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Path Double Covering Number of Product Graphs

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Abstract: A path partition or a path cover of a graph G is a collection P of paths in G such that every edge of G is in exactly one path in P. Various types of path covers such as Smarandache path k-cover, simple path covers have been studied by several authors by imposing conditions on the paths in the path covers. In this paper, We study the minimum number of paths which cover a graph such that each edge of the graph occurs exactly twice in two(distinct) paths.

Key Words: Path double cover; path double cover number; product graphs; Smarandache path k-cover.

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§1. Introduction

Let G be a simple graph and ψ be a collection of paths covering all the edges of G exactly twice. Then ψ is called a path double cover of G. The notion of path double cover was first introduced by J.A. Bondy in [4]. In the paper, he posed the following conjecture:

Every simple graph has a path double cover ψ such that each vertex of G occurs exactly twice as an end of a path of ψ.

The above conjecture was proved by Hao Li in [5] and the conjecture becomes a theorem now. The theorem implies that every simple graph of order p can be path double covered by at most p paths. Obviously, the reason We need p paths in a perfect path double cover is due to the requirement that every vertex must be an end vertex of a path exactly twice. If We drop this requirement, the number of paths need is less than p in general. In this paper, We shall investigate the following number:

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$$\gamma_2(G) = \min \{ |\psi| : \psi \text{ is a path double cover of } G \}$$

For convenience, We call $\gamma_2(G)$, the path double cover number of G throughout this paper.

We need the following definitions for our discussion. For two graphs G and H their cartesian product $G \times H$ has vertex set $V(G) \times V(H)$ in which (g_1, h_1) is adjacent to (g_2, h_2) if $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $h_1 = h_2$ and $g_1g_2 \in E(G)$. For the graphs G and H their wreath product $G * H$ has vertex set $V(G) \times V(H)$ in which (g_1, h_1) is joined to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. Similarly $G \circ H$, the weak product of graphs G and H has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. We define the vertex set of the weak product of graphs as in [6] and hence $V(G \circ H) = V_1 \cup V_2 \cup \ldots \cup V_n$ where $V_i = \{u^i_1, u^i_2, \ldots, u^i_m\}, 1 \leq i \leq n$, u^i_j stands for (v_i, u_j) and $V(H) = \{u_1, u_2, \ldots, u_m\}$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Clearly, for each edge $v_iv_j \in E(G)$ the subgraph of $G \circ K_m$ induced by $V_i \cup V_j$ is $K_{V_i,V_j} \setminus \alpha_1(V_i, V_j)$, where $\alpha_k(V_i, V_j)$ is a 1-factor given by $\alpha_k(V_i, V_j) = (u^i_1u^j_k, u^i_2u^j_{k+1}, u^i_3u^j_{k+2}, \ldots, u^i_mu^j_{k-1}), 1 \leq k \leq m$.

For our future reference we state some known results.

Theorem 1.1([3]) *If both G_1 and G_2 have Hamilton cycle decomposition and at least one of G_1 and G_2 is of odd order then $G_1 \circ G_2$ has a Hamilton cycle decomposition.*

Theorem 1.2([5]) *Let $m \geq 3$ and $n \geq 3$. The graph $C_m \times C_n$ can be decomposed into two Hamilton cycles if and only if at least one of the numbers m, n is odd.*

Theorem 1.3([7]) *Let $m \geq 3$and $n \geq 2$. If mis even, then $C_m \circ P_n$ consists of two connected components which are isomorphic to each other.*

Theorem 1.4([7]) *If $m = 4i + 2$, $i \geq 1$, and $n \geq 2$, then each connected component of the graph $C_m \circ P_n$ is isomorphic to $C_{m/2} \circ P_n$.*

A more general definition on graph covering using paths is given as follows.

Definition 1.5([2]) *For any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common. Thus if k = 1 and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G.*

§2. Main results

Lemma 2.1 *Let G be a graph with n pendent vertices. Then $\gamma_2(G) \geq n$.*

Proof Every pendent vertex is an end vertex of two different paths of a path double cover of G. Since there are n pendent vertices, We have $\gamma_2(G) \geq n$. □

Lemma 2.2 *If G is a graph with $\delta(G) \geq 2$, then $\gamma_2(G) \geq max(\delta(G) + 1, \Delta(G))$.*

Proof One can observe that the total degree of each vertex v of G in a path double cover is $2deg(v)$. If v is an external vertex of a path in a path double cover ψ of G then v is external in at least two different paths of ψ. So We have

$$\begin{aligned}|\psi| &\geq \{(2deg(v)-2)/2\}+2\\ &= deg(v)-1+2 = deg(v)+1 \geq \delta(G)+1\end{aligned}$$

