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Embeddings of Circular graph $C(2n+1,2)(n \geq 2)$ on the Projective Plane

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Abstract: Researches on embeddings of graphs on the projective plane have significance to determine the total genus distributions of graphs. Based on the embedding model of joint tree, this paper calculated the embedding number of the circular graph $C(2n+1,2)(n \geq 2)$ on the projective plane. Therefore, embeddings of K_5 on the projective plane is solved.

Key Words: Surface, genus, embeddings, joint tree, Smarandachely k-drawing.

AMS(2000): 05C15, 05C25

§1. Introduction

In this paper, a surface is a compact 2-dimensional manifold without boundary. It is orientable or nonorientable. Given a graph G and a surface S, a *Smarandachely k-drawing* of G on S is a homeomorphism ϕ: $G \to S$ such that $\phi(G)$ on S has exactly k intersections in $\phi(E(G))$ for an integer k. If $k=0$, i.e., there are no intersections between in $\phi(E(G))$, or in another words, each connected component of $S - \phi(G)$ is homeomorphic to an open disc, then G has an 2-cell embedding on S. Two embeddings $h: G \to S$ and $g: G \to S$ of G into a surface S are said to be equivalent if there is a homeomorphism $f: S \to S$ such that $f \circ h = g$.

Given a graph G, how many nonequivalent embeddings of G are there into a given surface is one of important problems in topological graph theory. It can be tracked back to the genus distributions or total genus distributions of graphs. Since Gross and Furst [1] had introduced these concepts, the genus distributions or total genus distributions of a few graph classes had been solved by scholars [2-7]. However, for many other graph classes, we have not solved the related problems temporarily. There are always relationships among the numbers of embeddings of a graph on different genus surfaces. Therefore, researching on embeddings of graphs on sphere,torus,projective plane,Klein bottle has special significance. The embedding model of joint tree [8] is a special method which had promoted the research on genus distributions or total distributions of graphs [9-12].Basing on this model,this paper calculated the embedding number of circular graph $C(2n+1,2)(n \geq 2)$ on the projective plane.

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§2. Related Knowledge and Lemmas

A surface can be represented by a polygon of even edges in the plane whose edges are pairwise identified and directed clockwise or counterclockwise. To distinguish the direction of each edge, we index each edge by "+"(always omitted) and "−" . For example, sphere, torus, projective plane, Klein bottle can be represented by $O_0 = aa^-$, $O_1 = aba^-b^-$, $N_1 = aa$, $N_2 = aabb$ respectively. In general,

$$O_p = \prod_{i=1}^{p} a_i b_i a_i^- b_i^-, N_q = \prod_{i=1}^{q} a_i a_i$$

denote respectively an orientable surface with genus p and a nonorientable surface with genus $q (p \geq 1, q \geq 1)$. Edge a is called a twisted edge if the directions of the identical edges a is the same. Otherwise edge a is called an untwisted edge. A nonorientable surface has at least one twist edge.

The following three operations don't change the type of a surface:

Operation 1 $Aaa^- \Leftrightarrow A$.

Operation 2 $AabBab \Leftrightarrow AcBc$.

Operation 3 $AB \Leftrightarrow (Aa)(a^-B)$.

Among the above three operations, the parentheses stand for cyclic order. A and B stand liner order and they aren't empty except operation 2. Actually the above operations determine a topological equivalence denoted $\sim$. Therefore, They introduce three relations of topological equivalence.

Relation 1 $AxByCx^-Dy^-E \sim ADCBExyx^-y^-$.

Relation 2 $AxBxC \sim AB^-Cxx$.

Relation 3 $Axxyzy^-z^- \sim Axxyyzz$.

Based on the above operations and relations, It is easy to obtain the following lemmas:

Lemma 2.1([8]) *Suppose S_1 is an orientable surface with genus p and S_2 is a nonorientable surface with genus q.*

(1) If $S = S_1xyx^-y^-$, Then S is an orientable surface with genus $p+1$;

(2) If $S = S_2xyx^-y^-$, Then S is a nonorientable surface with genus $q+2$;

(3) If $S = S_1xx$, Then S is a nonorientable surface with genus $2p+1$;

(4) If $S = S_2xx$, Then S is a nonorientable surface with genus $q+1$.

