

## On Pathos Semitotal and Total Block Graph of a Tree

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**Abstract:** In this communications, the concept of pathos semitotal and total block graph of a graph is introduced. Its study is concentrated only on trees. We present a characterization of those graphs whose pathos semitotal block graphs are planar, maximal outer planar, non-minimally non-outer planar, non-Eulerian and hamiltonian. Also, we present a characterization of graphs whose pathos total block graphs are planar, maximal outer planar, minimally non-outer planar, non-Eulerian, hamiltonian and graphs with crossing number one.

**Key Words:** Pathos, path number, Smarandachely block graph, semitotal block graph, Total block graph, pathos semitotal graph, pathos total block graph, pathos length, pathos point, inner point number.

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### §1. Introduction

The concept of pathos of a graph  $G$  was introduced by Harary [2], as a collection of minimum number of line disjoint open paths whose union is  $G$ . The path number of a graph  $G$  is the number of paths in pathos. A new concept of a graph valued functions called the semitotal and total block graph of a graph was introduced by Kulli [6]. For a graph  $G(p, q)$  if  $B = \{u_1, u_2, u_3, \dots, u_r; r \geq 2\}$  is a block of  $G$ , then we say that point  $u_1$  and block  $B$  are incident with each other, as are  $u_2$  and  $B$  and so on. If two distinct blocks  $B_1$  and  $B_2$  are incident with a common cut point, then they are adjacent blocks. The points and blocks of a graph are called its members. A *Smarandachely block graph*  $T_S^V(G)$  for a subset  $V \subset V(G)$  is such a graph with vertices  $V \cup \mathcal{B}$  in which two points are adjacent if and only if the corresponding members of  $G$  are adjacent in  $\langle V \rangle_G$  or incident in  $G$ , where  $\mathcal{B}$  is the set of blocks of  $G$ . The semitotal block graph of a graph  $G$  denoted by  $T_b(G)$  is defined as the graph whose point set is the union of set of points, set of blocks of  $G$  in which two points are adjacent if and only if members of  $G$  are incident, thus a Smarandachely block graph with  $V = \emptyset$ . The total block graph of a graph

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$G$  denoted by  $T_B(G)$  is defined as the graph whose point set is the union of set of points, set of blocks of  $G$  in which two points are adjacent if and only if the corresponding members of  $G$  are adjacent or incident, i.e., a Smarandachely block graph with  $V = V(G)$ . Stanton [11] and Harary [3] have calculated the path number for certain classes of graphs like trees and complete graphs.

All undefined terminology will conform with that in Harary [1]. All graphs considered here are finite, undirected and without loops or multiple lines.

The pathos semitotal block graph of a tree  $T$  denoted by  $P_{T_B}(T)$  is defined as the graph whose point set is the union of set of points, set of blocks and the set of path of pathos of  $T$  in which two points are adjacent if and only if the corresponding members of  $G$  are incident and the lines lie on the corresponding path  $P_i$  of pathos. Since the system of pathos for a tree is not unique, the corresponding pathos semitotal and pathos total block graph of a tree  $T$  is also not unique.

In Fig.1, a tree  $T$ , its semitotal block graph  $T_b(T)$  and their pathos semi total block  $P_{T_b}(T)$  graph are shown. In Fig. 2, a tree  $T$ , its semitotal block graph  $T_b(T)$  and their pathos total block  $P_{T_B}(T)$  graph are shown.

The line degree of a line  $uv$  in a tree  $T$ , pathos length, pathos point in  $T$  was defined by Muddebihal [10]. If  $G$  is planar, the inner point number  $i(G)$  of a graph  $G$  is the minimum number of points not belonging to the boundary of the exterior region in any embedding of  $G$  in the plane. A graph  $G$  is said to be minimally nonouterplanar if  $i(G) = 1$ , as was given by Kulli [4].

We need the following results to prove further results.

**Theorem [A]**[Ref.6] *If  $G$  is connected graph with  $p$  points and  $q$  lines and if  $b_i$  is the number of blocks to which  $v_i$  belongs in  $G$ , then the semitotal block graph  $T_b(G)$  has  $\left(\sum_{i=1}^p b_i\right) + 1$ , points and  $q + \left(\sum_{i=1}^p b_i\right)$  lines.*

**Theorem [B]**[Ref.6] *If  $G$  is connected graph with  $p$  points and  $q$  lines and if  $b_i$  is the number of blocks to which  $v_i$  belongs in  $G$ , then the total block graph  $T_B(G)$  has  $\left(\sum_{i=1}^p b_i\right) + 1$ , points and  $q + \sum_{i=1}^p \binom{b_i + 1}{2}$  lines.*

**Theorem [C]**[Ref.8] *The total block graph  $T_B(G)$  of a graph  $G$  is planar if and only if  $G$  is outerplanar and every cutpoint of  $G$  lies on atmost three blocks.*

**Theorem [D]** [Ref.7] *The total block graph  $T_B(G)$  of a connected graph  $G$  is minimally nonouter planar if and only if,*

- (1)  $G$  is a cycle, or
- (2)  $G$  is a path  $P$  of length  $n \geq 2$ , together with a point which is adjacent to any two adjacent points of  $P$ .