This is true for every path double cover of G. Hence $\gamma_2(G) \geq \delta(G)+1$. Let u be a vertex of degree Δ in G. We always have $|\psi| \geq 2deg(u)/2 = \Delta$. Hence $\gamma_2(G) \geq max(\delta(G)+1, \Delta(G))$.□

Corollary 2.3 *If G is a k-regular graph, then $\gamma_2(G) \geq k+1$ and for all other graphs $\gamma_2(G) \geq \Delta$.*

Proof We know that $\Delta(G) \geq \delta(G)$ and for a regular graph $\delta(G) = \Delta(G)$. Hence the result follows. □

Proposition 2.4 *Let$m \geq 3$.*

$$\gamma_2(C_m \circ K_2) = \begin{cases} 3 & \text{if m is odd;}\\ 6 & \text{if m is even.}\end{cases}$$

Proof The proof is divided into two cases.

Case (i) m is odd.

Since $C_m \circ K_2$ is a 2-regular graph, We have $\gamma_2(C_m \circ K_2) \geq 3$ by Corollary 2.3. Now We prove the other part.

The graph $C_m \circ K_2$ is a hamilton cycle C of length $2m$. (ie) $C =< u_1u_2u_3 \ldots u_{2m}u_1 >$. Then we take

$P_1 =< u_2u_3 \ldots u_{2m-1}u_{2m} >$;

$P_2 =< u_3u_4 \ldots u_{2m-1}u_{2m}u_1 >$;

$P_3 =< u_1u_2u_3 >$;

The above three paths form the path double cover for $C_m \circ K_2$ and $\gamma_2(C_m \circ K_2) \leq 3$. Hence $\gamma_2(C_m \circ K_2) = 3$.

Case (ii) m is even.

The graph $C_m \circ K_2$ can be factorized into two isomorphic components of cycle of even length m. Also since any cycle has minimum path double cover number as 3 (by using Case(i)). We have $\gamma_2(C_m \circ K_2) = 6$. □

Proposition 2.5 *Let $m, n \geq 3$. $\gamma_2(C_m \circ C_n) = 5$ if at least one of the numbers m and n is odd.*

Proof Since $C_m \circ C_n$ is a 4- regular graph, We have $\gamma_2(C_m \circ C_n) \geq 5$ by Corollary 2.3. Since at least one of the numbers m and n is odd, $C_m \circ C_n$ can be decomposed into two hamilton cycles C_1 and C_2 by Theorem 1.1. Let $u \in V(C_m \circ C_n)$. Since $deg(u) = 4$, there exist four vertices v_1, v_2, v_3 and v_4 adjacent with u and exactly two of them together with u are on C_1 and the other two together with u are on C_2. Without loss of generality assume that $< v_1uv_2 >$ and $< v_3uv_4 >$ lie on the cycles C_1 and C_2 respectively.

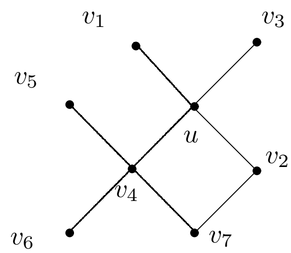


Fig. 1

Since $deg(v_4) = 4$, there are vertices v_5, v_6, v_7 together with u are adjacent with v_4 as in Fig.1. Now assume that $< v_5v_4v_7 >$ and $< uv_4v_6 >$ lie on C_1 and C_2 respectively. Let $C_i^{(1)}, C_i^{(2)}$ be the two copies of C_i $(i = 1, 2)$. If v_2v_7 is in C_1 then $\{(C_1^{(1)} - (v_2v_7)), (C_1^{(2)} - (v_4v_7)), (C_2^{(1)} - (uv_3)), (C_2^{(2)} - (uv_4)), (v_3uv_4v_7v_2)\}$ is a path double cover for $C_m \circ C_n$. Otherwise v_2v_7 is in C_2 and $\{(C_1^{(1)} - (v_1u)), (C_1^{(2)} - (v_4v_7)), (C_2^{(1)} - (uv_4)), (C_2^{(2)} - (v_2v_7)), (v_1uv_4v_7v_2)\}$ is a path double cover for $C_m \circ C_n$. Hence $\gamma_2(C_m \circ C_n) = 5$. For the remaining possibilities it is verified that $\gamma_2(C_m \circ C_n) = 5$ in a similar manner. □

Proposition 2.6 *Let* $m, n \geq 3$.

$$\gamma_2(P_m \circ C_n) = \begin{cases} 4 & \text{if n } \equiv 1 \text{ or } 3(\text{mod } 4) \\ 8 & \text{if n } \equiv 0 \text{ or } 2(\text{mod } 4) \end{cases}$$

Proof Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ and let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$.
Let $V_i = \{u_1^i, u_2^i, \ldots, u_n^i\}, 1 \leq i \leq m$ be the set of n vertices of $P_m \circ C_n$ that corresponds to the vertex v_i of P_m.

