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The H-Line Signed Graph of a Signed Graph

R.Rangarajan and M. S. Subramanya

Department of Studies in Mathematics of University of Mysore

Manasagangotri, Mysore 570 006, India

P. Siva Kota Reddy

Department of Mathematics of Acharya Institute of Graduate Studies

Soldevanahalli, Bangalore 560 090, India

Email: reddy_math@yahoo.com

Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. Given a connected graph H of order at least 3, the *H-Line Graph* of a graph $G = (V, E)$, denoted by $HL(G)$, is a graph with the vertex set E, the edge set of G where two vertices in $HL(G)$ are adjacent if, and only if, the corresponding edges are adjacent in G and there exists a copy of H in G containing them. Analogously, for a connected graph H of order at lest 3, we define the *H-Line signed graph* $HL(S)$ of a signed graph $S = (G, \sigma)$ as a signed graph, $HL(S) = (HL(G), \sigma')$, and for any edge e_1e_2 in $HL(S)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. In this paper, we characterize signed graphs S which are H-line signed graphs and study some properties of H-line graphs as well as H-line signed graphs.

Key Words: Smarandachely k-Signed graphs, Smarandachely k-Marked graphs, Signed graphs, Balance, Switching, H-Line signed graph.

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§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [8]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed*

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graph or a *marked graph*. We say that a signed graph is *connected* if its underlying graph is connected. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (See [9]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ.

The following characterization of balanced signed graphs is well known.

Theorem 1.1(E. Sampathkumar [12]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S.

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking μ of a signed graph S. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [22]) or *cycle isomorphic* (see [23]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [23]):

Theorem 1.2(T. Zaslavsky [23]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. H-Line Signed Graph of a Signed Graph

The line graph $L(G)$ of a nonempty graph $G = (V, E)$ is the graph whose vertices are the edges of G and two vertices are adjacent if and only if the corresponding edges are adjacent. The triangular line graph $\mathcal{T}(G)$ of a nonempty graph was introduced by Jerret [10] as a graph whose vertices are edges of G and two vertices are adjacent if and only if corresponding edges belongs to a common triangle. Triangular graphs were introduced to model a metric space defined on the edge set of a graph. These concepts were generalized in [5] as follows: Let H be a fixed connected graph of order at least 3. For a graph G of size the *H-line graph* of G, denoted by $HL(G)$,is the graph whose vertices are the edges of G and two vertices are adjacent the corresponding edges are adjacent and belong to a copy of H.If $H \cong P_3$ then $HL(G) = L(G)$ and so H-line graph is a generalization of line graphs. Clearly, if a graph is H free, then its H-line graph is trivial.

In [10], the authors introduced the notion of triangular line graph of a graph as follows: The *triangular line graph* of a $G = (V, E)$ denoted by $\mathcal{T}(G) = (V', E')$, whose vertices are the edges of G and two vertices are adjacent the corresponding edges belongs to a triangle in G. Clearly for any graph G, $\mathcal{T}(G) = K_3L(G)$.

Behzad and Chartrand [3] introduced the notion of *line signed graph* $L(S)$ of a given signed graph S as follows: $L(S)$ is a signed graph such that $(L(S))^u \cong L(S^u)$ and an edge e_ie_j in $L(S)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [7], is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [7] (See also E. Sampathkumar et al. [13,14]). For more operations on signed graphs see [15-20].

Proposition 2.1 *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

In [21], the authors extends the notion of triangular line graphs to triangular line signed graphs. We now extend the notion of H-line graph to the realm of signed graph as follows:

Let $S = (G, \sigma)$ ba a signed graph. For any fixed connected graph H of order at least 3, the H-line signed graph of S, denoted by $HL(S)$ is the signed graph $HL(S) = (HL(G), \sigma')$ whose underlying graph is $HL(G)$ and for any edge ee' in $HL(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. Further a signed graph S is said to be H-line signed graph if there exists a signed graph S' such that $HL(S') \cong S$.