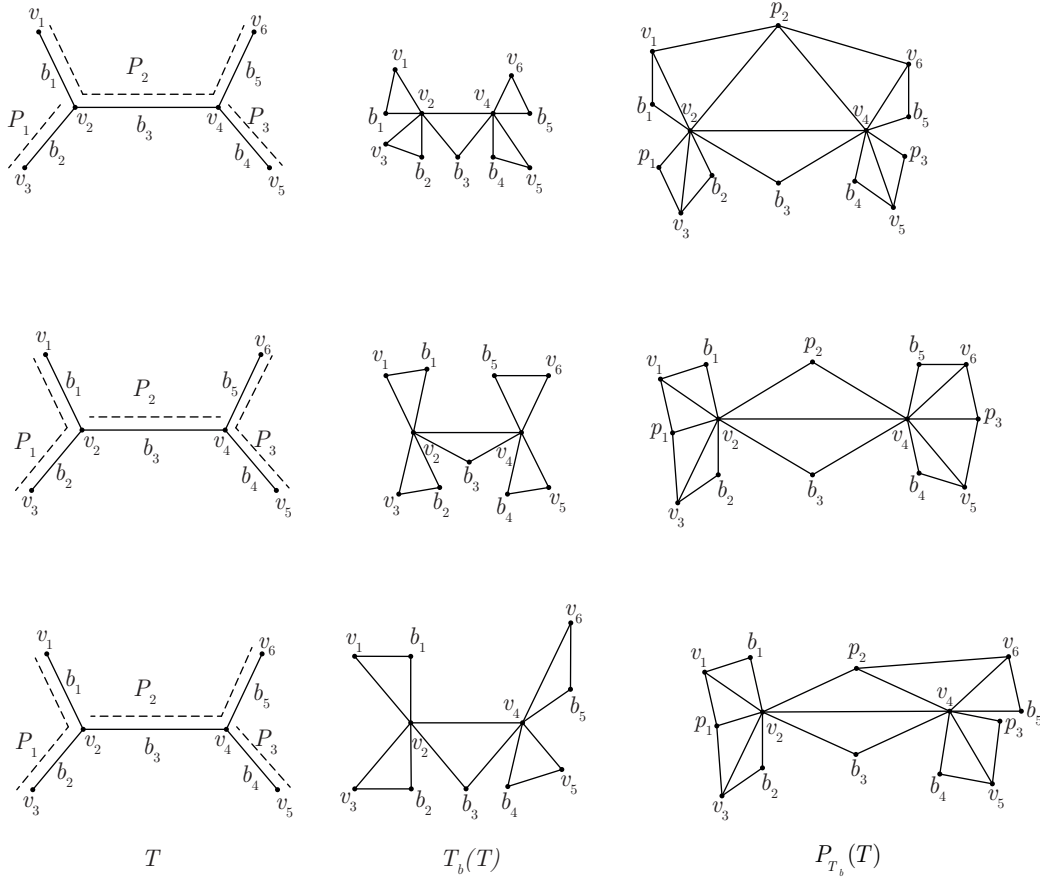


Figure 1:

**Theorem [E][Ref.9]** *The total block graph  $T_B(G)$  of a graph  $G$  crossing number 1 if and only if*

- (1)  *$G$  is outer planar and every cut point in  $G$  lies on at most 4 blocks and  $G$  has a unique cut point which lies on 4 blocks, or*
- (2)  *$G$  is minimally non-outer planar, every cut point of  $G$  lies on at most 3 blocks and exactly one block of  $G$  is theta-minimally non-outer planar.*

**Corollary [A][Ref.1]** *Every nontrivial tree contains at least two end points.*

**Theorem [F][Ref.1]** *Every maximal outerplanar graph  $G$  with  $p$  points has  $(2p - 3)$  lines.*

**Theorem [G][Ref.5]** *A graph  $G$  is a non empty path if and only if it is connected graph with  $p \geq 2$  points and  $\sum_{i=1}^p d_i^2 - 4p + 6 = 0$ .*

## §2. Pathos Semitotal Block Graph of a Tree

We start with a few preliminary results.

**Remark 1** The number of blocks in pathos semitotal block graph of  $P_{T_b}(T)$  of a tree  $T$  is equal to the number of pathos in  $T$ .

**Remark 2** If the degree of a pathos point in pathos semi total block graph  $P_{T_b}(T)$  of a tree  $T$  is  $n$ , then the pathos length of the corresponding path  $P_i$  of pathos in  $T$  is  $n - 1$ .

Kulli [6] developed the new concept in graph valued functions i.e., semi total and total block graph of a graph. In this article the number of points and lines of a semi total block graph of a graph has been expressed in terms of blocks of  $G$ . Now using this we have a modified theorem as shown below in which we have expressed the number of points and lines in terms of lines and degrees of the points of  $G$  which is a tree.

**Theorem 1** For any  $(p, q)$  tree  $T$ , the semitotal block graph  $T_b(T)$  has  $(2q + 1)$  points and  $3q$  lines.

*Proof* By Theorem [A], the number of points in  $T_b(G)$  is  $\left(\sum_{i=1}^p b_i\right) + 1$ , where  $b_i$  are the number of blocks in  $T$  to which the points  $v_i$  belongs in  $G$ . Since  $\sum b_i = 2q$ , for  $G$  is a tree. Thus the number of points in  $T_b(G) = 2q + 1$ . Also, by Theorem [A] the number of lines in  $T_b(G)$  are  $q + \left(\sum_{i=1}^p b_i\right)$ , since  $\sum b_i = 2q$  for  $G$  is a tree. Thus the number of lines in  $T_b(G)$  is  $q + 2q = 3q$ .  $\square$

In the following theorem we obtain the number of points and lines in  $P_{T_b}(T)$ .

**Theorem 2** For any non trivial tree  $T$ , the pathos semitotal block graph of a tree  $T$ , whose points have degree  $d_i$ , then the number of points in  $P_{T_b}(T)$  are  $(2q + k + 1)$  and the number of lines are  $\left(2q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$ , where  $k$  is the path number.

*Proof* By Theorem 1, the number of points in  $T_b(T)$  are  $2q + 1$ , and by definition of  $P_{T_b}(T)$ , the number of points in  $(2q + k + 1)$ , where  $k$  is the path number. Also by Theorem 1, the number of lines in  $T_b(T)$  are  $3q$ . The number of lines in  $P_{T_b}(T)$  is the sum of lines in  $T_b(T)$  and the number of lines which lie on the points of pathos of  $T$  which are to  $\left(-q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$ . Thus the number of lines in is equal to  $\left(3q + (-q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2)\right) = \left(2q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$ .

## §2. Planar Pathos Semitotal Block Graphs

A criterion for pathos semi total block graph to be planar is presented in our next theorem.

**Theorem 3** For any non trivial tree  $T$ , the pathos semitotal block graph  $P_{T_b}(T)$  of a tree  $T$  is planar.

*Proof* Let  $T$  be a non trivial tree, then in  $T_b(T)$  each block is a triangle. We have the following cases.

**Case 1** Suppose  $G$  is a path,  $G = P_n : u_1, u_2, u_3, \dots, u_n, n > 1$ . Further,  $V[T_b(T)] =$

$\{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$ , where  $b_1, b_2, b_3, \dots, b_{n-1}$  are the corresponding block points. In pathos semi total block graph  $P_{T_b}(T)$  of a tree  $T$ ,  $\{u_1b_1u_2w, u_2b_2u_3w, u_3b_3u_4w, \dots, u_{n-1}b_{n-1}u_nw\} \in V[P_{T_b}(T)]$ , each set  $\{u_{n-1}b_{n-1}u_nw\}$  forms an induced subgraph as  $K_4 - x$ . Hence one can easily verify that  $P_{T_b}(T)$  is planar.

