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Common Fixed Points for Pairs of Weakly Compatible Mappings

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Abstract: In this note we establish a common fixed point theorem for a quadruple of self mappings satisfying a common (E.A) property on a metric space satisfying weakly compatibility and a generalized Φ- contraction. Our results improve and extend some known results.

Key Words: Common fixed points, weakly compatible mappings, generalized Φ- contraction, a common (E.A) property, Smarandache metric multi-space.

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§1. Introduction

For an integer $n \geq 1$, a Smarandache metric multi-space $\widetilde{S}$ is a union $\bigcup\limits_{i=1}^{n} A_i$ of spaces $A_1, A_2, \cdots$, A_n, distinct two by two with metrics $\rho_1, \rho_2, \cdots, \rho_n$ such that (A_i, ρ_i) is a metric space for integers $1 \leq i \leq n$. In 1986, the notion of compatible mappings which generalized commuting mappings, was introduced by Jungck [3]. This has proven useful for generalization of results in metric fixed point theory for single-valued as well as multi-valued mappings. Further in 1998, the more general class of mappings called weakly compatible mappings was introduced by Jungck and Rhoades [4]. Recall that self mappings S and T of a metric space (X, d) are called weakly compatible if $Sx = Tx$ for some $x \in X$ implies that $STx = TSx$.

Recently Aamri et al. [1] introduced the following notion for a pair of maps as:

Definition 1.1 *Let S and T be two self mappings of a metric space (X, d). S and T are said to satisfy the property (E.A), if there exists a sequence $\{x_n\}$ in X such that* $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$, *for some* $t \in X$.

Most recently, Y. Liu et al. [5] defined a common property (E.A) for pairs of mappings as

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follows:

Definition 1.2 *Let $A, B, S, T : X \to X$. The pairs (A, S) and (B, T) satisfy a common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X.$$

If $B = A$ and $S = T$ in above, we obtain the definition of property (E.A).

Example 1.3 Let A, B, S and T be self maps on $X = [0, 1]$, with the usual metric $d(x, y) = |x - y|$, defined by:

$$Ax = \begin{cases} 1 - \frac{x}{2} & when \quad x \in [0, \frac{1}{2}), \\ 1 & when \quad x \in [\frac{1}{2}, 1]). \end{cases}$$

$$Sx = \begin{cases} 1 - 2x & when \quad x \in [0, \frac{1}{2}), \\ 1 & when \quad x \in [\frac{1}{2}, 1]). \end{cases}$$

$Bx = 1 - x$ and $Tx = 1 - \dfrac{x}{3}$, $\forall\, x \in X$. Let $\{x_n\}$ and $\{y_n\}$ be a sequences defined by $x_n = \dfrac{1}{n+1}$ and $y_n = \dfrac{1}{n^2+1}$, then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in X$. Thus a common (E.A) property is satisfied.

In this paper we prove some common fixed point theorems for a quadruple of weak compatible self mappings of a metric space satisfying a common (E.A) property, a special Smarandache metric multi-space $\bigcup_{i=1}^{n}(A_i, \rho_i)$ for $n = 1$ and a generalized Φ-contraction. These theorems extend and generalize results of Pathak et al. [6] and [7].

§2. Preliminaries

Now onwards, we denote by Φ the collection of all functions $\varphi : [0, \infty) \to [0, \infty)$ which are upper semi-continuous from the right, non-decreasing and satisfy $lim_{s\to t+}$ sup $\varphi(s) < t$, $\varphi(t) < t$ for all $t > 0$.

Let X denote a metric space endowed with metric d and let $\mathbb{N}$ denote the set of natural numbers.

