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Bounds on the Largest of Minimum Degree Laplician Eigenvalues of a Graph

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Abstract: In this paper we give three upper bounds for the largest of minimum degree Laplacian eigenvalues of a graph and also obtain a lower bound for the same.

Key Words: Minimum degree matrix, minimum degree Laplacian eigenvalues.

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§1. Introduction

Let $G = (V, E)$ be a simple, connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Assume that the vertices are ordered such that $d_1 \geq d_2 \geq \ldots \geq d_n$, where d_i is the degree of v_i for $i = 1, 2, \ldots, n$. The energy of G was first defined by I.Gutman [5] in 1978 as the sum of the absolute values of its eigenvalues. The energy of a graph has close links to Chemistry(see for instance [6]). The $n \times n$ matrix $m(G) = (d_{ij})$ is called the minimum degree matrix of G, where

$$d_{ij} = \begin{cases} min\{d_i, d_j\} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

This was introduced and studied in [1]. The characteristic polynomial of the minimum degree matrix $m(G)$ is defined by

$$\begin{aligned} \phi(G; \lambda) &= det(\lambda I - m(G)) \\ &= \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n, \end{aligned} \tag{1.1}$$

where I is the unit matrix of order n. The minimum degree Laplacian matrix of G is $L(G) = D(G) - m(G)$, where $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$. $L(G)$ is a real, symmetric matrix. The minimum degree Laplacian eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the graph G, assumed in the non increasing order, are the eigenvalues of $L(G)$. The Laplacian matrix of G is $L_1(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G. The eigenvalues of the laplacian matrix $L_1(G)$ are important in graph theory, because they have relations to numerous graph invariants including connectivity, expanding property, isoperimetric number, independence number, genus, diameter, mean distance, and bandwidth-type parameters of a graph(see, for example, [2,3,9,10]). In

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many applications one needs good bounds for the largest Laplacian eigenvalue (see for instance [2,3,9,10]). In this paper, we give three upper bounds and a lower bound for μ_1 the largest of minimum degree Laplacian eigenvalues of a graph.

§2. Main Results

In this section, we will give three upper bounds for μ_1 the largest of minimum degree Laplacian eigenvalues of a graph. We employ the following theorem to prove one of our main results.

Theorem 2.1([4]) *Let G be a simple graph with n vertices and m edges, and let $\Pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of G. Then,*

$$d_1^2 + d_2^2 + \ldots + d_n^2 \leq m(\frac{2m}{n-1} + n - 2).$$

Theorem 2.2 *Let G be a connected graph with n vertices and m edges. Then*

$$\mu_1 \leq \frac{2m + \sqrt{(n-1)\left[n(2|c_2| + m(\frac{2m}{n-1} + n - 2)) - 4m^2\right]}}{n},$$

where c_2 is the coefficient of λ^{n-2} in $det(\lambda I - m(G))$.

Proof Clearly

$$\mu_1 + \mu_2 + \ldots + \mu_n = Trace[L(G)] = \sum_{v \in V(G)} d_v, \tag{2.1}$$

$$\mu_1^2 + \mu_2^2 + ... + \mu_n^2 = 2|c_2| + \sum_{i=1}^{n} d_i^2. \tag{2.2}$$

By Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{2.3}$$

Putting $a_i = 1$ and $b_i = \mu_i$ for $i = 2, ..., n$ in (2.3), we get

$$\left(\sum_{i=1}^{n} \mu_i - \mu_1\right)^2 \leq (n-1)\left(\sum_{i=1}^{n} \mu_i^2 - \mu_1^2\right).$$

Using (2.1) and (2.2) in above inequality, we obtain

$$\left(\sum_{v \in V(G)} d_v - \mu_1\right)^2 \leq (n-1)\left[2|c_2| + \sum_{i=1}^{n} d_i^2\right] - (n-1)\mu_1^2.$$

After some simplifications, we deduce that

$$\left(n\mu_1 - \sum_{v \in V(G)} d_v\right)^2 + (n-1)\left(\sum_{v \in V(G)} d_v\right)^2 \leq n(n-1)\left[2|c_2| + \sum_{i=1}^{n} d_i^2\right].$$

i.e.,
$$n\mu_1 - \sum_{v\in V(G)} d_v \le \sqrt{(n-1)\left[n(2|c_2| + \sum_{i=1}^{n} d_i^2) - \left(\sum_{i=1}^{n} d_i\right)^2\right]}.$$

Therefore

$$\mu_1 \le \frac{\sum_{i=1}^{n} d_i + \sqrt{(n-1)\left[n\left(2|c_2| + \sum_{i=1}^{n} d_i^2\right) - \left(\sum_{i=1}^{n} d_i\right)^2\right]}}{n}. \tag{2.4}$$

Employing Theorem 2.1 and $\sum_{i=1}^{n} d_i = 2m$ in (2.4), we see that

$$\mu_1 \le \frac{2m + \sqrt{(n-1)\left[n(2|c_2| + m(\frac{2m}{n-1} + n - 2)) - 4m^2\right]}}{n}.$$

This completes the proof. □

The following theorem gives another type of upper bound for μ_1.

