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Generalizations of Poly-Bernoulli Numbers and Polynomials

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Abstract: The concepts of poly-Bernoulli numbers $B_n^{(k)}$, poly-Bernoulli polynomials $B_n^k(t)$ and the generalized poly-Bernoulli numbers $B_n^{(k)}(a,b)$ are generalized to $B_n^{(k)}(t,a,b,c)$ which is called the generalized poly-Bernoulli polynomials depending on real parameters *a,b,c*. Some properties of these polynomials and some relationships between B_n^k , $B_n^{(k)}(t)$, $B_n^{(k)}(a,b)$ and $B_n^{(k)}(t,a,b,c)$ are established.

Key Words: Poly-Bernoulli polynomial, Euler number, Euler polynomial.

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§1. Introduction

In this paper we shall develop a number of generalizations of the poly-Bernoulli numbers and polynomials, and obtain some results about these generalizations. They are fundamental objects in the theory of special functions.

Euler numbers are denoted with B_k and are the coefficients of Taylor expansion of the function $\frac{t}{e^t-1}$ as following:

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}.$$

The Euler polynomials $E_n(x)$ are expressed in the following series

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!}.$$

for more details, see [1]-[4].

In [10], Q.M.Luo, F.Oi and L.Debnath defined the generalization of Euler polynomials $E_k(x,a,b,c)$ which are expressed in the following series:

$$\frac{2c^{xt}}{b^t+a^t}=\sum_{k=0}^{\infty}E_k(x,a,b,c)\frac{t^k}{k!}.$$

where $a,b,c\in\mathcal{Z}^+$. They proved that

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I) for $a=1$ and $b=c=e$

$$E_k(x+1)=\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix} E_j(x) \tag{1}$$

and

$$E_k(x+1)+E_k(x)=2x^k. \tag{2}$$

II) for $a=1$ and $b=c$,

$$E_k(x+1,1,b,b)+E_k(x,1,b,b)=2x^k(\ln b)^k. \tag{3}$$

In[5], Kaneko introduced and studied poly-Bernoulli numbers which generalize the classical Bernoulli numbers. Poly-Bernoulli numbers $B_n^{(k)}$ with $k \in \mathcal{Z}$ and $n \in \mathcal{N}$ appear in the following power series:

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}, \qquad (*)$$

where $k \in \mathcal{Z}$ and

$$Li_k(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^k}. \qquad |z|<1.$$

So for $k \leq 1$,

$$Li_1(z)=-\ln(1-z), Li_0(z)=\frac{z}{1-z}, Li_{-1}=\frac{z}{(1-z)^2}, \dots \quad .$$

Moreover when $k \geq 1$, the left hand side of $(*)$ can be written in the form of "interated integrals"

$$\begin{aligned} e^t\frac{1}{e^t-1} &= \int_0^t \frac{1}{e^t-1}\int_0^t \dots \frac{1}{e^t-1}\int_0^t \frac{t}{e^t-1}dtdt\dots dt \\ &= \sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}. \end{aligned}$$

In the special case, one can see $B_n^{(1)}=B_n$.

Definition 1.1 *These poly-Bernoulli polynomials $B_n^{(k)}(t)$ are appeared in the expansion of $\frac{Li_k(1-e^{-x})}{1-e^{-x}}e^{xt}$ as follows:*

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}}e^{xt}=\sum_{n=0}^{\infty}\frac{B_n^{(k)}(t)}{n!}x^n \tag{4}$$

for more details, see $[6]-[11]$.

Proposition 1.1 (Kaneko theorem [6]) *The Poly-Bernoulli numbers of non-negative index k, satisfy the following*

$$B_n^{(k)}=(-1)^n\sum_{m=1}^{n+1}\frac{(-1)^{m-1}(m-1)!\left\{ \begin{matrix} n \\ m-1 \end{matrix} \right\}}{m^k}, \tag{5}$$

and for negative index $-k$, we have

$$B_n^{(-k)} = \sum_{j=0}^{min(n,k)} (j!)^2 \left\{ \begin{array}{c} n+1 \\ j+1 \end{array} \right\} \left\{ \begin{array}{c} k+1 \\ j+1 \end{array} \right\}, \tag{6}$$

where

$$\left\{ \begin{array}{c} n \\ m \end{array} \right\} = \frac{(-1)^m}{m!} \sum_{l=0}^{m} (-1)^l \left(\begin{array}{c} m \\ l \end{array} \right) l^n \quad m, n \geq 0 \tag{7}$$