Case (i) When $n \equiv 1$ or 3(mod 4), n must be odd.By Lemma 2.2, $\gamma_2(P_m \circ C_n) \geq 4$. Now we prove the other part. Take

$P_1 =< u_1^1u_2^2u_3^1u_4^2 \ldots u_{n-1}^2u_n^1u_1^2u_2^3u_3^2 \ldots u_{n-1}^mu_n^{m-1}u_1^m >= P_3$.

$P_2 =< u_1^1u_n^2u_{n-1}^1u_{n-2}^2 \ldots u_2^1u_1^2u_n^3u_{n-1}^2 \ldots u_2^{m-1}u_1^m >= P_4$.

Clearly $\{P_1, P_2, P_3, P_4\}$ is a path double cover for $P_m \circ C_n$ and $\gamma_2(P_m \circ C_n) \leq 4$. Hence $\gamma_2(P_m \circ C_n) = 4$.

Case (ii) When $n \equiv 2(\text{mod } 4)$, $n = 4i + 2$ where $i \geq 1$,then by Theorems 1.3 and 1.4, $P_m \circ C_n$ can be decomposed into two connected components and each of which is isomorphic to $P_m \circ C_{n/2}$. Now since $n/2$ is odd, using Case(i) We can have $\gamma_2(P_m \circ C_{n/2}) = 4$ and hence $\gamma_2(P_m \circ C_n) = 4$.

Case (*iii*) When $n \equiv 0(mod4)$, $n = 4i$ where $i \geq 1$.

Subcase (1) m is even.

$$P_1 =< u_1^1u_2^2u_3^1u_4^2 \ldots u_{n-1}^1u_n^2u_{n-1}^3u_{n-2}^4u_{n-3}^3u_{n-4}^4 \ldots u_2^4u_1^3u_n^4u_{n-1}^5 u_{n-2}^6u_{n-3}^5 \ldots u_1^{m-1}u_n^m >= P_5;$$

$$P_2 =< u_1^1u_n^2u_1^3u_2^2u_3^3u_4^2u_3^5 \ldots u_{n-2}^2u_{n-1}^3u_n^4u_1^5u_2^4u_3^5 \ldots u_{n-2}^4u_{n-1}^5 u_n^6u_1^7 \ldots u_{n-2}^{m-2}u_{n-1}^{m-1}u_n^m >= P_6;$$

$$P_3 =< u_n^1u_{n-1}^2u_{n-2}^1u_{n-3}^2 \ldots u_2^1u_1^2u_2^3u_3^4u_4^3u_5^4 \ldots u_{n-1}^4u_n^3u_1^4u_2^5 u_3^6u_4^5 \ldots u_{n-1}^6u_n^5u_1^6 \ldots u_{n-1}^mu_n^{m-1}u_1^m >= P_7;$$

$$P_4 =< u_n^1u_1^2u_n^3u_{n-1}^2u_{n-2}^3u_{n-3}^2 \ldots u_3^2u_2^3u_1^4u_n^5u_{n-1}^4u_{n-2}^5 \ldots u_4^{m-1}u_3^{m-2}u_2^{m-1}u_1^m >= P_8.$$

Subcase (2) m is odd.

$$P_1 =< u_1^1u_2^2u_3^1u_4^2 \ldots u_{n-1}^1u_n^2u_{n-1}^3u_{n-2}^4u_{n-3}^3u_{n-4}^4 \ldots u_2^4u_1^3u_n^4u_{n-1}^5 u_{n-2}^6u_{n-3}^5 \ldots u_n^{m-1}u_{n-1}^m >= P_5;$$

$$P_2 =< u_1^1u_n^2u_1^3u_2^2u_3^3u_4^2u_3^5 \ldots u_{n-2}^2u_{n-1}^3u_n^4u_1^5u_2^4u_3^5 \ldots u_{n-2}^4u_{n-1}^5 u_n^6u_1^7 \ldots u_{n-3}^mu_{n-2}^{m-1}u_{n-1}^m >= P_6;$$

$$P_3 =< u_n^1u_{n-1}^2u_{n-2}^1u_{n-3}^2 \ldots u_2^1u_1^2u_2^3u_3^4u_4^3u_5^4 \ldots u_{n-1}^4u_n^3u_1^4u_2^5 u_3^6u_4^5 \ldots u_{n-1}^6u_n^5u_1^6 \ldots u_1^{m-1}u_2^m >= P_7;$$