Lemma 2.2 *Suppose surface S is nonorientable and $S = AxByCx^-Dy^-$, then the nonrientable genus of S is not less than 3.*

Proof According to relation1, $S = AxByCx^-Dy^- \sim ADCBxyx^-y^-$. Let$S_2 = ADCB$, then S_2 is nonorientable and its genus is at least 1. Based on Lemma **??**, the nonorientable genus of surface S is not less than 3. □

Lemma 2.3 *Suppose surface S is nonorientable, if $S = AxByCyDx$ or $S = AxByCxDy^-$, then the nonorientable genus is not less than 2.*

Proof If $S = AxByCyDx$, according to relation 2,

$$S = AxByCyDx \sim AxBC^{-}Dxyy \sim AD^{-}CB^{-}yyxx.$$

According to Lemma 2.1,the nonorientable genus of S is not less than 2;

Suppose $S = AxByCxDy^{-}$, according to relation 2,

$$S = AxByCxDy^{-} \sim AC^{-}y^{-}B^{-}Dy^{-}xx \sim AC^{-}D^{-}Bxxy^{-}y^{-}.$$

According to Lemma 2.1, the nonorientable genus of S is not less than 2. □

The embedding model of joint tree may be introduced in the following way: Given a spanning tree T of a graph $G = (V, E)$, we split every cotree edge into two edges and label them by the identical letter. The two edges are called the semi-edges of the original cotree edge. The resulting graph is the joint tree of the original graph G. Suppose the number of cotree edges is β. Given a direction to every semi-edge so that the direction of each pair of semi-edges can be the same or opposite. Beginning with a vertex, we walk all over the edges of the joint tree by its rotation. Writing the letter of semi-edges of the original graph cotree edges by order. we obtain a polygon of 2β edges which is exactly the associated surface of the graph G. There is a 1 to 1 correspondence between the associated surfaces and the embeddings of graph G. Hence an embedding of a graph G on a surface can be exactly represented by an associate surface of the graph G.

§3. Main Conclusions

The first, we investigate the structure character of polygon representation of projective plane.

Definition 3.1 *If surface $S = AxByCxDy$, then x and y are said to be interlaced in S; if surface $S = AxBxCyDy$, then x and y are said to be parallel in S.*

According to Lemmas 2.2 and 2.3, it is easy to obtain the following theorem:

Theorem 3.1 *Suppose S is a projective plane. If two edges in the polygon representation of S are all twisted, then they must be interlaced; otherwise, they must be parallel.*

Definition 3.2 *Circular graph $C(2n+1,2)$ $(n \geq 2)$ is obtained by appending chords $\{u_j u_{j+2} \mid j = 1, 2, \cdots, 2n+1\}$ on the circle $u_1 u_2 \cdots u_{2n+1} u_1$. Figure 1 is the circular graph $C(7,2)$. $a_i = u_{2i-1}u_{2i+1} (i = 1, 2, \cdots, n)$ are called odd chords; $b_i = u_{2i}u_{2i+2} (i = 1, 2, \cdots, n-1)$ are called even chords. Specially, let $c_0 = u_{2n+1}u_1$, $a_0 = u_{2n+1}u_2$, $b_0 = u_{2n}u_1$. Denote the collection of odd chords by E_1, $E_1 = \{a_i \mid i = 1, 2, \cdots, n\}$; Denote the collection of even chords by E_2, $E_2 = \{b_i \mid i = 1, 2, \cdots, n-1\}$. The subscriptions of vertices are the Residue Class Modules of $2n+1$.*

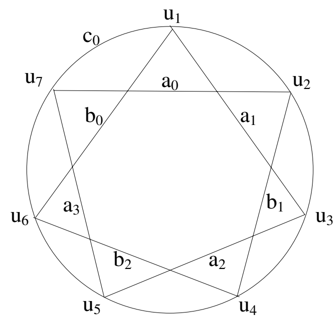

Figure 1: C(7,2)

There are some researches on embeddings of circular graphs in paper [13]. According to it, a circular graph can be embedded on the projective plane. But the embedding number and structure have not been investigated yet. In this paper, we calculated the embedding number of circular graphs on the projective plane.