Definition 1.2 *Let $a, b > 0$ and $a \neq b$. The generalized poly-Bernoulli numbers $B_n^{(k)}(a,b)$, the generalized poly-Bernoulli polynomials $B_n^{(k)}(t,a,b)$ and the polynomial $B_n^{(k)}(t,a,b,c)$ are appeared in the following series respectively.*

$$\frac{Li_k(1-(ab)^{-t})}{b^t - a^{-t}} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}(a,b)}{n!} t^n \quad |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{8}$$

$$\frac{Li_k(1-(ab)^{-t})}{b^t - a^{-t}} e^{xt} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}(x,a,b)}{n!} t^n \quad |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{9}$$

$$\frac{Li_k(1-(ab)^{-t})}{b^t - a^{-t}} c^{xt} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}(x,a,b,c)}{n!} t^n \quad |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{10}$$

§2. Main Theorems

We present some recurrence formulae for generalized poly-Bernoulli polynomials.

Theorem 2.1 *Let $x \in \mathbb{R}$ and $n \geq 0$. For every positive real numbers a, b and c such that $a \neq b$ and $b > a$, we have*

$$B_n^{(k)}(a,b) = B_n^{(k)} \left(\frac{-\ln b}{\ln a + \ln b} \right) (\ln a + \ln b)^n, \tag{11}$$

$$B_j^{(k)}(a,b) = \sum_{i=1}^{j} (-1)^{j-i} (\ln a + \ln b)^i (\ln b)^{j-i} \left(\begin{array}{c} j \\ i \end{array} \right) B_j^{(k)}, \tag{12}$$

$$B_n^{(k)}(x;a,b,c) = \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) (\ln c)^{n-l} B_l^{(k)}(a,b) x^{n-l}, \tag{13}$$

$$B_n^{(k)}(x+1;a,b,c) = B_n^{(k)}(x;ac,\frac{b}{c},c), \tag{14}$$

$$B_n^{(k)}(t) = B_n^{(k)}(e^{t+1}, e^{-t}), \tag{15}$$

$$B_n^{(k)}(x,a,b,c) = (\ln a + \ln b)^n B_n^{(k)}(\frac{-\ln b + x \ln c}{\ln a + \ln b}). \tag{16}$$

Proof Applying Definition 1.2, we prove formulae (11)-(16) as follows.

(1) For formula (11), we note that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}} &= \sum_{n=0}^{\infty}\frac{B_n^{(k)}(a,b)}{n!}t^n = \frac{1}{b^t}\left(\frac{Li_k(1-e^{-t\ln ab})}{1-e^{-t\ln ab}}\right)\\ &= e^{-t\ln b}\left(\frac{Li_k(1-e^{-t\ln ab})}{1-e^{-t(\ln ab)}}\right)\\ &= \sum_{n=0}^{\infty}B_n^{(k)}\left(\frac{-\ln b}{\ln a+\ln b}\right)(\ln a+\ln b)^n\frac{t^n}{n!}\end{aligned}$$

Therefore

$$B_n^{(k)}(a,b) = B_n^{(k)}\left(\frac{-\ln b}{\ln a+\ln b}\right)(\ln a+\ln b)^n.$$

(2) For formula (12), notice that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}} &= \frac{1}{b^t}\left(\frac{Li_k(1-(ab)^{-t}\ )}{1-e^{-t\ln ab}}\right)\\ &= \left(\sum_{k=0}^{\infty}\frac{(\ln b)^k}{k!}(-1)^k t^k\right)\left(\sum_{n=0}^{\infty}B_n^{(k)}\frac{(\ln a+\ln b)^n}{n!}t^n\right)\\ &= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}(-1)^{j-i}B_i^{(k)}\frac{(\ln a+\ln b)^i}{i!(j-i)!}(\ln b)^{j-i}\right)t^j.\end{aligned}$$

We have

$$B_j^{(k)}(a,b) = \sum_{i=0}^{j}(-1)^{j-i}(\ln a+\ln b)^i(\ln b)^{j-i}\begin{pmatrix} j\\ i\end{pmatrix}B_i^{(k)}.$$

(3) For formula (13), by calculation we know that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt} &= \sum_{n=0}^{\infty}B_n^{(k)}(x,a,b,c)\frac{t^n}{n!}\\ &= \left(\sum_{l=0}^{\infty}B_l^{(k)}(a,b)\frac{t^l}{l!}\right)\left(\sum_{i=0}^{\infty}\frac{(\ln c)^i t^i}{i!}x^i\right)\\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\frac{(\ln c)^{l-i}}{i!(l-i)!}B_i^{(k)}(a,b)x^{l-i}t^l\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}(\ln c)^{n-l}B_l^{(k)}(a,b)x^{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