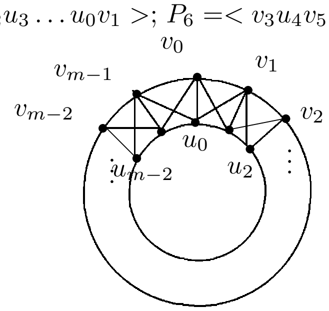
$$P_4 =< u_n^1u_1^2u_n^3u_{n-1}^2u_{n-2}^3u_{n-3}^2 \ldots u_3^2u_2^3u_1^4u_n^5u_{n-1}^4u_{n-2}^5 \ldots u_4^mu_3^{m-1}u_2^m >= P_8.$$

In both Subcases $\{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$ is the minimum path double cover for $P_m \circ C_n$ and hence $\gamma_2(P_m \circ C_n) = 8$ for $n \equiv 0(\text{mod } 4)$. □

Proposition 2.7 *Let $m \geq 3$, $\gamma_2(C_m * K_2) = 6$ if m is odd.*

Proof Since the graph is 5-regular, We have $\gamma_2(C_m * K_2) \geq 6$ by Corollary 2.3. Now We prove the converse part. Given m is odd, then take

$P_1 =< u_0u_1v_1v_2u_2u_3v_3 \ldots u_{m-2}v_{m-2}v_{m-1} >$; $P_2 =< u_0v_0v_1u_1u_2v_2v_3u_3 \ldots v_{m-2}u_{m-2}u_{m-1} v_{m-1} >$; $P_3 =< u_1u_2u_3 \ldots u_{m-1}u_0v_0v_{m-1}v_{m-2} \ldots v_2v_1 >$; $P_4 =< u_1u_0u_{m-1}v_{m-1}v_0v_1u_2v_3 >$; $P_5 =< u_3v_3u_4v_5 \ldots u_{m-1}v_0u_1v_2u_3 \ldots u_0v_1 >$; $P_6 =< v_3u_4v_5 \ldots u_{m-1}v_0u_1v_2u_3 \ldots u_0v_1u_2 >$.



$C_m * K_2$

Fig.2

$\{P_1, P_2, P_3, P_4, P_5, P_6\}$ is a path double cover for $C_m * K_2$ and $\gamma_2(C_m * K_2) \leq 6$. Hence $\gamma_2(C_m * K_2) = 6$. □

Proposition 2.8 *Let* $m \geq 3$. $\gamma_2(C_m * \bar{K}_2) = 5$ *if* m *is odd.*

Proof By Corollary 2.3, $\gamma_2(C_m * \bar{K}_2) \geq 5$. Given m is odd and take

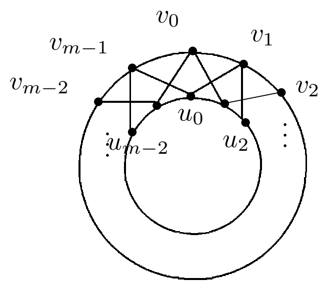
$P_1 =< u_0u_1v_2u_3v_4 \ldots u_{m-2}v_{m-1}v_0u_{m-1}v_{m-2}u_{m-3} \ldots v_3u_2v_1 >$;

$P_2 =< u_0u_1u_2 \ldots u_{m-1}v_0v_1v_2 \ldots v_{m-1} >$;

$P_3 =< u_{m-1}u_0v_{m-1}u_{m-2}v_{m-3} \ldots u_1v_0v_1u_3v_3 \ldots u_{m-3}v_{m-2} >$;

$P_4 =< v_1u_0u_{m-1}u_{m-2}u_{m-3}u_2u_1v_0v_{m-1}v_{m-2} >$;

$P_5 =< v_{m-1}u_0v_1v_2v_3 \ldots v_{m-2}u_{m-1} >$;



$C_m * \bar{K}_2$

Fig.3

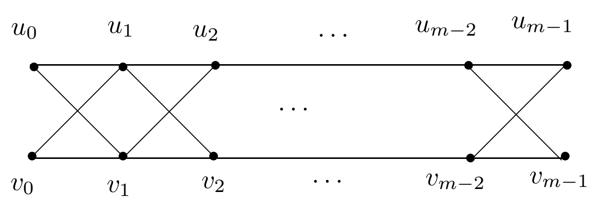
$\{P_1, P_2, P_3, P_4, P_5\}$ is a path double cover for $C_m * \bar{K}_2$ and $\gamma_2(C_m * \bar{K}_2) \leq 5$. Hence $\gamma_2(C_m * \bar{K}_2) = 5$. □

Proposition 2.9 $\gamma_2(P_m * = K_2) = 4$ *for* $m \geq 3$.