We now give a straightforward, yet interesting property of H-line signed graphs.

Theorem 2.2 *For any connected graph H of order at least 3 and for any signed graph $S = (G, \sigma)$, its H-line signed graph $HL(S)$ is balanced.*

Proof Let $\sigma^{'}$ denote the signing of $HL(S)$ and let the signing σ of S be treated as a marking of the vertices of $HL(S)$. Then by definition of $HL(S)$ we see that $\sigma'(e_1, e_2) = \sigma(e_1)\sigma(e_2)$, for every edge (e_1, e_2) of $HL(S)$ and hence, by Theorem 1.1, the result follows. □

Corollary 2.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $HL(S_1) \sim HL(S_2)$.*

The following result characterizes signed graphs which are H-line signed graphs.

Theorem 2.4 *A signed graph $S = (G, \sigma)$ is a H-line signed graph for some connected graph H of order at least 3 if, and only if, S is balanced signed graph and its underlying graph G is a*

H-line graph.

Proof Suppose that S is H-line signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $HL(S') \cong S$. Hence by definition $HL(G) \cong G'$ and by Theorem 2.2, S is balanced.

Conversely, suppose that $S = (G, \sigma)$ is balanced and G is H-line graph. That is there exists a graph G' such that $HL(G') \cong G$. Since S is balanced by Theorem 1.1, there exists a marking μ of vertices of S such that for any edge $uv \in G$, $\sigma(uv) = \mu(u)\mu(v)$. Also since $G \cong HL(G')$, vertices in G are in one-to-one correspondence with the edges of G'. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e' in G' to be the marking on the corresponding vertex in G. Then clearly $HL(S') \cong S$ and so S is H-line graph. □

For any positive integer k, the k^{th} iterated H-line signed graph, $HL^k(S)$ of S is defined as follows:

$$HL^0(S) = S,\ HL^k(S) = HL(HL^{k-1}(S)).$$

Corollary 2.5 *Given a signed graph $S = (G, \sigma)$ and any positive integer k, $HL^k(S)$ is balanced, for any connected graph H of order ≥ 3.*

In [6], the authors proved the following for a graph G its H-line graph $HL(G)$ is isomorphic to G then H is a path or a cycle. Analogously we have the following.

Theorem 2.6 *If a signed graph $S = (G, \sigma)$ satisfies $S \sim HL(S)$ then S is balanced and H is a cycle or a path.*

Theorem 2.7 *For any cycle C_k on $k \geq 3$ vertices, a connected graph G on $n \geq r$ vertices satisfies $C_kL(G) \cong G$ if, and only if, $G = C_k$.*

Proof Suppose that $C_kL(G) \cong G$. Then clearly, G must be unicyclic. Since $C_kL(G) \cong G$, we observe that G must contain a cycle C_k. Next, suppose that G contains a vertex of degree ≥ 3, then the vertex corresponding to the edge not on the cycle in $C_kL(G)$ will be isolated vertex. Hence G must be a cycle C_k.

Conversely, if $G = C_k$, then clearly for any two adjacent edges in C_k belongs to a copy of C_k and so $C_kL(G) \cong L(G)$. Since the line graph of any C_k is C_k itself, we have $C_kL(G) \cong G$.□

Corollary 2.8 *For any cycle C_k on $k \geq 3$ vertices, a graph G on $n \geq r$ vertices satisfies $C_kL(G) \cong G$ if, and only if, G is 2-regular and every component of G is C_k.*

In view of the above theorem we have,

Theorem 2.9 *For any cycle C_k on $k \geq 3$ vertices, a signed graph $S = (G, \sigma)$ connected graph G on $n \geq r$ vertices satisfies $C_kL(S) \sim S$ if, and only if, $G = C_k$.*

Theorem 2.10 *For a path P_k on $k \geq 3$ vertices a connected graph G on $n \geq r$ vertices which contains a cycle of length $r > k$ satisfies $P_kL(G) \cong L(G)$ if, and only if, $G = C_n$ and $n \geq k$.*