(4) For formula (14), calculation shows that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+1)t} &= \frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}.c^t\\ &= \frac{Li_k(1-(ab)^{-t})}{\left(\frac{b}{c}\right)^t-(ac)^{-t}}c^{xt} = \sum_{n=0}^{\infty}B_n^{(k)}(x;ac,\frac{b}{c},c)\frac{t^n}{n!}.\end{aligned}$$

(5) For formula (15), because of

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}}e^{xt} = \frac{Li_k(1-e^{-x})}{e^{-xt}-e^{-x-xt}} = \frac{Li_k(1-e^{-x})}{(e^{-t})^x-(e^{1+t})^{-x}},$$

so we get that

$$B_n^{(k)}(t) = B_n^{(k)}(e^{t+1}, e^{-t}).$$

(6) For formula (16), write

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(k)}(x,a,b,c)\frac{t^n}{n!} &= \frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt} = \frac{1}{b^t}\frac{Li_k(1-(ab)^{-t})}{(1-(ab)^{-t})}c^{xt}\\ &= e^{t(-\ln b+x\ln c)}\left(\frac{Li_k(1-e^{-t\ln ab})}{1-e^{-t(\ln ab)}}\right)\\ &= \sum_{n=0}^{\infty}(\ln a+\ln b)^n B_n^{(k)}\left(\frac{-\ln b+x\ln c}{\ln a+\ln b}\right)\frac{t^n}{n!}.\end{aligned}$$

So

$$B_n^{(k)}(x,a,b,c) = (\ln a+\ln b)^n B_n^{(k)}\left(\frac{-\ln b+x\ln c}{\ln a+\ln b}\right).$$

□

Theorem 2.2 *Let $x \in \mathbb{R}$, $n \geq 0$. For every positive real numbers a,b such that $a \neq b$ and $b > a > 0$, we have*

$$\begin{aligned}B_n^{(k)}(x+y,a,b,c) &= \sum_{l=0}^{\infty}\begin{pmatrix} n\\ l\end{pmatrix}(\ln c)^{n-l}B_l^{(k)}(x;a,b,c)y^{n-l}\\ &= \sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}(\ln c)^{n-l}B_l^{(k)}(y,a,b,c)x^{n-l}. \qquad (17)\end{aligned}$$

Proof Calculation shows that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+y)t} &= \sum_{n=0}^{\infty}B_n^{(k)}(x+y;a,b,c)\frac{t^n}{n!} = \frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}.c^{yt}\\ &= \left(\sum_{n=0}^{\infty}B_n^{(k)}(x;a,b,c)\frac{t^n}{n!}\right)\left(\sum_{i=0}^{\infty}\frac{y^i(\ln c)^i}{i!}t^i\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}y^{n-l}(\ln c)^{n-l}B_l^{(k)}(x,a,b,c)\right)\frac{t^n}{n!}.\end{aligned}$$

So we get

$$\begin{aligned}\frac{Li_k(1-(ab)^{t})}{b^t-a^{-t}}c^{(x+y)t} &= \frac{Li_k(1-(ab)^{-t}}{b^t-a^{-t}}c^{yt}c^{xt}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}x^{n-l}(\ln c)^{n-l}B_l^{(k)}(y,a,b,c)\right)\frac{t^n}{n!}.\end{aligned}$$

□

Theorem 2.3 *Let $x \in \mathbb{R}$ and $n \geq 0$.For every positive real numbers a,b and c such that $a \neq b$ and $b > a > 0$, we have*

$$B_n^{(k)}(x;a,b,c) = \sum_{l=0}^{n} \begin{pmatrix} n \\ l \end{pmatrix} (\ln c)^{n-l} B_l^{(k)} \left(\frac{-\ln b}{\ln a + \ln b} \right) (\ln a + \ln b)^l x^{n-l}, \qquad (18)$$

$$B_n^{(k)}(x;a,b,c) = \sum_{l=0}^{n} \sum_{j=0}^{l} (-1)^{l-j} \begin{pmatrix} n \\ l \end{pmatrix} \begin{pmatrix} l \\ j \end{pmatrix} (\ln c)^{n-l} (\ln b)^{l-j} (\ln a + \ln b)^j B_j^{(k)} x^{n-k}. \quad (19)$$

