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# Respectable Graphs

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**Abstract**: Two graphs $X$ and $Y$ are said to be respectable to each other if $\mathcal{A}(X) = \mathcal{A}(Y)$. In this study we explore some graph theoretic and algebraic properties shared by the respectable graphs.

**Key Words**: Adjacency algebra, coherent algebra, walk regular graphs, vertex transitive graphs.

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## §1. Introduction

Let $A(X)$ (or simply $A$, if $X$ is clear from the context) be the adjacency matrix of a graph $X$. The set of all polynomials in $A$ with coefficients from the field of complex numbers $\mathbb{C}$ forms an algebra called the adjacency algebra of $X$, denoted by $\mathcal{A}(X)$. Let $\dim(\mathcal{A}(X))$ denote the dimension of $\mathcal{A}(X)$ as a vector space over $\mathbb{C}$. It is easy to see that $\dim(\mathcal{A}(X))$ is equal to degree of the minimal polynomial of $A$. Since $\dim(\mathcal{A}(X))$ is also equal $|spec(A)|$ where $spec(A)$ denote the set of all distinct eigenvalues of $A$ and $|B|$ denote the cardinality of the set $B$.

**Definition** 1 *Two graphs $X$ and $Y$ are said to be respectable to each other if $\mathcal{A}(X) = \mathcal{A}(Y)$. In this case we say that either $X$ respects $Y$ or $Y$ respects $X$.*

A graph $Y$ is said to be a polynomial in a graph $X$ if $A(Y) \in \mathcal{A}(X)$. For example, $K_n$ the complete graph is a polynomial in every connected regular graph with $n$ vertices. By definition if $X$ respects $Y$, then $X$ is a polynomial in $Y$ and $Y$ is a polynomial in $X$. In this study we explore some graph theoretic and algebraic properties shared by respectable graphs. In the remaining part of this section we will give some preliminaries required for this paper.

For two vertices $u$ and $v$ of a connected graph $X$, let $d(u, v)$ denote the length of the shortest path from $u$ to $v$. Then the diameter of a connected graph $X = (V, E)$ is $\max\{d(u, v) : u, v \in V\}$. It is shown in Biggs [3] that if $X$ is a connected graph with diameter $\ell$, then $\ell + 1 \leq \dim(\mathcal{A}(X)) \leq n$.

A graph $X_1 = (V(X_1), E(X_1))$ is said to be *isomorphic* to a graph $X_2 = (V(X_2), E(X_2))$, written $X_1 \cong X_2$, if there is a one-to-one correspondence $\rho : V(X_1) \to V(X_2)$ such that $\{v_1, v_2\} \in E(X_1)$ if and only if $\{\rho(v_1), \rho(v_2)\} \in E(X_2)$. In such a case, $\rho$ is called an *isomor-*

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*phism* of $X_1$ and $X_2$. An isomorphism of a graph $X$ onto itself is called an *automorphism*. The collection of all automorphisms of a graph $X$ is denoted by $\mathrm{Aut}(X)$. It is well known that $\mathrm{Aut}(X)$ is a group under composition of two maps. It is easy to see that $\mathrm{Aut}(X) = \mathrm{Aut}(X^c)$, where $X^c$ is the complement of the graph X. If $X$ is a graph with $n$ vertices we can think $\mathrm{Aut}(X)$ as a subgroup of $S_n$. Under this correspondence, if a graph $X$ has $n$ vertices then $\mathrm{Aut}(X)$ consists of $n \times n$ permutation matrices and for each $g \in \mathrm{Aut}(X)$, $P_g$ will denote the corresponding permutation matrix.

## §2. Graph Theoretic Properties

In this section we will see some graph theoretical properties shared by the respectable graphs. The next result gives a method to check whether a given permutation matrix is an element of $\mathrm{Aut}(X)$ or not.

**Lemma** 2.1(Biggs [3]) *Let $A$ be the adjacency matrix of a graph $X$. Then $g \in Aut(X)$ is an automorphism of $X$ if and only if $P_g A = AP_g$.*

The following result is immediate from the above lemma, also given by Paul.M.Weichsel [7].