Now, let A, B, S and T be self-mappings of X such that

$$\begin{aligned} &[d^p(Ax, By) + a\ d^p(Sx, Ty)]d^p(Ax, By) \\ &\quad \le a\ max\{d^p(Ax, Sx)d^p(By, Ty), d^q(Ax, Ty)d^{q'}(By, Sx)\} \\ &\quad +max\{\varphi_1(d^{2p}(Sx, Ty)), \varphi_2(d^r(Ax, Sx)d^{r'}(By, Ty)), \\ &\quad \varphi_3(d^s(Ax, Ty)d^{s'}(By, Sx)), \\ &\quad \varphi_4(\frac{1}{2}[d^l(Ax, Ty)d^{l'}(Ax, Sx) + d^l(By, Sx))d^{l'}(By, Ty)\} \end{aligned} \tag{2.1}$$

for all $x, y \in X, \varphi_i \in \Phi(i = 1, 2, 3, 4), a, p, q, q', r, r', s, s', l, l' \geq 0$ and $2p = q + q' = r + r' = s + s' = l + l'$. The condition (2.1) is commonly called a generalized Φ-contraction.

§3. Main Results

The following theorems are our main results in this section.

Theorem 3.1 *Let A, B, S and T be self mappings of a metric space (X, d) satisfying (2.1).If the pairs (A, S) and (B, T) satisfy a common (E.A) property, are weakly compatible and that $T(X)$ and $S(X)$ are closed subsets of X, then A, B, S and T have a unique common fixed point in X.*

Proof. Since (A, S) and (B, T) satisfy a common property (E.A). Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in X$. Assume that $S(X)$ and $T(X)$ are closed subspaces of X. Then, $z = Su = Tv$ for some $u, v \in X$. Then by using (2.1) with $x = x_n$ and $y = v$, we have

$$\begin{aligned}
[d^p(Ax_n, Bv) + a\ d^p(Sx_n, Tv)]d^p(Ax_n, Bv) \leq a \max\{&d^p(Ax_n, Sx_n)d^p(Bv, Tv), \\
&d^q(Ax_n, Tv)d^{q'}(Bv, Sx_n)\} + max\{\varphi_1(d^{2p}(Sx_n, Tv)), \\
&\varphi_2(d^r(Ax_n, Sx_n)d^{r'}(Bv, Tv)), \varphi_3(d^s(Ax_n, Tv)d^{s'}(Bv, Sx_n)), \\
&\varphi_4(\frac{1}{2}[d^l(Ax_n, Tv)d^{l'}(Ax_n, Sx_n) + d^l(Bv, Sx_n))d^{l'}(Bv, Tv)]),
\end{aligned}$$

taking $\lim_{n\to\infty}$, we obtain

$$\begin{aligned}
[d^p(z, Bv) + a\ d^p(z, Tv)]d^p(z, Bv) \leq a \max\{&d^p(z, z)d^p(Bv, z), d^q(z, Tv)d^{q'}(Bv, z)\} \\
&+ max\{\varphi_1(d^{2p}(z, Tv)), \varphi_2(d^r(z, z)d^{r'}(Bv, z)), \\
&\varphi_3(d^s(z, Tv)d^{s'}(Bv, z)), \varphi_4(\frac{1}{2}[d^l(z, Tv)d^{l'}(z, z) \\
&+ d^l(Bv, z))d^{l'}(Bv, z)])\},
\end{aligned}$$

or
$$d^{2p}(z, Bv) \leq max\{\varphi_1(0), \varphi_2(0), \varphi_3(0), \varphi_4(\tfrac{1}{2}d^{l+l'}(Bv, z))\},$$
or
$$\begin{aligned}
d^{2p}(z, Bv) \leq max\{&\varphi_1(d^{2p}(z, Bv)), \varphi_2(d^{r+r'}(z, Bv), \\
&\varphi_3(d^{s+s'}(z, Bv)), \varphi_4(\tfrac{1}{2}d^{l+l'}(Bv, z))\}.
\end{aligned}$$

This together with a well known result of Chang [2] which states that if $\varphi_i \in \Phi$ where $i \in I$ (some indexing set), then there exists a $\varphi \in \Phi$ such that max $\{\varphi_i, i \in I\} \leq \varphi(t)$ for all $t > 0$; imply

$$d^{2p}(z, Bv) \leq \varphi(d^{2p}(z, Bv)) \ < d^{2p}(z, Bv),$$

a contradiction. This implies that $z = Bv$. Therefore $Tv = z = Bv$. Hence it follows by the weak compatibility of the pair (B, T) that $BTv = TBv$, that is $Bz = Tz$.

