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Plick Graphs with Crossing Number 1

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Abstract: In this paper, we deduce a necessary and sufficient condition for graphs whose plick graphs have crossing number 1. We also obtain a necessary and sufficient condition for plick graphs to have crossing number 1 in terms of forbidden subgraphs.

Key Words: Smarandache $\mathscr{P}$-drawing, drawing, line graph, plick graph, crossing number.

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§1. Introduction

All graphs considered here are finite, undirected and without loops or multiple edges. We refer the terminology of [2]. For any graph G, $L(G)$ denote the line graph of G.

A *Smarandache $\mathscr{P}$-drawing* of a graph G for a graphical property $\mathscr{P}$ is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$. A Smarandache $\mathscr{P}$-drawing is said to be *optimal* if $\mathscr{P} = G$ and it minimizes the number of crossings. A graph is planar if it can be drawn in the plane or on the sphere in such a way that no two of its edges intersect. *The crossing number* $cr(G)$ of a graph G is the least number of intersections of pairs of edges in any embedding of G in the plane. Obviously, G is planar if and only if $cr(G) = 0$. It is implicit that the edges in a drawing are Jordan arcs(hence, non-selfintersecting), and it is easy to see that a drawing with the minimum number of crossings(an optimal drawing) must be *good* drawing, that is, each two edges have at most one vertex in common, which is either a common end-vertex or a crossing. *Theta* is the result of adding a new edge to a cycle and it is denoted by θ. The corona G^+ of a graph G is obtained from G by attaching a path of length 1 to every vertex of G.

The *plick graph* $P(G)$ *of a graph* G is obtained from the line graph by adding a new vertex corresponding to each block of the original graph and joining this vertex to the vertices of the line graph which correspond to the edges of the block of the original graph(see[4]).

The following will be useful in the proof of our results.

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Theorem A([5]) *The line graph of a planar graph G is planar if and only if $\Delta(G) \leq 4$ and every vertex of degree 4 is a cut-vertex.*

Theorem B([3]) *Let G be a nonplanar graph. Then $cr(L(G)) = 1$ if and only if the following conditions hold:*

(1) $cr(G) = 1$;

(2) $\Delta(G) \leq 4$, *and every vertex of degree 4 is a cut-vertex of* G;

(3) *There exists a drawing of G in the plane with exactly one crossing in which each crossed edge is incident with a vertex of degree 2.*

Theorem C([3]) *The line graph of a planar graph G has crossing number one if and only if* (1) *or* (2) *holds:*

(1) $\Delta(G) = 4$ *and there is a unique non-cut-vertex of degree 4;*

(2) $\Delta(G) = 5$, *every vertex of degree 4 is a cut-vertex, there is a unique vertex of degree 5 and it has at most 3 incident edges in any block.*

Theorem D([4]) *The plick graph $P(G)$ of a graph G is planar if and only if G satisfies the following conditions:*

(1) $\Delta(G) \leq 4$, *and*

(2) *every block of G is either a cycle or a* K_2.

Theorem E([1]) *A graph has a planar ilne graph if and only if it has no subgraph homeomorphic to* $K_{3,3}$, $K_{1,5}$, $P_4 + K_1$ *or* $K_2 + \overline{K}_3$.

Remark 1([4]) For any graph, $L(G)$ is a subgraph of $P(G)$.

§2. Results

The following theorem supports the main theorem.

Theorem 1 *Let x be any edge of K_4. If G is homeomorphic to $K_4 - x$, then $cr(P(G)) = 1$.*

Proof We prove the theorem first for $G = (K_4 - x)$. One can see that the graph $P(K_4 - x)$ has 6 vertices and 13 edges. But a planar graph with 6 vertices has at most 12 edges. This shows that $P(K_4 - x)$ has crossing number at least 1. Figure 1, being drawing of $P(K_4 - x)$ concludes that $cr(P(K_4 - x)) = 1$. Suppose now G is the graph as in the statement. Referring to Figure 1, it is immediate to see that $cr(P(K_4 - x)) = 1$. □

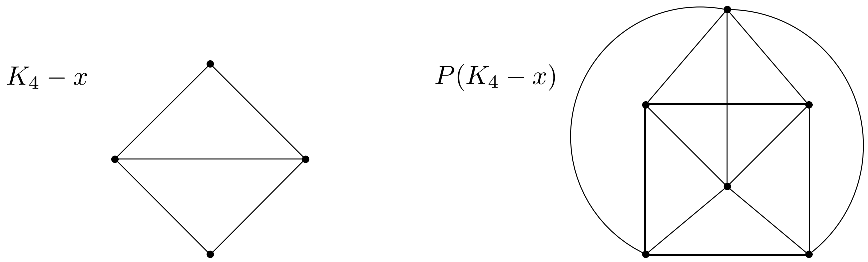

Figure 1

The following theorem gives a necessary and sufficient condition for graphs whose plick graphs have crossing number 1.