**Corollary** 2.2 *Let $X$ be a graph and $p(x)$ be a polynomial such that $p(X)$ is a graph. Then $Aut(X) \subseteq Aut(p(X))$.*

**Corollary** 2.3 *If the graph $X$ respects the graph $Y$, then $Aut(X) = Aut(Y)$.*

**Lemma** 2.4(Biggs [3]) *A graph $X$ is regular if and only if $A(X)J = JA(X)$, where $J$ is a matrix with each entry is 1.*

The following result shows that any graph which is a polynomial in a regular graph is regular.

**Corollary** 2.5 *Let $X$ be a regular graph. The any graph which is a polynomial in $X$ is also regular . In particular if $X$ respects $Y$, then $Y$ is regular.*

**Lemma** 2.6(Biggs [3]) *A graph $X$ is connected regular if and only if $J \in \mathcal{A}(X)$.*

**Corollary** 2.7 *If $X$ is a regular graph then $J$ is polynomial in either $A$ or $A^c$.*

*Proof* For every graph $X$, either $X$ or $X^c$ is connected. Hence the result follows from the above lemma. □

**Corollary** 2.8 *Let $X$ be a connected regular graph, then $X^c$ is connected if and only if $X$ respects $X^c$.*

*Proof* It is easy to verify that $X^c$ is also regular. Since $X$ is connected regular graph from Lemma 2.6 we have $J \in \mathcal{A}(X) \Rightarrow A(X^c) = J - I - A \in \mathcal{A}(X) \Rightarrow \mathcal{A}(X^c) \subseteq \mathcal{A}(X)$. Now it is

sufficient to prove that $X^c$ is connected if and only if $\mathcal{A}(X) \subseteq \mathcal{A}(X^c)$.
$X^c$ is connected $\Leftrightarrow J \in \mathcal{A}(X^c) \Leftrightarrow A \in \mathcal{A}(X^c) \Leftrightarrow \mathcal{A}(X) \subseteq \mathcal{A}(X^c)$. □

**Corollary** 2.9 *Let $X$ be a connected regular graph. If $X$ respects $Y$, then $Y$ is connected regular graph.*

We say that a graph $X$ is walk-regular if, for each $s$, the number of closed walks of length $s$ starting at a vertex $v$ is independent of the choice of $v$.

**Theorem** 2.10([6]) *Let $A$ be the adjacency matrix of a graph $X$. Then $X$ is walk-regular if and only if the diagonal entries of $A^s$ $\forall s$ are all equal.*

**Corollary** 2.11 *Let $X$ be a walk regular graph and $p(x)$ be a polynomial such that $p(X)$ is a graph. Then $p(X)$ is walk regular.*

*Proof* Let $A$ be the adjacency matrix of $X$. From the above theorem the diagonal entries of $A^s$ $\forall s$ are all equal, so as for every element in $\mathcal{A}(X)$. As one of the basis for $\mathcal{A}(X)$ is $\{I, A, A^2, \ldots A^{l-1}\}$ where $l$ is the degree of the minimal polynomial of $A$. □

From the above result we deduce that if $X$ be a walk regular graph and $X$ respects $Y$, then $Y$ is also walk regular graph.

Now we will see some symmetrical properties shared by the respectable graphs.

**Definition** 2 *A graph $X = (V, E)$ is said to be vertex transitive if its automorphism group acts transitively on $V$. That is for any two vertices $x, y \in V, \exists g \in G$ such that $g(x) = y$.*

**Definition** 3 *A graph $X = (V, E)$ is said to be* generously transitive *if its automorphism group acts generously transitively on $V(X)$, i.e., if any $x, y \in V$ then $\exists g \in Aut(X)$ such that $g(x) = y$ and $g(y) = x$.*

Every generously transitive graph is transitive. From the Corollary 2.2 we have the following result.

**Lemma** 2.12 *If $X$ is a generously transitive (or vertex transitive) graph and $Y$ is a polynomial in $X$, then $Y$ is also a generously transitive (or vertex transitive) graph.*

## §3. Algebraic Properties

Let $X$ be a graph with $n$ vertices and $A$ be the adjacency matrix of $X$. By graph algebra of $X$, we mean a matrix subalgebra of $M_n(\mathbb{C})$ which contains $A$. For example $M_n(\mathbb{C})$ and $\mathcal{A}(X)$ are graph algebras of $X$. If the graph $X$ respects $Y$, then in this section we will show that the following three graph algebras of $X$ and $Y$ will coincide.

