vertices without loss of generality. From vertices of all greater than or equal to 3, we are allowed only to consider all reflective edges as on the cotree.

If v_{i_1} and v_{i_2} are both reflective vertices in (1), their reflective edges are adjacent in G and $v_{i_1'} = v_{i_1}$ and $v_{i_2'} = v_{i_2}$, $[P_{v_{i_1,i_2}}]_{\rm eg} \cap [P_{v_{i_1',i_2'}}]_{\rm eg} = \emptyset$, but neither $v_{i_1'}$ nor $v_{i_2'}$ is a reflective vertex, then the transformation from C to

$$\triangle_{v_{i_1},v_{i_2}} C(V;e) = 0P_{0,i_1} v_{i_1} P_{i_1',i_2'} v_{i_2} P_{i_2,i_1'} v_{i_1'} P_{i_1,i_2} v_{i_2'} P_{i_2',0} 0. \tag{2}$$

is called an operation of *interchange segments* for $\{v_{i_1}, v_{i_2}\}$.

Example 2 In $C = C(V;e))$ of Example 1, $v_2 = 2$ and $v_4 = 3$ are two reflective vertices, their reflective edges α and β, $v_9 = 2$ and $v_15 = 3$. For interchange segments once on C, we have

$$\triangle_{2,3} C = 0P_{0,2} 2P_{9,15} 3P_{4,9} 2P_{2,4} 3P_{15,20} 0 \ (= C_1).$$

where

$$\begin{aligned} &P_{0,2} = a1\alpha \ (= P_{1;0,2});\\ &P_{9,15} = \delta 3\delta^{-1} 2\lambda 4\lambda^{-1} 2b^{-1} 0c3 \ (= P_{1;2,8});\\ &P_{4,9} = \beta^{-1} 1\gamma 4\gamma^{-1} 1a^{-1} 0b2 \ (= P_{1;8,13});\\ &P_{2,4} = \alpha^{-1} 1\beta \ (= P_{1;13,15});\\ &P_{15,20} = \sigma 4\sigma^{-1} 3c^{-1} 0d4d^{-1} \ (= P_{1;15,20}). \end{aligned}$$

Lemma 1 Polyhegon $\triangle_{v_i,v_j} C_{\rm eg}$ is orientable if, and only if, $C_{\rm eg}$ is orientable and the genus of $\triangle_{v_{i_1},v_{i_2}} C_{\rm eg}$ is exactly 1 greater than that of $C_{\rm eg}$.

Proof Because of the invariant of orientability for $\triangle$-operation on a polyhegon, the first statement is true.

In order to prove the second statement, assume cotree edges α and β are reflective edges at vertices, respectively,v_{i_1} and v_{i_2}. Because of

$$C_{\rm eg} = A\alpha\alpha^{-1} B\beta\beta^{-1} CDE$$

where

$$\begin{aligned} &A\alpha = [P_{0,i_1}]_{\rm eg};\ \alpha^{-1}B\beta = [P_{i_1,i_2}]_{\rm eg};\\ &\beta^{-1}C = [P_{i_2,i_1'}]_{\rm eg};\ D = [P_{i_1',i_2'}]_{\rm eg};\\ &E = [P_{i_2',i_\epsilon}]_{\rm eg}, \end{aligned}$$

we have

$$\begin{aligned} \triangle_{v_{i_1},v_{i_2}} C_{\rm eg} &= A\alpha D\beta^{-1} C\alpha^{-1} B\beta E\\ &\sim_{\rm top} ABCDE\alpha\beta\alpha^{-1}\beta^{-1},\ \text{(Theorem 3.3.3 in [5])}\\ &= C_{\rm eg}\alpha\beta\alpha^{-1}\beta^{-1}\ \text{(Transform 1, in §3.1 of [5])}. \end{aligned}$$

Therefore, the second statement is true. □

If interchange segments can be done on C successively for k times, then C is called a *k-tree travel*. Since one reflective edge is reduced for each interchange of segments on C and C has at most $m = \lfloor \beta/2 \rfloor$ reflective edges, we have $0 \leqslant k \leqslant m$ where $\beta = \beta(G)$ is the Betti number(or corank) of G. When $k = m$, C is also called *normal*.

