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The Forcing Weak Edge Detour Number of a Graph

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Abstract: For two vertices u and v in a graph $G = (V, E)$, the *distance* $d(u, v)$ *and detour distance* $D(u, v)$ are the length of a shortest or longest $u - v$ path in G, respectively, and the *Smarandache distance* $d_S^i(u, v)$ is the length $d(u, v) + i(u, v)$ of a $u - v$ path in G, where $0 \leq i(u, v) \leq D(u, v) - d(u, v)$. A $u - v$ path of length $d_S^i(u, v)$, if it exists, is called a *Smarandachely* $u - v$ i*-detour*. A set $S \subseteq V$ is called a *Smarandachely* i*-detour set* if every edge in G has both its ends in S or it lies on a Smarandachely i-detour joining a pair of vertices in S. In particular, if $i(u, v) = 0$, then $d_S^i(u, v) = d(u, v)$; and if $i(u, v) = D(u, v) - d(u, v)$, then $d_S^i(u, v) = D(u, v)$. For $i(u, v) = D(u, v) - d(u, v)$, such a Smarandachely i-detour set is called a *weak edge detour set* in G. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is a *weak edge detour basis* of G. For any weak edge detour basis S of G, a subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique weak edge detour basis containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing weak edge detour number* of S, denoted by $fdn_w(S)$, is the cardinality of a minimum forcing subset for S. The *forcing weak edge detour number* of G, denoted by $fdn_w(G)$, is $fdn_w(G) = min\{fdn_w(S)\}$, where the minimum is taken over all weak edge detour bases S in G. The forcing weak edge detour numbers of certain classes of graphs are determined. It is proved that for each pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there is a connected graph G with $fdn_w(G) = a$ and $dn_w(G) = b$.

Key Words: Smarandache distance, Smarandachely i-detour set, weak edge detour set, weak edge detour number, forcing weak edge detour number.

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§1. Introduction

For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest u–v path in G. A u–v path of length $d(u, v)$ is called a u–v *geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, $radG$ and the maximum eccentricity among the vertices of G is its *diameter*, $diamG$ of G. Two vertices u and v of G are *antipodal* if $d(u, v)$

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$= diamG$. For vertices u and v in a connected graph G, the *detour distance* $D(u,v)$ is the length of a longest u–v path in G. A u–v path of length $D(u,v)$ is called a *u–v detour*. It is known that the distance and the detour distance are metrics on the vertex set $V(G)$. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D G$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D G$ of G is the maximum detour eccentricity among the vertices of G. These concepts were studied by Chartrand et al. [2].

A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. A vertex v that belongs to every detour basis of G is a *detour vertex* in G. If G has a unique detour basis S, then every vertex in S is a detour vertex in G. These concepts were studied by Chartrand et al. [3].

In general, there are graphs G for which there exist edges which do not lie on a detour joining any pair of vertices of V. For the graph G given in Figure 1.1, the edge v_1v_2 does not lie on a detour joining any pair of vertices of V. This motivated us to introduce the concept of *weak edge detour set* of a graph [5].

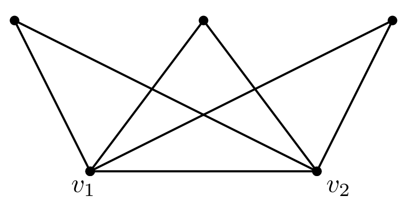

Figure 1: G

The *Smarandache distance* $d^i_S(u,v)$ is the length $d(u,v)+i(u,v)$ of a $u-v$ path in G, where $0 \leq i(u,v) \leq D(u,v) - d(u,v)$. A $u-v$ path of length $d^i_S(u,v)$, if it exists, is called a *Smarandachely $u-v$ i-detour*. A set $S \subseteq V$ is called a *Smarandachely i-detour set* if every edge in G has both its ends in S or it lies on a Smarandachely i-detour joining a pair of vertices in S. In particular, if $i(u,v) = 0$, then $d^i_S(u,v) = d(u,v)$ and if $i(u,v) = D(u,v) - d(u,v)$, then $d^i_S(u,v) = D(u,v)$. For $i(u,v) = D(u,v) - d(u,v)$, such a Smarandachely i-detour set is called a *weak edge detour set* in G. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is called a *weak edge detour basis* of G. A vertex v in a graph G is a *weak edge detour vertex* if v belongs to every weak edge detour basis of G. If G has a unique weak edge detour basis S, then every vertex in S is a weak edge detour vertex of G. These concepts were studied by A. P. Santhakumaran and S. Athisayanathan [5].