**Case 2** Suppose  $G$  is not a path. Then  $V[T_b(G)] = \{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$  and  $w_1, w_2, w_3, \dots, w_k$  be the pathos points. Since  $u_{n-1}u_n$  is a line and  $u_{n-1}u_n = b_{n-1} \in V[T_b(G)]$ . Then in  $P_{T_b}(G)$  the set  $\{u_{n-1}b_{n-1}u_nw\} \forall n > 1$ , forms  $K_4 - x$  as an induced subgraphs. Hence  $P_{T_b}(G)$  is planar.  $\square$

Further we develop the maximal outer planarity of  $P_{T_b}(G)$  in the following theorem.

**Theorem 4** *For any non trivial tree  $T$ , the pathos semitotal block graph  $P_{T_b}(T)$  of a tree  $T$  is maximal outer planar if and only if  $T$  is a path.*

*Proof* Suppose  $P_{T_b}(T)$  is maximal outer planar. Then  $P_{T_b}(T)$  is connected. Hence  $T$  is connected. If  $P_{T_b}(T)$ , is  $K_4 - x$ , then obviously  $T$  is  $k_2$ .

Let  $T$  be any connected tree with  $p \geq 2$ ,  $q$  lines  $b_i$  blocks and path number  $k$ , then clearly  $P_{T_b}(T)$  has  $(2q + k + 1)$  points and  $\left(2q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$  lines. Since  $P_{T_b}(T)$  is maximal outer planar, by Theorem [F], it has  $[2(2q + k + 1) - 3]$  lines. Hence,

$$\begin{aligned} 2 + 2q + \frac{1}{2} \sum_{i=1}^p d_i^2 &= 2(2q + k + 1) - 3 = 4q + 2k + 2 - 3 = 4q + 2k - 1 \\ \frac{1}{2} \sum_{i=1}^p d_i^2 &= 2q + 2k - 3 \\ \sum_{i=1}^p d_i^2 &= 4q + 4k - 6 \\ \sum_{i=1}^p d_i^2 &= 4(p - 1) + 4k - 6 \\ \sum_{i=1}^p d_i^2 &= 4p + 4k - 10. \end{aligned}$$

But for a path,  $k = 1$ .

$$\begin{aligned} \sum_{i=1}^p d_i^2 &= 4p + 4(1) - 10 = 4p - 6 \\ \sum_{i=1}^p d_i^2 - 4p + 6 &= 0. \end{aligned}$$

By Theorem [G], it follows that  $T$  is a non empty path. Thus necessity is proved.

For sufficiency, suppose  $T$  is a non empty path. We prove that  $P_{T_b}(T)$  is maximal outer planar. By induction on the number of points  $p_i \geq 2$  of  $T$ . It is easy to observe that  $P_{T_b}(T)$  of a path  $P$  with 2 points is  $K_4 - x$ , which is maximal outer planar. As the inductive hypothesis, let the pathos semitotal block graph of a non empty path  $P$  with  $n$  points be maximal outer planar. We now show that the pathos semitotal block graph of a path  $P'$  with  $(n + 1)$  points is maximal outer planar. First we prove that it is outer planar. Let the point and line sequence of the path

$P'$  be  $v_1, e_1, v_2, e_2, v_3, \dots, v_n, e_n, v_{n+1}$ , Where  $v_1v_2 = e_1 = b_1, v_2v_3 = e_2 = b_2, \dots, v_{n-1}v_n = e_{n-1} = b_{n-1}, v_nv_{n+1} = e_n = b_n$ .

The graphs  $P, P', T_b(P), T_b(P'), P_{T_b}(P)$  and  $P_{T_b}(P')$  are shown in the figure 2. Without loss of generality  $P' - v_{n+1} = P$ .

By inductive hypothesis,  $P_{T_b}(P)$  is maximal outer planar. Now the point  $v_{n+1}$  is one more point more in  $P_{T_b}(P')$  than  $P_{T_b}(P)$ . Also there are only four lines  $(v_{n+1}, v_n)(v_n, b_n)(b_n, v_{n+1})$  and  $(v_{n+1}, K_1)$  more in  $P_{T_b}(P')$ . Clearly the induced subgraph on the points  $v_{n+1}, v_n, b_n, K_1$  is not  $K_4$ . Hence  $P_{T_b}(P')$  is outer planar.

We now prove that  $P_{T_b}(P')$  is maximal outer planar. Since  $P_{T_b}(P)$  is maximal outer planar, it has  $2(2q + k + 1) - 3$  lines. The outer planar graph  $P_{T_b}(P')$  has  $2(2q + k + 1) - 3 + 4 = 2(2q + k + 1 + 2) - 3$

$$= 2[(2q + 1) + (k + 1) + 1] - 3 \text{ lines.}$$

By Theorem [F],  $P_{T_b}(P')$  is maximal outer planar. □

The next theorem gives a non-minimally non-outer planar  $P_{T_b}(T)$ .

**Theorem 5** *For any non trivial tree  $T$ , the pathos semitotal block graph  $P_{T_b}(T)$  of a tree  $T$  is non-minimally non-outer planar.*

*Proof* We have the following cases.

**Case 1** Suppose  $T$  is a path, then  $\Delta(T) \leq 2$ , then by Theorem 4,  $P_{T_b}(T)$  is maximal outer planar.

**Case 2** Suppose  $T$  is not a path, then  $\Delta(T) \geq 3$ , then by theorem 3,  $P_{T_b}(T)$  is planar. On embedding  $P_{T_b}(T)$  in any plane, the points with degree greater than two of  $T$  forms the cut points. In  $P_{T_b}(T)$  which lie on at least two blocks. Since each block of  $P_{T_b}(T)$  is a maximal outer planar than one can easily verify that  $P_{T_b}(T)$  is outer planar. Hence for any non trivial tree with  $\Delta(T) \geq 3$ ,  $P_{T_b}(T)$  is non minimally non-outer planar. □

In the next theorem, we characterize the non-Eulerian  $P_{T_b}(T)$ .

**Theorem 6** *For any non trivial tree  $T$ , the pathos semitotal block graph  $P_{T_b}(T)$  of a tree  $T$  is non-Eulerian.*

*Proof* We have the following cases.

