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Smarandachely Precontinuous maps and Preopen Sets in Topological Vector Spaces

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Abstract: It is shown that linear functional on topological vector spaces are Smarandachely precontinuous. Prebounded, totally prebounded and precompact sets in topological vector spaces are identified.

Key Words: Smarandachely Preopen set, precompact set, Smarandachely precontinuous map.

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§1. Introduction

N. Levine [7] introduced the theory of semi-open sets and the theory of α-sets for topological spaces. For a systematic development of semi-open sets and the theory of α-sets one may refer to [1], [2], [4], [5] and [9]. The notion of preopen sets for topological spaces was introduced by S. N. Mashour, M. E. Abd El-Moncef and S.N. El-Deep in [8]. These concepts above are closely related. It is known that, in a topological space, a set is preopen and semi-open if and only if it is an α-set [10], [11]. Our object in section 3 is to define a prebounded set, totally prebounded set, and precompact set in a topological vector space. In Sections 3 and 4 we identify them. Moreover, in Section 2, we show that every linear functional on a topological vector space is precontinuous and deduce that every topological vector space is a prehausdorff space.

§2. Precontinuous maps

We recall the following definitions [2], [8].

Definition 2.1 *Let X be a topological space. A subset S of X is said to be Smarandachely preopen if there exists a set $U \subset \mathrm{cl}(S)$ such that $S \subset \mathrm{int}(\mathrm{cl}(S) \cup U)$. A Smarandachely preneighbourhood of the point $x \in X$ is any Smarandachely preopen set containing x. Particularly, a*

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Smarandachely $\varnothing$-preopen set S is usually called a preopen set.

Definition 2.2 *Let X and Y be topological spaces and $f : X \to Y$. The function f is said to be Smarandachely precontinuous if the inverse image $f^{-1}(B)$ of each open set B in Y is a Smarandachely preopen set in X. The function f is said to be Smarandachely preopen if the image $f(A)$ of every open set A in X is Smarandachely preopen in Y. Particularly, if we replace each Smarandachely preopen by preopen, f is called to be precontinuous.*

The following lemma is obvious.

Lemma 2.1 *Let X and Y be topological vector spaces and $f : X \to Y$ linear. The function f is preopen if and only if, for every open set U containing $0 \in X$, $0 \in Y$ is an interior point of* $\mathrm{cl}(f(U))$.

The following two theorems are known but we include the proofs for convenience of the reader.

Theorem 2.1 *Let X, Y be topological vector spaces and let Y have the Baire property, that is, whenever $Y = \cup_{n=1}^{\infty} B_n$ with closed sets B_n, there is is N such that $\mathrm{int}(B_N)$ is nonempty. Let $f : X \to Y$ be linear and $f(X) = Y$. Then f is preopen.*

Proof Let $U \subset X$ be a neighborhood of 0. There is a neighborhood V of 0 such that $V - V \subset U$. Since V is a neighborhood of 0 we have $X = \cup_{n=1}^{\infty} nV$. It follows from linearity and surjectivity of f that $Y = \cup_{n=1}^{\infty} nf(V)$. Since Y has the Baire property, there is N such that $\mathrm{cl}(Nf(V)) = N\mathrm{cl}(f(V))$ contains an open set S which is not empty. Then $\mathrm{cl}(f(V))$ contains the open set $T = \frac{1}{N}S$. It follows that

$$T - T \subset \mathrm{cl}(f(V)) - \mathrm{cl}(f(V)) \subset \mathrm{cl}(f(V) - f(V)) = \mathrm{cl}(f(V - V)) \subset \mathrm{cl}(f(U)).$$

The set $T - T$ is open and contains 0. Therefore, $0 \in Y$ is an interior point of $\mathrm{cl}(f(U))$. From Lemma 2.1 we conclude that f is preopen. □

Note that f can be any linear surjective map. It is not necessary to assume that f is continuous or precontinuous.