- The *commutant algebra* of a graph $Z$ is the set all matrices over $\mathbb{C}$ which commutes with adjacency matrix of $Z$.
- The *coherent closure* of a graph $Z$ is the smallest coherent algebra containing the adjacency matrix of $Z$.

- The *centralizer algebra* of a graph $Z$ is the set all matrices which commute with all automorphisms of $Z$.

### 3.1 Coherent Closure of a Graph

**Definition** 4 *Hadamard product of two $n \times n$ matrices $A$ and $B$ is denoted by $A \odot B$ and is defined as $(A \odot B)_{xy} = A_{xy}B_{xy}$.*

**Definition** 5 *Two $n \times n$ matrices $A$ and $B$ are said to be* disjoint *if their Hadamard product is the zero matrix.*

**Definition** 6 *A sub algebra of $M_n(\mathbb{C})$ is called coherent if it contains the matrices $I$ and $J$ and if it is closed under conjugate-transposition and Hadamard multiplication.*

The following result is well known.

**Theorem** 3.1 *Every coherent algebra contains unique basis of disjoint 0-1 matrices.*

We call the unique basis containing disjoint 0-1 matrices as a standard basis.

**Corollary** 3.2 *Every $0,1$-matrix in a coherent algebra is sum of one or more matrices in its standard basis.*

*Proof* Let $\mathcal{M}$ be a coherent algebra over $\mathbb{C}$ with standard basis $\{M_1, \dots M_t\}$. Let $B \in \mathcal{M}$ be a $0,1$-matrix. Then $B = \sum_{i=1}^{t} a_i M_i$ where $a_i \in \mathbb{C}$. $B = B \odot B = \sum_{i=1}^{t} a_i^2 M_i \Rightarrow a_i^2 = a_i$. Hence the result follows. □

**Observation** 3.3 The intersection of coherent algebras is again a coherent algebra.

**Definition** 7 *Let $X = (V, E)$ be a graph with adjacency matrix $A$ then any coherent algebra which contains $A$ is called coherent algebra of $X$.*

**Definition** 8 *If $X = (V, E)$ be a graph and $A$ is its adjacency matrix then coherent closure of $X$, denoted by $\langle\langle A \rangle\rangle$ or $\mathcal{CC}(X)$, is the smallest coherent algebra containing $A$.*

Since $A(X^c) = J - I - A(X)$ consequently $A(X), A(X^c) \in \mathcal{CC}(X) \cap \mathcal{CC}(X^c)$, hence we get the following lemma.

**Lemma** 3.4 *For every graph $X$, $\mathcal{CC}(X) = \mathcal{CC}(X^c)$.*

**Lemma** 3.5 *If the graph $X$ respects $Y$, then $\mathcal{CC}(X) = \mathcal{CC}(Y)$.*

*Proof* Since $X$ respects $Y$, we have $\mathcal{A}(X) = \mathcal{A}(Y) \subseteq \mathcal{CC}(Y)$. Consequently $\mathcal{CC}(Y)$ is a coherent algebra containing $A(X)$ but by definition $\mathcal{CC}(X)$ is the smallest coherent algebra containing $A(X)$. So $\mathcal{CC}(X) \subseteq \mathcal{CC}(Y)$. Similarly we can prove $\mathcal{CC}(Y) \subseteq \mathcal{CC}(X)$. Hence the result follows. □

Clearly, the converse of this result is not true as $\mathcal{CC}(X) = \mathcal{CC}(X^c)$, but $X$ need not respect $X^c$.