**Case 1** Suppose  $T$  is a path with 2 points, then  $P_{T_b}(T) = K_4 - x$ , which is non-Eulerian. If  $T$  is a path with  $p > 2$  points. Then in  $T_b(T)$  each block is a triangle such that they are in sequence with the vertices of  $T_b(T)$  as  $\{v_1, b_1, v_2, v_1\}$  an induced subgraph as a triangle  $T_b(T)$ . Further  $\{v_2, b_2, v_3, v_2\}, \{v_3, b_3, v_4, v_3\}, \dots, \{v_{n-1}, b_n, v_n, v_{n-1}\}$ , in which each set form a triangle as an induced subgraph of  $T_b(T)$ . Clearly one can easily verify that  $T_b(T)$  is Eulerian. Now this path has exactly one pathos point say  $k_1$ , which is adjacent to  $v_1, v_2, v_3, \dots, v_n$  in  $P_{T_b}(T)$  in which all the points  $v_1, v_2, v_3, \dots, v_n \in P_{T_b}(T)$  are of odd degree. Hence  $P_{T_b}(T)$  is non-Eulerian.

**Case 2** Suppose  $\Delta(T) \geq 3$ . Assume  $T$  has a unique point of degree  $\geq 3$  and also assume that  $T = K_{1,n}$ . Then in  $T_b(T)$  each block is a triangle, such that the number of blocks which are  $K_3$

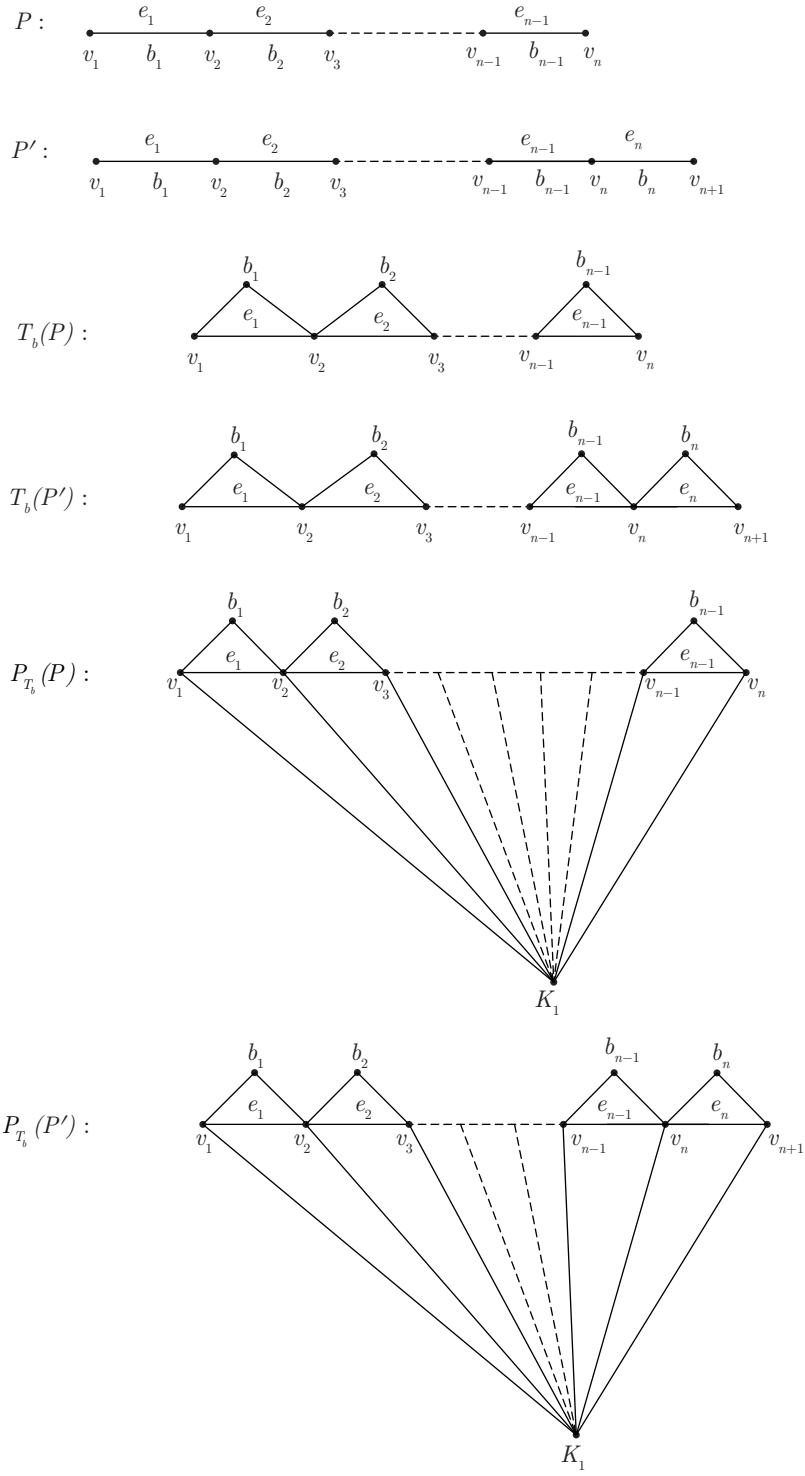


Figure 2:

are  $n$  with a common cut point  $k$ . Since the degree of a vertex  $k = 2n$ . One can easily verify that  $T_b(K_{1,3})$  is Eulerian. To form  $P_{T_b}(T)$ ,  $T = K_{1,n}$ , the points of degree 2 and the point  $k$  are joined by the corresponding pathos point which give  $(n+1)$  points of odd degree in  $P_{T_b}(T)$ . Hence  $P_{T_b}(T)$  is non-Eulerian.  $\square$

In the next theorem we characterize the hamiltonian  $P_{T_b}(T)$ .

**Theorem 7** *For any non trivial tree  $T$ , the pathos semitotal block graph  $P_{T_b}(T)$  of a tree  $T$  is hamiltonian if and only if  $T$  is a path.*

*Proof* For the necessity, suppose  $T$  is a path and has exactly one path of pathos. Let  $V[T_b(T)] = \{u_1, u_2, u_3, \dots, u_n\} \cup \{b_1, b_2, b_3, \dots, b_{n-1}\}$ , where  $b_1, b_2, b_3, \dots, b_{n-1}$  are block points of  $T$ . Since each block is a triangle and each block consists of points as  $B_1 = \{u_1, b_1, u_2\}$ ,  $B_2 = \{u_2, b_2, u_3\}$ ,  $\dots$ ,  $B_m = \{u_m, b_m, u_{m+1}\}$ . In  $P_{T_b}(T)$  the pathos point  $w$  is adjacent to  $\{u_1, u_2, u_3, \dots, u_n\}$ . Hence  $V[P_{T_b}(T)] = \{u_1, u_2, u_3, \dots, u_n\} \cup \{b_1, b_2, b_3, \dots, b_{n-1}\} \cup w$  form a cycle as  $w, u_1, b_1, u_2, b_2, u_3, \dots, u_{n-1}, b_{n-1}, u_n, w$ . Containing all the points of  $P_{T_b}(T)$ . Clearly  $P_{T_b}(T)$  is hamiltonian. Thus necessity is proved.