Proof By Lemma 2.2, $\gamma_2(P_m * \bar{K}_2) \geq 4$. If m is odd then take

$P_1 =< u_0v_1u_2v_3u_4 \ldots v_{m-2}u_{m-1}u_{m-2}v_{m-3} \ldots v_2u_1v_0 >= P_3$;

$P_2 =< u_0u_1u_2 \ldots u_{m-2}v_{m-1}v_{m-2} \ldots v_1v_0 >= P_4$.



$P_m * \bar{K}_2$

Fig.4

If m is even then take

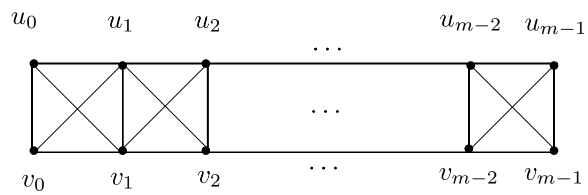
$P_1 =< u_0v_1u_2v_3u_4 \ldots u_{m-2}v_{m-1}v_{m-2}u_{m-3} \ldots v_2u_1v_0 >= P_3$;

$P_2 =< u_0u_1u_2 \ldots u_{m-1}v_{m-2}v_{m-3} \ldots v_1v_0 >= P_4$.

Clearly $\{P_1, P_2, P_3, P_4\}$ is a path double cover for $P_m * \bar{K}_2$ and $\gamma_2(P_m * \bar{K}_2) \leq 4$. Hence $\gamma_2(P_m * \bar{K}_2) = 4$. □

Proposition 2.10 *$\gamma_2(P_m * K_2) = 5$ for $m \geq 3$.*

Proof By Lemma 2.2, $\gamma_2(P_m * K_2) \geq 5$.



$P_m * K_2$

Fig.5

If m is even then take

$P_1 =< u_0v_1u_1v_2u_2v_3 \ldots u_{m-2}v_{m-1} >$;

$P_2 =< u_0v_0u_1v_1u_2v_2 \ldots v_{m-2}u_{m-1} >$;

$P_3 =< u_0v_1u_2v_3u_4 \ldots v_{m-3}u_{m-2}v_{m-1}u_{m-1}v_{m-2}u_{m-3} \ldots u_2v_1u_0 >$;

$P_4 =< u_0u_1u_2u_3 \ldots u_{m-1}v_{m-1}v_{m-2} \ldots v_1v_0 >$;

$P_5 =< u_{m-1}u_{m-2}u_{m-3} \ldots u_1u_0v_0v_1 \ldots v_{m-2}v_{m-1} >$.

If m is odd then take

$P_1 =< u_0v_1u_1v_2u_2 \ldots u_{m-2}v_{m-1} >$;

$P_2 =< u_0v_0u_1v_1u_2v_2 \ldots v_{m-2}u_{m-1} >$;

$P_3 =< u_0v_1u_2v_3u_4v_4 \ldots v_{m-2}u_{m-1}v_{m-1}u_{m-2} \ldots v_2u_1v_0 >$;

$P_4 =< u_0u_1u_2 \ldots u_{m-1}v_{m-1}v_{m-2} \ldots v_1v_0 >$;

$P_5 =< u_{m-1}u_{m-2} \ldots u_1u_0v_0v_1 \ldots v_{m-1} >$.

$\{P_1, P_2, P_3, P_4, P_5\}$ is a path double cover for $P_m * K_2$ and $\gamma_2(P_m * K_2) \leq 5$. Hence $\gamma_2(P_m * K_2) = 5$. □

Proposition 2.11 *Let $m \geq 3$. $\gamma_2(C_m \times P_3) = 5$ if m is odd.*

Proof Let $V(C_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(P_3) = \{u_1, u_2, u_3\}$. Let $V_i = \{u_1^i, u_2^i, u_3^i\}, 1 \leq i \leq m$ be the set of 3 vertices of $C_m \times P_3$ that corresponds to a vertex v_i of C_m. Now we construct the paths for $C_m \times P_3$.

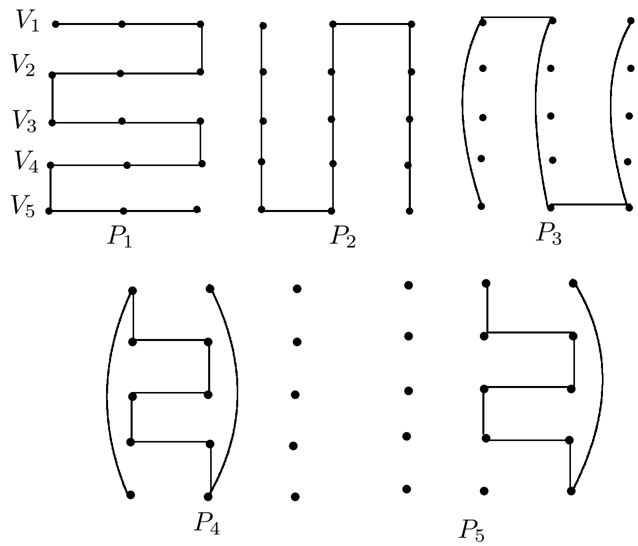
$P_1 =< u_1^1u_2^1u_3^1u_3^2u_2^2u_1^2u_1^3u_2^3u_3^3 \ldots u_1^mu_2^mu_3^m >$;