For a k-tree travel $C_k(V; e, e^{-1})$ of G, graph G_k is defined as

$$G_k = T \bigcup [E_{\rm ref} \bigcap E_{\bar{T}} - \sum_{j=1}^{k} \{e_j, e'_j\}] \tag{3}$$

where T is a spanning tree, $[X]$ represents the edge induced subgraph by edge subset X, and $e \in E_{\rm ref}$, $e \in E_{\bar{T}}$, $\{e_j, e'_j\}$ are, respectively, reflective edge, cotree edge, pair of reflective edges for interchange segments.

Example 3 On C_1 in Example 2, $v_{1;3} = 3$ and $v_{1;5} = 4$ are two reflective vertices, $v_{1;8} = 3$ and $v_{1;10} = 4$. By doing interchange segments on C_1, obtain

$$\triangle_{3,4} C_1 = 0P_{1;0,10}3P_{1;17,19}4P_{1;12,15}3P_{1;10.12}4P_{1;19,20}0 \ (= C_2)$$

where

$$P_{1;0,10} = a1\alpha 2b^{-1}0c3\beta^{-1}1\gamma 4\gamma^{-1}1a^{-1}0b2\delta (= P_{2;0,10});$$
$$P_{1;17,19} = c^{-1}0d (= P_{2;10,12});$$
$$P_{1;12.17} = \alpha^{-1}2\alpha^{-1}1\beta 3\sigma 4\sigma^{-1} (= P_{2;12,17});$$
$$P_{1;10,12} = \delta^{-1}2\lambda (= P_{2;17,19});$$
$$P_{1;19,20} = d^{-1} (= P_{2;19,20}).$$

Because of $[P_{2;6,16}]_{\rm eg} \cap [P_{2;12,19}]_{\rm eg} \neq \emptyset$ for $v_{2;12} = 4$ and $v_{2;19} = 4$, only $v_{2;6} = 4$ and $v_{2;16} = 4$ with their reflective edges γ and σ are allowed for doing interchange segments on C_2. The protracted tree $\breve{T}$ in Fig.1(b) provides a 2-tree travel C, and then a 1-tree travel as well.

However, if interchange segments are done for pairs of cotree edges as $\{\beta, \gamma\}$, $\{\delta, \lambda\}$ and $\{\alpha, \sigma\}$ in this order, it is known that C is also a 3-tree travel.

On C of Example 1, the reflective vertices of cotree edges β and γ are, respectively, $v_4 = 3$ and $v_6 = 4$, choose $4' = 15$ and $6' = 19$, we have

$$\triangle_{4,6} C = 0P_{1;0,4}3P_{1;4,8}4P_{1;8,17}3P_{1;17,19}4P_{1;19,20}0 (= C_1)$$

where

$$P_{1;0,4} = P_{0,4}; \ P_{1;4,8} = P_{15,19}; \ P_{1;8,17} = P_{6,15};$$
$$P_{1;17,19} = P_{4,6}; \ P_{1;19,20} = P_{19,20}.$$

On C_1, subindices of the reflective vertices for reflective edges δ and λ are 5 and 8, choose $5' = 17$ and $8' = 19$, find

$$\triangle_{5,8} C_1 = 0P_{2;0,5}3P_{2;5,7}4P_{2;7,16}3P_{2;16,19}4P_{2;19,20}0 (= C_2)$$

where

$$P_{2;0,12} = P_{1;0,12};\ P_{2;12,14} = P_{1;17,19};\ P_{2;14,17} = P_{1;14,17};$$
$$P_{2;17,19} = P_{1;12,14};\ P_{2;19,20} = P_{1;19,20}.$$

On C_2, subindices of the reflective vertices for reflective edges α and σ are 2 and 5, choose $2' = 18$ and $5' = 19$, find

$$\triangle_{5,8}C_2 = 0P_{3;0,2}3P_{3;2,3}4P_{3;3,16}3P_{3;16,19}4P_{3;19,20}0(= C_3)$$

where

$$P_{3;0,2} = P_{2;0,2};\ P_{2;2,3} = P_{2;18,19};\ P_{3;3,16} = P_{2;5,18};$$
$$P_{3;16,19} = P_{2;2,5};\ P_{3;19,20} = P_{2;19,20}.$$

Because of $\beta(K_5) = 6$, $m = 3 = \lfloor \beta/2 \rfloor$. Thus, the tree-travel C is normal.