Theorem 2 *A graph G has a plick graph with crossing number 1 if and only if G is planar and one of the following holds:*

(1) $\Delta(G) = 3$, *G has exactly two non-cut-vertices of degree 3 and they are adjacent.*

(2) $\Delta(G) = 4$, *every vertex of degree 4 is a cut-vertex of G, there exists exactly one theta as a block in G such that at least one vertex of theta is a non-cut- vertex of degree 2 or 3 and every other block of G is either a cycle or a K_2.*

(3) $\Delta(G) = 5$, *G has a unique cut-vertex of degree 5 and every block of G is either a cycle or a K_2.*

Proof Suppose $P(G)$ has crossing number one. Then by Remark 1, and Theorem B, G is planar. By Theorem D, $\Delta(G) \leq 4$, then at least one block of G is neither a cycle nor a K_2.

Suppose $\Delta(G) \leq 6$. Then $K_{1,6}$ is a subgraph of G. Clearly $L(K_{1,6}) = K_6$. It is known that $cr(K_6) = 3$. By Remark 1, K_6 is a subgraph of $P(G)$ and hence $cr(P(G)) > 1$, a contradiction. This implies that $\Delta(G) \leq 5$. If $\Delta(G) \leq 2$, then $P(G)$ is planar, again a contradiction. Thus $\Delta(G) = 3$ or 4 or 5.

We now consider the following cases:

Case 1. Suppose $\Delta(G) = 3$. Then by Theorem D and since $cr(P(G)) = 1$, G has a non-cut-vertex of degree 3. Clearly G contains a subgraph homeomorphic to $K_4 - x$, so that there exist at least two non-cut-vertex of degree3. More precisely, there is an even number, say $2n$, of non-cut-vertex of degree 3. Now suppose G has at least two diagonal edges. Then there are two subcases to consider depending on whether 2 diagonal edges exist in one cycle or in two different edge disjoint cycles.

Subcase 1.1 If two diagonal edges exist in one cycle of G. Then G has a subgraph homeomorphic from K_4. The graph $P(K_4)$ has 7 vertices and 18 edges. It is known that a planar graph with 7 vertices has at most 15 edges. This shows that $P(K_4)$ must have crossing number exceeding 1 and hence $P(G)$ has crossing number greater than 1, a contradiction.

Subcase 1.2 If two diagonal edges exist in two different edge-disjoint cycles of G. Then by

Theorem 1, we see that for every subgraph of G homeomorphic to $K_4 - x$, there corresponds at least one crossing of G. Hence $P(G)$ has at least 2 crossings, a contradiction.

Hence G has exactly two non-cut-vertices of degree 3 and every other vertex of degree 3 is a cut-vertex.

Suppose a graph G has two non-cut-vertices of degree 3 and they are not adjacent. Then G contains a subgraph homeomorphic to $K_{2,3}$. On drawing $P(K_{2,3})$ in a plane one can see that $cr(P(K_{2,3})) = 2$. Since $P(K_{2,3})$ is a subgraph of $P(G)$, $P(G)$ has crossing number exceeding 1, a contradiction(see Figure 2).

Therefore, we conclude that G contains exactly two non-cut-vertices of degree 3 and these are adjacent. This proves (1).

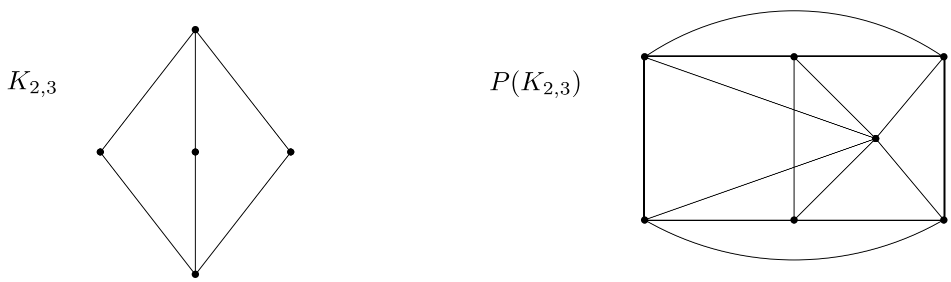

Figure 2

Case 2. Assume $\Delta(G) = 4$. We show first that every vertex of degree 4 is a cut-vertex. On the contrary suppose that G has non-cut-vertex v of degree 4. Then by Theorem C, $cr(L(G)) \geq 1$. The vertex u_1 in $P(G)$ corresponding to the block which contains a non-cut-vertex of degree 4 is adjacent to every vertex of $L(G)$. We obtain the drawing of $P(G)$ with 3 crossings.