For the sufficiency, suppose  $P_{T_b}(T)$  is hamiltonian, now we consider the following cases.

**Case 1** Assume  $T$  is a path. Then  $T$  has at least one point with  $\deg v \geq 3$ ,  $\forall v \in V(T)$ , assume that  $T$  has exactly one point  $u$  such that degree  $u > 2$ , then  $G = T = K_{1,n}$ . Now we consider the following subcases of Case 1.

**Subcase 1.1** For  $K_{1,n}$ ,  $n > 2$  and  $n$  is even, then in  $T_b(T)$  each block is  $k_3$ . The number of path of pathos are  $\frac{n}{2}$ . Since  $n$  is even we get  $\frac{n}{2}$  blocks. Each block contains two lines of  $\langle K_4 - x \rangle$ , which is a non line disjoint subgraph of  $P_{T_b}(T)$ . Since  $P_{T_b}(T)$  has a cut point, one can easily verify that there does not exist any hamiltonian cycle, a contradiction.

**Subcase 1.2** For  $K_{1,n}$ ,  $n > 2$  and  $n$  is odd, then the number of path of pathos are  $\frac{n+1}{2}$ , since  $n$  is odd we get  $\frac{n-1}{2} + 1$  blocks in which  $\frac{n-1}{2}$  blocks contains two times of  $\langle K_4 - x \rangle$  which is nonlinear disjoint subgraph of  $P_{T_b}(T)$  and remaining block is  $\langle K_4 - x \rangle$ . Since  $P_{T_b}(T)$  contain a cut point, clearly  $P_{T_b}(T)$  does not contain a hamiltonian cycle, a contradiction. Hence the sufficient condition.

### §3. Pathos Total Block Graph of a Tree

A tree  $T$ , its total block graph  $T_B(T)$ , and their pathos total block graphs  $P_{T_B}(T)$  are shown in the Fig.3. We start with a few preliminary results.

**Remark 3** For any non trivial path, the inner point number of the pathos total block graph  $P_{T_B}(T)$  of a tree  $T$  is equal to the number of cut points in  $T$ .

**Remark 4** The degree of a pathos point in  $P_{T_B}(T)$  is  $n$ , then the pathos length of the corresponding path  $P_i$  of pathos in  $T$  is  $n - 1$ .

**Remark 5** For any non trivial tree  $T$ ,  $P_{T_B}(T)$  is a block.

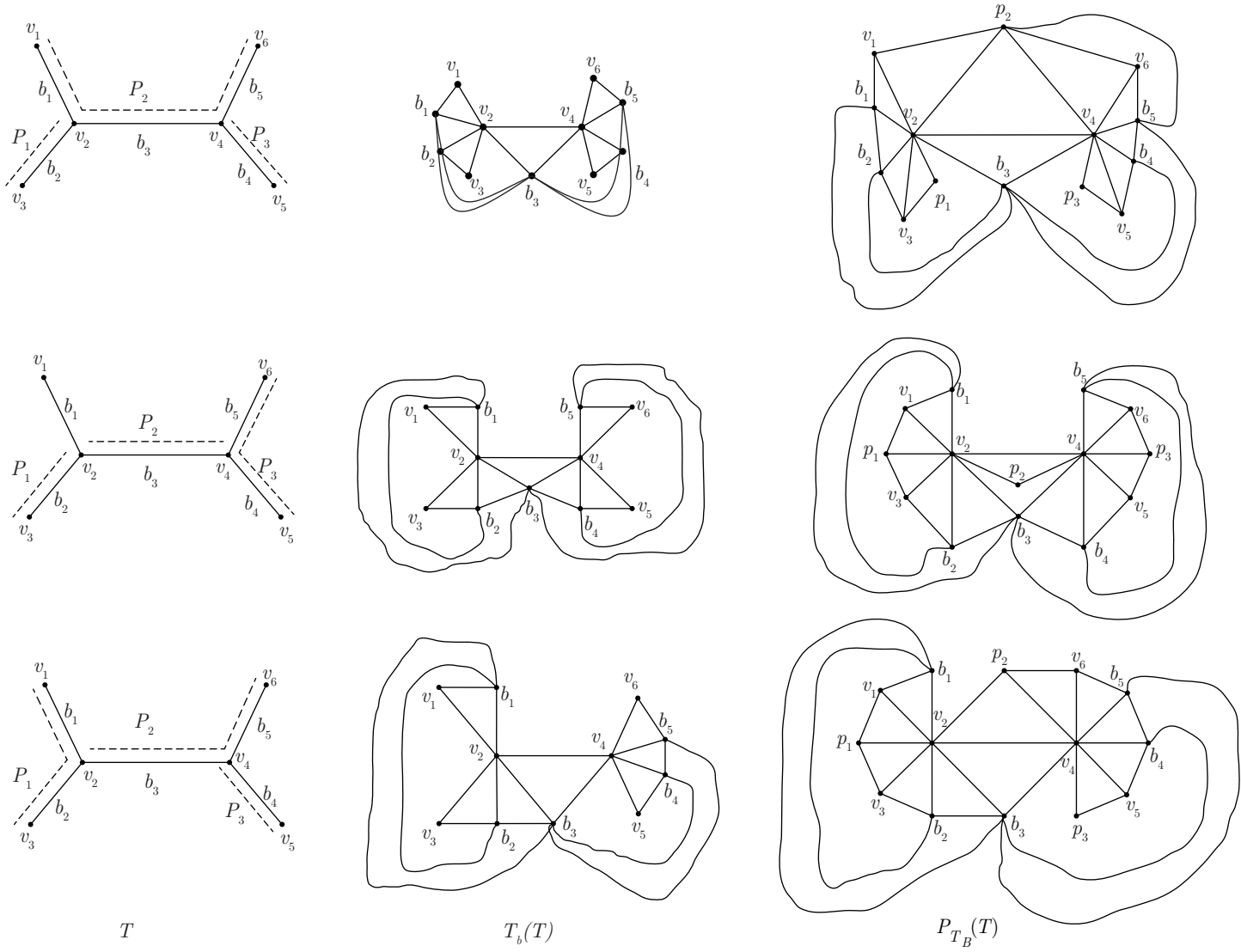


Figure 3:

Also in Kulli [4], developed the number of points and lines of a total block graph of a graph

has been expressed in terms of blocks of  $G$ . Now using this we have a modified theorem as shown below in which we have expressed the number of points and lines in terms of lines and degrees of the points of  $G$  which is a tree.