This example tells us the problem of determining the maximum orientable genus of a graph can be transformed into that of determining a k-tree travel of a graph with k maximum as shown in [4].

Lemma 2 Among all k-tree travel of a graph G, the maximum of k is the maximum orientable genus $\gamma_{\max}(G)$ of G.

Proof In order to prove this lemma, the following two facts have to be known(both of them can be done via the finite recursion principle in §1.3 of [5]!).

Fact 1 In a connected graph G considered, there exists a spanning tree such that any pair of cotree edges whose fundamental circuits with vertex in common are adjacent in G.

Fact 2 For a spanning tree T with Fact 1, there exists an orientation such that on the protracted tree $\breve{T}$, no two articulate subvertices(articulate vertices of T) with odd out-degree of cotree have a path in the cotree.

Because of that if two cotree edges for a tree are with their fundamental circuits without vertex in common then they for any other tree are with their fundamental circuits without vertex in common as well, Fact 1 enables us to find a spanning tree with number of pairs of adjacent cotree edges as much as possible and Fact 2 enables us to find an orientation such that the number of times for dong interchange segments successively as much as possible. From Lemma 1, the lemma can be done. □

§3. Tree-Travel Theorems

The purpose of what follows is for characterizing the embeddability of a graph on a surface of genus not necessary to be zero via k-tree travels.

Theorem 1 *A graph G can be embedded into an orientable surface of genus k if, and only if, there exists a k-tree travel $C_k(V;e)$ such that G_k is planar.*

Proof Necessity. Let $\mu(G)$ be an embedding of G on an orientable surface of genus k. From Lemma 2, $\mu(G)$ has a spanning tree T with its edge subsets E_0, $|E_0| = \beta(G) - 2k$, such that $\hat{G} = G - E_0$ is with exactly one face. By successively doing the inverse of interchange segments for k times, a k-tree travel is obtained on $\hat{G}$. Let K be consisted of the k pairs of cotree edge subsets. Thus, from Operation 2 in §3.3 of [5], $G_k = G - K = \hat{G} - K + E_0$ is planar.

Sufficiency. Because of G with a k-tree travel $C_k(V;e)$, Let K be consisted of the k pairs of cotree edge subsets in successively doing interchange segments for k times. Since $G_k = G - K$ is planar, By successively doing the inverse of interchange segments for k times on $C_k(V;e)$ in its planar embedding, an embedding of G on an orientable surface of genus k is obtained. □

Example 4 In Example 1, for $G = K_5$, C is a 1-tree travel for the pair of cotree edges α and β. And, $G_1 = K_5 - \{\alpha, \beta\}$ is planar. Its planar embedding is

$$\begin{aligned}
&[4\sigma^{-1}3c^{-1}0d4]_{\rm eg} = (\sigma^{-1}c^{-1}d);\\
&[4d^{-1}0a1\gamma 4]_{\rm eg} = (d^{-1}a\gamma);\\
&[3\sigma 4\lambda^{-1}2\delta 3]_{\rm eg} = (\sigma\lambda^{-1}\delta);\ [0c3\delta^{-1}2b^{-1}0]_{\rm eg} = (c\delta^{-1}b^{-1});\\
&[2\lambda 4\gamma^{-1}1a^{-1}0b2]_{\rm eg} = (\lambda\gamma^{-1}a^{-1}b).
\end{aligned}$$