We choose path $u_1u_2...u_{2n}u_{2n+1}$ as the spanning tree of the circular graph $C(2n+1,2)$ $(n \geq 2)$. Then by splitting each cotree edge, we obtain the joint tree. The two edges by splitting one cotree edge are called semi-edges of the original cotree edge. The upside of the spanning tree is the side which the semi-edge a_0 incident with vertex u_{2n+1} is placed. The other side is called the underside of the spanning tree. Considering the special positions of cotree edges c_0, a_0, b_0, we discuss the embedding of circular graph $C(2n+1,2)(n \geq 2)$ on the projective plane basing on whether the three cotree edges are twisted.

First, according to Lemmas 2.2 and 2.3, if the associated surface of circular graph $C(2n+1,2)(n \geq 2)$ is projective plane, then we have the following claims:

Claim 1 There are at most three twisted edges in $E_1 \cup E_2$.

In fact, if there are more than three twisted edges in $E_1 \cup E_2$, there will exist two twisted edges and they are parallel in the associated surface. It contradicts to Theorem 3.1.

Claim 2 Each semi-edges pair of an untwisted edge must be placed on the same side of the spanning tree.

In fact, if a semi-edges pair of an untwisted edge are placed on the distinct sides of the spanning tree, the untwisted edge and c_0 must be interlaced in the associated surface of graph G. It contradicts to Theorem 3.1.

Claim 3 If a_{i-1}, a_i(or b_{i-1}, b_i) are two untwisted edges in E_1(orE_2) and they are placed on distinct sides of the spanning tree, then b_{i-1}(ora_i) is twisted and its two semi-edges must be placed on distinct side of the spanning tree.

As is shown in Figure 2, a_{i-1}, a_i are two untwisted edges and placed on the two sides of the spanning tree respectively. If b_{i-1} is not twisted, then it must be interlaced with one of the three edges a_{i-1}, a_i, c_0. If b_{i-1} is twisted but its two semi-edges are placed on the same side of the spanning tree, it will be interlaced with a_{i-1} or a_i. The two cases all contradict to Theorem 3.1. Similarly we can prove the case of the two edges b_{i-1}, b_i.

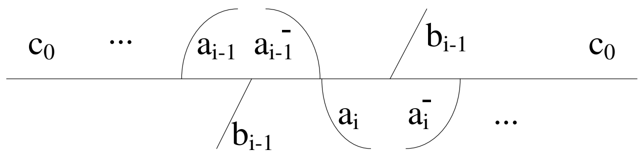

Figure 2: The side-transferring of untwisted neighbor edges

Theorem 3.2 *The embedding number of a circular graph $C(2n+1,2)(n \geq 2)$ on the projective plane is $8n+6$.*

Proof There are two embedding cases when considering whether c_0 is twisted.

Case 1 c_0 is untwisted

Because c_0 is untwisted, each semi-edges pair of a twisted edge must be placed on the same side of the spanning tree. Otherwise, it will be interlaced with c_0 and contract to Theorem 3.1. On the other side, every two twisted edges must be interlaced in the associated surface. all the twisted edges are placed on the same side. According to Claim3, there are no side-transferring case of neighbor untwisted edges in E_1 or E_2. Otherwise, there must exist a twisted edge that its semi-edges pair are placed on the two distinct side of the spanning tree respectively. It contracts to the above discussion. According to whether a_0, b_0 are twisted edges, The embeddings can be classified into four subcases:

Subcase 1.1 a_0 and b_0 are all untwisted

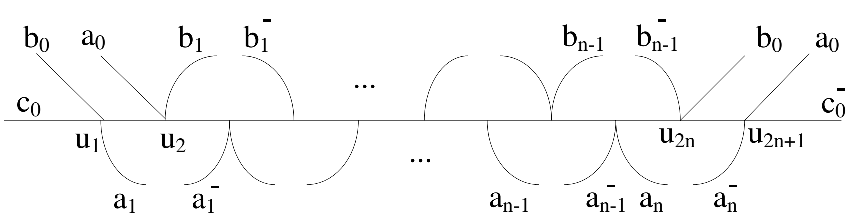

Figure 3: The joint tree of Subcase 1.1

As shown in Figure 3, a_0 and b_0 can only be placed on the same side of the spanning tree. If there are twisted edges in $E_1 \cup E_2$. they can only be a_1 or a_n. Suppose a_1 is twisted, then it must be on the upside of the spanning tree. Furthermore, b_1 can only be placed on the underside and also is a_n. Corresponding b_{n-1} is on the upside. Therefore, the sequence of untwisted edges $b_1b_2\cdots b_{n-1}$ will shift sides at one vertex. It contradicts to the above discussion. Then a_1 can't be a twisted edge, so is a_n in the same way. Then there are no twisted edges in $E_1 \cup E_2$. According to claim 3 and the above discussion, the untwisted edges sequence $b_1b_2\cdots b_{n-1}$ must be on the upside of the spanning tree while another untwisted edges sequence $a_1a_2\cdots a_n$ must be on the underside. Beginning at semi-edge c_0 incident to vertex u_1. Walk along all the joint tree edges by its rotation, we get the associated surface:

$$S = c_0b_0a_0b_1b_1^-b_2b_2^- \cdots b_{n-1}b_{n-1}^-b_0a_0c_0^-a_n^-a_na_{n-1}^-a_{n-1}\cdots a_1^-a_1$$
$$\sim b_0a_0b_0a_0 \sim N_1.$$

Considering the symmetry of the two sides of the spanning tree, the embedding number of Subcase 1.1 is 2.

Subcase 1.2 a_0 is twisted, b_0 is untwisted

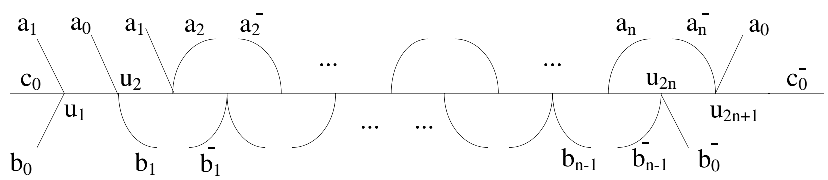

Figure 4: The joint tree of Subcase 1.2

Similarly, according to Theorem 3.1, a_0 and b_0 can only be placed on the distinct side of the spanning tree(as shown in Figure 4. If there is no twisted edge in $E_1 \cup E_2$, then a_n can only be placed on the upside because the untwisted edge can only be placed on the underside of the spanning tree. Then the sequence of untwisted edges $a_1a_2\cdots a_n$ will shift sides at one vertex. It contradicts to the above discussion. So there is no twisted edges in $E_1 \cup E_2$.

Each twisted edge in $E_1 \cup E_2$ and a_0 must be interlaced and they are placed on the same side. Then, the twisted edges in $E_1 \cup E_2$ can only be the following edges:a_1, b_1, a_n. a_1 and a_n can't all be twisted edges, otherwise they will be parallel. However there are at least one twisted edge among them, otherwise the sequence of untwisted edges $a_1a_2\cdots a_n$ will shift sides. If a_n is twisted, then it can only be placed on the upside and be interlaced with a_0. So b_1 and a_1 must be untwisted. Furthermore, a_1 must be placed on the underside while b_1 must be placed on the upside. Therefore, the untwisted edge b_{n-1} can only be placed on the underside. It indicate that the untwisted edges sequence $b_1b_2\cdots b_{n-1}$ shift sides at one vertex. It contradicts to Claim 3. So a_1 must be twisted and a_n is untwisted. If b_1 is also twisted, Then it will be placed on the upside. So a_2 will be placed on the underside while a_n will be placed on the upside. It is to say that the untwisted edges sequence $a_2a_3\cdots a_n$ will shift sides and contradicts to Claim 3.

Based on the above discussion, a_1 is the only twisted edge in $E_1 \cup E_2$. According to Theorem 3.1 and Claims 1,2,3, the rotations of the joint tree are only fixed. The associated surface is

$$S = c_0a_1a_0a_1a_2a_2^- \cdots a_na_n^-a_0c_0^-b_0^-b_{n-1}^-b_{n-1}\cdots b_1b_1^-b_0$$
$$\sim a_1a_0a_1a_0 \sim N_1.$$

So the embedding number of Subcase 1.2 on the projective plane is also 2.