Assume now G has at least two blocks each of which is a theta. By Theorem 1 and case 1 of this theorem, we see that for every subgraph of G homeomorphic to $K_4 - x$, there correspond to at least 2 crossings of G, a contradiction.

Suppose there exists exactly one theta S as a block in G such that none of its vertices is a non-cut-vertex of degree 2 or 3. Assume all vertices of theta S have degree 4 in G. Then by Theorem A, $L(S)$ is planar. Let v_1 be the vertex of $L(G)$ corresponding to the chord of a cycle C of theta. The vertex w_1 in $P(G)$ corresponding to the block theta S is adjacent to every vertex of $L(C)$ without crossings. In $P(G) - v_1w_1$, the vertex w_1 is adjacent to every vertex of $L(S) - v_1$ without crossings. By the definition of $P(G)$, the vertices v_1 and w_1 are adjacent in $P(G)$. The edge v_1w_1 crosses at least two edges of $L(G)$. On drawing of $P(G)$ in the plane, it has at least two crossings, a contradiction. This proves that $\Delta(G) = 4$, there exist exactly one theta as a block in G such that at least one vertex of theta is either a non-cut-vertex of degree 2 or 3.

Suppose every block of G different from theta block is neither a cycle nor a K_2. It implies that G has a block which is a subgraph homeomorphic to $K_4 - x$. By Cases 1 and 2 of this theorem, we see that for every subgraph of G homeomorphic to $K_4 - x$, there corresponds at least one crossing of G. Hence $P(G)$ has at least 2 crossings, a contradiction.

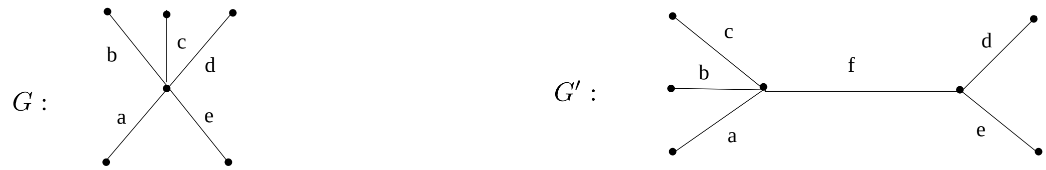

Figure 3

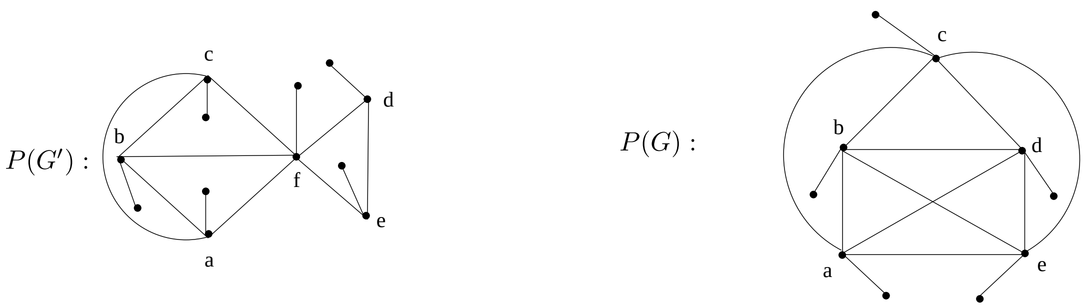

Figure 4

Case 3. Assume $\Delta(G) = 5$. Suppose G has at least two vertices of degree 5. Then by Theorem C, $L(G)$ has crossing number at least 2. By Remark 1, $cr(P(G)) \geq 2$, which is a contradiction. Thus G has a unique vertex of degree 5.

Suppose G has a vertex v of degree 5 and at least one block of G is neither a cycle nor a K_2. Then some block of G has a subgraph homeomorphic to $K_4 - x$. By Case 1 of this theorem $cr(P(K_4 - x)) \geq 1$ and the 5 edges incident to v form K_5 as a subgraph in $P(G)$. Hence $cr(P(K_4 - x)) \geq 2$, a contradiction.