### 3.2 Centralizer Algebra of a Graph

**Definition** 9 *Let $G$ be a subset of $n \times n$ permutation matrices forming a group. Then $\mathcal{V}_{\mathbb{C}}(G) = \{A \in M_n(\mathbb{C}) : PA = AP\ \forall P \in G\}$ forms an algebra over $\mathbb{C}$ called the centralizer algebra of the group $G$.*

**Definition** 10 *If $G$ is a group acting on a set $V$, then $G$ also acts on $V \times V$ by $g(x, y) = (g(x), g(y))$. The orbits of $G$ on $V \times V$ are called orbitals. In the context of graphs, the orbitals of graph $X$ are orbitals of its automorphism group $Aut(X)$ acting on the vertex set of $X$. That is, the orbitals are the orbits of the arcs/non-arcs of the graph $X = (V, E)$. The number of orbitals is called the rank of $X$.*

An orbital can be represented by a $0-1$ matrix $M$ where $M_{ij}$ is 1 if $(i, j)$ belongs to the orbital. We can associate directed graphs to these matrices. If the matrices are symmetric, then these can be treated as undirected graphs.

**Observation 3.6**

- The '1' entries of any orbital matrix are either all on the diagonal or all are off diagonal.
- The orbitals containing 1's on the diagonal will be called diagonal orbitals.

**Definition** 11 *The centralizer algebra of a graph $X$ denoted by $\mathcal{V}(X)$ is the centralizer algebra of its automorphism group.*

**Theorem** 3.7([4]) *$\mathcal{V}_{\mathbb{C}}(G)$ is a coherent algebra and orbitals of* Aut$X$ *acting on the vertex set of $X$ form its unique 0-1 matrix basis.*

$\mathcal{V}(X) = \mathcal{V}(X^c)$ follows from the fact that $\mathrm{Aut}(X) = \mathrm{Aut}(X^c)$ . $\mathcal{CC}(X)$ is the smallest coherent algebra of $X$ and $\mathcal{V}(X)$ is a coherent algebra of $X$ so $\mathcal{CC}(X) \subseteq \mathcal{V}(X)$. So for any graph $X$ we have $\mathcal{A}(X) \subseteq \mathcal{CC}(X) \subseteq \mathcal{V}(X)$. The following result follows from the Corollary 2.3.

**Lemma** 3.8 *If the graph $X$ respects the graph $Y$, then $\mathcal{V}(X) = \mathcal{V}(Y)$.*

Now we will see a consequence of above result. For that we need the following definition.

**Definition** 12(Robert A.Beezer [1]) *A graph $X = (V, E)$ is* orbit polynomial *graph if each orbital matrix is member of $\mathcal{A}(X)$. That is each orbital matrix is a polynomial in $A$.*

**Lemma** 3.9 *$X$ is an orbit polynomial graph if and only if $\mathcal{A}(X) = \mathcal{V}(X)$.*

If $X$ is an orbit polynomial graph, then we have $\mathcal{A}(X) = \mathcal{CC}(X) = \mathcal{V}(X)$.

**Corollary** 3.10 *Let $X$ be an orbit polynomial graph and suppose $X$ respects the graph $Y$, then $Y$ is also an orbit polynomial graph.*

**Corollary** 3.11 *If $X$ is an orbit polynomial graph and $X^c$ is connected then $X^c$ is orbit polynomial graph.*

If $X$ is an orbit polynomial graph and $X^c$ is connected, then we have $\mathcal{A}(X) = \mathcal{A}(X^c) = \mathcal{CC}(X) = \mathcal{CC}(X^c) = \mathcal{V}(X) = V(X^c)$.

### 3.3 Commutant algebra of a graph

The commutant algebra of graph $X$, denoted by $C[X]$ is the set of all matrices which commutes with $A$. It is shown in (Davis [5]) that dim($C[A]$)=sum of the squares of the multiplicities of eigenvalues of $A$. Hence the following lemma.

**Lemma** 3.12 $\mathcal{A}(X) = C[X]$ *if and only if all eigenvalues of $X$ are distinct.*

**Lemma** 3.13 *If the graph $X$ respects the graph $Y$ then $C[X] = C[Y]$.*

*Proof* Notice that

$$B \in C[X] \Leftrightarrow BA(X) = A(X)B \Leftrightarrow BA(Y) = A(Y)B \Leftrightarrow B \in C[Y].$$

We get the result. □

## §4 Polynomial Equivalence

Let $\mathcal{G}_n$ be the set of all graphs with $n$ vertices. We define a relation $\mathcal{R}$ on $\mathcal{G}_n$ as $X\mathcal{R}Y \Leftrightarrow X$ respects $Y$. It is easy to see that $\mathcal{R}$ is an equivalence relation on $\mathcal{G}_n$. Now for a given graph $X$, our objective is to find the equivalence class $[X]$ under the equivalence relation $\mathcal{R}$. First we identify a set $[X]$ with a set in polynomial algebra $\mathbb{C}[x]$. For that we need the following notations and definitions.