$P_2 =< u_1^1u_1^2u_1^3 \ldots u_1^mu_2^mu_2^{m-1} \ldots u_2^2u_2^1u_3^1u_3^2u_3^m >$;

$P_3 =< u_1^mu_1^1u_2^1u_2^mu_3^mu_3^1 >$;

$P_4 =< u_1^mu_1^1u_1^2u_2^2u_2^3u_1^3u_1^4u_2^4 \ldots u_2^{m-1}u_2^mu_2^1 >$;

$P_5 =< u_2^1u_2^2u_3^2u_3^3u_2^3u_2^4 \ldots u_3^mu_3^1 >$.



$C_5 \times P_3$

Fig.6

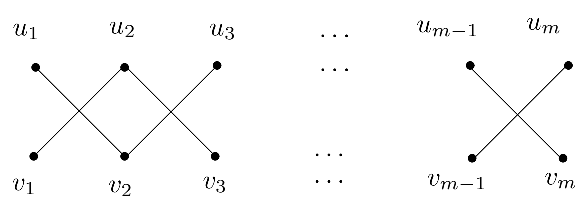
Clearly $\{P_1, P_2, P_3, P_4, P_5\}$ is a path double cover for $C_m \times P_3$ (See Fig.6) and $\gamma_2(C_m \times P_3) \leq 5$. Hence $\gamma_2(C_m \times P_3) = 5$. □

Proposition 2.12 *$\gamma_2(P_m \circ K_2) = 4$ for $m \geq 2$.*

Proof By Theorem 2.1, $\gamma_2(P_m \circ K_2) \geq 4$. If m is even then take

$P_1 =< u_1v_2u_3v_4u_5 \ldots v_{m-2}u_{m-1}v_m >= P_3$;

$P_2 =< v_1u_2v_3u_4v_5 \ldots u_{m-2}v_{m-1}u_m >= P_4$.



$P_m \circ K_2$

Fig.7

If m is odd then take

$P_1 =< u_1v_2u_3v_4u_5 \ldots u_{m-2}v_{m-1}u_m >= P_3$;

$P_2 =< v_1u_2v_3u_4v_5 \ldots v_{m-2}u_{m-1}v_m >= P_4$.

$\{P_1, P_2, P_3, P_4\}$ is a path double cover for $P_m \circ K_2$ and $\gamma_2(P_m \circ K_2) \leq 4$. Hence $\gamma_2(P_m \circ K_2) = 4$. □

Proposition 2.13 *For the complete bipartite graph $K_{m,n}$, $\gamma_2(K_{m,n}) = \max\{m, n\}$.*

Proof Let (A, B) be the bipartition of $K_{m,n}$ where $A = \{u_0, u_1, \ldots, u_{m-1}\}$, $B = \{v_0, v_1, \ldots, v_{n-1}\}$.

Case (i) $m \leq n$.

By Corollary 2.3, $\gamma_2(K_{m,n}) \geq n$. Let $P_i =< v_i u_1 v_{i+1} u_2 \ldots u_{m-1} v_{i+m-1} u_0 v_{i+m} >$, where the indices i are taken modulo n. $\psi = \{P_i : 0 \leq i \leq n-1\}$ is clearly a path double cover for $K_{m,n}$ with n paths. Hence $\gamma_2(K_{m,n}) = n$.

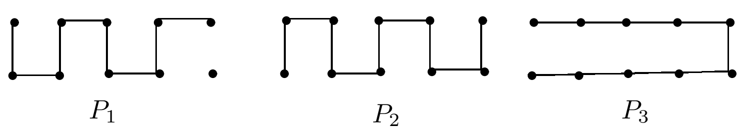
Case (ii) $m > n$.

By Corollary 2.3, $\gamma_2(K_{m,n}) \geq m$.Let $P_i =< u_i v_1 u_{i+1} v_2 \ldots v_{n-1} u_{i+n-1} v_0 u_{i+n} >$, where the indices i are taken modulo m. $\psi = \{P_i : 0 \leq i \leq m-1\}$ is clearly a path double cover for $K_{m,n}$ with m paths. Hence $\gamma_2(K_{m,n}) = m$. This completes the proof. □

Proposition 2.14 *Let* $m, n \geq 2$.