Theorem 2.3 *Let G be connected graph with n vertices and m edges. Then*

$$\mu_1 \le \sqrt{2d_1^2 + 4m - 2d_n^3(n - d_1)}.$$

Proof Suppose that $X = (x_1, x_2, x_3, \ldots, x_n)^T$ be an eigenvector with unit length corresponding to μ_1. Then

$$L(G)X = \mu_1 X.$$

Hence, for $u \in V(G)$,

$$\mu_1 x_u = d_u x_u - \sum_{\substack{v \in V(G) \\ v \neq u}} d_{uv} x_v.$$

Here x_u we mean x_i if $u = v_i$. Therefore

$$\mu_1 x_u = \sum_{vu\in E(G)} (x_u - min(d_u, d_v)x_v). \tag{2.5}$$

By Cauchy-Schwarz inequality, we have

$$\mu_1^2 x_u^2 \le \left(\sum_{vu\in E(G)} 1^2\right)\left(\sum_{vu\in E(G)} (x_u - min(d_u, d_v)x_v)^2\right)$$

$$= d_u \left[\sum_{vu\in E(G)} x_u^2 + \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2 - 2x_u min(d_u, d_v)x_v\right].$$

Observe that

$$-2x_u \sum_{vu\in E(G)} min(d_u, d_v)x_v \le d_u x_u^2 + \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2. \tag{2.6}$$

Hence,

$$\mu_1^2 x_u^2 \leq d_u \left[\sum_{vu\in E(G)} x_u^2 + \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2 + d_u x_u^2 + \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2 \right].$$

i.e.,

$$\mu_1^2 x_u^2 \leq 2d_u^2 x_u^2 + 2d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2. \tag{2.7}$$

Consequently,

$$\begin{aligned}
\mu_1^2 &= \mu_1^2 \sum_{u\in V(G)} x_u^2 \\
&\leq \sum_{u\in V(G)} [2d_u^2 x_u^2 + 2d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2] \\
&= 2 \sum_{u\in V(G)} d_u^2 x_u^2 + 2 \sum_{u\in V(G)} d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2.
\end{aligned}$$

Thus

$$\mu_1^2 \leq 2d_1^2 + 2 \sum_{u\in V(G)} d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2. \tag{2.8}$$

Now let $v \nsim u$ mean that u and v are not adjacent. Then

$$\begin{aligned}
&\sum_{u\in V(G)} d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2 \\
&= \sum_{u\in V(G)} d_u [1 - \sum_{v\nsim u} min(d_u, d_v)^2 x_v^2] = 2m - \sum_{u\in V(G)} d_u \sum_{v\nsim u} min(d_u, d_v)^2 x_v^2 \\
&= 2m - \left(\sum_{u\in V(G)} d_u min(d_u, d_v)^2 x_u^2 + \sum_{u\in V(G)} d_u \sum_{v\nsim u, v\neq u} min(d_u, d_v)^2 x_v^2 \right) \\
&\leq 2m - \left(d_n^2 \sum_{u\in V(G)} d_u x_u^2 + \sum_{u\in V(G)} d_n \sum_{v\nsim u, v\neq u} d_n^2 x_v^2 \right) \\
&= 2m - \left(d_n^2 \sum_{u\in V(G)} d_u x_u^2 + \sum_{u\in V(G)} d_n^3 (n - d_u - 1) x_u^2 \right) \\
&= 2m - \left(d_n^2 \sum_{u\in V(G)} d_u x_u^2 + d_n^3 \sum_{u\in V(G)} n x_u^2 - d_n^3 \sum_{u\in V(G)} d_u x_u^2 - d_n^3 \sum_{u\in V(G)} x_u^2 \right) \\
&\leq 2m - d_n^3 \sum_{u\in V(G)} (n - d_1) x_u^2 \\
&= 2m - d_n^3 (n - d_1).
\end{aligned}$$

Hence, employing this in (2.8) we have

$$\mu_1^2 \leq 2d_1^2 + 4m - 2d_n^3 (n - d_1).$$

Therefore

$$\mu_1 \leq \sqrt{2d_1^2 + 4m - 2d_n^3(n - d_1)}.$$

Theorem 2.4 *Let G be a connected graph then*

$$\mu_1 \leq max\left(\sqrt{2\left(d_u^2 + d_1^2 m_u d_u\right)} \;\; : \;\; u \in V(G)\right).$$

Proof From (2.7) we have

$$\mu_1^2 x_u^2 \leq 2d_u^2 x_u^2 + 2d_u \sum_{vu \in E(G)} min(d_u, d_v)^2 x_v^2.$$

