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Graphoidal Tree d - Cover

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Abstract: In [1] Acharya and Sampathkumar defined a graphoidal cover as a partition of edges into internally disjoint (not necessarily open) paths. If we consider only open paths in the above definition then we call it as a graphoidal path cover [3]. Generally, a Smarandache graphoidal tree (k, d)-cover of a graph G is a partition of edges of G into trees $T_1, T_2, \cdots, T_l$ such that $|E(T_i) \cap E(T_j)| \leq k$ and $|T_i| \leq d$ for integers $1 \leq i, j \leq l$. Particularly, if $k = 0$, then such a tree is called a graphoidal tree d-cover of G. In [3] a graphoidal tree cover has been defined as a partition of edges into internally disjoint trees. Here we define a graphoidal tree d-cover as a partition of edges into internally disjoint trees in which each tree has a maximum degree bounded by d. The minimum cardinality of such d-covers is denoted by $\gamma_T^{(d)}(G)$. Clearly a graphoidal tree 2-cover is a graphoidal cover. We find $\gamma_T^{(d)}(G)$ for some standard graphs.

Key Words: Smarandache graphoidal tree (k, d)-cover, graphoidal tree d-cover, graphoidal cover.

AMS(2000): 05C70

§1. Introduction

Throughout this paper G stands for simple undirected graph with p vertices and q edges. For other notations and terminology we follow [2]. A Smarandache graphoidal tree (k, d)-cover of G is a partition of edges of G into trees $T_1, T_2, \cdots, T_l$ such that $|E(T_i) \cap E(T_j)| \leq k$ and $|T_i| \leq d$ for integers $1 \leq i, j \leq l$. Particularly, if $k = 0$, then such a cover is called a graphoidal tree d-cover of G. A graphoidal tree d-cover $(d \geq 2)$ $\mathscr{F}$ of G is a collection of non-trivial trees in G such that

(i) Every vertex is an internal vertex of at most one tree;

(ii) Every edge is in exactly one tree;

(iii) For every tree $T \in \mathscr{F}, \Delta(T) \leq d$.

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Let $\mathscr{G}$ denote the set of all graphoidal tree d-covers of G. Since $E(G)$ is a graphoidal tree d-cover, we have $\mathscr{G} \neq \emptyset$. Let $\gamma_T^{(d)}(G) = \min_{\mathscr{J} \in \mathscr{G}} |\mathscr{J}|$. Then $\gamma_T^{(d)}(G)$ is called the graphoidal tree d-covering number of G. Any graphoidal tree d-cover of G for which $|\mathscr{J}| = \gamma_T^{(d)}(G)$ is called a minimum graphoidal tree d-cover.

A graphoidal tree cover of G is a collection of non-trivial trees in G satisfying (i) and (ii). The minimum cardinality of graphoidal tree covers is denoted by $\gamma_T(G)$. A graphoidal path cover (or acyclic graphoidal cover in [5]) is a collection of non-trivial path in G such that every vertex is an internal vertex of at most one path and every edge is in exactly one path. Clearly a graphoidal tree 2-cover is a graphoidal path cover and a graphoidal tree d-cover ($d \geq \Delta$) is a graphoidal tree cover. Note that $\gamma_T(G) \leq \gamma_T^{(d)}(G)$ for all $d \geq 2$. It is observe that $\gamma_T^{(d)}(G) \geq \Delta - d + 1$.

§2. Preliminaries

Theorem 2.1([4]) $\gamma_T(K_p) = \lceil \frac{p}{2} \rceil$.

Theorem 2.2([4]) $\gamma_T(K_{n,n}) = \lceil \frac{2n}{3} \rceil$.

Theorem 2.3([4]) *If $m \leq n < 2m-3$, then $\gamma_T(K_{m,n}) = \lceil \frac{m+n}{3} \rceil$. Further more, if $n > 2m-3$, then $\gamma_T(K_{m,n}) = m$.*

Theorem 2.4([4]) $\gamma_T(C_m \times C_n) = 3$ *if* $m, n \geq 3$.

Theorem 2.5([4]) $\gamma_T(G) \leq \lceil \frac{p}{2} \rceil$ *if* $\delta(G) \geq \frac{p}{2}$.

§3. Main results

We first determine a lower bound for $\gamma_T(d)(G)$. Define $n_d = \min_{\mathscr{J} \in \mathscr{G}_d} n_{\mathscr{J}}$, where $\mathscr{G}_d$ is a collection of all graphoidal tree d-covers and $n_{\mathscr{J}}$ is the number of vertices which are not internal vertices of any tree in $\mathscr{J}$.

Theorem 3.1 *For $d \geq 2$, $\gamma_T(d)(G) \geq q - (p - n_d)(d-1)$.*

Proof Let Ψ be a minimum graphoidal tree d-cover of G such that n vertices of G are not internal in any tree of Ψ.

Let k be the number of trees in Ψ having more than one edge. For a tree in Ψ having more than one edge, fix a root vertex which is not a pendant vertex. Assign direction to the edges of the k trees in such a way that the root vertex has in degree zero and every other vertex has in degree 1. In Ψ, let l_1 be the number of vertices of out degree d and l_2 the number of vertices of out degree less than or equal to $d-1$ (and > 0) in these k trees. Clearly $l_1 + l_2$ is the number of internal vertices of trees in Ψ and so $l_1 + l_2 = p - n$. In each tree of Ψ there is at most one vertex of out degree d and so $l_1 \leq k$. Hence we have

$$\begin{aligned}\gamma_T^{(d)} &\geq k+q-(l_1 d+l_2(d-1)) = k+q-(l_1+l_2)(d-1)l_1 \\ &= k+q-(p-n_\Psi)(d-1)-l_1 \geq q-(p-n_d)(d-1).\end{aligned}$$

□

Corollary 3.2 $\gamma_T^{(d)}(G) \geq q-p(d-1)$.