By recovering $\{\alpha, \beta\}$ to G and then doing interchange segments once on C, obtain C_1. From C_1 on the basis of a planar embedding of G_1, an embedding of G on an orientable surface of genus 1(the torus) is produced as

$$\begin{aligned}
&[4\sigma^{-1}3c^{-1}0d4]_{\rm eg} = (\sigma^{-1}c^{-1}d);\ [4d^{-1}0a1\gamma 4]_{\rm eg} = (d^{-1}a\gamma);\\
&[3\sigma 4\lambda^{-1}2\delta 3\beta^{-1}1a^{-1}0b2\alpha^{-1}1\beta 3]_{\rm eg} = (\sigma\lambda^{-1}\delta\beta^{-1}a^{-1}b2\alpha^{-1}\beta);\\
&[0c3\delta^{-1}2b^{-1}0]_{\rm eg} = (c\delta^{-1}b^{-1});\ [2\lambda 4\gamma^{-1}1\alpha 2]_{\rm eg} = (\lambda\gamma^{-1}\alpha).
\end{aligned}$$

Similarly, we further discuss on nonorientable case. Let $G = (V, E)$, T a spanning tree, and

$$C(V;e) = 0P_{0,i}v_iP_{i,j}v_jP_{j,2\epsilon}0 \tag{4}$$

is the travel obtained from 0 along the boundary of protracted tree $\breve{T}$. If v_i is a reflective vertex and $v_j = v_i$, then

$$\widetilde{\triangle}_\xi C(V;e) = 0P_{0,i}v_iP_{i,j}^{-1}v_jP_{j,2\epsilon}0 \tag{5}$$

is called what is obtained by doing a *reverse segment* for the reflective vertex v_i on $C(V;e)$.

If reverse segment can be done for successively k times on C, then C is called a $\widetilde{k}$*-tree travel*. Because of one reflective edge reduced for each reverse segment and at most β reflective edges on C, we have $0 \leqslant k \leqslant \beta$ where $\beta = \beta(G)$ is the Betti number of G(or corank). When $k = \beta$, C(or G) is called *twist normal*.

Lemma 3 A connected graph is twist normal if, and only if, the graph is not a tree.

Proof Because of trees no cotree edge themselves, the reverse segment can not be done, this leads to the necessity. Conversely, because of a graph not a tree, the graph has to be with a circuit, a tree-travel has at least one reflective edge. Because of no effect to other reflective edges after doing reverse segment once for a reflective edge, reverse segment can always be done for successively $\beta = \beta(G)$ times, and hence this tree-travel is twist normal. Therefore, sufficiency holds. □

Lemma 4 Let C be obtained by doing reverse segment at least once on a tree-travel of a graph. Then the polyhegon $[\triangle_i C]_{\text{eg}}$ is nonorientable and its genus

$$\widetilde{g}([\triangle_\xi C]_{\text{eg}}) = \begin{cases} 2g(C)+1, & \text{when } C \text{ orientable;} \\ \widetilde{g}(C)+1, & \text{when } C \text{ nonorientable.} \end{cases} \tag{6}$$

Proof Although a tree-travel is orientable with genus 0 itself, after the first time of doing the reverse segment on what are obtained the nonorientability is always kept unchanged. This leads to the first conclusion. Assume C_{eg} is orientable with genus $g(C)$(in fact, only $g(C) = 0$ will be used!). Because of

$$[\triangle_i C]_{\text{eg}} = A\xi B^{-1}\xi C$$

where $[P_{0,i}]_{\text{eg}} = A\xi$, $[P_{i,j}]_{\text{eg}} = \xi^{-1}B$ and $[P_{j,\epsilon}]_{\text{eg}} = C$, From (3.1.2) in [5]

$$[\triangle_i C]_{\text{eg}} \sim_{\text{top}} ABC\xi\xi.$$

Noticing that from Operation 0 in §3.3 of [5], $C_{rseg} \sim_{\text{top}} ABC$, Lemma 3.1.1 in [5] leads to

$$\widetilde{g}([\triangle_\xi C]_{\text{eg}}) = 2g([C]_{\text{eg}}) + 1 = 2g(C) + 1.$$

Assume C_{eg} is nonorientable with genus $g(C)$. Because of

$$C_{\text{eg}} = A\xi\xi^{-1}BC \sim_{\text{top}} ABC,$$

$\widetilde{g}([\triangle_\xi C]_{\text{eg}}) = \widetilde{g}(C) + 1$. Thus, this implies the second conclusion. □

As a matter of fact, only reverse segment is enough on a tree-travel for determining the nonorientable maximum genus of a graph.