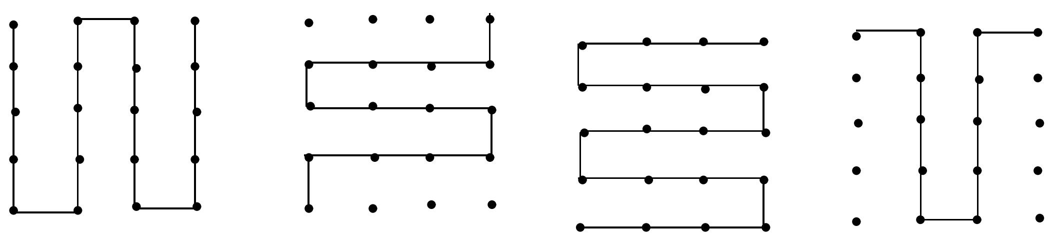
$$\gamma_2(P_m \times P_n) = \begin{cases} 3 & \text{if m=2 or n=2;} \\ 4 & \text{otherwise.} \end{cases}$$

Proof By Lemma 2.2, $\gamma_2(P_m \times P_n) \geq 3$ if $m = 2$ or $n = 2$ and $\gamma_2(P_m \times P_n) \geq 4$ if $m, n \geq 3$. The reverse inclusion follows from Fig.8 and Fig.9.



$P_2 \times P_5$

Fig.8



$P_5 \times P_4$

Fig.9

□

Proposition 2.15 *Let* $m \geq 3$, $n \geq 3$. $\gamma_2(C_m \times C_n) = 5$ *if at least one of the numbers* m *and* n *is odd.*

Proof Since $C_m \times C_n$ is a 4-regular graph, We have $\gamma_2(C_m \times C_n) \geq 5$ by corollary 2.3. Since at least one of the numbers m and n is odd, $C_m \times C_n$ can be decomposed into two hamilton cycles C_1 and C_2 by Theorem 1.2. Let $v \in V(C_m \times C_n)$. Since $deg(v) = 4$, there exist four vertices u_1, u_2, u_3 and u_4 adjacent with v and exactly two of them together with v are on C_1 and the other two together with v are on C_2. Without loss of generality assume that $< u_1vu_2 >$ and $< u_3vu_4 >$ lie on C_1 and C_2 respectively. Since $deg(u_4) = 4$, there are vertices u_5, u_6, u_7 together with v are adjacent with u_4 as in Fig.10. As before assume that $(u_5u_4u_6)$ and (vu_4u_7) lie on C_1 and C_2 respectively. Let $C_i^{(1)}, C_i^{(2)}$ be the two copies of $C_i (i = 1, 2)$. If u_2u_6 is in C_1 then $\{(C_1^{(1)} - (u_2u_6)), (C_1^{(2)} - (u_4u_6)), (C_2^{(1)} - (vu_3)), (C_2^{(2)} - (vu_4)), (u_3vu_4u_6u_2)\}$ is a path double cover for $C_m \times C_n$. Otherwise u_2u_6 is in C_2 and $\{(C_1^{(1)} - (u_1v)), (C_1^{(2)} - (u_4u_6)), (C_2^{(1)} - (vu_4)), (C_2^{(}2) - (u_2u_6)), (u_1vu_4u_6u_2)\}$ is a path double cover for $C_m \times C_n$. Hence $\gamma_2(C_m \times C_n) = 5$. For the remaining possibilities it is verified that $\gamma_2(C_m \times C_n) = 5$ in a similar manner.

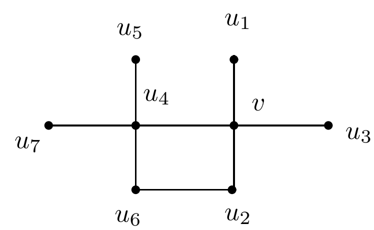


Fig.10

□

Proposition 2.16 $\gamma_2(C_m \times K_2) = 4$ *for* $m \geq 3$.

Proof By Corollary 2.3, $\gamma_2(C_m \times K_2) \geq 4$. Now We prove the other part. If m is odd then take

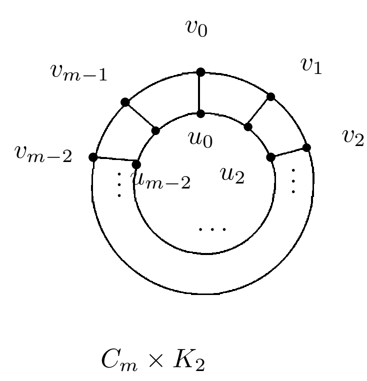
$P_1 =< u_0u_1v_1v_2u_2u_3v_3 \ldots u_{m-2}v_{m-2}v_{m-1} >$;

$P_2 =< u_0v_0v_1u_1u_2v_2v_3u_3 \ldots v_{m-2}u_{m-2}u_{m-1}v_{m-1} >$.