Now we determine graphoidal tree d-covering number of a complete graph.

Theorem 3.3 *For any integer* $p \geq 4$,

$$\gamma_T^{(d)}(K_p) = \begin{cases} \frac{p(p-2d+1)}{2} & if\ d < \frac{p}{2}; \\ \lceil \frac{p}{2} \rceil & if\ d \geq \frac{p}{2}. \end{cases}$$

Proof Let $d \geq \frac{p}{2}$. We know that $\gamma_T^{(d)}(K_p) \geq \gamma_T(K_p) = \lceil \frac{p}{2} \rceil$ by Theorem 2.1.

Case (i) Let p be even, say $p = 2k$. We write $V(K_p) = \{0, 1, 2, \cdots, 2k-1\}$. Consider the graphoidal tree cover $\mathscr{J}_1 = \{T_1, T_2, \cdots, T_k\}$, where each T_i ($i = 1, 2, \cdots, k$) is a spanning tree with edge set defined by

$$\begin{aligned}E(T_i) &= \{(i-1, j) : j = i, i+1, \cdots, i+k-1\} \\ &\cup \{(k+i-1, s) : s \equiv j(\mathrm{mod} 2k), j = i+k, i+k+1, \cdots, i+2k-2\}.\end{aligned}$$

Now $|\mathscr{J}_1| = k = \frac{p}{2}$. Note that $\Delta(T_i) = k \leq d$ for $i = 1, 2, \cdots, k$ and hence $\gamma_T(d)(K_p) = \lceil \frac{p}{2} \rceil$.

Case (ii) Let p be odd, say $p = 2k+1$. We write $V(K_p) = \{0, 1, 2, \cdots, 2k\}$. Consider the graphoidal tree cover $\mathscr{J}_2 = \{T_1, T_2, \cdots, T_{k+1}\}$ where each T_i ($i = 1, 2, \cdots, k$) is a tree with edge set defined by

$$\begin{aligned}E(T_i) &= \{(i-1, j) : j = i, i+1, \cdots, i+k-1\} \\ &\cup \{(k+i-1, s) : s \equiv j(\mathrm{mod} 2k+1), j = i+k, i+k+1, \cdots, i+2k-1\}.\end{aligned}$$

$$E(T_{k+1}) = \{(2k, j) : j = 0, 1, 2, \cdots, k-1\}.$$

Now $|\mathscr{J}_2| = k = \frac{p}{2}$. Note that the degree of every internal vertex of T_i is either k or $k+1$ and so $\Delta(T_i) \leq d$, $i = 1, 2, \cdots, k+1$. Hence $\gamma_T^{(d)}(K_p) = \lceil \frac{p}{2} \rceil$ if $d \geq \frac{p}{2}$.

Let $d < \frac{p}{2}$. By Corollary 3.2,

$$\gamma_T^{(d)}(K_p) \geq q+p-pd = \frac{p(p-1)}{2}+p-pd = \frac{p(p-2d+1)}{2}.$$

Remove the edges from each T_i in $\mathscr{J}_1$ (or $\mathscr{J}_2$) when p is even (odd) so that every internal vertex is of degree d in the new tree T_i' formed by this removal. The new trees so formed together with the removed edges form $\mathscr{J}_3$.

If p is even, then $\mathscr{J}_3$ is constructed from $\mathscr{J}_1$ and

$$|\mathscr{J}_3| = k + q - k(2d-1) = k + \frac{2k(2k-1)}{2} - k(2d - 10 = k(2k - 2d + 1) = \frac{p(p-2d+1)}{2}.$$

If p is odd, then $\mathscr{J}_3$ is constructed from $\mathscr{J}_2$ and

$$|\mathscr{J}_3| = k+1+q-k(2d-1)-d = k+1+\frac{2k(2k+1)}{2}-2kd+k-d = (2k+1)(1+k-d) = \frac{p(p-2d+1)}{2}.$$

Hence $\gamma_T^{(d)}(K_p) = \frac{p(p+1-2d)}{2}$. $\square$

The following examples illustrate the above theorem.

Examples 3.4 Consider K_6. Take $d = 3 = \frac{p}{2}$ and $V(K_6) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$.

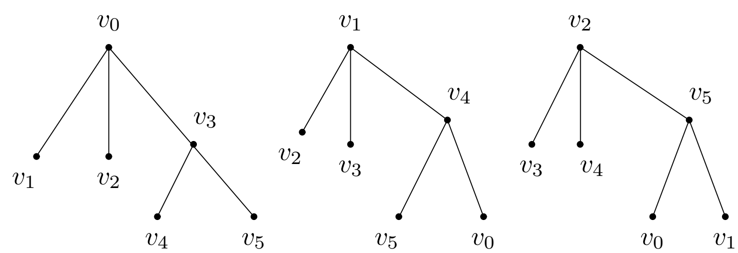

Fig. 1

Whence $\gamma_T^{(3)}(K_6) = 3$. Take $d = 2 < \frac{p}{2}$.

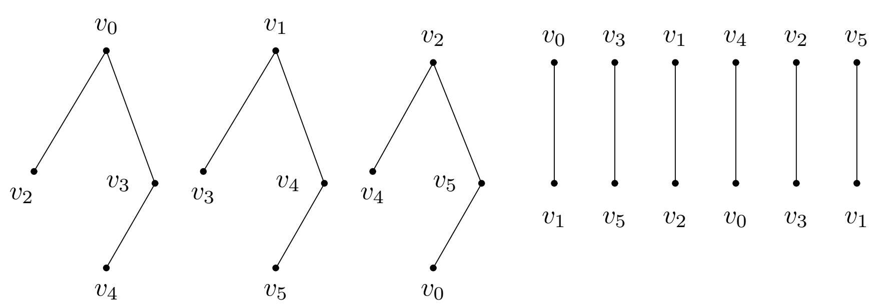

Fig.2

Whence $\gamma_T^{(2)}(K_6) = \frac{6}{2}(6 + 1 - 2 \times 2) = 9$.

