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Duality Theorems of Multiobjective Generalized Disjunctive Fuzzy Nonlinear Fractional Programming

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Abstract: This paper is concerned with the study of duality conditions to convex-concave generalized multiobjective fuzzy nonlinear fractional disjunctive programming problems for which the decision set is the union of a family of convex sets. The Lagrangian function for such problems is defined and the Kuhn-Tucker Saddle and Stationary points are characterized. In addition, some important theorems related to the Kuhn-Tucker problem for saddle and stationary points are established. Moreover, a general dual problem is formulated together with weak; strong and converse duality theorems are proved.

Key Words: Generalized multiobjective fractional programming; Disjunctive programming; Convexity; Concavity; fuzzy parameters Duality.

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§1. Introduction

Fractional programming models have been became a subject of wide interest since they provide a universal apparatus for a wide class of models in corporate planning, agricultural planning, public policy decision making, and financial analysis of a firm, marine transportation, health care, educational planning, and bank balance sheet management. However, as is obvious, just considering one criterion at a time usually does not represent real life problems well because almost always two or more objectives are associated with a problem. Generally, objectives conflict with each other; therefore, one cannot optimize all objectives simultaneously. Non-differentiable fractional programming problems play a very important role in formulating the set of most preferred solutions and a decision maker can select the optimal solution.

Chang in [8] gave an approximate approach for solving fractional programming with absolute-value functions. Chen in [10] introduced higher-order symmetric duality in non-differentiable multiobjective programming problems. Benson in [6] studied two global optimization problems, each of which involves maximizing a ratio of two convex functions, where at least one of the two convex functions is quadratic form. Frenk in [12] gives some general results of the above Benson problem. The Karush-Kuhn-Tucker conditions in an optimization problem with interval-valued objective function are derived by Wu in [29].

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Balas introduced Disjunctive programs in [3, 4,]. The convex hull of the feasible points has been characterized for these programs with a class of problems that subsumes pure mixed integer programs and for many other non-convex programming problems in [5]. Helbig presented in [17, 18] optimality criteria for disjunctive optimization problems with some of their applications. Gugat studied in [15, 16] an optimization a problem having convex objective functions, whose solution set is the union of a family of convex sets. Grossmann proposed in [14] a convex nonlinear relaxation of the nonlinear convex disjunctive programming problem. Some topics of optimizing disjunctive constraint functions were introduced in [28] by Sherali. In [7], Ceria studied the problem of finding the minimum of a convex function on the closure of the convex hull of the union of a finite number of closed convex sets. The dual of the disjunctive linear fractional programming problem was studied by Patkar in [25]. Eremin introduced in [11] disjunctive Lagrangian function and gave sufficient conditions for optimality in terms of their saddle points. A duality theory for disjunctive linear programming problems of a special type was suggested by Gon?alves in [13].

Liang In [21] gave sufficient optimality conditions for the generalized convex fractional programming. Yang introduced in [30] two dual models for a generalized fractional programming problem. Optimality conditions and duality were considered in [23] for nondifferentiable, multiobjective programming problems and in [20, 22] for nondifferentiable, nonlinear fractional programming problems. Jain et al in [19] studied the solution of a generalized fractional programming problem. Optimality conditions in generalized fractional programming involving nonsmooth Lipschitz functions are established by Liu in [23]. Roubi [26] proposed an algorithm to solve generalized fractional programming problem. Xu [31] presented two duality models for a generalized fractional programming and established its duality theorems. The necessary and sufficient optimality conditions to nonlinear fractional disjunctive programming problems for which the decision set is the union of a family of convex sets were introduced in [1]. Optimality conditions and duality for nonlinear fractional disjunctive minimax programming problems were considered in [2]. In this paper we define the Langrangian function for the nonlinear generalized disjunctive multiobjective fractional programming problem and investigate optimality conditions. For this class of problems, the Mond-Weir and Schaible type of duality are proposed. Weak, strong and converse duality theorems are established for each dual problem.

§2. Problem Statement

Assume that $N = \{1, 2, \cdots, p\}$ and $\mathcal{K} = \{1, 2, \cdots, q\}$ are arbitrary nonempty index sets. For $i \in N$, let $g_j^i : \mathbf{R}^n \to \mathbf{R}$ be a vector map whose components are convex functions, $g_j^i(x) \leq 0$, $1 \leq j \leq m$. Suppose that $f_r^{ik},\ h_r^{i+m+k} : \mathbf{R}^{n+q} \to \mathbf{r}$ are convex and concave functions for $i \in N,\ k \in \mathcal{K},\ r = 1, \cdots, s$ respectively, and $h_r^{ik}(x, \widetilde{b}_r) > 0$. Here, these $\widetilde{a}_r, \widetilde{b}_r,\ r = 1, 2, \cdots, m$ represent the vectors of fuzzy parameters in the objectives functions. These fuzzy parameters are assumed to be characterized as fuzzy numbers [4].