To illustrate these concepts, we consider the graph G given in Figure 1.2. The sets $S_1 = \{u,x\}$, $S_2 = \{u,y\}$ and $S_3 = \{u,z\}$ are the detour bases of G so that $dn(G) = 2$ and the sets $S_4 = \{u,v,y\}$ and $S_5 = \{u,x,z\}$ are the weak edge detour bases of G so that $dn_w(G) = 3$. The vertex u is a detour vertex and also a weak edge detour vertex of G.

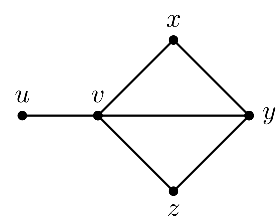

Figure 2: G

The following theorems are used in the sequel.

Theorem 1.1([5]) *For any graph G of order $p \geq 2$, $2 \leq dn_w(G) \leq p$.*

Theorem 1.2([5]) *Every end-vertex of a non-trivial connected graph G belongs to every weak edge detour set of G. Also if the set S of all end-vertices of G is a weak edge detour set, then S is the unique weak edge detour basis for G.*

Theorem 1.3([5]) *If T is a tree with k end-vertices, then $dn_w(T) = k$.*

Theorem 1.4([5]) *Let G be a connected graph with cut-vertices and S a weak edge detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

Throughout this paper G denotes a connected graph with at least two vertices.

§2. Forcing Weak Edge Detour Number of a Graph

First we determine the weak edge detour numbers of some standard classes of graphs so that their forcing weak edge detour numbers will be determined.

Theorem 2.1 *Let G be the complete graph K_p $(p \geq 3)$ or the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$. Then a set $S \subseteq V$ is a weak edge detour basis of G if and only if S consists of any two vertices of G.*

Proof Let G be the complete graph $K_p(p \geq 3)$ and $S = \{u, v\}$ be any set of two vertices of G. It is clear that $D(u, v) = p - 1$. Let $xy \in E$. If $xy = uv$, then both its ends are in S. Let $xy \neq uv$. If $x \neq u$ and $y \neq v$, then the edge xy lies on the u–v detour $P : u, x, y, \ldots, v$ of length $p - 1$. If $x = u$ and $y \neq v$, then the edge xy lies on the u–v detour $P : u = x, y, \ldots, v$ of length $p - 1$. Hence S is a weak edge detour set of G. Since $|S| = 2$, S is a weak edge detour basis of G.

Now, let S be a weak edge detour basis of G. Let S' be any set consisting of two vertices of G. Then as in the first part of this theorem S' is a weak edge detour basis of G. Hence $|S| = |S'| = 2$ and it follows that S consists of any two vertices of G.

Let G be the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$. Let X and Y be the bipartite sets of G with $|X| = m$ and $|Y| = n$. Let $S = \{u, v\}$ be any set of two vertices of G.

Case 1 Let $u \in X$ and $v \in Y$. It is clear that $D(u, v) = 2m - 1$. Let $xy \in E$. If $xy = uv$, then both of its ends are in S. Let $xy \neq uv$ be such that $x \in X$ and $y \in Y$. If $x \neq u$ and $y \neq v$, then the edge xy lies on the u–v detour $P : u, y, x, \ldots, v$ of length $2m - 1$. If $x = u$ and $y \neq v$, then the edge xy lies on the u–v detour $P : u = x, y, \ldots, v$ of length $2m - 1$. Hence S is a weak edge detour set of G.

Case 2 Let $u, v \in X$. It is clear that $D(u, v) = 2m - 2$. Let $xy \in E$ be such that $x \in X$ and $y \in Y$. If $x \neq u$, then the edge xy lies on the u–v detour $P : u, y, x, \ldots, v$ of length $2m - 2$. If $x = u$, then the edge xy lies on the u–v detour $P : u = x, y, \ldots, v$ of length $2m - 2$. Hence S is a weak edge detour set of G.

Case 3 Let $u, v \in Y$. It is clear that $D(u, v) = 2m$. Then, as in Case 2, S is a weak edge detour set of G. Since $|S| = 2$, it follows that S is a weak edge detour basis of G.