Consider K_7. Take $d = 4 = \lceil \frac{p}{2} \rceil$ and $V(K_7) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$.

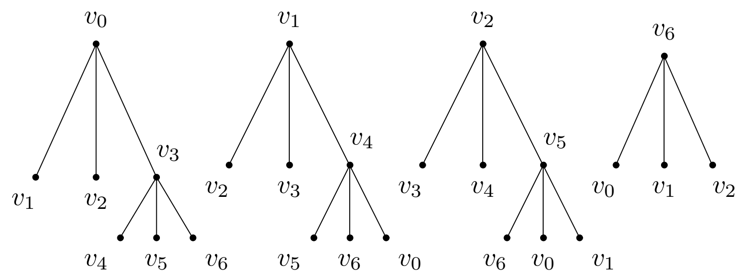

Fig.3

Whence, $\gamma_T^{(4)} = 4 = \lceil \frac{p}{2} \rceil$. Now take $d = 3 < \lceil \frac{p}{2} \rceil$.

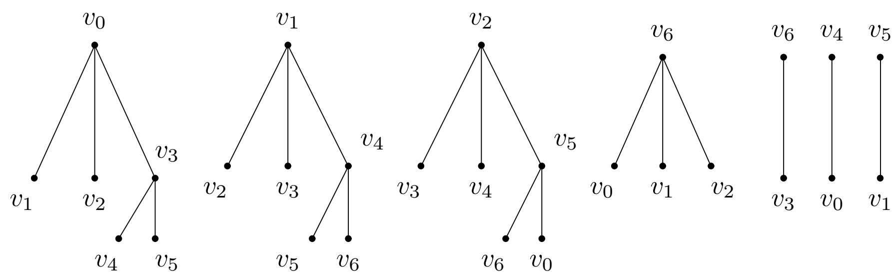

Fig.4

Therefore, $\gamma_T^{(3)}(K_7) = \frac{7}{2}(7 + 1 - 2 \times 3) = 7$.

We now turn to some cases of complete bipartite graph.

Theorem 3.5 *If $n, m \geq 2d$, then $\gamma_T^{(d)}(K_{m,n}) = p + q - pd = mn - (m+n)(d-1)$.*

Proof By theorem 3.2, $\gamma_T^{(d)}(K_{m,n}) \geq p + q - pd = mn - (m+n)d + m + n$. Consider $G = K_{2d,2d}$. Let $V(G) = X_1 \cup Y_1$, where $X_1 = \{x_1, x_2, \cdots, x_{2d}\}$ and $Y_1 = \{y_1, y_2, \cdots, y_{2d}\}$. Clearly $deg(x_i) = deg(y_j) = 2d$, $1 \leq i, j \leq 2d$. For $1 \leq i \leq d$, we define

$$T_i = \{(x_i, y_j) : 1 \leq j \leq d\}, \quad T_{d+i} = \{(x_{i+d}, y_j) : d + 1 \leq j \leq 2d\}$$

$$T_{2d+i} = \{(y_i, x_j) : d + 1 \leq j \leq 2d\} \text{ and } T_{3d+i} = \{(y_{i+d}, x_j) : 1 \leq j \leq d\}.$$

Clearly, $\mathscr{J} = \{T_1, T_2, \cdots, T_{4d}\}$ is a graphoidal tree d-cover for G. Now consider $K_{m,n}, m, n \geq 2d$. Let $V(K_{m,n}) = X \cup Y$, where $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. Now for $4d + 1 \leq i \leq 4d + m - 2d = m + 2d$, we define $T_i = \{(x_{i-2d}, y_j) : 1 \leq j \leq d\}$. For $m + 2d + 1 \leq i \leq m + n$, we define $T_i = \{(y_{i-m}, x_j) : 1 \leq j \leq d\}$. Then $\mathscr{J}' = \{T_1, T_2, \cdots, T_{4d}, T_{4d+1}, \cdots, T_{m+2d}, T_{m+2d+1}, \cdots, T_{m+n}\} \cup \{E(G) - [E(T_i) : 1 \leq i \leq m + n]\}$ is a graphoidal tree d-cover for $K_{m,n}$. Hence $|\mathscr{J}'| = p + q - pd$ and so $\gamma_T^{(d)}(K_{m,n}) \leq p + q - pd = mn - (m+n)(d-1)$ for $m, n \geq 2d$. □

The following example illustrates the above theorem.

Example 3.6 Consider $K_{8,10}$ and take $d = 4$.