$\mathbb{C}[A]$ denote the set of all matrices which are polynomials in the square matrix $A$. It is easy to see that $\mathbb{C}[A] \cong \mathbb{C}[x]/\langle p(x)\rangle$ where $\langle p(x)\rangle$ is the ideal in $\mathbb{C}[x]$ generated by $p(x)$, which is the minimal polynomial of $A$. Consequently if $B \in \mathbb{C}[A]$, then there exists a unique polynomial $f_B(x)$ called representor polynomial of $B$ such that $\deg(f_B(X)) \leq \deg(p(x))$ and $f_B(A) = B$.

**Definition** 13 *Let $A$ be a square matrix and $f(x)$ be a polynomial. We say that $f(x)$ respects spec(A) if $f(\lambda_i) \neq f(\lambda_j)$ for $\lambda_i$ and $\lambda_j$ distinct eigenvalues of $A$.*

The following result is given by Paul M.Weichsel [7].

**Lemma** 4.1 *Let $A$ be diagonalizable matrix over a field and $f(x) \in \mathbb{C}[x]$. Then $f(x)$ respects spec(A) if and only if there exists a polynomial $g(x) \in \mathbb{C}[x]$ such that $g(f(A)) = A$.*

*Proof* Let $B = f(A)$. Clearly $\mathbb{C}[B] \subseteq \mathbb{C}[A]$. Since $A$ is diagonalizable, so is $B$. Consequently $A$ is a polynomial in $B$ if and only if $\mathbb{C}[B] = \mathbb{C}[A]$ if and only if $|spec(A)| = |spec(B)|$ if and only if $f(x)$ respects $spec(A)$. □

Now let $A$ be the adjacency matrix of a graph $X$ and we denote

$$F_X = \{f(x) \in \mathbb{C}[x] | \deg(f(x)) \leq \deg(p(x)) \text{ and } f(A) \text{ is a 0,1-matrix}\},$$
$$H_X = \{g(x) \in F_X | g(x) \text{ respects } spec(A)\}.$$

Now one can easily verify that finding the set $[X]$ is equivalent to finding the set $H_X$. By definition, in order to find $H_X$ we need to find $F_X$ but for a given graph $X$ finding $F_X$ seems difficult. Let $X$ be a graph with the property $\mathcal{A}(X) = \mathcal{CC}(X)$, then from Corollary 3.2 it is easy to evaluate $F_X$. Distance regular graphs and orbit polynomial graphs satisfy $\mathcal{A}(X) = \mathcal{CC}(X)$ for details one can refer Robert A.Beezer [2] and Paul M.Weichsel [7].

The following theorem shows that if $X$ is a connected vertex transitive graph with a prime number of vertices then $X$ respects $Y$ if and only if $\text{Aut}(X) = \text{Aut}(Y)$.

**Theorem** 4.2(Robert A.Beezer [2]) *Suppose that $X$ is a connected, vertex transitive graph with a prime number of vertices. Let $p(x)$ be a polynomial such that $p(X)$ is a connected graph, and $Aut(X) = Aut(p(X))$. Then $p(x)$ respects $spec((A(X))$.*

**Comments** In spite of these results, there are many properties which are not shared by the respectable graphs. We illustrate few of them with examples. Let $C_n$ denote the cycle graph with $n$ vertices, then $C_n$ respects $C_n^c$ for $n \geq 5$. It is known that $C_{2n}$ is bipartite for every $n$, but $C_{2n}^c$ is not bipartite for $n \geq 3$. For $n \geq 3$, $C_{2n}$ is Eulerain graph but $C_{2n}^c$ is not. $C_n$ is planar graph $\forall n$ where as $C_n^c$ is not planar for $n \geq 9$ as every finite, simple, planar graph has a vertex of degree less than 6. Petersen graph is not Hamiltonian graph but from Dirac's theorem its compliment (respects Petersen graph) is a Hamiltonian graph.

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