We consider the generalized disjunctive multiobjective convex-concave fractional program problem as in the following form:

$$\text{GDFFVOP}(i) \qquad \inf_{x\in Z_i}\max_{k\in K}\left\{\frac{f_r^{ik}(x,\widetilde{a}_r)}{h_r^{ik}(x,\widetilde{b}_r)},\quad r=1,2,\cdots,s\right\}, \tag{1}$$

$$\text{Subject to} \qquad \mathrm{x}\in \mathrm{Z_i},\ \mathrm{i}\in \mathrm{N}, \tag{2}$$

where $Z_i=\{x\in \mathbf{R}^n:\ g_j^i(x)\le 0,\ j=1,2,\cdots,m\}$. Assume that $Z_i\neq\emptyset$ for $i\in N$.

Definition 1([1]) *The α-level set of the fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$ are defined as the ordinary set $S_\alpha(\widetilde{a},\widetilde{b})$ for which the degree of their membership functions exceeds level α:*

$$S_\alpha(\widetilde{a},\widetilde{b})=\{(a,b)\in \mathbf{R}^{2m}|\mu_{ar}(a_r)\ge\alpha,\ r=1,2,\cdots,m\}.$$

For a certain degree of α, the GDFVOP(i) problem can be written in the ordinary following form [11].

Lemma 1([7]) *Let $\alpha^k,\beta^k,\ k\in\mathcal{K}$ be real numbers and $\alpha^k>0$ for each $k\in K$. Then*

$$\max_{k\in K}\frac{\beta^k}{\alpha^k}\ge\frac{\sum\limits_{k\in K}\beta^k}{\sum\limits_{k\in K}\alpha^k}. \tag{3}$$

By using Lemma 1 and from [9] The generalized multiobjective fuzzy fractional problem GDFFVOP(i) may be reformulated [3] as in the following two forms:

GDFFNLP(i,t,α):

$$\inf_{i\in N}\inf_{x\in Z_i(S)}\left\{\frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K}t^kf_r^{ik}(x,a_r)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K}t^kh_r^{ik}(x,b_r)},\ (a_r,b_r)\in S_\alpha(\widetilde{a},\widetilde{b}),r=1,2,\cdots,m\right\}, \tag{4}$$

where $t^k\in\mathbf{R}_+^q$. Denote by

$$M_i=\inf_{x\in Z_i}\frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K}t^kf_r^{ik}(x,a_r)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K}t^kh_r^{ik}(x,b_r)},\ (a_r,b_r)\in S_\alpha(\widetilde{a},\widetilde{b}),r=1,2,\cdots,m$$

the minimal value of GDFFNLP(i,t,α), and let

$$P_i=\left\{x\in Z_i:\ \frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K}t^kf_r^{ik}(x,a_r)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K}t^kh_r^{ik}(x,b_r)}=M_i,\ i\in N\right\}$$

be the set of solutions of GDFFNLP(i,t,α). The generalized multiobjective disjunctive fuzzy fractional programming problem is formulated as:

$$\text{GDFFNLP}(t,\alpha): \qquad \inf_{i\in N}\inf_{x\in Z}\left\{\frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K}t^kf_r^{ik}(x,a_r)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K}t^kh_r^{ik}(x,b_r)}\right\}, \tag{5}$$

where $t^k \in \mathbf{R}^q_+, k \in \mathcal{K}$ and $Z = \bigcup_{i \in N} Z_i$ is the feasible solution set of problem GDFFNLP(t, α). For problem GDFFNLP(t, α), we assume the following sets:

(I) $M = \inf_{i \in N} M_i$ is the minimal value of GDFFNLP(t, α).

(II) $Z^* = \left\{ x \in Z : \exists i \in I(X),\ \inf_i \frac{\sum_{r=1}^{s} \sum_{k=1}^{K} t^k f_r^{ik}(x, a_r)}{\sum_{r=1}^{s} \sum_{k=1}^{K} t^k h_r^{ik}(x, b_r)} = M \right\}$ is set of these of solutions on the problem GDFFNLP(t, α), where $I = \{i \in I' : x \in Z\}$, $I' = \{i \in N :\ Z^* \neq \emptyset\}$ and $I' = \{1, 2, \cdots, a\} \subset N$. Problem GDFFVOP$(t, \alpha)$ may be reformulated in the following form:

GDFFNLP(t, α, d):

$$\inf_{i \in I} \inf_{x \in Z} \left\{ F^i(x, t, d^i, a, b) = \sum_{r=1}^{s} \sum_{k=1}^{K} t^k f_r^{ik}(x, a_r) - d^i \sum_{r=1}^{s} \sum_{k=1}^{K} t^k h_r^{ik}(x, b_r) \right\}, \tag{6}$$

where

$$d^i = \frac{\sum_{r=1}^{s} \sum_{k=1}^{K} t^k f_r^{ik}(x, a_r)}{\sum_{r=1}^{s} \sum_{k=1}^{K} t^k h_r^{ik}(x, b_r)} > 0, \quad i \in I.$$

We define the Lagrangian functions of problems GDFFNLP(t, α, d) and GDFFNLP(t, α) [21, 24, and 25] in the following forms:

$$GL^i(x, \lambda^i, a, b) = F^i(x, t, d^i, a, b) + \lambda \sum_{j=1}^{m} \lambda_j^i g_j^i(x) \tag{7}$$

and

$$L^i(x, u, \lambda^i, a, b) = \frac{u^i \sum_{r=1}^{s} \sum_{k=1}^{K} t^k f_r^{ik}(x, a_r) + \sum_{j=1}^{m} \lambda_j^i g_j^i(x)}{u^i \sum_{r=1}^{s} \sum_{k=1}^{K} t^k h_r^{ik}(x, b_r)}, \tag{8}$$

where $\lambda_j^i \geq 0$ and $u^i \geq 0$, $i \in I$ are Lagrangian multipliers. Then the Lagrangian functions $GL(x, \lambda, a, b)$ and $L(x, u, \lambda, a, b)$ of GDFFNLP(t, α, d) are defined by:

$$GL(x, \lambda, a, b) = \inf_{i \in I} GL^i(x, \lambda^i, a, b) = \inf_{i \in I} \left\{ F^i(x, t, d^i, a, b) + \sum_{j=1}^{m} \lambda_j^i g_j^i(x) \right\} \tag{9}$$

and

$$L(x, u, \lambda, a, b) = \inf_{i \in I} L^i(x, u, \lambda^i, a, b) = \inf_{i \in I} \left\{ \frac{u^i \sum_{r=1}^{s} \sum_{k=1}^{K} t^k f_r^{ik}(x, a_r) + \sum_{j=1}^{m} \lambda_j^i g_j^i(x)}{u^i \sum_{r=1}^{s} \sum_{k=1}^{K} t^k h_r^{ik}(x, b_r)} \right\}, \tag{10}$$

where $x \in Z$, $t^k \in \mathbf{R}^q_+$, $u \in \mathbf{R}^q_+$ and $\lambda \in \mathbf{R}^q_+$ are Lagrangian multipliers, respectively.

§3. Optimality Theorems with Differentiability

Definition 3.1 *A point $(x^0, \lambda^0, a^), b^))$ in R^{n+p+2m} with $\lambda^) \geq 0$ is said to be a GL-saddle point of problem GDFFNLP(t, α, d) if and only if*

$$GL(x^0, \lambda, a, b) \leq GL(x^0, \lambda^0, a^0, b^0) \leq GL(x, \lambda^0, a^0, b^0) \tag{11}$$

for all with $x \in \mathbf{R}^{n+p}$ and $\lambda \in \mathbf{R}^m_+$.

Definition 3.1 *A point $(^0, u^0, \lambda^0)$ in $\mathbf{R}^{n+p+m}$, with $u^0 \geq 0$ and $\lambda^0 \geq 0$ is said to be an L-saddle point of problem GDFFNLP(t, α) if and only if*

$$L(x^0, \lambda, a, b) \leq L(x^0, \lambda^0, a^0, b^0) \leq L(x, \lambda^0, a^0, b^0) \tag{12}$$

for all with $x \in \mathbf{R}^{n+p}$, $u \in \mathbf{R}^m_+$ and $\lambda \in \mathbf{R}^m_+$.

The proof of the following theorems follows as in [3].

Theorem 3.1(Sufficient Optimality Criteria) *If for $d^{0i} \geq 0$ the point $(x^0, u^0, \lambda^0, a^0, b^0)$ is a saddle point of $GL(x, \lambda, a, b)$ and $F^i(x, t, d^{0i}, a^0, b^0)$, $g^i_j(x)$ are bounded and convex functions. Then x^0 is a minimal solution for the problem GDFFNLP(t, d).*

Corollary 3.1 *If the point $(x^0, u^0, \lambda^0, a^0, b^0)$ is a saddle point of $L(x, u, \mu)$ and $F^i(x, t, d^i, a^0, b^0)$, $g^i_j(x)$ are bounded and convex functions. Then x^0 is a minimal solution for the problem GDFFNLP(t, α).*

The proof is follows similarly as proof of Theorem 3.1.