**Theorem 8** *For any non trivial  $(p, q)$  tree whose points have degree  $d_i$ , the number of points and lines in total block graph of a tree  $T$  are  $(2q + 1)$  and  $\left(2q + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$ .*

*Proof* By Theorem [B], the number of points in  $T_b(T)$  is  $\left(\sum_{i=1}^b b_i\right) + 1$ , where  $b_i$  are the number of blocks in  $T$  to which the points  $v_i$  belongs in  $G$ . Since  $\sum b_i = 2q$ , for  $G$  is a tree. Thus the number of points in  $T_B(G) = 2q + 1$ . Also, by Theorem [B], the number of lines in  $T_B(G)$  are  $q + \sum_{i=1}^b \binom{b_i + 1}{2} = \left(\sum_{i=1}^b b_i\right) + \left(\frac{1}{2} \sum_{i=1}^p d_i^2\right) = \left(2q + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$ , for  $G$  is a tree.  $\square$

In the following theorem we obtain the number of points and lines in  $P_{T_B}(T)$ .

**Theorem 9** *For any non trivial tree  $T$ , the pathos total block graph  $P_{T_B}(T)$  of a tree  $T$ , whose points have degree  $d_i$ , then the number of points in  $P_{T_B}(T)$  are  $(2q + k + 1)$  and the number of lines are  $\left(q + 2 + \sum_{i=1}^p d_i^2\right)$ , where  $k$  is the path number.*

*Proof* By Theorem 7, the number of points in  $T_B(T)$  are  $2q + 1$ , and by definition of  $P_{T_B}(T)$ , the number of points in  $P_{T_B}(T)$  are  $(2q + k + 1)$ , where  $k$  is the path number in  $T$ . Also by Theorem 7, the number of lines in  $T_B(T)$  are  $\left(2q + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$ . The number of lines in  $P_{T_B}(T)$  is the sum of lines in  $T_B(T)$  and the number of lines which lie on the points of pathos of  $T$  which are to  $\left(-q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$ . Thus the number of lines in  $P_{T_B}(T)$  is equal to  $\left(2q + \frac{1}{2} \sum_{i=1}^p d_i^2\right) + \left(-q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2\right) = \left(q + 2 + \sum_{i=1}^p d_i^2\right)$ .  $\square$

#### §4. Planar Pathos Total Block Graphs

A criterion for pathos total block graph to be planar is presented in our next theorem.

**Theorem 10** *For any non trivial tree  $T$ , the pathos total block graph  $P_{T_B}(T)$  of a tree  $T$  is planar if and only if  $\Delta(T) \leq 3$ .*

*Proof* Suppose  $P_{T_B}(T)$  is planar. Then by Theorem [C], each cut point of  $T$  lie on at most 3 blocks. Since each block is a line in a tree, now we can consider the degree of cutpoints instead of number of blocks incident with the cut points. Now suppose if  $\Delta(T) \leq 3$ , then by Theorem [C],  $T_B(T)$  is planar. Let  $\{b_1, b_2, b_3, \dots, b_{p-1}\}$  be the blocks of  $T$  with  $p$  points such that  $b_1 = e_1, b_2 = e_2, \dots, b_{p-1} = e_{p-1}$  and  $P_i$  be the number of pathos of  $T$ . Now  $V[P_{T_B}(T)] = V(G) \cup \{b_1, b_2, \dots, b_{p-1}\} \cup \{P_i\}$ . By Theorem [C], and by the definition of pathos, the embedding of  $P_{T_B}(T)$  in any plane gives a planar  $P_{T_B}(T)$ .

Suppose  $\Delta(T) \geq 4$  and assume that  $P_{T_B}(T)$  is planar. Then there exists at least one point

of degree 4, assume that there exists a vertex  $v$  such that  $\deg v = 4$ . Then in  $T_B(T)$ , this point together with the block points form  $k_5$  as an induced subgraph. Further the corresponding pathos point are adjacent to the  $V(T)$  in  $T_B(T)$  which gives  $P_{T_B}(T)$  in which again  $k_5$  as an induced subgraph, a contradiction to the planarity of  $P_{T_B}(T)$ . This completes the proof.  $\square$

The following theorem results the maximal outer planar  $P_{T_B}(T)$ .

**Theorem 11** *For any non trivial tree  $T$ , the pathos total block graph  $P_{T_B}(T)$  of a tree  $T$  is maximal outer planar if and only if  $T = k_2$ .*

*Proof* Suppose  $T = k_3$  and  $P_{T_B}(T)$  is maximal outer planar. Then  $T_B(T) = k_4$  and one can easily verify that,  $i[P_{T_B}(T)] > 1$ , a contradiction. Further we assume that  $T = K_{1,2}$  and  $P_{T_B}(T)$  is maximal outer planar, then  $T_B(T)$  is  $W_p - x$ , where  $x$  is outer line of  $W_p$ . Since  $K_{1,2}$  has exactly one pathos, this point together with  $W_p - x$  gives  $W_{p+1}$ . Clearly,  $P_{T_B}(T)$  is non maximal outer planar, a contradiction. For the converse, if  $T = k_2$ ,  $T_B(T) = k_3$  and  $P_{T_B}(T) = K_4 - x$  which is a maximal outer planar. This completes the proof of the theorem.  $\square$

Now we have a pathos total block graph of a path  $p \geq 2$  point as shown in the below remarks, and also a cycle with  $p \geq 3$  points.

**Remark 6** For any non trivial path with  $p$  points,  $i[P_{T_B}(T)] = p - 2$ .

**Remark 7** For any cycle  $C_p$ ,  $p \geq 3$ ,  $i[P_{T_B}(C_p)] = p - 1$ .

The next theorem gives a minimally non-outer planar  $P_{T_B}(T)$ .

**Theorem 12** *For any non trivial tree  $T$ , the pathos total block graph  $P_{T_B}(T)$  of a tree  $T$  is minimally non-outer planar if and only if  $T$  is a path with 3 points.*

*Proof* Suppose  $P_{T_B}(T)$  is minimally non-outer planar. Assume  $T$  is not a path. We consider the following cases.