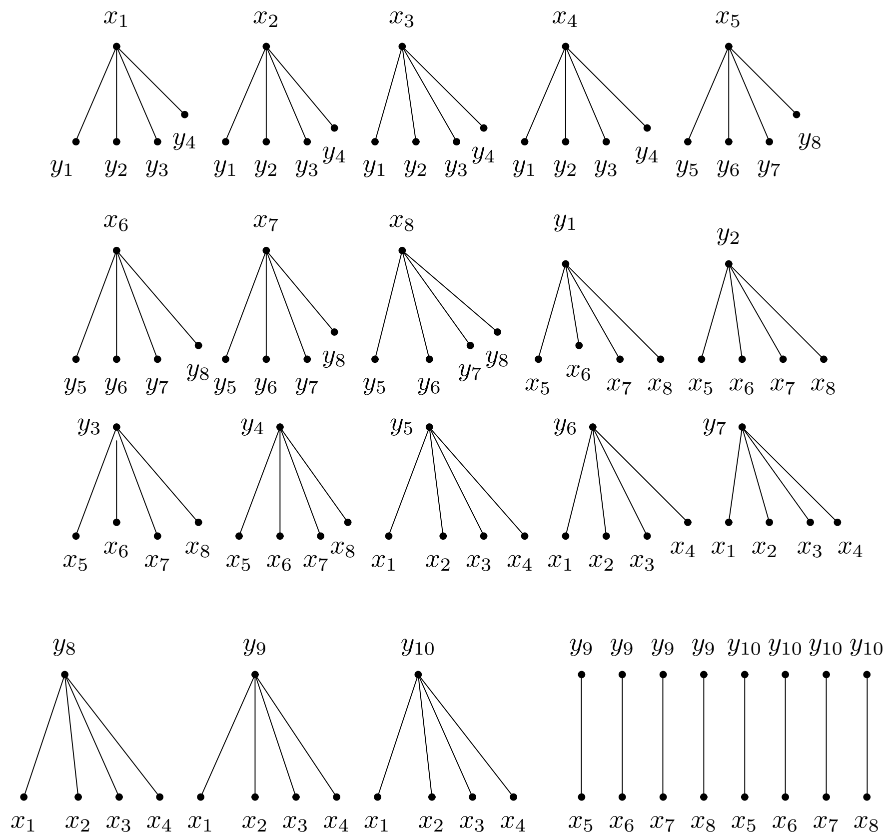

Fig.5

Whence, $\gamma_T^{(4)} = 18 + 80 - 18 \times 4 = 26$.

Theorem 3.7 $\gamma_T^{(d)}(K_{2d-1,2d-1}) = p + q - pd = 2d - 1$.

Proof By Theorem 3.2, $\gamma_T^{(d)}(K_{2d-1,2d-1}) \geq p + q - pd = 2d - 1$. For $1 \leq i \leq d-1$, we define

$$T_i = \{(x_i, y_j) : 1 \leq j \leq d\} \cup \{(y_i, x_{d+j}) : 1 \leq j \leq d-1\} \cup \{(x_{d+i}, y_{d+j}) : 1 \leq j \leq d-1\}.$$

Let $T_d = \{(x_d, y_j) : 1 \leq j \leq d\} \cup \{(y_d, x_{d+j}) : 1 \leq j \leq d-1\}$. For $d+1 \leq i \leq 2d-1$, we define $T_i = \{(y_i, x_j) : 1 \leq j \leq d\}$. Clearly $\mathscr{J} = \{T_1, T_2, \cdots, T_{2d-1}\}$ is a graphoidal tree d-cover of G and so

$$\gamma_T^{(d)}(G) \leq 2d - 1 = (2d-1)(2d-1-2(d-1)) = q + p - pd.$$

$\square$

The following example illustrates the above theorem.

Example 3.8 Consider $K_{9,9}$ and $d = 5$.

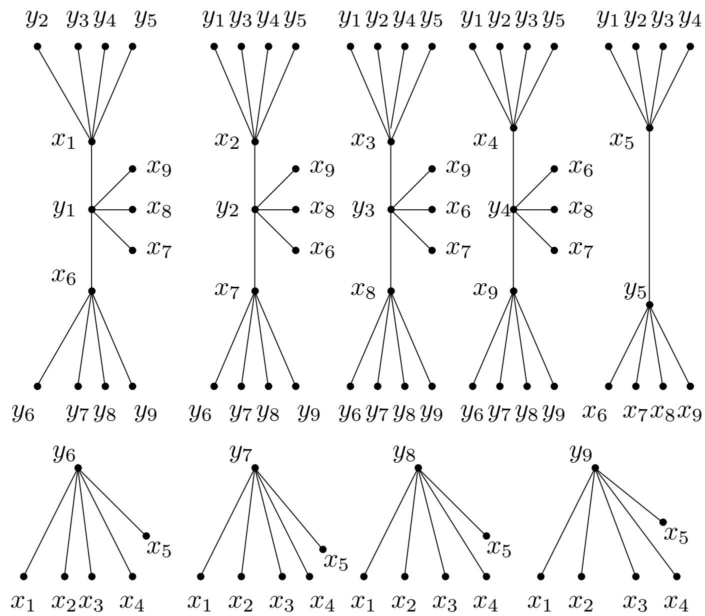

Fig.6

Thereafter, $\gamma_T^{(5)}(K_{9,9}) = 81 + 18 - 90 = 9$.

Lemma 3.9 $\gamma_T^{(d)}(K_{3r,3r}) \leq 2r$, *where* $d \geq 2r$ *and* $r \geq > 1$.

Proof Let $V(K_{3r,3r}) = X \cup Y$, where $X = \{x_1, x_2, \cdots, x_{3r}\}$ and $Y = \{y_1, y_2, \cdots, y_{3r}\}$.

Case (i) r is even.