Proof The result follows if $k = 3$, since $P_3L(G) = L(G)$. Assume that $k \geq 4$. Clearly G must contain at least k vertices. Suppose that $P_kL(G) \cong L(G)$ and G contains a cycle of length $r \geq k$. Then number of vertices in G and number of edges are equal. Hence G must be unicyclic. Since G contains a cycle of length $r > k$, then any two adjacent edges in C of G belongs to a common P_k. Hence $P_kL(G)$ also contains a cycle of length r. Next, suppose that G contains a vertex of degree ≥ 3, then the vertex corresponding to the edge not on the cycle in $P_kL(G)$ will be adjacent to two adjacent vertices forming a C_3 and so $HL(G)$ is not unicyclic. Hence G must be the cycle C_n .

Conversely, if $G = C_n$ and $n \geq k$, then clearly any two adjacent edges in C_k belongs to a copy of C_k and so $P_kL(G) \cong L(G)$. Since the line graph of C_n is C_n itself, $P_kL(G) \cong L(G)$. □

Corollary 2.11 *For any path P_k on $k \geq 3$ vertices, a graph G on $n \geq r$ vertices satisfies $P_kL(G) \cong G$ if, and only if, G is 2-regular and every component of G is C_r, for some $r \geq k$.*

Analogously, we have the following for signed graphs:

Corollary 2.12 *For any path P_k on $k \geq 3$ vertices, a signed graph $S = (G, \sigma)$ on $n \geq r$ vertices satisfies $P_kL(S) \sim S$ if, and only if, S is balanced and every component of G is C_r, for some $r \geq k$.*

In [10], the authors prove that for any graph G, $\mathcal{T}(G) \cong L(G)$ if, and only if, $G = K_n$. Analogously, we have the following:

Theorem 2.13 *A graph G of order n satisfies $K_rL(G) \cong L(G)$ for some $r \leq n$ if, and only if, $G = K_n$.*

Proof The result is trivial if $k = n$. Suppose that $K_rL(G) \cong L(G)$ and G is not complete for some $r \leq n - 1$. Then there exists at least two nonadjacent vertices u and v in G. Now for any vertex w, the edges uw and vw are adjacent and hence the corresponding vertices are adjacent. But the edges uw and vw can not be adjacent in $K_rL(G)$ since any set of r vertices containing u and v can not induce complete subgraph K_r. Whence, the condition is necessary.

For sufficiency, suppose $G = K_n$ for some $n \geq r$. Then for any two adjacent vertices in L(G), the corresponding edges adjacent edges in G belongs to some K_r. Hence they are also adjacent in $K_rL(G)$ and any two nonadjacent vertices in $L(G)$ remain nonadjacent.This completes the proof. □

Analogously, we have the following result for signed graphs:

Theorem 2.14 *A signed graph $S = (G, \sigma)$ satisfies $K_rL(S) \sim L(S)$, for some $3 \leq k \leq |V(G)|$ if, and only if, S is a balanced on a complete graph.*

§3. Triangular Line Signed Graphs and (0, 1, -1) Matrices

Matrices are very good models to represent a graph. In general given a matrix $A = (a_{ij})$ of order $m \times n$ one can associate many graphs with it (see [11]. On the other hand given any graph G we can associate many matrices such adjacency matric, incidence matrix etc (see [8]). Analogously, given a matrix with entries one can associate many signed graphs (See [11]). In this section, we give a relation between the notion of triangular line graph and some graph associated with $\{0,1\}$-matrices. Also we extend this to triangular signed graphs and some signed graphs associated with matrices whose entries are $-1, 0$, or 1.

Given a $(0,1)$-matrix A, the term graph $T(A)$ of A was defined as follows (See [2]): The vertex set of $T(A)$ consists of m row labels $r_1, r_2, ..., r_m$ and n column labels $c_1, c_2, ..., c_n$ of A and the edge set consists of the unordered pairs $r_i c_j$ for which $a_{ij} \neq 0$.