Theorem 2.2 *Let X, Y be topological vector spaces, and let X have the Baire property. Then every linear map $f : X \to Y$ is precontinuous.*

Proof Let $G = \{(x, f(x)) : x \in X\}$ be the graph of f. The projections $\pi_1 : G \to X$ and $\pi_2 : G \to Y$ are continuous. The projection $\pi_1 : G \to X$ is bijective. It follows from Theorem **??** that π_1 is preopen. Therefore, the inverse mapping π_1^{-1} is precontinuous. Then $f = \pi_2 \circ \pi_1^{-1}$ is precontinuous. □

Theorem 2.2 shows that many linear maps are automatically precontinuous. Therefore, it is natural to ask for an example of a linear map which is not precontinuous.

Let $X = C[0, 1]$ be the vector space of real-valued continuous functions on $[0, 1]$ equipped with the norm

$$\|f\|_1 = \int_0^1 |f(x)|\,dx.$$

Let $Y = C[0,1]$ be equipped with the norm

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|.$$

Lemma 2.2 *The identity operator $T : X \to Y$ is not precontinuous.*

Proof Let $U = \{f \in C[0,1] : \|f\|_\infty < 1\}$ which is an open subset of Y. Let $\mathrm{cl}(U)$ be the closure of U in X. We claim that

$$\mathrm{cl}(U) \subset \{f \in C[0,1] : \|f\|_\infty \leqslant 1\}. \tag{2.1}$$

For the proof, consider a sequence $f_n \in U$ and a function $f \in C[0,1]$ such that $\{f_n\}$ converges to f in X. Suppose that there is $x_0 \in [0,1]$ such that $f(x_0) > 1$. By continuity of f, there are $a < b$ and $\delta > 0$ such that $0 \leqslant a \leqslant x_0 \leqslant b \leqslant 1$ and $f(x) > 1+\delta$ for $x \in (a,b)$. Then, as $n \to \infty$,

$$(b-a)\delta \leqslant \int_a^b |f_n(x) - f(x)|\,dx \leqslant \int_0^1 |f_n(x) - f(x)|\,dx \to 0$$

which is a contradiction. Therefore, $f(x) \leqslant 1$ for all $x \in [0,1]$. Similarly, we show that $f(x) \geqslant -1$ for all $x \in [0,1]$. Now $0 \in U = T^{-1}(U)$ but U is not preopen in X. We see this as follows. Suppose that U is preopen in X. The sequence $g_n(x) = 2x^n$ converges to 0 in X. Therefore, $g_n \in \mathrm{cl}(U)$ for some n and (2.1) implies $2 = \|g_n\|_\infty \leqslant 1$ which is a contradiction. □

We can improve Theorem 2.2 for linear functionals.

Theorem 2.3 *Let f be a linear functional on a topological vector space X. If V is a preopen subset of $\mathbb{R}$ then $f^{-1}(V)$ is a preopen subset of X. In particular, f is precontinuous.*

Proof We distinguish the cases that f is continuous or discontinuous.

Suppose that f is continuous. If $f(x) = 0$ for all $x \in X$ the statement of the theorem is true. Suppose that f is onto. We choose $u \in X$ such that $f(u) = 1$. Let V be a preopen subset of $\mathbb{R}$, and set $U := f^{-1}(V)$. Let $x \in U$ so $f(x) \in V$. Since V is preopen, there is $\delta > 0$ such that

$$I := (f(x)-\delta, f(x)+\delta) \subset \mathrm{cl}(V). \tag{2.2}$$

Since f is continuous, $f^{-1}(I)$ is an open subset of X containing x. We claim that

$$f^{-1}(I) \subset \mathrm{cl}(U). \tag{2.3}$$

In order to prove (2.3), let $y \in f^{-1}(I)$ so $f(y) \in I$. By (2.2), there is a sequence $\{t_n\}$ in V converging to $f(y)$. Set

$$y_n := y + (t_n - f(y))u.$$

We have $f(y_n) = t_n \in V$ so $y_n \in U$. Since X is a topological vector space, y_n converges to y. This establishes (2.3). It follows that U is preopen.