Now, let S be a weak edge detour basis of G. Let S' be any set consisting of two vertices of G. Then as in the first part of the proof of $K_{m,n}$, S' is a weak edge detour basis of G. Hence $|S| = |S'| = 2$ and it follows that S consists of any two vertices adjacent or not. □

Theorem 2.2 *Let G be an odd cycle of order $p \geq 3$. Then a set $S \subseteq V$ is a weak edge detour basis of G if and only if S consists of any two adjacent vertices of G.*

Proof Let $S = \{u, v\}$ be any set of two adjacent vertices of G. It is clear that $D(u, v) = p-1$. Then every edge $e \neq uv$ of G lies on the u–v detour and both the ends of the edge uv belong to S so that S is a weak edge detour set of G. Since $|S| = 2$, S is a weak edge detour basis of G.

Now, assume that S is a weak edge detour basis of G. Let S' be any set of two adjacent vertices of G. Then as in the first part of this theorem S' is a weak edge detour basis of G. Hence $|S| = |S'| = 2$. Let $S = \{u, v\}$. If u and v are not adjacent, then since G is an odd cycle, the edges of u–v geodesic do not lie on the u–v detour in G so that S is not a weak edge detour set of G, which is a contradiction. Thus S consists of any two adjacent vertices of G. □

Theorem 2.3 *Let G be an even cycle of order $p \geq 4$. Then a set $S \subseteq V$ is a weak edge detour basis of G if and only if S consists of any two adjacent vertices or two antipodal vertices of G.*

Proof Let $S = \{u, v\}$ be any set of two vertices of G. If u and v are adjacent, then $D(u, v) = p - 1$ and every edge $e \neq uv$ of G lies on the u–v detour and both the ends of the edge uv belong to S. If u and v are antipodal, then $D(u, v) = p/2$ and every edge e of G lies on a u–v detour in G. Thus S is a weak edge detour set of G. Since $|S| = 2$, S is a weak edge detour basis of G.

Now, assume that S is a weak edge detour basis of G. Let S' be any set of two adjacent vertices or two antipodal vertices of G. Then as in the first part of this theorem S' is a weak edge detour basis of G. Hence $|S| = |S'| = 2$. Let $S = \{u, v\}$. If u and v are not adjacent and u and v are not antipodal, then the edges of the u–v geodesic do not lie on the u–v detour in G so that S is not a weak edge detour set of G, which is a contradiction. Thus S consists of any two adjacent vertices or two antipodal vertices of G. □

Corollary 2.4 *If G is the complete graph K_p $(p \geq 3)$ or the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$ or the cycle C_p $(p \geq 3)$, then $dn_w(G) = 2$.*

Proof This follows from Theorems 2.1, 2.2 and 2.3. □

Every connected graph contains a weak edge detour basis and some connected graphs may contain several weak edge detour bases. For each weak edge detour basis S in a connected graph G, there is always some subset T of S that uniquely determines S as the weak edge detour basis containing T. We call such subsets "forcing subsets" and we discuss their properties in this section.

Definition 2.5 *Let G be a connected graph and S a weak edge detour basis of G. A subset $T \subseteq S$ is called a* forcing subset *for S if S is the unique weak edge detour basis containing T. A forcing subset for S of minimum cardinality is a* minimum forcing subset *of S. The* forcing weak edge detour number *of S, denoted by $fdn_w(S)$, is the cardinality of a minimum forcing subset for S. The* forcing weak edge detour number *of G, denoted by $fdn_w(G)$, is $fdn_w(G) = min\ \{fdn_w(S)\}$, where the minimum is taken over all weak edge detour bases S in G.*

Example 2.6 For the graph G given in Figure 2.1(a), $S = \{u, v, w\}$ is the unique weak edge detour basis so that $fdn_w(G) = 0$. For the graph G given in Figure 2.1(b), $S_1 = \{u, v, x\}$, $S_2 = \{u, v, y\}$ and $S_3 = \{u, v, w\}$ are the only weak edge detour bases so that $fdn_w(G) = 1$. For the graph G given in Figure 2.1(c), $S_4 = \{u, w, x\}$, $S_5 = \{u, w, y\}$, $S_6 = \{v, w, x\}$ and $S_7 = \{v, w, y\}$ are the four weak edge detour bases so that $fdn_w(G) = 2$.