Now, we shall show that z is a common fixed point of B and T. For this put $x = x_n$ and $y = z$ in (2.1), we have

$$[d^p(Ax_n, Bz) + a\ d^p(Sx_n, Tz)]d^p(Ax_n, Bz) \leq a\ \max\{d^p(Ax_n, Sx_n)d^p(Bz, Tz),$$
$$d^q(Ax_n, Tz)d^{q'}(Bz, Sx_n)\} + max\{\varphi_1(d^{2p}(Sx_n, Tz)),$$
$$\varphi_2(d^r(Ax_n, Sx_n)d^{r'}(Bz, Tz)), \varphi_3(d^s(Ax_n, Tz)d^{s'}(Bz, Sx_n)),$$
$$\varphi_4(\frac{1}{2}[d^l(Ax_n, Tz)d^{l'}(Ax_n, Sx_n) + d^l(Bz, Sx_n))d^{l'}(Bz, Tz)\}.$$

Letting $n \to \infty$ with the help of the fact that $lim_{n\to\infty}Ax_n = z = lim_{n\to\infty}Sx_n$ and $Bz = Tz$, we get

$$[d^p(z, Bz) + ad^p(z, Tz)]d^p(z, Bz) \leq a\ max\{d^p(z, z)d^p(Bz, z), d^q(z, Tz)d^{q'}(Bz, z)\}$$
$$+ max\{\varphi_1(d^{2p}(z, Tz)), \varphi_2(d^r(z, z)d^{r'}(Bz, z)), \varphi_3(d^s(z, Tz)d^{s'}(Bz, z)),$$
$$\varphi_4(\frac{1}{2}[d^l(z, Tz)d^{l'}(z, z) + d^l(Bz, z))d^{l'}(Bz, z)])\},$$

or
$$d^{2p}(z, Bz) + a\ d^{2p}(z, Bz) \leq a\ d^{q+q'}(Bz, z) + max\{\varphi_1(d^{2p}(z, Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z, Bz)), \varphi_4(0)\},$$
or
$$(1 + a)d^{2p}(z, Bz) \leq a\ d^{q+q'}(Bz, z)\} + max\{\varphi_1(d^{2p}(z, Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z, Bz)), \varphi_4(0)\},$$
or
$$d^{2p}(z, Bz) \leq \frac{a}{1+a}d^{q+q'}(Bz, z) + \frac{1}{1+a}max\{\varphi_1(d^{2p}(z, Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z, Bz)), \varphi_4(0)\} < d^{2p}(z, Bz),$$

a contradiction. So $z = Bz = Tz$. Thus z is a common fixed point of B and T.

Similarly we can prove that z is a common fixed point of A and S. Thus z is the common fixed point of A, B, S and T. The uniqueness of z as a common fixed point of A, B, S and T can easily be verified. □

Remark 3.3 Our Theorem 3.1 extends theorem 2.1 of Pathak et al. [6].

In Theorem 3.1, if we put $a = 0$ and $\varphi_i(t) = ht$ $(i = 1, 2, 3, 4)$, where $0 < h < 1$, we get the following corollary:

Corollary 3.4 *Let A, B, S and T be self mappings of a metric space X. If the pairs (A, S) and (B, T) satisfy a common (E.A) property and*

$$d^{2p}(Ax, By) \leq h\ max\{d^{2p}(Sx, Ty), d^r(Ax, Sx)d^{r'}(By, Ty), d^s(Ax, Ty)$$
$$d^{s'}(By, Sx)), \frac{1}{2}[d^l(Ax, Ty)d^{l'}(Ax, Sx) + d^l(By, Sx))d^{l'}(By, Ty)\} \quad (2.2)$$

for all $x, y \in X, \varphi_i \in \Phi$ (i=1,2,3,4), a, p, q, $q', r, r', s, s', l, l' \geq 0$ and $2p = q + q' = r + r' = s + s' = l + l'$. If the pairs (A, S) and (B, T) are weakly compatible and that $T(X)$ and $S(X)$ are closed, then A, B, S and T have a unique common fixed point in X.