Suppose now that f is not continuous. By [3, Corollary 22.1], $N(f) = \{x \in X : f(x) = 0\}$ is not closed. Therefore, there is $y \in \text{cl}(N(f))$ such that $y \notin N(f)$ so $f(y) \neq 0$. Let x be any vector in X. There is $t \in \mathbb{R}$ such that $f(x) = tf(y)$ and so $x - ty \in N(f)$. It follows that $x \in \text{cl}(N(f))$. We have shown that $N(f)$ is dense in X. Let $a \in \mathbb{R}$. There is $y \in X$ such that $f(y) = a$. Then $f^{-1}(\{a\}) = y + N(f)$ and so the closure of $f^{-1}(\{a\})$ is $y + \text{cl}(N(f)) = X$. Therefore, $f^{-1}(\{a\})$ is dense for every $a \in \mathbb{R}$. Let V be a preopen set in $\mathbb{R}$. If V is empty then $f^{-1}(V)$ is empty and so is preopen. If V is not empty choose $a \in V$. Then $f^{-1}(V) \supset f^{-1}(\{a\})$ and so $f^{-1}(V)$ is dense. Therefore, $f^{-1}(V)$ is preopen. □

§3. Subsets of topological vector spaces

In this section our principal goal is to define prebounded sets, totally prebounded sets and precompact sets in a topological vector space, and to find relations between them. We begin this section with some definitions.

Definition 3.1 *A subset E of a topological vector space X is said to be prebounded if for every preneighbourhood V of 0 there exists $s > 0$ such that $E \subset tV$ for all $t > s$.*

Definition 3.2 *A subset E of a topological vector space X is said to be totally prebounded if for every preneighbourhood U of 0 there exists a finite subset F of X such that $E \subset F + U$.*

Definition 3.3 *A subset E of a topological vector space X is said to be precompact if every preopen cover of E admits a finite subcover.*

Lemma 3.1 *Every precompact set in a topological vector space X is totally prebounded.*

Proof Let E be precompact. Let V be preopen with $0 \in V$. Then the collection $\{x + V : x \in E\}$ is a cover of E consisting of preopen sets. There are $x_1, x_2, \ldots, x_n \in E$ such that $E \subset \bigcup\limits_{i=1}^{n} \{x_i + V\}$. Therefore, E is totally prebounded. □

Lemma 3.2 *In a topological vector space X the singleton $\{0\}$ is the only prebounded set.*

Proof It is enough to show that every singleton $\{u\}$, $u \neq 0$, is not prebounded. Let $V = X - \left\{\frac{1}{n}u : n \in \mathbb{N}\right\}$. The closure of V is X so V is preopen . But $\{u\}$ is not subset of nV for $n = 1, 2, 3, \ldots$ Therefore, $\{u\}$ is not prebounded. □

Theorem 3.1 *If E is a prebounded subset of a topological vector space X, then E is totally prebounded. The converse statement is not true.*

Proof This follows from the fact that every finite set is totally prebounded, and by using Lemma 3.2. □

§4. Applications of Theorem 2.3

We need the following known lemma.

Lemma 4.1 *If U, V are two vector spaces, and W is a linear subspace of U and $f : W \to V$ is a linear map. then there is a linear map $g : U \to V$ such that $f(x) = g(x)$ for all $x \in W$.*

Proof We choose a basis A in W and then extend to a basis $B \supset A$ in U. We define $h(a) = f(a)$ for $a \in A$ and $h(b)$ arbitrary in V for $b \in B - A$. There is a unique linear map $g : U \to V$ such that $g(b) = h(b)$ for $b \in B$. Then $g(x) = f(x)$ for all $x \in W$. □

We obtain the following result.