Given a $(0,1)$-matrix A of order $m \times n$, the graph $G_t(A)$ can be constructed as follows: The vertex set of $G_t(A)$ consists of non-zero entries of A and the edge set consists of distinct pairs of vertices (a_{ij}, a_{kr}) that lie in the same row (i=k) with $a_{ir} \neq 0$ or or same column(j=r) with $a_{kj} \neq 0$. The following result relates the connects the two notions the term graph and G_t graph of a given matrix A:

Theorem 3.1 *For any* $(0,1)$*-matrix A,* $G_t(A) = \mathcal{T}(T(A))$.

Let $A = (a_{ij})$ be any $m \times n$ matrix in which each entry belongs to the set $\{-1, 0, 1\}$; we shall call such a matrix a $(0, \pm 1)$-matrix. The notion of term graph of a $(0,1)$-matrix can be easily extended to term signed graph of a $(0, \pm 1)$-matrix A as follows (see [2]): The vertex set of $T(A)$ consists of m row labels $r_1, r_2, ..., r_m$ and n column labels $c_1, c_2, ..., c_n$ of A, the edge set consists of the unordered pairs $r_i c_j$ for which $a_{ij} \neq 0$ and the sign of the edge $r_i c_j$ is the sign of the nonzero entry a_{ij}.

Next, given any $(0, \pm 1)$-matrix A a *triangular matrix signed graph* $Sg_t(A)$ of A can be constructed as follows: The vertex set of $Sg_t(A)$ is consists of nonzero entries of A and edge set consists of distinct pairs of vertices (a_{ij}, a_{kr}) that lie in the same row $(i = k)$ with $a_{ir} \neq 0$ or same column $(j = r)$ with $a_{kj} \neq 0$; the sign of an edge uv in $Sg(A)$ is defined as the product of sings of the entries of A that correspond to $u = a_{ij}$ and $v = a_{kr}$.

The following is a observation whose proof follows from the definition of triangular line graph and the facts just mentioned above:

Theorem 3.2 *For any* $(0, \pm 1)$ *matrix A,* $Sg_t(A) \cong \mathcal{T}(T_g(A))$.

The *Kronecker product* or *tensor product* of two signed graphs S_1 and S_2, denoted by $S_1 \bigotimes S_2$ is defined (see [2]) as follows:The vertex set of $(S_1 \bigotimes S_2)$ is $V(S_1) \times V(S_2)$, the edge set is $E(S_1 \bigotimes S_2) := \{((u_1, v_1), (u_2, v_2)) : u_1u_2 \in E(S_1),\ v_1v_2 \in E(S_2)\}$ and the sign of the edge $(u_1, v_1)(u_2, v_2)$ is the product of the sign of u_1u_2 in S_1 and the sign of v_1v_2 in S_2. In the following result, $A(S)$ will denote the usual adjacency matrix of the given signed graph S and $A \bigotimes B$ denotes the standard tensor product of the given matrices A and B.

Theorem 3.3(M. Acharya [2]) *For any two signed graphs* $S_1 and S_2$*,* $A(S_1 \bigotimes S_2) = A(S_1) \bigotimes A(S_2)$.

Theorem 3.4 *For any signed graph S,* $T(A(S)) = S \bigotimes K_2^+$*, where* K_2^+ *denotes the complete graph* K_2 *with its only edge treated as being positive.*

The *adjacency signed graph* $\eth(S)$ of a given signed graph S is the matrix signed graph $Sg(A(S))$ of the adjacency matrix $A(S)$ of S [2].

Theorem 3.5(M. Acharya [2]) *For any signed graph S,* $\eth(S) = L(S \bigotimes K_2^+)$.

Analogously we define *triangular adjacency signed graph* of $A(S)$, the adjacency matrix of S denoted by $\eth_t(S)$ as the signed graph $Sg_t(A(S))$. We have the following result.

Theorem 3.6 *For any signed graph S,* $\eth_t(S) = \mathcal{T}(S \bigotimes K_2^+)$.

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