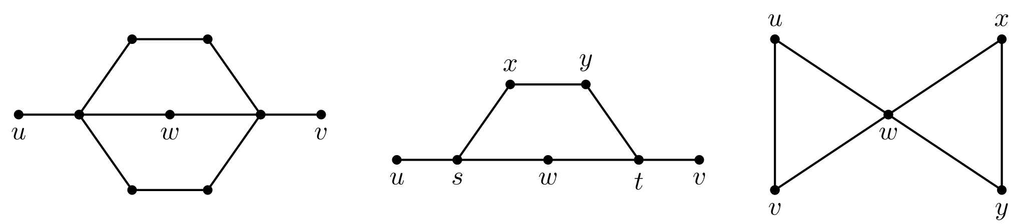

Figure 3: G

The following theorem is clear from the definitions of weak edge detour number and forcing weak edge detour number of a connected graph G.

Theorem 2.7 *For every connected graph G, $0 \leqslant fdn_w(G) \leqslant dn_w(G)$.*

Remark 2.8 The bounds in Theorem 2.7 are sharp. For the graph G given in Figure 2.1(a), $fdn_w(G) = 0$. For the cycle C_3, $fdn_w(C_3) = dn_w(C_3) = 2$. Also, all the inequalities in Theorem 2.7 can be strict. For the graph G given in Figure 2.1(b), $fdn_w(G) = 1$ and $dn_w(G) = 3$ so that $0 < fdn_w(G) < dn_w(G)$.

The following two theorems are easy consequences of the definitions of the weak edge detour number and the forcing weak edge detour number of a connected graph.

Theorem 2.9 *Let G be a connected graph. Then*

a) $fdn_w(G) = 0$ *if and only if G has a unique weak edge detour basis,*

b) $fdn_w(G) = 1$ *if and only if G has at least two weak edge detour bases, one of which is a unique weak edge detour basis containing one of its elements, and*

c) $fdn_w(G) = dn_w(G)$ *if and only if no weak edge detour basis of G is the unique weak edge detour basis containing any of its proper subsets.*

Theorem 2.10 *Let G be a connected graph and let $\mathscr{F}$ be the set of relative complements of the minimum forcing subsets in their respective weak edge detour bases in G. Then $\bigcap_{F\in\mathscr{F}} F$ is the set of weak edge detour vertices of G. In particular, if S is a weak edge detour basis of G, then no weak edge detour vertex of G belongs to any minimum forcing subset of S.*

Theorem 2.11 *Let G be a connected graph and W be the set of all weak edge detour vertices of G. Then $fdn_w(G) \leqslant dn_w(G) - |W|$.*

Proof Let S be any weak edge detour basis S of G. Then $dn_w(G) = |S|$, $W \subseteq S$ and S is the unique weak edge detour basis containing $S - W$. Thus $fdn_w(S) \leqslant |S - W| = |S| - |W| = dn_w(G) - |W|$. □

Remark 2.12 The bound in Theorem 2.11 is sharp. For the graph G given in Figure 2.1(c), $dn_w(G) = 3$, $|W| = 1$ and $fdn_w(G) = 2$ as in Example 2.6. Also, the inequality in Theorem 2.11 can be strict. For the graph G given in Figure 2.2, the sets $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_2, v_3\}$ are the two weak edge detour bases for G and $W = \emptyset$ so that $dn_w(G) = 2$, $|W| = 0$ and $fdn_w(G) = 1$. Thus $fdn_w(G) < dn_w(G) - |W|$.

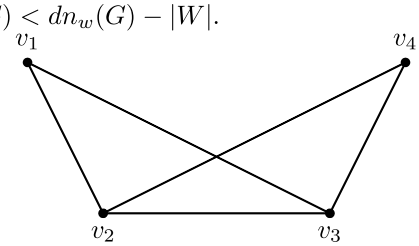

Figure 4: G

In the following we determine $fdn_w(G)$ for certain graphs G.

Theorem 2.13 a) *If G is the complete graph $K_p\,(p \geq 3)$ or the complete bipartite graph $K_{m,n}\,(2 \leq m \leq n)$, then $dn_w(G) = fdn_w(G) = 2$.*

b) *If G is the cycle $C_p\,(p \geq 4)$, then $dn_w(G) = fdn_w(G) = 2$.*

c) *If G is a tree of order $p \geq 2$ with k end-vertices, then $dn_w(G) = k$, $fdn_w(G) = 0$.*

Proof a) By Theorem 2.1, a set S of vertices is a weak edge detour basis if and only if S consists of any two vertices of G. For each vertex v in G there are two or more vertices adjacent with v. Thus the vertex v belongs to more than one weak edge detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any weak edge detour basis of G. Thus the result follows.

b) By Theorems 2.2 and 2.3, a set S of two adjacent vertices of G is a weak edge detour basis of G. For each vertex v in G there are two vertices adjacent with v. Thus the vertex v

belongs to more than one weak edge detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any weak edge detour basis of G. Thus the result follows.

c) By Theorem 1.3, $dn_w(G) = k$. Since the set of all end-vertices of a tree is the unique weak edge detour basis, the result follows from Theorem 2.9(a). $\square$

The following theorem gives a realization result.