Assumption 3.1 Let $F^i(x, y, d^i, a, b) = 0$ be a convex function on $Conv\ Z$ ($Z = \bigcup\limits_{i \in I}$). If for all $x \in Conv\ Z$, the functions $F^i(x, t^0, d^{0i}, a^0, b^0) - F^i(x^0, t^0, d^{0i}, a^0, b)$, $x^0 \in Conv\ Z$, $i \in I$, $t^0 \in \mathbf{R}^q_+$ and $(a^0, b^0) \in \mathbf{R}^{2m}$ are bounded, then $\inf\limits_{i \in I} \left\{F^i(x, t^0, d^{0i}, a^0, b) - F^i(x^0, t^0, d^{0i}, a^0, b)\right\}$ is a convex function on $Conv\ Z$.

Proposition 3.1 *Under the Assumption 3.1, and if the system*

$$\left.\begin{array}{l} \inf\limits_{i \in I} F^i(x, t^0, d^{0i}, a^0, b) - F^i(x^0, t^0, d^{0i}, a^0, b^0) < 0, \\ g^i_j(x) \leq 0 \ for\ at\ least\ one\ i \in I \end{array}\right\}$$

has no solution on $Conv\ Z$, then $\exists \lambda^0 \in \mathbf{R}_+$, $\lambda^{0i} \in \mathbf{R}^m_+$, $(\lambda^0, \lambda^{0i}) \geq 0$ and $t^0 \in \mathbf{R}^q_+$ such that

$$\mu^0 \inf_{i \in i} F^i(x, t^0, d^{0i}, a^0, b^0) + \inf_{i \in i} \sum_{j=1}^{m} \mu^{0i}_j g^i_j(x) \geq 0$$

for $\forall x \in Conv\ Z$.

Corollary 3.2 *With Assumption 3.1, $g^i_j(x)$, $i \in I$, $j = 1, 2, \cdots, m$ satisfy the CQ and x^0 is an optimal solution of problem GDFFNLP(t, α), then there exists $u^0 \geq 0$ and $\lambda^0 \geq 0$ such that $(x^0, t^0, \lambda^0, a^0, b^0)$ is a saddle point of $L(x^0, t^0, \lambda^0, a^0, b^0)$.*

The proof is follows similarly as proof of Theorem 3.2.

§4. Optimality Theorems without Differentiability

Definition 4.1 *The point* $(x^0, \lambda^0, a^0, b^0)$, $x^0 \in x \in \mathbf{R}^{n+p}$, $\lambda^0, a^0, b^0 \in \mathbf{R}^{3m}$, *if they exist such that*

$$\nabla_x GL(x^0, \lambda^0, a^0, b^0) \geq 0, \qquad x^0 \nabla_x GL(x^0, \lambda^0, a^0, b^0) = 0, \tag{13}$$

$$\nabla_{\lambda x} GL(x^0, \lambda^0, a^0, b^0) \geq 0, \qquad \lambda^0 \nabla_\lambda GL(x^0, \lambda^0, a^0, b^0) = 0, \tag{14}$$

$$\sum_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0) = 0, \quad \lambda_j^i \geq 0, \qquad i \in I, \quad j = 1, 2, \cdots, m. \tag{15}$$

is could Kuhn- Tucker stationary point of problem GDFFNLP(t^0, α^0, d^0). *Or, equivalently,*

$$\nabla_x \inf_{i \in I} \left\{ F^i(x^0, t^0, d^{0i}, a^0, b^0) + \lambda_j^{0i} g_j^i(x^0) \right\} = 0, \quad i \in I, \tag{16}$$

$$g_j^i(x^0) \leq 0, \quad i \in I, \quad j = 1, 2, \cdots, m, \tag{17}$$

$$\sum_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0) = 0, \quad \lambda_j^i \geq 0, \qquad i \in I, \quad j = 1, 2, \cdots, m. \tag{18}$$

Definition 4.2 *The point* $(x^0, u^0, \lambda^0, a^0, b^0)$, $x \in \mathbf{R}^{n+p+2m}$, $u \in \mathbf{R}_+^q$ *and* $\lambda \in \mathbf{R}_+^m$, *if they exist such that*

$$\nabla_x L(x^0, u^0, \lambda^0, a^0, b^0) \geq 0, \qquad x^0 \nabla_x L(x^0, u^0, \lambda^0, a^0, b^0) = 0, \tag{19}$$

$$\nabla_u L(x^0, u^0, \lambda^0, a^0, b^0) \geq 0, \qquad u^0 \nabla_\lambda L(x^0, u^0, \lambda^0, a^0, b^0) = 0, \tag{20}$$

$$\nabla_\mu L(x^0, u^0, \lambda^0, a^0, b^0) \geq 0, \qquad \mu^0 \nabla_\lambda L(x^0, u^0, \lambda^0, a^0, b^0) = 0, \tag{21}$$