For $1 \leq s \leq r$, we define

$$\begin{aligned} T_s &= \{(x_s, y_{s+i}) : 0 \leq i \leq r-1\} \cup \{(x_s, y_{2r+s})\} \cup \{(x_{r+s}, y_{2r+s})\} \cup \{(y_{2r+s}, x_{2r+s})\} \\ &\cup \{(x_i, y_{2r+s}) : 1 \leq i \leq r, i \neq s\} \cup \{(x_{r+s}, y_i) : r+s \leq i \leq 3r, i \neq 2r+s\} \\ &\cup \{(x_{r+s}, y_i) : 1 \leq i \leq s-1, s \neq 1\} \end{aligned}$$

and

$$\begin{aligned} T_{r+s} &= \{(y_s, x_{s+i}) : 1 \leq i \leq r\} \cup \{(y_s, x_{2r+s})\} \cup \{(y_{r+s}, x_{2r+s})\} \\ &\cup \{(y_i, x_{2r+s}) : 1 \leq i \leq r, i \neq s, 2r+1 \leq i \leq 3r, i \neq 2r+s\} \\ &\cup \{(y_{r+s}, x_i) : r+s+1 \leq i \leq 3r, 1 \leq i \leq s, i \neq 2r+s\}. \end{aligned}$$

Then $\mathscr{J}_1 = \{T_1, T_2, \cdots, T_{2r}\}$ is a graphoidal tree d-cover for $K_{3r,3r}$, $\Delta(T_i) \leq 2r$ and $d \geq 2r$. So we have, $\gamma_T^{(d)}(_{K3r,3r}) \leq 2r$.

Case (ii) r is odd.

For $1 \leq s \leq r$, we define

$$\begin{aligned} T_s &= \{(x_s, y_{s+i}) : 0 \leq i \leq 2r-1\} \cup \{(y_{r+s}, x_i) : r+1 \leq i \leq 3r, i \neq r+s\} \\ &\cup \{(x_{2r+s}, y_i) : 2r+s \leq i \leq 3r\} \cup \{(x_{2r+s}, y_i) : 1 \leq i \leq s-1, s \neq 1\} \end{aligned}$$

$$\begin{aligned} T_{r+s} &= \{(y_s, x_{s+i}) : 1 \leq i \leq 2r\} \cup \{(x_{r+s}, y_i) : 2r+1 \leq i \leq 3r; i = r+s\} \\ &\cup \{(y_{2r+s}, x_i) : 2r+s+1 \leq i \leq 3r, s \neq r\} \cup \{(y_{2r+s}, x_i) : 1 \leq i \leq s\}. \end{aligned}$$

Clearly $\Delta(T_i) \leq 2r$ for each i. In this case also $\mathscr{J}_2 = \{T_1, T_2, \cdots, T_{2r}\}$ is a graphoidal tree d-cover for $K_{3r,3r}$ and so $\gamma_T^{(d)}(K_{3r,3r}) \leq 2r$ when r is odd. $\square$

The following example illustrates the above lemma for $r = 2, 3$. Consider $K_{6,6}$ and $K_{9,9}$.

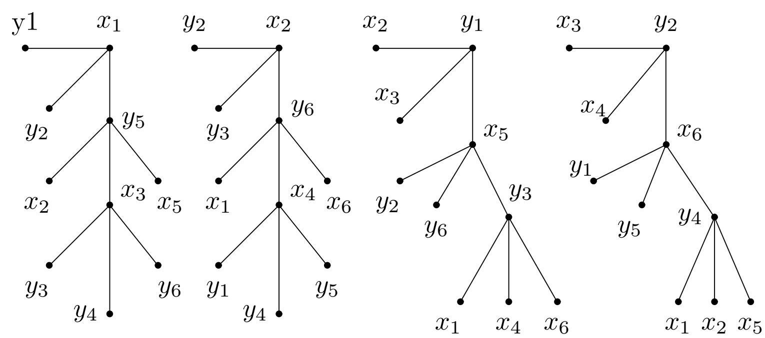

Fig.7

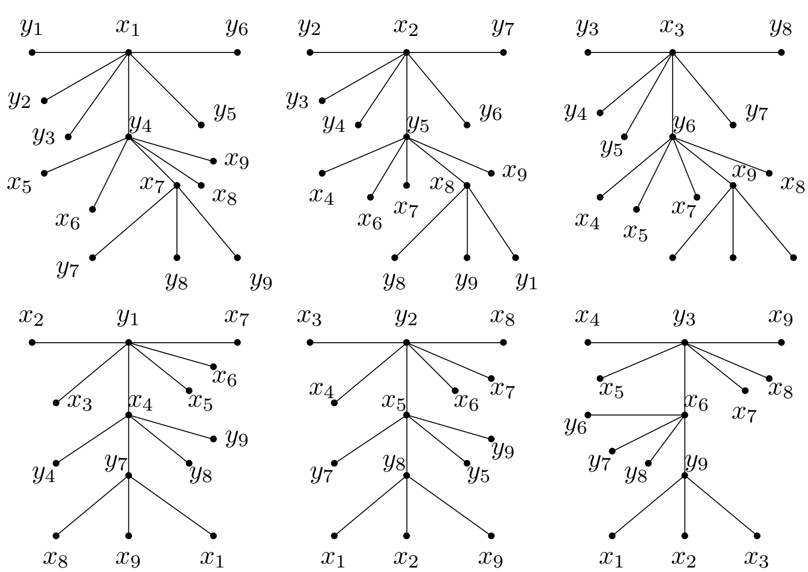

Fig.8

Theorem 3.10 *$\gamma_T^{(d)}(K_{n,n}) = \lceil \frac{2n}{3} \rceil$ for $d \geq \lceil \frac{2n}{3} \rceil$ and $n > 3$.*

Proof By Theorem 2.2, $\lceil \frac{2n}{3} \rceil = \gamma_T(K_{n,n})$ and $\gamma_T(K_{n,n}) \leq \gamma_T^{(d)}(K_{n,n})$, it follows that $\gamma_T^{(d)}(K_{n,n}) \geq \lceil \frac{2n}{3} \rceil$ for any n. Hence the result is true for $n \equiv 0(\text{mod}3)$. Let $n \equiv 1(\text{mod}3)$ so that $n = 3r + 1$ for some r. Let $\mathscr{J}_1 = \{T_1', T_2', \cdots, T_{2r}'\}$ be a minimum graphoidal tree d-cover for $K_{3r,3r}$ as in Lemma 3.9. For $1 \leq i \leq r$, we define

$$T_i = T_i' \cup \{(x_i, y_{3r+1})\},$$

$$T_{r+i} = T_{r+i}' \cup \{(y_i, x_{3r+1})\} \text{ and}$$

$$T_{2r+1} = \{(x_{3r+1}, y_{r+i}) : 1 \leq i \leq 2r + 1\} \cup \{(y_{3r+1}, x_{r+i}) : 1 \leq i \leq 2r\}.$$