Subcase 1.3 a_0 is untwisted, b_0 is twisted

Similarly, a_0 and b_0 can only be placed on the distinct side of the spanning tree. discussed in the same way with Subcase 1.2, a_n is the only twisted edges in $E_1 \cup E_2$. The joint tree is shown in Figure 5:

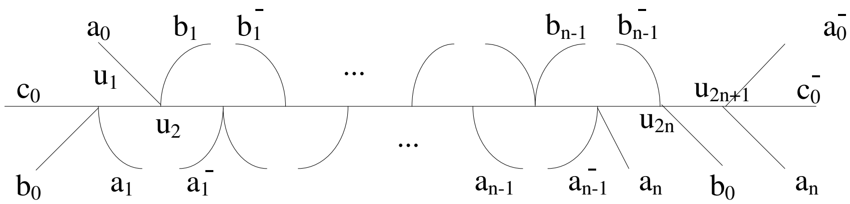

Figure 5; joint tree of subcase 1.3

The associated surface is

$$
\begin{aligned}
S &= c_0 a_0 b_1 b_1^- \cdots b_{n-1} b_{n-1}^- a_0^- c_0^- a_n b_0 a_n a_{n-1}^- a_{n-1} \cdots a_1^- a_1 b_0 \\
&\sim a_n b_0 a_n b_0 \sim N_1.
\end{aligned}
$$

The embedding number of Subcase 1.3 is 2.

Subcase 1.4 a_0, b_0 are all untwisted

As shown in Figure 6, a_0, b_0 can only be placed on the distinct side respectively, otherwise they are interlaced and contradict to Theorem 3.1. Furthermore, a_1 must be placed on the underside and a_n must be placed on the upside. In correspondence, b_1 is on the upside and b_{n-1} is on the underside. Because the associated surface is projective plane, so there are at least one twisted edge in $E_1 \cup E_2$.

If there is only one twisted edge in $E_1 \cup E_2$ and it is $a_i (1 \leq i \leq n)$, then the untwisted sequence $b_1 b_2 \cdots b_{n-1}$ will shift sides at one vertex and contradiction happens. similarly is the case that $b_i (1 \leq i \leq n-1)$ is the only twisted edge. So there are at least two twisted edges in $E_1 \cup E_2$.

If there are two twisted edges in $E_1 \cup E_2$, then the twisted edges pair must be the following combinations: $\{a_i, b_i\}$, $\{a_i, b_{i-1}\}$, $\{a_i, a_{i+1}\}$, $\{b_i, b_{i+1}\}$. If the twisted edge pair are $a_i, a_{i+1} (1 \leq i \leq n-1)$, Then the untwisted edges sequence $b_1 b_2 \cdots b_{n-1}$ will shift sides. Similarly, if the twisted edges pair are $b_i, b_{i+1} (1 \leq i \leq n-2)$, the untwisted edges sequence $a_1 a_2 \cdots a_n$ will shift sides. According to Claim 3, contradiction happens.

If the twisted edges pair is $a_i, b_i (1 \leq i \leq n-1)$, according to Theorem 3.1, they are on the underside. The joint tree is shown in Figure 6.

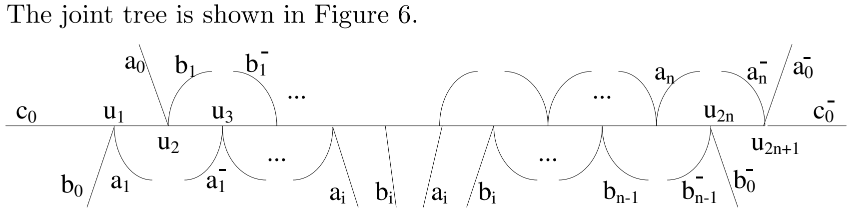

Figure 6: The joint tree of embedding Subcase 1.4 (a_i, b_i is twisted)

The associated surface

$$\begin{aligned} S &= c_0a_0b_1b_1^- \cdots b_{i-1}b_{i-1}^-a_{i+1}a_{i+1}^- \cdots a_na_n^-a_0^- \\ &\quad c_0^-b_0^-b_{n-1}^-b_{n-1} \cdots b_{i+1}^-b_{i+1}b_ia_ib_ia_ia_{i-1}^-a_{i-1} \cdots a_1^-a_1b_0 \\ &\sim b_ia_ib_ia_i \sim N_1 \end{aligned}$$

and the embedding number of this case is $2(n-1)$.