$$\sum_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0) = 0, \quad \lambda_j^i \geq 0, \qquad i \in I, \quad j = 1, 2, \cdots, m. \tag{22}$$

is could Kuhn- Tucker stationary point of problem GDFFNLP(t^0, α^0). *Or, equivalently,*

$$\nabla_x \inf_{i \in I} \left\{ \frac{u^{0i} \sum\limits_{r=1}^{s} \sum\limits_{k=1}^{K} t^{0k} f_r^{ik}(x^0, a^0) + \sum\limits_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0)}{u^{0i} \sum\limits_{r=1}^{s} \sum\limits_{k=1}^{K} t^{0k} h_r^{ik}(x^0, b^0)} \right\} = 0, \tag{23}$$

$$g_j^i(x^0) \leq 0, \quad i \in I, \quad j = 1, 2, \cdots, m, \tag{24}$$

$$\sum_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0) = 0, \quad \lambda_j^i \geq 0, \qquad i \in I, \quad j = 1, 2, \cdots, m. \tag{25}$$

The proof of the following theorem follows as in [3].

Theorem 4.1 *Assume that* $F^i(x, t, d^i, a, b)$, $g_j^i(x)$, $i \in I$, $j = 1, 2, \cdots, m$ *are convex differentiable functions on Conv S. If* $F^i(x, t, d^i, a, b)$ *and* $g_j^i(x)$ *are bounded functions for each* $x \in Cov\ S$ *and* $g_j^i(x)$ *satisfy CQ for* $i \in I$, *then* x^0 *is an optimal solution of GDFFNLP*(t, α, d) *if and only if there are Lagrange multipliers* $\lambda^0 \in \mathbf{R}^{p+m}$, $\lambda \geq 0$ *such that (13)-(15) are satisfied.*

Corollary 4.1 *Suppose that $F^i(x,t,d^i,a,b)$, $g^i_j(x)$, $i \in I$, $j = 1,2,\cdots,m$ are convex differentiable functions on Conv S. If $F^i(x,t,d^i,a,b)$ and $g^i_j(x)$ are bounded functions for each $x \in Cov\ S$ and $g^i_j(x)$ satisfy CQ for $i \in I$, then x^0 is an optimal solution of GDFFNLP(t,α,d) if and only if there are Lagrange multipliers $u^0 \geq 0$, $\lambda \geq 0$, $u \in \mathbf{R}^q_+$ and $\lambda^0 \in \mathbf{R}^{p+m}$ such that (19)-(22) are satisfied.*

The proof is follows similarly as the proof of Theorem 4.1.

Theorem 4.2 *Assume that $F^i(x,t,d^i,a,b)$ is a pseudoconvex function at $x \in Conv\ S$ and that $\sum\limits_{j=1}^{m} \lambda^i_j g^i_j(x)$ is a quasiconvex function. If $F^i(x,t,d^i)$ and $g^i_j(x)$ are bounded functions for each $x \in Conv\ S$, and if the equations (28)-(30) are satisfied for $tin\mathbf{R}^k_+$ and $\lambda^0 \in \mathbf{R}^{p+m}_+$, then x^0 is an optimal solution of GDFFNL(t,α,d).*

Corollary 4.2 *Assume that $F^i(x,t,d^i,a,b)$ is a pseudoconvex function at $x \in Conv\ S$ and that $\sum\limits_{j=1}^{m} \lambda^i_j g^i_j(x)$ is a quasiconvex function. If $F^i(x,t,d^i,a,b)$ and $g^i_j(x)$ are bounded functions for each $x \in Conv\ S$ and there exists $u^0 \in \mathbf{R}^k_+$ and $\lambda^0 \in \mathbf{R}^{p+m}_+$ such that equations (16)-(18) are satisfied, then x^0 is an optimal solution of GDFFNLP(t,α,d).*

The proof is follows similarly as proof of Theorem 4.2.

§5. Duality Using Mond-Weir Type

According to optimality Theorems 4.1 and 4.2, we can formulate the Mond-Weir type dual (M-WDGF) of the disjunctive fractional minimax problem GDFFNLP(t,α,d) as follows:

$$\text{M}-\text{WDGF} \quad \max_{y\in\mathbf{R}^n}\sup_{i\in I}\left(H^i(y,t,\alpha,D,a,b) = \sum_{r=1}^{s}\sum_{k=1}^{K} t^k f^{ik}_r(y,a) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h^{ik}_r(y,b)\right), \tag{26}$$

where

$$D^i = \frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k f^{ik}_r(y,a)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k h^{ik}_r(y,b)} > 0, \qquad i \in I.$$