Clearly $\mathscr{J}_2 = \{T_1, T_2, \cdots, T_{2r+1}\}$ is a graphoidal tree d-cover for $K_{3r+1,3r+1}$, as $\Delta(T_i) \leq 2r + 1 = \lceil \frac{2n}{3} \rceil \leq d$ for each i. Hence $\gamma_T^{(d)}(K_{n,n}) = \gamma_T^{(d)}(K_{3r+1,3r+1}) \leq 2r + 1 = \lceil \frac{2n}{3} \rceil$.

Let $n \equiv 2(\text{mod}3)$ and $n = 3r+2$ for some r. Let $\mathscr{J}_3$ be a minimum graphoidal tree d-cover for $K_{3r+1,3r+1}$ as in the previous case. Let $\mathscr{J}_3 = \{T_1, T_2, \cdots, T_{2r+1}\}$. For $1 \leq i \leq r$, we define

$$T_i' = T_i \cup \{(x_i, y_{3r+2})\},$$

$$T_{r+i}' = T_{r+i} \cup \{(y_i, x_{3r+2})\},$$

$$T_{2r+1}' = T_{2r+1},$$

$$T_{2r+2}' = \{(x_{3r+2}, x_{r+i}) : 1 \leq i \leq 2r + 2\} \cup \{(y_{3r+2}, x_{r+i}) : 1 \leq i \leq 2r + 1\}.$$

Clearly, $\mathscr{J}_4 = \{T_1', T_2', \cdots, T_{2r+2}'\}$ is a graphoidal tree d-cover for $K_{3r+2,3r+2}$, as $\Delta(T_i') \leq 2r+2 = \lceil \frac{2n}{3} \rceil \leq d$ for each i. Hence $\gamma_T^{(d)}(K_{n,n}) = \gamma_T^{(d)}(K_{3r+2,3r+2}) \leq 2r+2 = \lceil \frac{2n}{3} \rceil$. Therefore, $\gamma_T^{(d)}(K_{n,n}) = \lceil \frac{2n}{3} \rceil$ for every n. $\square$

Now we turn to the case of trees.

Theorem 3.11 *Let G be a tree and let $U = \{v \in V(G) : deg(v) - d > 0\}$. Then $\gamma_T^{(d)}(G) = \sum\limits_{v \in V(G)} \chi_U(v)(deg(v) - d) + 1$, where $d \geq 2$ and $\chi_U(v)$ is the characteristic function of U.*

Proof The proof is by induction on the number of vertices m whose degrees are greater than d. If $m = 0$, then $\mathscr{J} = G$ is clearly a graphoidal tree d-cover. Hence the result is true in this case and $\gamma_T^{(d)}(G) = 1$. Let $m > 0$. Let $u \in V(G)$ with $deg_G(u) = d + s$ ($s > 0$). Now decompose G into $s + 1$ trees $G_1, G_2, \cdots, G_s, G_{s+1}$ such that $deg_{G_i}(u) = 1$ for $1 \leq i \leq s$, $deg_{G_{s+1}}(u) = d$. By induction hypothesis,

$$\gamma_T^{(d)}(G_i) = \sum_{deg_{G_i}(v) > d} (deg_{G_i} - d) + 1 = k_i, \quad 1 \leq i \leq s + 1.$$

Now $\mathscr{J}_i$ is the minimum graphoidal tree d-cover of G_i and $|\mathscr{J}_i| = k_i$ for $1 \leq i \leq s + 1$. Let $\mathscr{J} = \mathscr{J}_1 \cup \mathscr{J}_2 \cup \cdots \cup \mathscr{J}_{s+1}$.

Clearly $\mathscr{J}$ is a graphoidal tree d-cover of G. By our choice of u, u is internal in only one tree T of $\mathscr{J}$. More over, $deg_T(u) = d$ and $deg_{G_i}(v) = deg_G(v)$ for $v \neq u$ and $v \in V(G_i)$ for $1 \leq i \leq s + 1$. Therefore,

$$
\begin{aligned}
\gamma_T^{(d)} &\leq |\mathscr{J}| = \sum_{i=1}^{s+1} k_i = \sum_{i=1}^{s+1}\left[\sum_{deg_{G_i}(v)>d}(deg_{G_i}(v)-d)+1\right] \\
&= \sum_{i=1}^{s+1}\left[\sum_{deg_{G_i}(v)>d}(deg_{G_i}(v)-d)\right]+s+1 = \sum_{deg_G(v)>d,v\neq u}(deg_G(v)-d)+s+1 \\
&= \sum_{deg_G(v)>d,v\neq u}(deg_G(v)-d)+(deg_G(u)-d)+1 = \sum_{deg_G(v)>d}(deg_G(v)-d)+1 \\
&= \sum_{v\in V(G)}\chi_U(v)(deg_G(v)-d)+1.
\end{aligned}
$$

For each $v \in V(G)$ and $deg_G(v) > d$ there are at least $deg_G(v) - d + 1$ subtrees of G in any graphoidal tree d-cover of G and so $\gamma_T^{(d)}(G) \geq \sum\limits_{deg_G(v)>d}(deg_G(v) - d) + 1$. Hence $\gamma_T^{(d)}(G) = \sum\limits_{v\in V(G)}\chi_U(v)(deg_G(v) - d) + 1$. □