Especially when

$$\max\{d^{2p}(Sx,Ty), d^r(Ax,Sx)d^{r'}(By,Ty), d^s(Ax,Ty)d^{s'}(By,Sx)),$$
$$\frac{1}{2}[d^l(Ax,Ty)d^{l'}(Ax,Sx) + d^l(By,Sx))d^{l'}(By,Ty)\} = d^{2p}(Sx,Ty),$$

it generalizes Corollary 3.9 of Pathak et al. [7].

In Theorem 3.1, if we take $S = T = I_X$ (the identity mapping on X), then we have the following corollary:

Corollary 3.5 *Let A and B be self mappings of a complete metric space X satisfying the following condition:*

$$[d^p(Ax,By) + a\ d^p(x,y)]d^p(Ax,By) \le a\ max\{d^p(Ax,x)d^p(By,y),$$
$$d^q(Ax,y)d^{q'}(By,x)\} + max\{\varphi_1(d^{2p}(x,y)), \varphi_2(d^r(Ax,x)d^{r'}(By,y)),$$
$$\varphi_3(d^s(Ax,y)d^{s'}(By,x)), \varphi_4(\tfrac{1}{2}[d^l(Ax,y)d^{l'}(Ax,x) + d^l(By,x))d^{l'}(By,y)\}$$

for all $x,y \in X, \varphi_i \in \Phi$ $(i = 1,2,3,4), a,p,q,q',r,r',s,s',l,l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$, then A and B have a unique common fixed point in X.

As an immediate consequences of Theorem 3.1 with $S = T$, we have the following:

Corollary 3.6 *Let A, B, and S be self-mappings of X such that (A,S) and (B,S) satisfy a common (E.A) property and*

$$d^{2p}(Ax,By) \le a\ max\{d^p(Ax,Sx)d^p(By,Sy), d^q(Ax,Sy)d^{q'}(By,Sx)\}$$
$$+ max\{\varphi_2(d^r(Ax,Sx)d^{r'}(By,Sy)), \varphi_3(d^s(Ax,Sy)d^{s'}(By,Sx)),$$
$$\varphi_4(\tfrac{1}{2}[d^l(Ax,Sy)d^{l'}(Ax,Sx) + d^l(By,Sx))d^{l'}(By,Sy)\} \qquad (2.3)$$

for all $x,y \in X, \varphi_i \in \Phi$ $(i = 1,2,3,4), a,p,q,q',r,r',s,s',l,l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$. If the pairs (A,S) and (B,S) are weakly compatible and that $S(X)$ is closed, then A, B and S have a unique common fixed point in X.

Theorem 3.7 *Let S, T and A_n $(n \in \mathbb{N})$ be self mappings of a metric space (X,d). Suppose further that the pairs (A_{2n-1}, S) and (A_{2n}, T) are weakly compatible for any $n \in \mathbb{N}$ and satisfying a common (E.A) property. If $S(X)$ and $T(X)$ are closed and that for any $i \in N$, the following condition is satisfied for all $x,y \in X$*

$$[d^p(A_ix, A_{i+1}y) + a\ d^p(Sx,Ty)]d^p(A_ix, A_{i+1}y)$$
$$\le a\max\{d^p(A_ix,Sx)d^p(A_{i+1}y,Ty),$$
$$d^q(A_ix,Ty)d^{q'}(A_{i+1}y,Sx)\} + max\{\varphi_1(d^{2p}(Sx,Ty)),$$
$$\varphi_2(d^r(A_ix,Sx)d^{r'}(A_{i+1}y,Ty)), \varphi_3(d^s(A_ix,Ty)d^{s'}(A_{i+1}y,Sx)),$$
$$\varphi_4(\frac{1}{2}[d^l(A_ix,Ty)d^{l'}(A_ix,Sx) + d^l(A_{i+1}y,Sx))d^{l'}(A_{i+1}y,Ty)\}$$

where $\varphi_i \in \Phi(i = 1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l, \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$, then S, T and $A_n(n \in \mathbb{N})$ have a common fixed point in X.

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