Problem (M-WDGF) satisfies the following conditions:

$$\sup_{i\in I}\nabla_y\left\{H(y,t,D,a,b) + \sum \lambda^i_j g^i_j(y)\right\} = 0, \tag{27}$$

$$\sum_{j=1}^{m} \lambda^i_j g^i_j(y) = 0, \quad \lambda^i_j \geq 0, \quad i \in I, \quad j = 1,2,\cdots,m, \tag{28}$$

$$\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f^{ik}_r(y,a) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h^{ik}_r(y,b) \geq 0, \quad i \in I, \quad D^i > 0. \tag{29}$$

Theorem 5.1(Weak Duality) *Let x be feasible for GDFFNLP(t,α,d) and (u,λ,t,a,b) be feasible for (M-WDGFD). If for all feasible (y,λ,t,a,b), $H^i(y,t,\alpha,D,a,b)$ are pseudoconvex* *for each* $i \in I$*, and* $\sum_{j=1}^{m} \lambda_j^i g_j^i(y)$ *are quasiconvex for* $i \in I$*, then* $\inf(GDFFNLP(t,\alpha,d)) \geq \sup(M - WDGF)$.

Proof If not, then there must be that

$$\inf_{i\in I}\inf_{x\in Z}\left(\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(x) - d^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(x)\right) < \sup_{i\in I}\sup_{y\in \mathbf{R}^n}\left(\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y)\right).$$

Hence, for $i \in I$, we get that

$$\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(x) - d^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(x) < \sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y). \tag{30}$$

and by the pseudoconvexity of $H^i(y,t,D)$, (30) implies that

$$(x-y)^t \nabla_x \left(\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y)\right) < 0. \tag{31}$$

Equation (31) implies that

$$\sup_{i\in I}\sup_{y\in \mathbf{R}^n}\left((x-y)^t \nabla_x \left(\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y)\right)\right) < 0. \tag{32}$$

From equation (27) and inequality (32) it follows that

$$\sup_{i\in I}\left\{(x-y)^t \nabla_x \sum_{j=1}^{m} \mu_j^i g_j^i(y)\right\} > 0. \tag{33}$$

By (26), inequality (33) implies that

$$\sup_{i\in I}\sum_{j=1}^{m} \mu_j^i g_j^i(x) > \sup_{i\in I}\sum_{j=1}^{m} \mu_j^i g_j^i(u) > 0.$$

Then $\sum_{j=1}^{m} \mu_j^i g_j^i(x) > 0$, which contradicts the assumption that x is feasible with respect to GDFFNLP(t,α,d). □

Theorem 5.2(Strong Duality) *If* x^0 *is an optimal solution of GDFFNLP*(t,α,d) *and CQ is satisfied, then there exists* $(y^0,\lambda^0,t^0,a^0,b^0) \in \mathbf{R}^{n+m}$ *is feasible for (M-WDGF) and the corresponding value of* $\inf(GDFFNLP(t,\alpha,d)) = \sup(M - WDGF)$.

Proof Since x^0 is an optimal solution of DGFFNLP(t^0,α^0,d^0) and satisfy CQ, then there is a positive integer $\lambda_j^{*i} \geq 0$, $i \in I$, $j = 1,2,\cdots,m$ such that Kuhn-Tucker conditions (27)-(29)

are satisfied. Assume that $\lambda^0 = \tau^{-1}\lambda^*$ in the Kuhn-Tucker stationary point conditions. It follows that $(y^0, \lambda^0, t^0, a^0, b^0)$ is feasible for (M-WDGF). Hence

$$\inf_{i\in I}\left(\frac{\sum_{k=1}^{K} t^{0k} f_r^{ik}(x^0, a^0)}{\sum_{r=1}^{s}\sum_{k=1}^{K} t^{0k} h_r^{ik}(x^0, b^0)}\right) = \sup_{i\in I}\left(\frac{\sum_{k=1}^{K} t^{0k} f_r^{ik}(y^0, a^0)}{\sum_{r=1}^{s}\sum_{k=1}^{K} t^{0k} h_r^{ik}(y^0, b^0)}\right). \qquad \square$$

Theorem 5.3(Converse Duality) *Let x^0 be an optimal solution of DGFFNLP(t^0, α^0, d^0) and CQ is satisfied. If (y^*, μ^*) is an optimal solution of (M-WDFD) and $H^i(y^*, t^*, D^*)$ is strictly pseudoconvex at y^*, then $y^* = x^0$ is an optimal solution of GDFFNLP(t, α, d).*