If m is even then take

$P_1 =< u_0u_1v_1v_2u_2u_3v_3 \ldots v_{m-2}u_{m-2}u_{m-1} >$;

$P_2 =< u_0v_0v_1u_1u_2v_2v_3u_3 \ldots u_{m-2}v_{m-2}v_{m-1}u_{m-1} >$.



$C_m \times K_2$

Fig.11

Let $P_3 =< u_1u_2 \ldots u_{m-1}u_0v_0v_{m-1}v_{m-2} \ldots v_2v_1 >$ and

$P_4 =< u_1u_0u_{m-1}v_{m-1}v_0v_1 >$.

$\{P_1, P_2, P_3, P_4\}$ is a path double cover for $C_m \times K_2$ and $\gamma_2(C_m \times K_2) \leq 4$. Hence $\gamma_2(C_m \times K_2) = 4$. □

Proposition 2.17 *$\gamma_2(K_{2n+1} \circ K_2) = 2n + 1$, for $n \geq 1$.*

Proof Let $V(K_{2n+1}) = \{v_0, v_1, v_2, \ldots, v_{2n}\}$ and let $V(K_2) = \{u_0, u_1\}$. Let $V_i = \{u_0^i, u_1^i\}$, $0 \leq i \leq 2n$ be the set of $2n + 1$ vertices of $K_{2n+1} \circ K_2$ that corresponds to a vertex v_i of K_{2n+1}. Define for $1 \leq i \leq 2n$, $H_i =< v_0v_iv_{i+1}v_{i-1}v_{i+2}v_{i-2} \ldots v_{n+i-1}v_{n+i+1}v_{n+i}v_0 >$ (H_i is nothing but the Walecki's Hamilton cycle decomposition[1]) where the indices are taken modulo $2n$ except 0. Clearly $\{H_1, H_2, \ldots, H_{2n}\}$ is a cycle double cover for K_{2n+1}. Then

$$\begin{aligned} K_{2n+1} \circ K_2 &= (H_1 \oplus H_2 \oplus \ldots \oplus H_{2n}) \circ K_2 \\ &= H_1 \circ K_2 \oplus H_2 \circ K_2 \oplus \ldots \oplus H_{2n} \circ K_2 \end{aligned}$$

where $H_i \circ K_2 (1 \leq i \leq 2n)$ is a cycle double cover for $K_{2n+1} \circ K_2$. Now remove an edge e_i from $H_i \circ K_2 (1 \leq i \leq 2n)$ so that $< e_1e_2e_3 \ldots e_{2n} >$ form a path (See Example 2.18). Hence $\gamma_2(K_{2n+1} \circ K_2) = 2n + 1$. Since $\delta(K_{2n+1} \circ K_2) = 2n$, We have $\gamma_2(K_{2n+1} \circ K_2) \geq 2n + 1$, by Lemma 2.2. Hence $\gamma_2(K_{2n+1} \circ K_2) = 2n + 1$. □

The following example illustrates the above theorem.

Example 2.18 Consider $K_5 \circ K_2$, $H_i =< v_0v_iv_{i+1}v_{i-1}v_{i+2}v_0 >, 1 \leq i \leq 4$. The cycle double covers for $K_{2n+1} \circ K_2$ are

$H_1 \circ K_2 =< u_1^0u_2^1u_1^2u_2^4u_1^3u_2^0u_1^1u_2^2u_1^4u_2^3u_1^0 >$;

$H_2 \circ K_2 =< u_1^0u_2^2u_1^3u_2^1u_1^4u_2^0u_1^2u_2^3u_1^1u_2^4u_1^0 >$;

$H_3 \circ K_2 =< u_1^0u_2^3u_1^4u_2^2u_1^1u_2^0u_1^3u_2^4u_1^2u_2^1u_1^0 >$ and

$H_4 \circ K_2 =< u_1^0u_2^4u_1^1u_2^3u_1^2u_2^0u_1^4u_2^1u_1^3u_2^2u_1^0 >$.

Now remove the edges $(u_1^0u_2^1), (u_2^1u_1^4), (u_1^4u_2^3)$ and $(u_2^3u_1^1)$ from$H_1 \circ K_2$, $H_2 \circ K_2, H_3 \circ K_2$ and $H_4 \circ K_2$ respectively so that $< u_1^0u_2^1u_1^4u_2^3u_1^1 >$ form a path. Hence $\gamma_2(K_5 \circ K_2) = 5$.

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