If the twisted edges pair is $a_i, b_{i-1}(2 \le i \le n)$, then they are on upside. The associated surface

$$\begin{aligned} S &= c_0a_0b_1b_1^- \cdots b_{i-2}b_{i-2}^-b_{i-1}a_ib_{i-1}a_ia_{i+1}a_{i+1}^- \cdots a_na_n^-a_0^- \\ &\quad c_0^-b_0^-b_{n-1}^-b_{n-1} \cdots b_i^-b_ia_{i-1}^-a_{i-1} \cdots a_1^-a_1b_0 \\ &\sim b_{i-1}a_ib_{i-1}a_i \sim N_1 \end{aligned}$$

and the embedding number of this case is also $2(n-1)$.

If there are three twisted edges in $E_1 \cup E_2$, Then the twisted edges must be the following two combinations: $\{a_i, a_{i+1}, b_i\}$ and $\{b_i, b_{i+1}, a_{i+1}\}$. Suppose $a_i, a_{i+1}, b_i(1 \le i \le n-2)$ are twisted edges and placed on the underside of the spanning tree. The untwisted edges a_n must be placed on the upside. According to Claim3, the untwisted edges sequence $a_n \cdots a_{i+2}$ are on the upside. Therefore, the untwisted edge b_{i+1} will be interlaced with a_{i+1} or a_{i+2}. It contradicts Theorem 3.1. Suppose $a_i, a_{i+1}, b_i(2 \le i \le n-1)$ are placed on the upside of the spanning tree, similarly, the untwisted edges sequence $a_1 \cdots a_{i-1}$ must be placed on the underside. Therefore, the untwisted edge b_{i-1} must be interlaced with a_{i-1} or a_i. It contradicts Theorem 3.1. Similarly, If b_i, b_{i+1}, a_{i+1} are twisted edges, contradiction will also happen.

So the embedding number of the Subcase1.4 on the projective plane is $4n-4$. The embedding number of the Case 1 on the projective plane is $4n+2$.

Case 2 c_0 is twisted

In this case, semi-edges pair of each twisted edge can only be placed on the distinct side. Otherwise, the twisted edge and c_0 will be parallel and contradicts to Theorem 3.1. There are at most two twisted edges in $E_1 \cup E_2$, otherwise there will exist two twisted edges and they are parallel in the associated surface. According to whether a_0 and b_0 are twisted, the embedding can be classified into four subcases.

Subcase 2.1 a_0, b_0 are all twisted

If there are twisted edges in $E_1 \cup E_2$, they can only be the following combinations: $a_i, a_{i+1}(1 \le i \le n-1)$ or $b_i, b_{i+1}(1 \le i \le n-2, n > 2)$. In fact, among the untwisted edges sequence $b_1b_2 \cdots b_{n-1}$, b_1, b_{n-1} are all on the underside. If the sequence shift sides, then it will shift sides two times continuously and $a_i, a_{i+1}(1 \le i \le n-1)$ will be twisted edges. similarly, $b_i, b_{i+1}(1 \le i \le n-2, n > 2)$ may be twisted edges in the same way.

If there are no twisted edges in $E_1 \cup E_2$, the untwisted edges sequence $a_1a_2 \cdots a_n$ must be placed on the upside while the the untwisted edges sequence $b_1b_2 \cdots b_{n-1}$ must be placed on the underside. the associated surface

$$
\begin{aligned}
S &= c_0b_0a_1a_1^-a_2a_2^- \cdots a_na_n^-a_0c_0b_0^-b_{n-1}^-b_{n-1}\cdots b_1^-b_1a_0 \\
&\sim c_0b_0a_0c_0b_0a_0 \sim N_1.
\end{aligned}
$$

The embedding number of this subcase on the projective plane is 2.

If $a_i, a_{i+1}(1 \leq i \leq n-1)$ are twisted edges, the joint tree is shown in Figure 7.