Theorem 4.1 *Every topological vector space X is a prehausdorff space, that is, for each $x, y \in X$, $x \neq y$, there exists a preneighbourhood U of x and a preneighbourhood V of y such that $U \cap V = \emptyset$.*

Proof Let $x, y \in X$ and $x \neq y$. If x, y are linearly dependent we choose a linear functional on the span of $\{x, y\}$ such that $f(x) < f(y)$. If x, y are linearly independent we set $f(sx+ty) = t$. By Lemma 4.1 we extend f to a linear functional g with $g(x) < g(y)$. Choose $c \in (g(x), g(y))$ and define $U = g^{-1}((-\infty, c))$, and $V = g^{-1}((c, \infty))$. Then, using Theorem 2.3, U, V are preopen. Also U and V are disjoint and $x \in U$, $y \in V$. □

We now determine totally prebounded subsets in $\mathbb{R}$. The result may not be surprising but the proof requires some care.

Lemma 4.2 *A subset of $\mathbb{R}$ is totally prebounded if and only if it is finite.*

proof It is clear that a finite set is totally prebounded. Let E be a countable (finite or infinite) subset of $\mathbb{R}$ which is totally prebounded. Let $A := \{x - y : x, y \in E\}$. The set A is countable. We define a sequence $\{u_n\}$ of real numbers inductively as follows. We set $u_1 = 0$. Then we choose $u_2 \in (-1, 0)$ such that $u_2 - u_1 \notin A$. Then we choose $u_3 \in (0, 1)$ such that $u_3 - u_i \notin A$ for $i = 1, 2$. Then we choose $u_4 \in (-1, -\frac{1}{2})$ such that $u_4 - u_i \notin A$ for $i = 1, 2, 3$. Continuing in this way we construct a set $U = \{u_n : n \in \mathbb{N}\} \subset (-1, 1)$ such that every interval of the form $(m2^{-k}, (m + 1)2^{-k})$ with $-2^k \leqslant m < 2^k$, $k \in \mathbb{N}$, contains at least one element of U, and such that $0 \in U$ and $u - v \notin A$ for all $u, v \in U$, $u \neq v$. Then $\mathrm{cl}(U) = [-1, 1]$ so U is a preneighborhood of 0. Since E is totally prebounded, there is a finite set F such that $E \subset F + U$. If $z \in F$ and $x, y \in E$ lie in $z + U$ then $x = z + u$, $y = z + v$ with $u, v \in U$. It follows that $u - v = x - y \in A$ and, by construction of U, $u = v$. Therefore, $x = y$ and so each set $z + U$, $z \in F$, contains at most one element of E. Therefore, E is finite. We have shown that every countable set which is totally prebounded is finite. It follows hat every totally prebounded set is finite. □

Combining several of our results we can now identify totally prebounded and precompact subset of any topological vector space.

Theorem 4.2 *Let X be a topological vector space. A subset of X is totally prebounded if and only if it is finite. Similarly, a subset of X is precompact if and only it is finite.*

Proof Every finite set is totally prebounded. Conversely, suppose that E is a totally prebounded subset of X. Let f be a linear functional on X. It follows easily from Theorem 2.3 that $f(E)$ is a totally prebounded subset of $\mathbb{R}$. By Lemma 4.2, $f(E)$ is finite. It follows that E is finite as we see as follows. Suppose that E contains a sequence $\{x_n\}_{n=1}^{\infty}$ which is linearly independent. Then, using Lemma 4.1, we can construct a linear functional f on X such that $f(x_n) \neq f(x_m)$ if $n \neq m$. This is a contradiction so E must lie in a finite dimensional subspace Y of X. We choose a basis $y_1, \ldots, y_k$ in Y, and represent each $x \in E$ in this basis

$$x = f_1(x)y_1 + \cdots + f_k(x)y_k.$$

Every f_j is a linear functional on Y so $f_j(E)$ is a finite set for each $j = 1, 2, \ldots, k$. It follows that E is finite.

Clearly, every finite set is precompact. Conversely, by Lemma 3.1, a precompact subset of X is totally prebounded, so it is finite. □

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