Proof Let x^0 be an optimal solution of DGFFNLP(t^0, α^0, d^0) and CQ is satisfied. Assume that $y^* \neq x^0$. Then $(y*, \mu^*)$ is an optimal solution of (M-WDGF). Whence,

$$\inf_{i\in I}\inf_{k\in K} F^i(x^0, t^0, d^{0i}) = \sup_{i\in I}\sup_{k\in K} H^i(y^*, t^*, D^{*i}) \tag{34}$$

Because (y^*, μ^*) is feasible with respect to (M-WDGF), it follows that

$$\sum_{j=1}^{m} \mu_j^{*i} g_j^i(x^0) \leq \sum_{j=1}^{m} \mu_j^{*i} g_j^i(y^*).$$

Quasiconvexity of $\sum_{j=1}^{m} \mu_j^{*i} g_j^i(x)$ implies that

$$\sup_{i\in i}(x^0 - y^*)\sum_{j=1}^{m} \nabla_x \mu_j^{*i} g_j^i(y^*) \leq 0. \tag{35}$$

From (34) and (35), it follows that

$$\sup_{i\in i}(x^0 - y^*)\nabla_y H^i(y^*, t^*, D^{*i}) \geq 0. \tag{36}$$

From (36) and the strict pseudoconvexity of at y^*, it follows that

$$\sup_{i\in i} \nabla_x F^i(x^0, t^0, d^{0i}) > \sup_{i\in i} \nabla_y H^i(y^*, t^*, D^{*i}).$$

This contradicts to (35). Hence $y^* = x^0$ is an optimal solution of DGFFNLP(t^0, α^0, d^0). $\square$

§6. **Duality Using Schaible Formula**

The Schaible dual of GDFFNLP(t, α, d) has been formulated in [27] as follows:

(SGD) $\qquad \max_{(y,\mu)\in \mathbf{R}^{n+m}} D,$

where $(y, \mu) \in \mathbf{R}^n \times \mathbf{R}_+^m$ satisfying:

$$\sup_{i \in I} \nabla_x \left\{ \sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) + \sum_{j=1}^{m} \mu_j^i g_j^i(y) \right\} = 0, \tag{37}$$

$$\sum_{j=1}^{m} \mu_j^i g_j^i(y) \geq 0, \quad i \in I, \tag{38}$$

$$\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) \geq 0, \quad i \in i, \tag{39}$$

$$D^i \geq 0 \quad \text{and} \quad \mu_j^i \geq 0, \quad i \in I, \quad j = 1, 2, \cdots, m. \tag{40}$$

Theorem 6.1(Weak Duality) *Let x be feasible with respect to GDFFNLP(t, α, d). If for all feasible (y, μ), $\sup_{i \in I} H^i(y, t, d)$ is pseudoconvex at u and $\sup_{i \in I} \sum_{j=1}^{K} \mu_j^i g_j^i(y)$ is quasiconvex, then $\inf GDFFNLP(t, \alpha, d) \geq \sup(SGD)$.*

Proof For each $i \in I$, suppose that

$$\frac{\sum_{k=1}^{K} t^k f^{ik}(y)}{\sum_{k=1}^{K} t^k h^{ik}(y)} < D^i.$$

Hence, for each $y \in \mathbf{R}^n$ and $i \in I$, we get that

$$\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) < 0.$$

Therefore,

$$\sup_{i \in I} \left(\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) \right) < 0. \tag{41}$$

From (39) and (41) with $t \neq 0$, we have

$$\left(\sum_{k=1}^{K} t^k f^{ik}(x) - d^i \sum_{k=1}^{K} t^k h^{ik}(x) \right) < \left(\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) \right).$$

By the pseudoconvexity of $\sup_{i \in I} H^i(y, t, D)$ at u, it follows that

$$(x - y)^T \left(\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) \right) < 0. \tag{42}$$

Consequently, (38) and (42) yield that

$$(x - y)^T \sum_{j=1}^{m} \mu_j^i \nabla_x g_j^i(y) > 0. \tag{43}$$

and, by the quasiconvexity of $\sum\limits_{j=1}^{m} \mu_j^i g_j^i(y)$, inequality (43) implies that

$$\sum_{j=1}^{m} \mu_j^i g_j^i(x) > \sum_{j=1}^{m} \mu_j^i g_j^i(y). \tag{44}$$

From inequalities (38) and (44) it follows that

$$\sum_{j=1}^{m} \mu_j^i g_j^i(x) > 0. \tag{45}$$

But, from the feasibility of $x \in S$ and $\mu_j^i \geq 0$, $i \in I$, $j = 1, 2, \cdots, m$, (1) implies that $\sum\limits_{j=1}^{m} \mu_j^i g_j^i(x) \leq 0$, this contradicts (45). Hence,

$$\frac{\sum\limits_{k=1}^{K} t^k f^{ik}(y)}{\sum\limits_{k=1}^{K} t^k h^{ik}(y)} \geq D^i,$$

i.e., $\inf GDFFNLP(t, \alpha, d) \geq \sup(SGD)$. □

Theorem 6.2(Strong Duality) *Let x^0 be an optimal solution of GDFFNLP(t, α, d) so that CQ is satisfied. Then there exists (y^0, μ^0) is feasible for (SDD) and the corresponding value of* $\inf GDFFNLP(t, \alpha, d) = \sup(SDD)$. *If, in addition, the hypotheses of Theorem 6.1 are satisfied, then (x^0, μ^0) is an optimal solution of (SDD).*