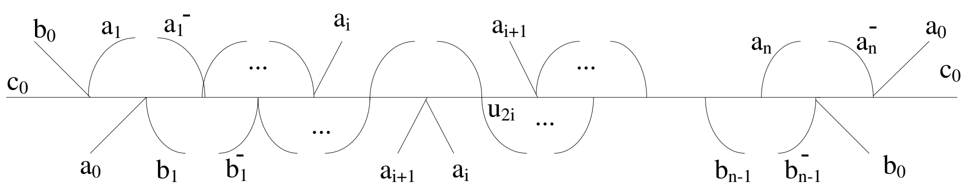

Figure 7: The joint tree of Subcase2.1(a_i, a_{i+1} are twisted)

The associated surface

$$
\begin{aligned}
s &= c_0b_0a_1a_1^- \cdots a_{i-1}a_{i-1}^-a_ib_ib_i^-a_{i+1}a_{i+2}a_{i+2}^- \cdots a_na_n^-a_0 \\
&\quad c_0b_0b_{n-1}^-b_{n-1}\cdots b_{i+1}^-b_{i+1}a_ia_{i+1}b_{i-1}^-b_{i-1}\cdots b_1^-b_1a_0 \\
&\sim c_0b_0a_ia_{i+1}a_0c_0b_0a_ia_{i+1}a_0 \sim N_1
\end{aligned}
$$

and the embedding number of this subcase on the projective plane is $2(n-1)$.

If $b_i, b_{i+1}(1 \leq i \leq n-2, n > 2)$ are twisted edges, the joint tree is shown in Figure 8.

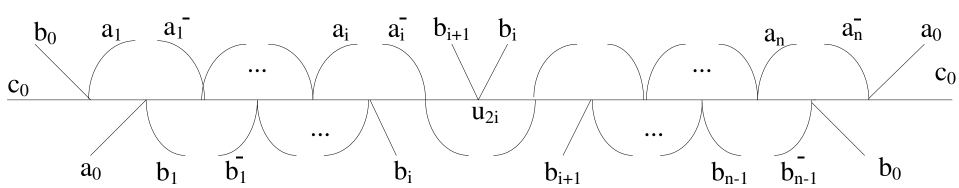

Figure 8: The joint tree of Subcase 2.1(b_i, b_{i+1} is twisted)

The associated surface

$$
\begin{aligned}
S &= c_0b_0a_1a_1^- \cdots a_ia_i^-b_{i+1}b_ia_{i+2}a_{i+2}^- \cdots a_na_n^-a_0 \\
&\quad c_0b_0b_{n-1}^-b_{n-1}\cdots b_{i+1}a_{i+1}a_{i+1}^-b_ib_{i-1}^-b_{i-1}\cdots b_1b_1^-a_0 \\
&\sim c_0b_0b_{i+1}b_ia_0c_0b_0b_{i+1}b_ia_0 \sim N_1
\end{aligned}
$$

and the embedding number of this subcase on the projective plane is $2(n-2) = 2n-4$.

So The embedding number of subcase 1.2 on the projective plane is $4n-4$.

Subcase 2.2 a_0 is twisted, b_0 is untwisted

As shown in Figure 9, the semi-edges pair of a_0 must be placed on the two distinct sides and b_0 be placed on the upside.

If there is no twisted edges in $E_1 \cup E_2$, then a_1 and a_n can only be placed on the distinct side. Then the untwisted edges sequence $a_1a_2 \cdots a_n$ will shift sides and contradict to Claim3. So there are twisted edges in $E_1 \cup E_2$. However, the twisted edges in $E_1 \cup E_2$ can only be a_1, a_n, b_{n-1}. Suppose a_n is twisted, then a_1, b_{n-1} are untwisted. Then the untwisted edges sequence $a_1a_2 \cdots a_{n-1}$ must be placed on the upside of the spanning tree. Therefore b_{n-1} must be on the underside and interlaced with a_n. Contradiction happens.