Proof Applying Theorems 2.1 and 2.2, we know that

$$B_n^{(k)}(x;a,b,c) = \sum_{l=0}^{n} \begin{pmatrix} n \\ l \end{pmatrix} (\ln c)^{n-l} B_l^{(k)}(a,b) x^{n-l}$$

and

$$B_n^{(k)}(a,b) = B_n^{(k)}(\frac{-lnb}{\ln a + \ln b})(\ln a + \ln b)^n$$

Then the relation (18) follow if we combine these formulae. The proof for (19) is similar. □

Now,we give some results about derivatives and integrals of the generalized poly-Bernoulli polynomials in the following theorem.

Theorem 2.4 *Let $x \in \mathbb{R}$.If a, b and $c > 0$, $a \neq b$ and $b > a > 0$,For any non-negative integer l and real numbers α and β we have*

$$\frac{\partial^l B_n^{(k)}(x,a,b,c)}{\partial x^l} = \frac{n!}{(n-l)!} (\ln c)^l B_{n-l}^{(k)}(x,a,b,c) \qquad (20)$$

$$\int_\alpha^\beta B_n^{(k)}(x,a,b,c)dx = \frac{1}{(n+1)\ln c} [B_{n+1}^{(k)}(\beta,a,b,c) - B_{n+1}^{(k)}(\alpha,a,b,c)] \qquad (21)$$

Proof Applying induction on n, these formulae (20) and (21) can be proved. □

In [9], GI-Sang Cheon investigated the classical relationship between Bernoulli and Euler polynomials, in this paper we study the relationship between the generalized poly-Bernoulli and Euler polynomials.

Theorem 2.5 *For $b > 0$ we have*

$$B_n^{(k_1)}(x+y,1,b,b) = \frac{1}{2} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} [B_n^{(k_1)}(y,1,b,b) + B_n^{(k_1)}(y+1,1,b,b)] E_{n-k}(x,1,b,b).$$

Proof We know that

$$B_n^{(k_1)}(x+y,1,b,b)=\sum_{k=0}^{\infty}\left(\begin{array}{c}n\\k\end{array}\right)(\ln b)^{n-k}B_k^{(k_1)}(y;1,b,b)x^{n-k}$$

and

$$E_k(x+y,1,b,b)+E_k(x,1,b,b)=2x^k(\ln b)^k$$

So, we obtain

$$\begin{aligned}
B_n^{(k_1)}(x+y,1,b,b) &= \frac{1}{2}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)(\ln b)^{n-k}B_k^{(k_1)}(y;1,b,b)\\
&\times \left[\frac{1}{(\ln b)^{n-k}}(E_{n-k}(x,1,b,b)+E_{n-k}(x+1,1,b,b))\right]\\
&= \frac{1}{2}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)B_k^{(k_1)}(y;1,b,b)\\
&\times \left[E_{n-k}(x,1,b,b)+\sum_{j=0}^{n-k}\left(\begin{array}{c}n-k\\j\end{array}\right)E_j(x,1,b,b)\right]\\
&= \frac{1}{2}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)B_k^{(k_1)}(y;1,b,b)E_{n-k}(x,1,b,b)\\
&+ \frac{1}{2}\sum_{j=0}^{n}\left(\begin{array}{c}n\\j\end{array}\right)E_j(x;1,b,b)\sum_{k=0}^{n-j}\left(\begin{array}{c}n-j\\k\end{array}\right)B_k^{(k_1)}(y,1,b,b)\\
&= \frac{1}{2}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)B_k^{(k_1)}(y;1,b,b)E_{n-k}(x,1,b,b)\\
&+ \frac{1}{2}\sum_{j=0}^{n}\left(\begin{array}{c}n\\j\end{array}\right)B_{n-j}^{(k_1)}(y+1;1,b,b)E_j(x,1,b,b)
\end{aligned}$$

So we have

$$B_n^{(k_1)}(x+y,1,b,b)=\frac{1}{2}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)[B_n^{(k_1)}(y,1,b,b)+B_n^{(k_1)}(y+1,1,b,b)]E_{n-k}(x,1,b,b).$$

□

Corollary 2.1 *In Theorem* 2.5*, if* $k_1=1$ *and* $b=e$*, then*

$$B_n(x)=\sum_{(k=0),(k\neq 1)}^{n}\left(\begin{array}{c}n\\k\end{array}\right)B_kE_{n-k}(x).$$

For more details see [7].

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