Conversely, suppose G is a planar graph satisfying (1) or (2) or (3). Then by Theorem D, $P(G)$ has crossing number at least 1. We now show that its crossing number is at most 1. First suppose (1) holds. Then G has exactly one block, say H, homeomorphic to $K_4 - x$ which contains 2 adjacent non-cut-vertices of degree 3. By Theorem 1, $cr(P(H)) = 1$. By Theorem D, all other remaining blocks of G have a planar plick graph. Hence $P(G)$ has crossing number 1.

Assume (2) holds. Let u be a cut-vertex of degree 4. The vertex u has a non-cut-vertex of degree 3 in a block for otherwise, G does contain a subgraph homeomorphic to $K_4 - x$ which is impossible. By virtue of Theorem 1, for a non-cut-vertex of G of degree 3, there corresponds one crossing in $P(G)$. However $P(G)$ can not have more than one crossing since the removal of any edge in a block containing u, yields a graph H such that $P(H)$ is planar by Theorem D. It follows easily that $P(G)$ has crossing number 1.

Suppose (3) holds. The edges at the vertex v of the degree 5 can be split into sets of sizes

2 and 3 so that no edges in different sets are in the same block. Transform G to G' as in Figure 3. Then $P(G')$ is again planar. Thus $P(G)$ can be drawn with only one crossing as shown in Figure 4. □

§3. Forbidden Subgraphs

By using Theorem 2, we now characterize graphs whose plick graphs have crossing number 1 in terms of forbidden subgraphs.

Theorem 3 *The plick graph of a connected graph G has crossing number 1 if and only if G has no subgraphs homeomorphic from any one of the graphs of Figure 5 or G has subgraph θ^+ such that none of the vertices of theta have non-cut-vertices of degree 2 or 3.*

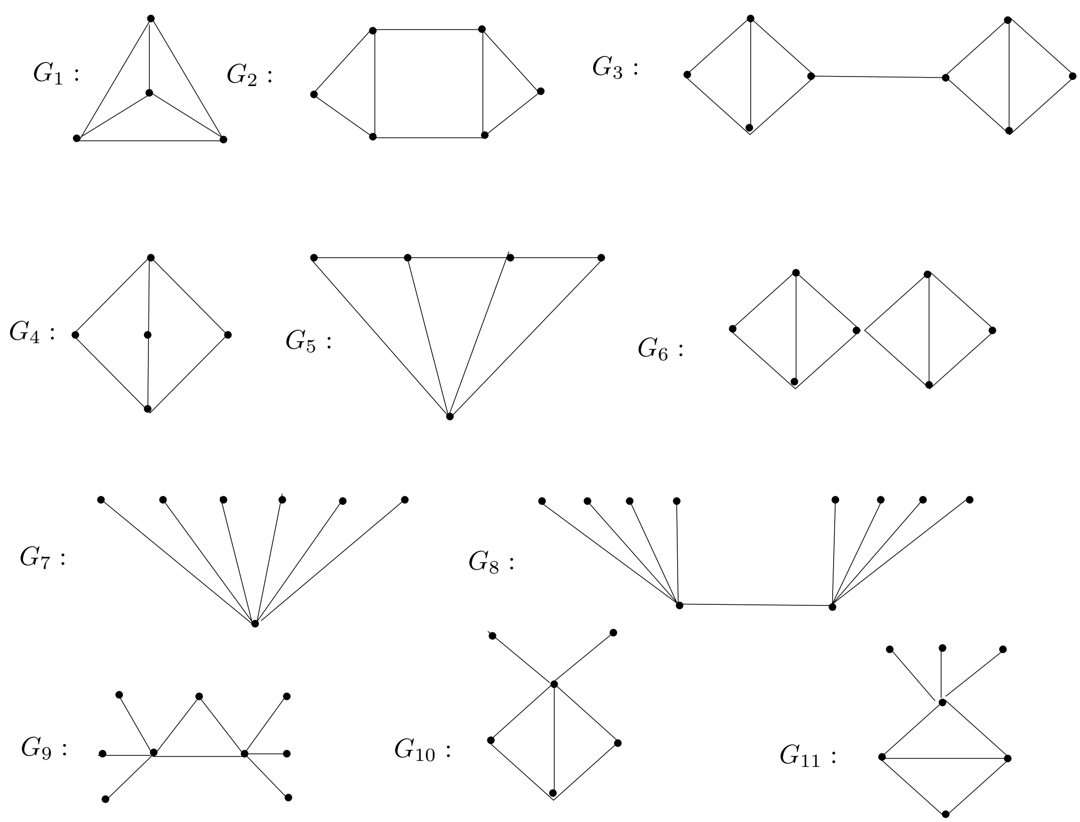

Figure 5

Proof Suppose G has a plick graph with crossing number one. We now show that all graphs homeomorphic from any one of the graphs of Figure 5 or a subgraph θ^+ such that none of the vertices of theta have non-cut-vertices of degree 2 or 3, have no plick graph with crossing number one. This result follows from Theorem 2, since graphs homeomorphic from G_1, G_2 or G_3 have more than two non-cut-vertices of degree three. Graphs homeomorphic from G_4 have two non-cut-vertices of degree 3 which are not adjacent. Graphs homeomorphic from