Corollary 3.12 *Let G be a tree in which degree of every vertex is either greater than or equal to d or equal to one. Then $\gamma_T^{(d)}(G) = m(d-1)-p(d-2)-1$, where m is the number of vertices of degree 1 and $d \geq 2$.*

Proof Since all the vertices of G other than pendant vertices have degree d we have,

$$
\begin{aligned}
\gamma_T^{(d)} &= \sum_{v\in V(G)}\chi_U(v)(deg_G(v)-d)+1 = \sum_{v\in V(G)}\chi_U(v)(deg_G(v)-d)+md-m+1 \\
&= 2q-dp+md-m+1 = 2p-2-dp+md-m+1 \quad (\text{as } q=p-1) \\
&= m(d-1)-p(d-2)-1.
\end{aligned}
$$

□

Recall that $n_d = \min\limits_{\mathscr{J}\in\mathscr{G}_d} n_{\mathscr{J}}$ and $n = \min\limits_{\mathscr{J}\in\mathscr{G}} n_{\mathscr{J}}$, where $\mathscr{G}_d$ is the collection of all graphoidal tree d-covers of G, $\mathscr{G}$ is the collection of all graphoidal tree covers of G and $n_{\mathscr{J}}$ is the number of vertices which are not internal vertices of any tree in $\mathscr{J}$. Clearly $n_d = n$ if $d \geq \Delta$. Now we prove this for any $d \geq 2$.

Lemma 3.13 *For any graph G, $n_d = n$ for any integer $d \geq 2$.*

Proof Since every graphoidal tree d-cover is also a graphoidal tree cover for G, we have $n \leq n_d$. Let $\mathscr{J} = \{T_1, T_2, \cdots, T_m\}$ be any graphoidal tree cover of G. Let Ψ_i be a minimum graphoidal tree d - cover of T_i $(i = 1, 2, \cdots, m)$. Let $\Psi = \bigcup\limits_{i=1}^{m}\Psi_i$. Clearly Ψ is a graphoidal tree d-cover of G. Let n_Ψ be the number of vertices which are not internal in any tree of Ψ. Clearly $n_\Psi = n_{\mathscr{J}}$. Therefore, $n_d \leq n_\Psi = n_{\mathscr{J}}$ for $\mathscr{J} \in \mathscr{G}$, where $\mathscr{G}$ is the collection of graphoidal tree covers of G and so $n_d \leq n$. Hence $n = n_d$. □

We have the following result for graphoidal path cover. This theorem is proved by S.

Arumugam and J. Suresh Suseela in [5]. We prove this, by deriving a minimum graphoidal path cover from a graphoidal tree cover of G.

Theorem 3.14 $\gamma_T^{(2)}(G) = q - p + n_2$.

Proof From Theorem 3.1 it follows that $\gamma_T^{(2)}(G) \geq q - p + n_2$. Let $\mathscr{J}$ be any graphoidal tree cover of G and $\mathscr{J} = \{T_1, T_2, \cdots, T_k\}$. Let Ψ_i be a minimum graphoidal tree d-cover of T_i $(i = 1, 2, \cdots, k)$. Let m_i be the number of vertices of degree 1 in T_i $(i = 1, 2, \cdots, k)$. Then by Theorem 3.12 it follows that $\gamma_T^{(2)}(T_i) = m_i - 1$ for all $i = 1, 2, \cdots, k$. Consider the graphoidal tree 2-cover $\Psi_{\mathscr{J}} = \bigcup\limits_{i=1}^{k} \Psi_i$ of G. Now

$$\begin{aligned} |\Psi_{\mathscr{J}}| = \sum_{i=1}^{k} |\Psi_i| &= \sum_{i=1}^{k}(m_i - 1) = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} q_i - \sum_{i=1}^{k} p_i \\ &= q - \sum_{i=1}^{k} p_i + \sum_{i=1}^{k} m_i. \end{aligned}$$

Notice that

$$\begin{aligned} \sum_{i=1}^{k} p_i &= \sum_{i=1}^{k}(\text{numbers of internal vertices and pendant vertices of } T_i) \\ &= p - n_{\mathscr{J}} + \sum_{i=1}^{k} m_i. \end{aligned}$$

Therefore, $|\Psi_{\mathscr{J}}| = q - p + n$. Choose a graphoidal tree cover $\mathscr{J}$ of G such that $n_{\mathscr{J}} = n$. Then for the corresponding $\Psi_{\mathscr{J}}$ we have $|\Psi_{\mathscr{J}}| = q - p + n = q - p + n_2$, as $n_2 = n$ by Lemma 3.13. □

Corollary 3.15 *If every vertex is an internal vertex of a graphoidal tree cover, then* $\gamma_T^{(2)}(G) = q - p$.

Proof Clearly $n = 0$ by definition. By Lemma 3.13, $n_2 = n$. So we have $n_2 = 0$. □

J. Suresh Suseela and S. Arumugam proved the following result in [5]. However, we prove the result using graphoidal tree cover.