If a_1 is twisted, then a_n, b_{n-1} are untwisted. The untwisted edges sequences $b_1b_2 \cdots b_{n-1}$ and $a_2a_3 \cdots a_n$ are placed on the upside and underside respectively. The joint tree is shown in Figure 9:

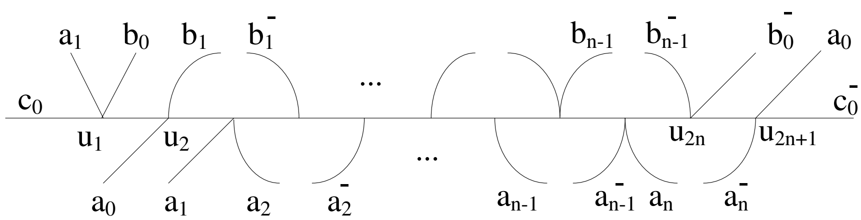

Figure 9: The joint tree of subcase 2.2(a_1 is twisted)

The associated surface

$$S = c_0a_1b_0b_1b_2^- \cdots b_{n-1}b_{n-1}^- b_0^- a_0c_0a_n^- a_na_{n-1}^- a_{n-1} \cdots a_2^- a_2a_0$$
$$\sim c_0a_1a_0c_0a_1a_0 \sim N_1.$$

If b_{n-1} is twisted, then a_1, a_n are untwisted. The joint tree is shown in Figure 10.

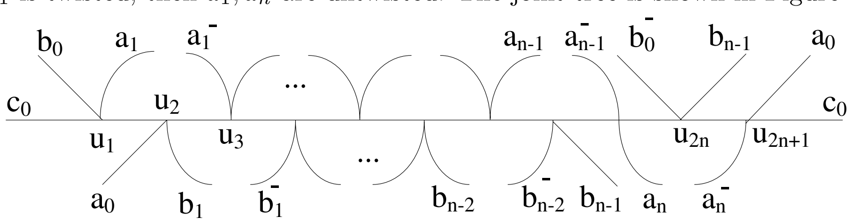

Figure 10: The joint tree of subcase2.2(b_{n-1} is twisted)

The associated surface

$$S = c_0b_0a_1a_1^- a_2a_2^- \cdots a_{n-1}a_{n-1}^- b_0^- b_{n-1}a_0c_0a_n^- a_nb_{n-1}b_{n-2}^- b_{n-2} \cdots b_1^- b_1a_0$$
$$\sim c_0b_{n-1}a_0c_0b_{n-1}a_0 \sim N_1$$

and the embedding number of subcase2.2 on the projective plane is 4.

Subcase 2.3 a_0 is untwisted, b_0 is twisted

Similarly, in this case, the twisted edges in $E_1 \cup E_2$ can only be b_1 or a_n. If b_1 is twisted, the associated surface

$$S = c_0b_0b_1a_0a_2a_2^- a_3a_3^- \cdots a_na_n^- a_0^- c_0b_0b_{n-1}^- b_{n-1}b_{n-2}^- b_{n-2} \cdots b_2^- b_2b_1$$
$$\sim c_0b_0b_1c_0b_0b_1 \sim N_1.$$

If a_n is twisted, the associated surface

$$S = c_0b_0a_0b_1b_1^-b_2b_2^- \cdots b_{n-1}b_{n-1}^-a_0^-a_nc_0b_0a_na_{n-1}^-a_{n-1}a_{n-2}^-a_{n-2}\cdots a_1^-a_1$$
$$\sim c_0b_0a_nc_0b_0a_n \sim N_1$$

and the embedding number of Subcase 2.3 on the projective plane is 4.

Subcase 2.4 a_0, b_0 are all untwisted

a_0 and b_0 must be placed on the distinct side of the spanning tree. If there are twisted edges in $E_1 \cup E_2$, then the semi-edges of the twisted edge must be placed on the distinct side. It will be interlaced with a_0 and b_0. So the edges in $E_1 \cup E_2$ are all untwisted. However, the untwisted edges a_1 and a_n can only be placed on the distinct side. Then the untwisted edges sequence $a_1a_2\cdots a_n$ will shift sides at one vertex. Contradiction happens. So Subcase 2.4 can't be embedded on the projective plane.

Then the embedding number of Case 2 on the projective plane is $4n+4$.

Based on the above discussion, the embedding number of circular graph $C(2n+1)(n \geq 2)$ on the projective plane is $8n+6$. □

Let $n = 2$, we obtain the following corollary:

Corollary 3.1 *The embedding number of complete graph K_5 on the projective plane is* 22.

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