G_5 have a vertex of degree 4 which is a non-cut-vertex. Graphs homeomorphic from G_6 have more than one theta. θ^+ has exactly one block which is a theta and none of its vertices have non-cut-vertices of degree 2 or 3. Graphs homeomorphic from G_7 have $\Delta(G_7) > 5$. Graphs homeomorphic from G_8 or G_9 have two or more vertices of degree 5. Graphs homeomorphic to G_{10} or G_{11} have a block which is neither a cycle nor a K_2.

Conversely, suppose G is a graph which does not contain a subgraph homeomorphic from any one of the graphs of Figure 5 or G has exactly one subgraph theta as a block such that none of the vertices of theta have non-cut-vertices of degree 2 or 3. First we prove condition (1) of Theorem 2. Suppose G contains more than two non-cut-vertices of degree 3. Then it is easy to see that G is a planar graph with at least 2 diagonal edges. Now consider 2 cases depending on whether the 2 diagonal edges exist in one block or in two different blocks.

Case 1. Suppose two diagonal edges exist in one block of G, then G has a subgraph homeomorphic from G_1 or G_2.

Case 2. Suppose two diagonal edges exist in two different blocks of G, then G has a subgraph homeomorphic from G_3.

In each case we have a contradiction. Hence G has at most two non-cut-vertices of degree 3. Suppose G has exactly two nonadjacent non-cut-vertices of degree 3. Then there exist 3 disjoint paths between these two non-cut-vertices of degree 3. Clearly G contains a subgraph homeomorphic from G_4, a contradiction. Thus G has exactly two adjacent non-cut-vertices of degree 3.

Since G does not contain a subgraph homeomorphic from G_7 i.e, $K_{1,6}$, $\Delta(G) \leq 5$. Also since $\Delta(G) \geq 4$, if it follows that $\Delta(G) = 4$ or 5.

Suppose G has a vertex v of degree 4. We prove that v is a cut-vertex. If not, let a, b, c and d be the vertices of G adjacent to v. Then there exist paths between every pair of vertices of a, b, c and d not containing v. Then it is proved in Theorem E, G has a subgraph homeomorphic from G_5, this is a contradiction. Thus v is a cut-vertex and every vertex of degree 4 is a cut-vertex.

Suppose that a cut-vertex of degree 4 lies on two blocks, each of which is a theta. Then G has a subgraph homeomorphic from G_6. This is a contradiction. G has exactly one block which is a theta such that at least one vertex of theta is either a non-cut-vertex of degree 2 or 3, for otherwise a forbidden subgraph has exactly one theta as a block such that none of the vertices of theta have non-cut-vertices of degree 2 or 3 would appear in G.

Suppose G has two vertices v_1 and v_2 of degree 5. Since G is a connected, v_1 and v_2 are connected by a path P and let (v_1, a_i) and (v_2, b_j), $i, j = 1, 2, 3, 4$, be edges of G. We consider the following possibilities.

If $a_i \neq b_j$ for $i, j = 1, 2, 3, 4$, then G contains a subgraph homeomorphic from G_8, a contradiction.

If there exists a path between a vertex of a_i and a vertex of b_j, then G has a subgraph homeomorphic from G_9, a contradiction.

If $a_i = b_j$, for $i, j = 1, 2$, then clearly G contains a subgraph homeomorphic from G_{10}, a contradiction.

This proves that G has exactly one vertex v of degree 5.

Suppose G has a vertex v of degree 5. We show that v is a cut-vertex. If possible let us assume that G has a non-cut-vertex of degree 5. In this case Greenwell and Hemminger showed in [1] that G must contain a subgraph homeomorphic from G_5, a contradiction.

Suppose G has a unique cut-vertex v of degree 5 and it lies on blocks, one block which is neither a cycle nor a K_2. Then G contains a subgraph homeomorphic from G_{10} or G_{11}.

Thus Theorem 2 implies that G has a plick graph with crossing number one. □

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