Lemma 5 Any connected graph, except only for trees, has its Betti number as the nonorientable maximum genus.

Proof From Lemmas 3-4, the conclusion can soon be done. □

For a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V;e)$ on G, the graph $G_{\widetilde{k}}$ is defined as

$$G_{\widetilde{k}} = T \bigcup [E_{\text{ref}} - \sum_{j=1}^{k} \{e_j\}] \tag{7}$$

where T is a spanning tree, $[X]$ the induced graph of edge subset X, and $e \in E_{\text{ref}}$ and $\{e_j, e'_j\}$, respectively, a reflective edge and that used for reverse segment.

Theorem 2 *A graph G can be embedded into a nonorientable surface of genus k if, and only if, G has a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V;e)$ such that $G_{\widetilde{k}}$ is planar.*

Proof From Lemma 3, for k, $1 \leqslant k \leqslant \beta(G)$, any connected graph G but tree has a $\widetilde{k}$-tree travel.

Necessity. Because of G embeddable on a nonorientable surface $S_{\widetilde{k}}$ of genus k, let $\widetilde{\mu}(G)$ be an embedding of G on $S_{\widetilde{k}}$. From Lemma 5, $\widetilde{\mu}(G)$ has a spanning tree T with cotree edge set E_0, $|E_0| = \beta(G) - k$, such that $\widetilde{G} = G - E_0$ has exactly one face. By doing the inverse of reverse segment for k times, a $\widetilde{k}$-tree travel of $\widetilde{G}$ is obtained. Let K be a set consisted of the k cotree edges. From Operation 2 in §3.3 of [5], $G_{\widetilde{k}} = G - K = \widetilde{G} - K + E_0$ is planar.

Sufficiency. Because of G with a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V; e)$, let K be the set of k cotree edges used for successively dong reverse segment. Since $G_{\widetilde{k}} = G - K$ is planar, by successively doing reverse segment for k times on $C_{\widetilde{k}}(V; e)$ in a planar embedding of $G_{\widetilde{k}}$, an embedding of G on a nonorientable surface $S_{\widetilde{k}}$ of genus k is then extracted. □

Example 5 On $K_{3,3}$, take a spanning tree T, as shown in Fig.2(a) by bold lines. In (b), given a protracted tree $\breve{T}$ of T. From $\breve{T}$, get a tree-travel

$$C = 0P_{0,11}2P_{11,15}2P_{15,0}0 \ (= C_0)$$

where $v_0 = v_{18}$ and

$$P_{0,11} = c4\delta5\delta^{-1}4\gamma3\gamma^{-1}4c^{-1}0d2e3\beta1\beta^{-1}3e^{-1};$$
$$P_{11,15} = d^{-1}0a1b5\alpha;$$
$$P_{15,0} = \alpha^{-1}5b^{-1}1a^{-1}.$$

Because of $v_{15} = 2$ as the reflective vertex of cotree edge α and $v_{11} = v_{15}$,