**Case 1** Suppose  $T$  is a tree with  $\Delta(T) \geq 3$ . Then there exists at least one point of degree at least 3. Assume  $v$  be a point of degree 3. Clearly,  $T = K_{1,3}$ . Then by the Theorem [D],  $i[T_B(T)] > 1$  since  $T_B(T)$  is a subgraph of  $P_{T_B}(T)$ . Clearly  $i[P_{T_B}(T)] \geq 2$  a contradiction.

**Case 2** Suppose  $T$  is a closed path with  $p$  points, then it is a cycle with  $p$  points. By Theorem [D],  $P_{T_B}(T)$  is minimally non-outer planar. By Remark 7,  $i[P_{T_B}(T)] > 1$ , a contradiction.

**Case 3** Suppose  $T$  is a closed path with  $p \geq 4$  points, clearly by Remark 6,  $i[P_{T_B}(T)] > 2$ , a contradiction.

Conversely, suppose  $T$  is a path with 3 points, clearly by Remark 6,  $i[P_{T_B}(T)] = 1$ . This gives the required result.  $\square$

In the following theorem we characterize the non-Eulerian  $P_{T_B}(T)$ .

**Theorem 13** *For any non trivial tree  $T$ , the pathos total block graph  $P_{T_B}(T)$  of a tree  $T$  is non-Eulerian.*

*Proof* We consider the following cases.

**Case 1** Suppose  $T$  is a path. For  $p = 2$  points, then  $P_{T_B}(T) = K_4 - x$ , which is non-Eulerian. For  $p = 3$  points, then  $P_{T_B}(T)$  is a wheel, which is non-Eulerian.

For  $p \geq 4$  we have a path  $P : v_1, v_2, v_3, \dots, v_p$ . Now in path each line is a block. Then  $v_1v_2 = e_1 = b_1, v_2v_3 = e_2 = b_2, \dots, v_{p-1}v_p = e_{p-1} = b_{p-1}$ ,  $\forall e_{p-1} \in E(G)$ , and  $\forall b_{p-1} \in V[T_B(P)]$ . In  $T_B(P)$ , the degree of each point is even except  $b_1$  and  $b_{p-1}$ . Since the path  $P$  has exactly one pathos which is a point of  $P_{T_B}(P)$  and is adjacent to the points  $v_1, v_2, v_3, \dots, v_p$ , of  $T_B(P)$  which are of even degree, gives as an odd degree points in  $P_{T_B}(P)$  including odd degree points  $b_1$  and  $b_2$ . Clearly  $P_{T_B}(P)$  is non-Eulerian.

**Case 2** Suppose  $T$  is not a path. We consider the following subcases,

**Subcase 2.1** Assume  $T$  has a unique point degree  $\geq 3$  and  $T = K_{1,n}$ , where  $n$  is odd. Then in  $T_B(T)$  each block is a triangle such that there are  $n$  number of triangles with a common cut point  $k$  which has a degree  $2n$ . Since the degree of each point in  $T_B(K_{1,n})$  is Eulerian. To form  $P_{T_B}(T)$  where  $T = K_{1,n}$ , the points of degree 2 and the point  $k$  are joined by the corresponding pathos point which gives  $(n + 1)$  points of odd degree in  $P_{T_B}(K_{1,n})$ .  $P_{T_B}(K_{1,n})$  has  $n$  points of odd degree. Hence  $P_{T_B}(T)$  non-Eulerian.

Assume that  $T = K_{1,n}$ , where  $n$  is even, then in  $T_B(T)$  each block is a triangle, which are  $2n$  in number with a common cut point  $k$ . Since the degree of each point other than  $k$  is either 2 or  $(n + 1)$  and the degree of the point  $k$  is  $2n$ . One can easily verify that  $T_B(K_{1,n})$  is non-Eulerian. To form  $P_{T_B}(T)$  where  $T = K_{1,n}$ , the points of degree 2 and the point  $k$  are joined by the corresponding pathos point which gives  $(n + 2)$  points of odd degree in  $P_{T_B}(T)$ . Hence  $P_{T_B}(T)$  non-Eulerian.

**Subcase 2.2** Assume  $T$  has at least two points of degree  $\geq 3$ . Then  $V[T_B(T)] = V(G) \cup \{b_1, b_2, b_3, \dots, b_p\}$ ,  $\forall e_p \in E(G)$ . In  $T_B(T)$ , each endpoint has degree 2 and these points are adjacent to the corresponding pathos points in  $P_{T_B}(T)$  gives degree 3, From Case 1, Tree  $T$  has at least 4 points and by Corollary [A],  $P_{T_B}(T)$  has at least two points of degree 3. Hence  $P_{T_B}(T)$  is non-Eulerian.  $\square$

In the next theorem we characterize the hamiltonian  $P_{T_B}(T)$ .

**Theorem 14** For any non trivial tree  $T$ , the pathos total block graph  $P_{T_B}(T)$  of a tree  $T$  is hamiltonian.

*Proof* We consider the following cases.

**Case 1** Suppose  $T$  is a path with  $\{u_1, u_2, u_3, \dots, u_n\} \in V(T)$  and  $b_1, b_2, b_3, \dots, b_m$  be the number of blocks of  $T$  such that  $m = n - 1$ . Then it has exactly one path of pathos. Now point set of  $T_B(T)$  is  $V[T_B(T)] = \{u_1, u_2, \dots, u_n\} \cup \{b_1, b_2, \dots, b_m\}$ . Since given graph is a path then in  $T_B(T)$ ,  $b_1 = e_1, b_2 = e_2, \dots, b_m = e_m$ , such that  $b_1, b_2, b_3, \dots, b_m \subset V[T_B(T)]$ . Then by the definition of total block graph  $\{u_1, u_2, \dots, u_m\} \cup \{b_1, b_2, \dots, b_{m-1}, b_m\} \cup \{b_1, u_1, b_2u_2, \dots, b_mu_{n-1}, b_mu_n\}$  form line set of  $T_B(T)$ [see Fig. 4].