Proof The proof is similar to that of Theorem 5.2. □

Theorem 6.3(Converse Duality) *Suppose that $x^{)}$ is an optimal solution of GDFFNLP(t, α, d) and $g_j^i(x)$ satisfy CQ. Let the hypotheses of the above Theorem 6.1 hold. If (y^*, μ^*) is an optimal solution of (SDD) and is strictly pseudocovex at y^*, then $y^* = x^0$ is an optimal solution of DGFFNLP(t^0, α^0, d^0).*

Proof Assume that $y^* \neq x^0$, x^0 is an optimal solution DGFFNLP(t^0, α^0, d^0) and try to find a contraction. From Theorem 4.2, for each $i \in I$, it follows that

$$\frac{\sum\limits_{k=1}^{K} t^{0k} f^{ik}(x^0)}{\sum\limits_{k=1}^{K} t^{0k} h^{ik}(x^0)} = d^{0i}. \tag{46}$$

Applying (1) with (38) we get that

$$\sum_{j=1}^{m} \mu_j^{*i} g_j^i(x^0) \leq \sum_{j=1}^{m} \mu_j^{*i} g_j^i(y^*).$$

By quasiconvexity of $\sum\limits_{j=1}^{m} \mu_j^{*i} g_j^i(x)$ and for each $i \in I$, it follows that

$$(x^0 - y^*) \sum_{j=1}^{m} \nabla_x \mu_j^{*i} g_j^i(y^*) \leq 0. \tag{47}$$

From (37) and (47) it follows that

$$(x^0 - y^*)\nabla_x \left(\sum_{k=1}^{K} t^{*k} f^{ik}(y^*) - D^{*i} \sum_{k=1}^{K} t^{*k} h^{ik}(y^*) \right) \leq 0. \tag{48}$$

From (39), (48) and the strict pseudoconvexity of $\left(\sum\limits_{k=1}^{K} t^{*k} f^{ik}(y) - D^{*i} \sum\limits_{k=1}^{K} t^{*k} h^{ik}(y) \right)$ for each $i \in I$ at y^*, it follows that

$$\left(\sum_{k=1}^{K} t^{0k} f^{ik}(x^0) - d^{0i} \sum_{k=1}^{K} t^{0k} h^{ik}(x^0) \right) > \left(\sum_{k=1}^{K} t^{*k} f^{ik}(y^*) - D^{*i} \sum_{k=1}^{K} t^{*k} h^{ik}(y^*) \right). \tag{49}$$

Inequality (49) implies that

$$\left(\sum_{k=1}^{K} t^{0k} f^{ik}(x) - d^{0i} \sum_{k=1}^{K} t^{0k} h^{ik}(x) \right) > 0, \quad i \in I. \tag{50}$$

i.e., for each $i \in I$ it is follows that

$$\frac{\sum\limits_{k=1}^{K} t^{0k} f^{ik}(x)}{\sum\limits_{k=1}^{K} t^{0k} h^{ik}(x)} > d^{0i}. \tag{51}$$

Consequently,

$$\frac{\sum\limits_{k=1}^{K} t^{0k} f^{ik}(x^0)}{\sum\limits_{k=1}^{K} t^{0k} h^{ik}(x^0)} \geq \frac{\sum\limits_{k=1}^{K} t^{0k} f^{ik}(x)}{\sum\limits_{k=1}^{K} t^{0k} h^{ik}(x)} > d^{0i}, \tag{52}$$

contradicts to that (46). So that $y^* = x^0$ is an optimal solution of DGFFNLP(t^0, α^0, d^0). $\square$

§7. Conclusion

This paper addresses the solution of generalized multiobjective disjunctive programming problems, which corresponds to minmax continuous optimization problems that involve disjunctions with convex-concave nonlinear fractional objective functions. We use Dinkelbach‘s global approach for finding the maximum of this problem. We first describe the Kuhn-Tucker saddle point of nonlinear disjunctive fractional minmax programming problems by using the decision set that is the union of a family of convex sets. Also, we discuss necessary and sufficient optimality conditions for generalized nonlinear disjunctive fractional minmax programming problems. For the class of problems, we study two duals; we propose and prove weak, strong and converse duality theorems.

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