Theorem 2.14 *For each pair a, b of integers with $0 \leqslant a \leqslant b$ and $b \geqslant 2$, there is a connected graph G with $fdn_w(G) = a$ and $dn_w(G) = b$.*

Proof The proof is divided into two cases following.

Case 1: $a = 0$. For each $b \geqslant 2$, let G be a tree with b end-vertices. Then $fdn_w(G) = 0$ and $dn_w(G) = b$ by Theorem 2.13(c).

Case 2: $a \geqslant 1$. For each $i\,(1 \leqslant i \leqslant a)$, let $F_i : u_i, v_i, w_i, x_i, u_i$ be the cycle of order 4 and let $H = K_{1,b-a}$ be the star at v whose set of end-vertices is $\{z_1, z_2, \ldots, z_{b-a}\}$. Let G be the graph obtained by joining the central vertex v of H to both vertices u_i, w_i of each $F_i\,(1 \leqslant i \leqslant a)$. Clearly the graph G is connected and is shown in Figure 2.3.

Let $W = \{z_1, z_2, \ldots, z_{b-a}\}$ be the set of all $(b-a)$ end-vertices of G. First, we show that $dn_w(G) = b$. By Theorems 1.2 and 1.4, every weak edge detour basis contains W and at least one vertex from each $F_i\,(1 \leqslant i \leqslant a)$. Thus $dn_w(G) \geqslant (b-a) + a = b$. On the other hand, since the set $S_1 = W \cup \{v_1, v_2, \ldots, v_a\}$ is a weak edge detour set of G, it follows that $dn_w(G) \leqslant |S_1| = b$. Therefore $dn_w(G) = b$.

Next we show that $fdn_w(G) = a$. It is clear that W is the set of all weak edge detour vertices of G. Hence it follows from Theorem 2.11 that $fdn_w(G) \leqslant dn_w(G) - |W| = b - (b-a) = a$. Now, since $dn_w(G) = b$, it is easily seen that a set S is a weak edge detour basis of G if and only if S is of the form $S = W \cup \{y_1, y_2, \ldots, y_a\}$, where $y_i \in \{v_i, x_i\} \subseteq V(F_i)\,(1 \leqslant i \leqslant a)$. Let T be a subset of S with $|T| < a$. Then there is a vertex $y_j\,(1 \leqslant j \leqslant a)$ such that $y_j \notin T$. Let $s_j \in \{v_j, x_j\} \subseteq V(F_j)$ distinct from y_j. Then $S' = (S - \{y_j\}) \cup \{s_j\}$ is a weak edge detour basis that contains T. Thus S is not the unique weak edge detour basis containing T. Thus $fdn_w(S) \geqslant a$. Since this is true for all weak edge detour basis of G, it follows that $fdn_w(G) \geqslant a$ and so $fdn_w(G) = a$. $\square$

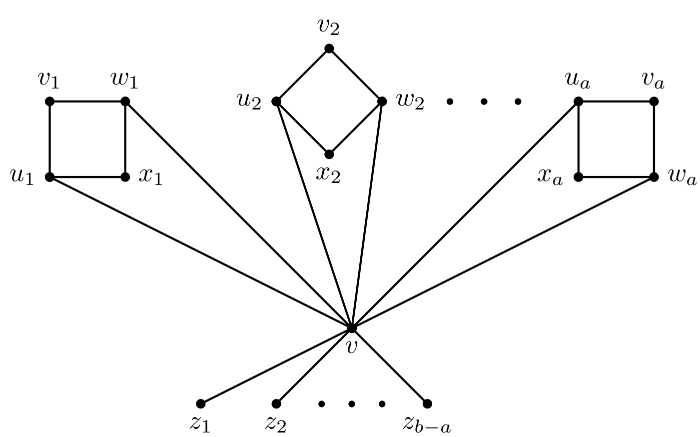

Figure 5: G

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