Theorem 3.16 *Let G be a unicyclic graph with r vertices of degree* 1*. Let C be the unique cycle of G and let m denote the number of vertices of degree greater than* 2 *on C. Then*

$$\gamma_T^{(2)}(G) = \begin{cases} 2 & if\ m = 0, \\ r + 1 & m = 1,\ deg(v) \geq 3 \text{ where } v \text{ is the unique vertex of degree} > 2 \text{ on } C, \\ r & oterwise. \end{cases}$$

Proof By Lemma 3.13 and Theorem 3.14, we have $\gamma_T^{(2)}(G) = q-p+n$. We have $q(G) = p(G)$ for unicyclic graph. So we have $\gamma_T^{(2)}(G) = n$. If $m = 0$, then clearly $\gamma_T^{(2)}(G) = 2$. Let $m = 1$ and let v be the unique vertex of degree > 2 on C. Let $e = vw$ be an edge on C. Clearly $\mathscr{J} = G - e, e$ is a minimum graphoidal tree cover for G and so $n \leq r + 1$. Since there is a vertex of C which is not internal in a tree of a graphoidal tree cover, we have $n = r + 1$. When $m = 1$, $\gamma_T^{(2)}(G) = r+1$. Let $m \geq 2$. Let v and w be vertices of degree greater than 2 on C such that all vertices in a (v, w) - section of C other than v and w have degree 2. Let P denote this (v, w)-section. If P has length 1. Then $P = (v, w)$. Clearly $\mathscr{J} = G - P, P$ is a graphoidal tree cover of G. Also $n = r$ and so $\gamma_T^{(2)}(G) = r$ when $m \geq 2$. Hence we get the theorem. $\square$

Theorem 3.17 *Let G be a graph such that $\gamma_T^{(G)} \leq \delta(G) - d + 1$ ($\delta(G) > d \geq 2$). Then $\gamma_T^{(d)}(G) = q - p(d-1)$.*

Proof By Theorem 3.2, $\gamma_T^{(d)}(G) \geq q-p(d-1)$. Let $\mathscr{J}$ be a minimum graphoidal tree cover of G. Since $\delta > \gamma_T(G)$, every vertex is an internal vertex of a tree in a graphoidal tree cover $\mathscr{J}$. Moreover, since $\delta \geq d+\delta_T(G)-1$ the degree of each internal vertex of a tree in $\mathscr{J}$ is $\geq d$. Let Ψ_i be a minimum graphoidal tree d-cover of T_i ($i = 1, 2, \cdots, k$). Let m_i be the number of vertices of degree 1 in T_i ($i = 1, 2, \cdots, k$). Then by Corollary 3.12, for $i = 1, 2, \cdots, k$ we have

$$\gamma_T^{(2)}(T_i) = -p_i(d-2) + m_i(d-1) - 1.$$

Consider the graphoidal tree d-cover $\Psi_T = \bigcup\limits_{i=1}^{k} \Psi_i$ of G.

$$\begin{aligned}
|\Psi_T| &= |\bigcup_{i=1}^{k} \Psi_i| = \sum_{i=1}^{k}(m_i(d-1) - p_i(d-2) - 1) \\
&= \sum_{i=1}^{k}(m_i(d-1) - p_i(d-2) + q_i - p_i) \\
&= \sum_{i=1}^{k}[(m_i - p_i)(d-1) + q_i] \\
&= (d-1)\sum_{i=1}^{k}(m_i - p_i) + \sum_{i=1}^{k} q_i \\
&= (d-1)\sum_{i=1}^{k}(m_i - p_i) + q.
\end{aligned}$$

Notice that

$$\begin{aligned}
\sum_{i=1}^{k} p_i &= \sum_{i=1}^{k}(\text{numbers of internal vertices and pendant vertices of } T_i) \\
&= p + \sum_{i=1}^{k} m_i.
\end{aligned}$$

Therefore, $|\Psi_T| = -(d-1) + q$. In other words, $\gamma_T^{(d)}(G) \leq q - p(d-1)$. Hence, $\gamma_T^{(d)}(G) = q - p(d-1)$ □

Corollary 3.18 *Let G be a graph such that $\delta(G) = \lceil \frac{p}{2} \rceil + k$ where $k \geq 1$. Then $\gamma_T^{(d)}(G) = q - p(d-1)$ for $d \leq k+1$.*

Proof $\delta(G) - d + 1 = \lceil \frac{p}{2} \rceil + k - d + 1 \geq \lceil \frac{p}{2} \rceil \geq \gamma_T(G)$ by Theorem 2.5. Applying Theorem 3.17, $\gamma_T^{(d)}(G) = q - p(d-1)$. □

Corollary 3.19 *Let G be an r-regular graph, where $r > \lceil \frac{p}{2} \rceil$. Then $\gamma_T^{(d)}(G) = q - p(d-1)$ for $d \leq r + 1 - \lceil \frac{p}{2} \rceil$.*

Proof Here $\delta(G) = r$ and so the result follows from Corollary 3.18. □

Corollary 3.20 $\gamma_T^{(d)}(K_{m,n}) = q - p(d-1)$, *where* $2 \leq d \leq \frac{2m-n}{3}$ *and* $6 \leq m \leq n \leq 2m-6$.

Proof Consider

$$\begin{aligned} \delta(G) - d + 1 &\geq m - \frac{2m-n}{3} + 1 = \frac{3m-2m+n}{3} + 1 \\ &= \frac{m+n}{3} + 1 \geq \lceil \frac{m+n}{3} \rceil = \gamma_T(K_{m,n}). \end{aligned}$$

Hence by Corollary 3.18, $\gamma_T^{(d)}(K_{m,n}) = q - p(d-1)$. □

Theorem 3.21 $\gamma_T^{(d)}(C_m \times C_n) = 3$ *for* $d \geq 4$ *and* $\gamma_T^{(2)}(C_m \times C_n) = q - p$.

Proof For $d \geq \Delta(G) = 4$, $\gamma_T^{(d)}(C_m \times C_n) = \gamma_T(C_m \times C_n) = 3$ by Theorem 2.14. Since $\delta(C_m \times C_n) = 4$ and $\gamma_T(C_m \times C_n) = 3$, we have $\gamma_T(C_m \times C_n) = \delta(G) - d + 1$ when $d = 2$. Applying Theorem 3.17, $\gamma_T^{(2)}(C_m \times C_n) = q - p$. □

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