Now this path has exactly one pathos say  $w$ . In forming pathos total block graph of a path, the pathos  $w$  becomes a point, then  $V[P_{T_B}(T)] = \{u_1, u_2, \dots, u_n\} \cup \{b_1, b_2, \dots, b_m\} \cup \{w\}$  and

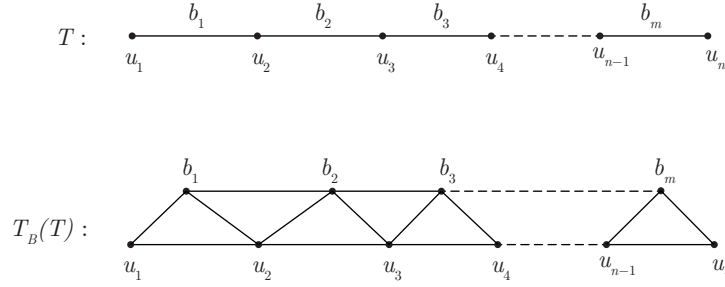


Figure 4:

$w$  is adjacent to all the points  $\{u_1, u_2, \dots, u_m\}$  shown in the Fig.5.

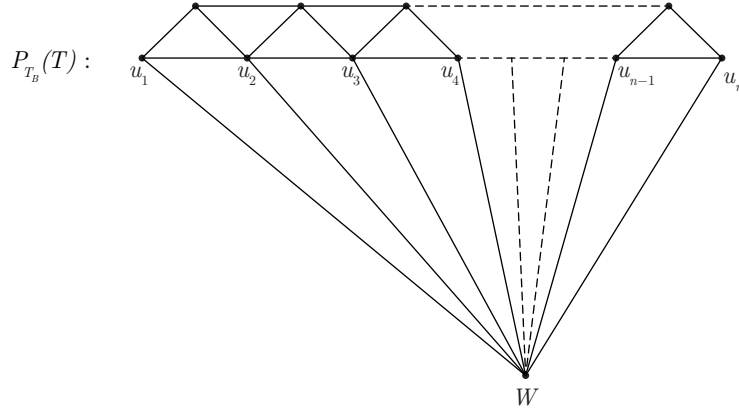


Figure 5:

In  $P_{T_B}(T)$ , the hamiltonian cycle  $w, u_1, b_1, u_2, b_2, u_3, b_3, \dots, u_{n-1}, b_m, u_n, w$  exist. Clearly the pathos total block graph of a path is hamiltonian graph.

**Case 2** Suppose  $T$  is not a path. Then  $T$  has at least one point with degree at least 3. Assume that  $T$  has exactly one point  $u$  such that  $\text{degree} > 2$ . Now we consider the following subcases of case 2.

**Subcase 2.1** Assume  $T = K_{1,n}$ ,  $n > 2$  and is odd. Then the number of paths of pathos are  $\frac{n+1}{2}$ . Let  $V[T_B(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{n-1}\}$ . By the definition of  $P_{T_B}(T)$ ,  $V[P_{T_B}(T)] = \{u_1, u_2, \dots, u_n, b_1 b_2, \dots, b_{n-1}\} \cup \{p_1, p_2, \dots, p_{n+1/2}\}$ . Then there exists a cycle containing the points of  $P_{T_B}(T)$  as  $p_1, u_1, b_1, b_2, u_3, p_2, u_2, b_3, u_4, \dots, p_1$  and is a hamiltonian cycle. Hence  $P_{T_B}(T)$  is a hamiltonian.

**Subcase 2.2** Assume  $T = K_{1,n}$ ,  $n > 2$  and is even. Then the number of path of pathos are  $\frac{n}{2}$ , then  $V[T_B(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{n-1}\}$ . By the definition of  $P_{T_B}(T)$ ,  $V[P_{T_B}(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{n-1}\} \cup \{p_1, p_2, \dots, p_{n/2}\}$ . Then there exist a cycle containing the points of  $P_{T_B}(T)$  as  $p_1, u_1, b_1, b_2, u_3, p_2, u_4, b_3, b_4, \dots, p_1$  and is a hamiltonian cycle. Hence  $P_{T_B}(T)$  is a hamiltonian.

Suppose  $T$  is neither a path or a star. Then  $T$  contains at least two points of degree  $> 2$ . Let  $u_1, u_2, u_3, \dots, u_n$  be the points of degree  $\geq 2$  and  $v_1, v_2, v_3, \dots, v_m$  be the end points of  $T$ . Since end block is a line in  $T$ , and denoted as  $b_1, b_2, \dots, b_k$ , then tree  $T$  has  $p_i$  pathos points,  $i > 1$  and each pathos point is adjacent to the point of  $T$  where the corresponding pathos lie on the points of  $T$ . Let  $\{p_1, p_2, \dots, p_i\}$  be the pathos points of  $T$ . Then there exists a cycle  $C$  containing all the points of  $P_{T_B}(T)$  as  $p_1, v_1, b_1, b_2, v_2, p_2, u_1, b_3, u_2, p_3, v_3, b_4, \dots, v_{n-1}, b_{n-1}, b_n, v_n, \dots, p_1$ . Hence  $P_{T_B}(T)$  is a hamiltonian cycle. Hence  $P_{T_B}(T)$  is a hamiltonian graph.  $\square$

In the next theorem we characterize  $P_{T_B}(T)$  in terms of crossing number one.

**Theorem 15** For any non trivial tree  $T$ , the pathos total block graph  $P_{T_B}(T)$  of a tree  $T$  has crossing number one if and only if  $\Delta(T) \leq 4$ , and there exist a unique point in  $T$  of degree 4.

*Proof* Suppose  $P_{T_B}(T)$  has crossing number one. Then it is non-planar. Then by Theorem 10, we have  $\Delta(T) \geq 4$ . We now consider the following cases.

**Case 1** Assume  $\Delta(T) = 5$ . Then by Theorem [E],  $T_B(T)$  is non-planar with crossing number more than one. Since  $T_B(T)$  is a subgraph of  $P_{T_B}(T)$ . Clearly  $cr(P_{T_B}(T)) > 1$ , a contradiction.

**Case 2** Assume  $\Delta(T) = 4$ . Suppose  $T$  has two points of degree 4. Then by Theorem [E],  $T_B(T)$  has crossing number at least two. But  $T_B(T)$  is a subgraph of  $P_{T_B}(T)$ . Hence  $cr(P_{T_B}(T)) > 1$ , a contradiction.

Conversely, suppose  $T$  satisfies the given condition and assume  $T$  has a unique point  $v$  of degree 4. The lines which are blocks in  $T$  such that they are the points in  $T_B(T)$ . In  $T_B(T)$ , these block points and a point  $v$  together forms an induced subgraph as  $K_5$ . In forming  $P_{T_B}(T)$ , the pathos points are adjacent to at most two points of this induced subgraph. Hence in all these cases the  $cr(P_{T_B}(T)) = 1$ . This completes the proof.  $\square$

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