Thus

$$\begin{aligned}\mu_1^2 \sum_{u \in V(G)} x_u^2 &\leq 2 \sum_{u \in V(G)} d_u^2 x_u^2 + 2 \sum_{u \in V(G)} d_u \sum_{vu \in E(G)} min(d_u, d_v)^2 x_v^2. \\ &\leq 2 \sum_{u \in V(G)} d_u^2 x_u^2 + 2d_1^2 \sum_{u \in V(G)} d_u \sum_{vu \in E(G)} x_v^2 \\ &= 2\left[\sum_{u \in V(G)} d_u^2 x_u^2 + d_1^2 \sum_{u \in V(G)} x_u^2 \sum_{vu \in E(G)} d_v\right] \\ &= 2\left[\sum_{u \in V(G)} d_u^2 x_u^2 + d_1^2 \sum_{u \in V(G)} x_u^2 m_u d_u\right]\end{aligned}$$

where m_u = average degree of the vertices adjacent to u.
So,

$$\mu_1 \leq \sqrt{2 \sum_{u \in V(G)} \left(d_u^2 + d_1^2 m_u d_u\right) x_u^2}.$$

Hence

$$\mu_1 \leq \max\left\{\sqrt{2\left(d_u^2 + d_1^2 m_u d_u\right)} \;\; : \;\; u \in V(G)\right\}.$$

§3. Lower Bonud for Spectral Radius of Graphs

In this section we establish a lower bound for the spectral radius μ_1 of G.

Lemma 3.1([7][8]) *Let M be real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Given a partition $\{1, 2, \ldots, n\} = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_m$ with $|\Delta_i| = n_i > 0$, consider the corresponding blocking $M = (M_{ij})$, so that M_{ij} is an $n_i \times n_j$ block. Let e_{ij} be the sum of the entries in M_{ij} and put $B = (\frac{e_{ij}}{n_i})$ i.e.,($\frac{e_{ij}}{n_i}$ is an average row sum in M_{ij}). let $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_m$ be the eigenvalues of B. Then the inequalities*

$$\lambda_i \geq \gamma_i \geq \lambda_{n-m+i} \qquad (i = 1, 2, \ldots, m)$$

hold. Moreover, if for some integer k, $1 \leq k \leq m, \lambda_i = \gamma_i$ for $i = 1, 2, \ldots, k$ and $\lambda_{n-m+i} = \gamma_i$ for $i = k+1, k+2, \ldots, m$, then all the blocks M_{ij} have constant row and column sums.

Let G be a connected graph with n vertices and m edges. Let $V_1 = \{v_1, v_2, \ldots v_{n_1}\}$ and $V_2 = \{v_{n_1+1}, v_{n_1+2} \ldots v_n\}$ be two partitions of vertices of graph G. Let

$$r_1 = \frac{1}{n_1} \sum_{\substack{i,\, j\, =\, 1 \\ i \neq j}}^{n_1} min(d(v_i), d(v_j)), \quad r_2 = \frac{1}{n - n_1} \sum_{\substack{i,\, j\, =\, 1 \\ i \neq j}}^{n-n_1} min(d(v_{n_1+i}), d(v_{n_1+j})),$$

$$k_1 = \frac{-1}{n_1} \sum_{\substack{i,\, j\, =\, 1 \\ i \neq j}}^{n-n_1} min(d(v_i), d(v_{n_1+j})), \quad k_2 = \frac{-1}{n - n_1} \sum_{\substack{i\, =\, 1 \\ j\, =\, 1,2,\ldots,n \\ i \neq j}}^{n-n_1} min(d(v_{n_1+i}), d(v_j)),$$

$$d_1 = \frac{1}{n_1} \sum_{v \in V_1} d(v), \qquad d_2 = \frac{1}{n - n_1} \sum_{v \in V_2} d(v),$$

where $d(v)$ is the degree of the vertex v of G. Now we prove the following theorem.

Theorem 3.2 *Let G be a connected graph with n vertices and m edges, then*

$$\mu_1 \geq \frac{1}{2}\{d_2 + d_1 - r_2 - r_1 + \sqrt{(d_2 - d_1 - r_2 + r_1)^2 - 4k_1k_2}\}.$$

Proof Rewrite $L(G)$ as

$$L(G) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

For $1 \leq i,\, j \leq 2$, let e_{ij} be the sum of the entries in L_{ij} and put $B = (e_{ij}/n_i)$. Then

$$B = \begin{pmatrix} d_1 - r_1 & k_1 \\ k_2 & d_2 - r_2 \end{pmatrix},$$

and so

$$|\lambda I - B| = \begin{vmatrix} \lambda - (d_1 - r_1) & -k_1 \\ -k_2 & \lambda - (d_2 - r_2) \end{vmatrix}.$$

Therefore we have

$$\lambda = \frac{1}{2}\{d_2 + d_1 - r_2 - r_1 \pm \sqrt{(d_2 - d_1 - r_2 + r_1)^2 - 4k_1k_2}\}.$$

Thus by Lemma 3.1 we get

$$\mu_1 \geq \frac{1}{2}\{d_2 + d_1 - r_2 - r_1 + \sqrt{(d_2 - d_1 - r_2 + r_1)^2 - 4k_1k_2}\}. \qquad \square$$

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