$$\triangle_3 C_0 = 0P_{1;0,11}2P_{1;11,15}2P_{1;15,0}0 \ (= C_1)$$

where

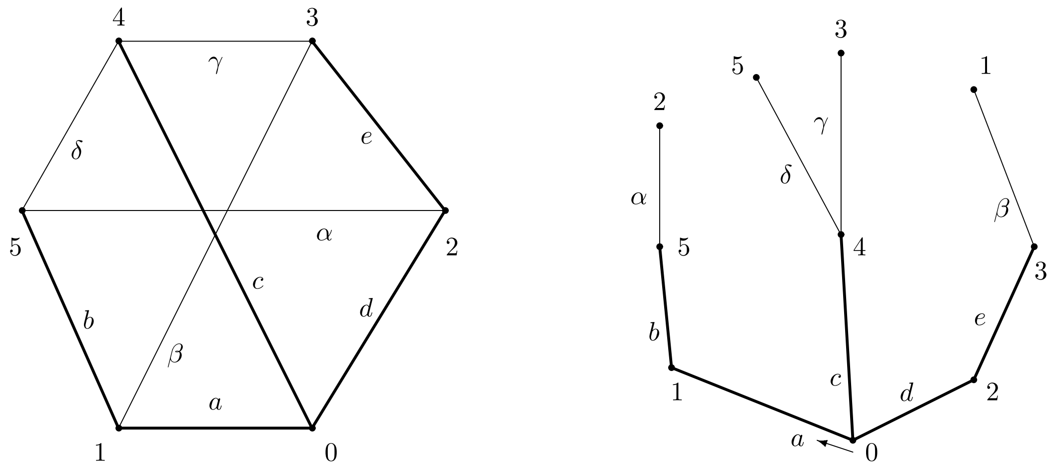
$$P_{1;0,11} = P_{0,11} = c4\delta5\delta^{-1}4\gamma3\gamma^{-1}4c^{-1}0d2e3\beta1\beta^{-1}3e^{-1};$$
$$P_{1;11,15} = P_{11,15}^{-1} = \alpha^{-1}5b^{-1}1a^{-1}0d;$$
$$P_{1;15,0} = P_{15,0} = \alpha^{-1}5b^{-1}1a^{-1}.$$

Since $G_{\widetilde{1}} = K_{3,3} - \alpha$ is planar, from C_0 we have its planar embedding

$$f_1 = [5P_{16,0}0P_{0,2}0]_{\rm eg} = (b^{-1}a^{-1}c\delta);$$
$$f_2 = [3P_{4,8}3]_{\rm eg} = (\gamma^{-1}c^{-1}de);$$
$$f_3 = [1P_{13,14}5P_{2,4}3P_{8,9}1]_{\rm eg} = (\delta^{-1}\gamma\beta b);$$
$$f_4 = [1P_{9,13}1]_{\rm eg} = (\beta^{-1}e^{-1}d^{-1}a).$$

By doing reverse segment on C_0, get C_1. On this basis, an embedding of $K_{3,3}$ on the projective plane(*i.e.*, nonorientable surface $S_{\widetilde{1}}$ of genus 1) is obtained as

$$\begin{cases} \widetilde{f_1} = [5P_{1;16,0}0P_{1;0,2}0]_{\rm eg} = f_1 = (b^{-1}a^{-1}c\delta); \\ \widetilde{f_2} = [3P_{1;4,8}3]_{\rm eg} = f_2 = (\gamma^{-1}c^{-1}de); \\ \widetilde{f_3} = [1P_{1;9,11}2P_{1;11,13}1]_{\rm eg} = be^{-1}e^{-1}\alpha^{-1}b^{-1}); \\ \widetilde{f_4} = [0P_{1;14,15}2P_{1;15,16}5P_{1;2,4}3P_{1;8,9}1P_{1;13,14}0]_{\rm eg} \\ \quad = (d\alpha^{-1}\delta^{-1}\gamma\beta a^{-1}). \end{cases}$$



(a) T (b) $\check{T}$

Fig.2

§4. **Research Notes**

A. For the embeddability of a graph on the torus, double torus *etc* or in general orientable surfaces of genus small, more efficient characterizations are still necessary to be further contemplated on the basis of Theorem 1.

B. For the embeddability of a graph on the projective plane(1-crosscap), Klein bottle(2-crosscap), 3-crosscap *etc* or in general nonorientable surfaces of genus small, more efficient characterizations are also necessary to be further contemplated on the basis of Theorem 2.

C. Tree-travels can be extended to deal with all problems related to embedings of a graph on surfaces as joint trees in a constructive way.

References

[1] Archdeacon, D., A Kuratowski theorem for the projective plane, *J. Comb. Theory*, 5 (1981), 243–246.

[2] Liu, Yi. and Liu, Y.P., A characterization of the embeddability of graphs on the surface of given genus, *Chin. Ann. Math.*, 17B(1996), 457–462.

[3] Liu, Y.P., The nonorientable maximum genus of a graph(in Chinese with English abstract), *Scintia Sinica*, Special Issue on Math. I(1979), 191–201.

[4] Liu, Y.P., The maximum orientable genus of a graph, *Scientia Sinica*, Special Issue on Math. II(1979), 41–55.

[5] Liu, Y.P., *Topological Theory on Graphs*, USTC Press, 2008.