International J.Math. Combin. Vol.2 (2009), 90-102

DECOMPOSITION OF GRAPHS INTO INTERNALLY DISJOINT TREES

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Abstract: A Smarandache graphoidal tree (k,d)-cover of a graph G is a partition of edges of G into trees $T_1, T_2, \cdots, T_l$ such that $|E(T_i) \cap E(T_j)| \leq k$ and $|T_i| \leq d$ for integers $1 \leq i, j \leq l$. In this paper we investigate the garphoidal tree covering number $\gamma_T(G)$, i.e., Smarandache graphoidal tree $(0,\infty)$-cover of complete graphs, complete bipartite graphs and products of paths and cycles. In [5] M.F.Foregger, define a parameter $z'(G)$ as the minimum number of subsets into which the vertex set of G can be partitioned so that each subset induces a tree. In this paper we also establish the relation $z'(G) \leq \gamma_T(G)$.

Key Words: Smarandache graphoidal tree (k,d)-cover, graphoidal tree cover, complete graph, complete bipartite graph, product of path and cycle.

AMS(2000): 05C70.

§1. Introduction

By a graph we mean a finite, undirected graphs without loops and multiple edges. Terms not defined here are used in the sense of Harary [6]. Any vertex of a graph H of degree greater than 1 is called an internal vertex of H. A *Smarandache graphoidal tree* (k,d)*-cover* of a graph G is a partition of edges of G into trees $T_1, T_2, \cdots, T_l$ such that $|E(T_i) \cap E(T_j)| \leq k$ and $|T_i| \leq d$ for integers $1 \leq i, j \leq l$. Particularly, a Smarandache graphoidal tree $(0,\infty)$-cover, usually called a *graphoidal tree cover* of G is a collection of non C trivial trees in G such that

(i) every vertex is an internal vertex of at most one tree;

(ii) every edge is in exactly one tree.

Let $\mathscr{G}$ denote the set of all graphoidal tree covers of G. Since $E(G)$ is a graphoidal tree cover, we have $\mathscr{G} \neq \emptyset$. We define the graphoidal tree covering number of a graph G to be the minimum number of trees in anygraphoidal tree cover of G, and denote it by $\gamma_T(G)$. Any

[1]Received March 16, 2009. Accepted June 15, 2009.

graphoidal tree cover $\mathscr{J}$ of G for which $|\mathscr{J}| = \gamma_T(G)$ is called a minimum graphoidal tree cover.

Example 1.1 Consider a graph G given in the Fig.1.

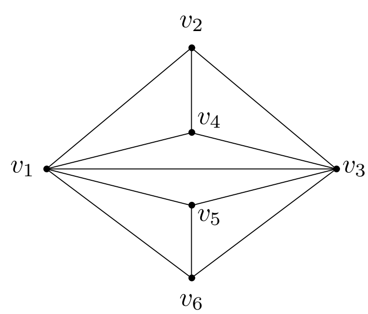


Fig.1

Let T_1, T_2 and T_3 be the trees in Fig.2.

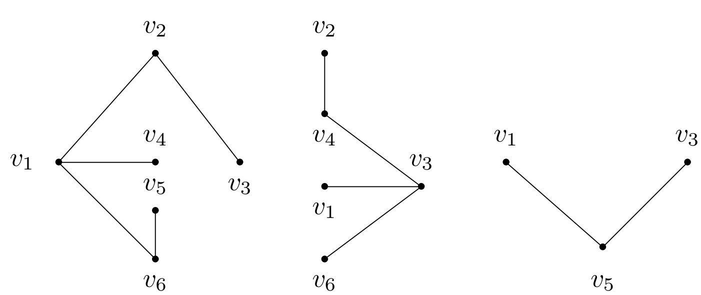


Fig.2

It is easy to see that the graph G in Fig.1 cannot be covered by two trees. Since the three trees shown in Fig.2 form a graphoidal tree cover, $\gamma_T(G) = 3$.

Observation 1.2 If $\deg(v) > \gamma_T(G)$, then v is an internal vertex in some tree in every minimum graphoidal tree cover.

Observation 1.3 For a (p, q) graph G, $\gamma_T(G) \geq \lceil \frac{q}{p-1} \rceil$.

Observation 1.4 $\gamma_T(G) > \frac{\delta(G)}{2}$ if $\delta(G) > 0$.

§2. Preliminaries

$\tau(G)$ is the minimum number of subsets into which the edge set $E(G)$ of G can be partitioned so that each subset forms a tree. A cyclically 4-edge connected graph is one in which the removal of no three edges will disconnect the graph into two components such that each component contains a cycle. We state some preliminary results from [2] and [4].

Theorem 2.1([4]) $\tau(K_n) = \lceil \frac{n}{2} \rceil$.

Theorem 2.2([2]) *If G is a 2-connected cubic graph with p vertices, $p \geq 8$, then $\tau(G) \leq \lfloor \frac{p}{4} \rfloor$.*

Theorem 2.3([2]) *If G is a 3-connected cubic graph with p vertices, $p \geq 12$, then $\tau(G) \leq \lfloor \frac{p}{6} \rfloor$.*

Theorem 2.4([2]) *If G is a cyclically 4-edge connected cubic graph with p vertices, $8 \leq p \leq 16$ then $\tau(G) = 2$.*

§3. Complete and complete bipartite graphs

We first determine the graphoidal tree covering number of a complete graphs.

Theorem 3.1 $\gamma_T(K_n) = \lceil \frac{n}{2} \rceil$.

Proof From observation 1.4, it follows that $\gamma_T(K_n) \geq \lceil \frac{n}{2} \rceil$. We give a construction for the reverse inclusion. First, let n be even, say $n = 2k$. For $i = 1, 2, \cdots, k$, let T_i be the tree shown in Fig.3 (subscripts modulo n). The Standard Rotation Method shows that this is true. (see Fig.4 for the case $n = 8$).

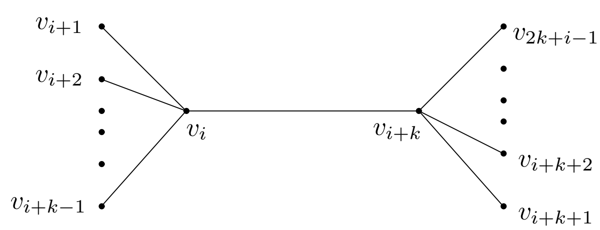


Fig.3

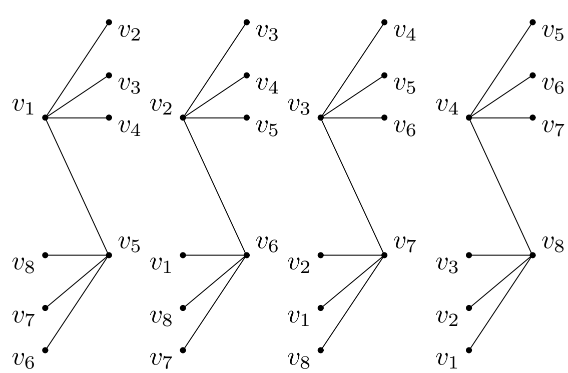


Fig.4

For n odd, delete one vertex from each tree in the decomposition given for K_{n+1}. The result is clearly a graphoidal tree cover for K_n, once isolated vertices are removed. □

We now turn to the case of complete bipartite graphs, beginning with a general result on the diameter of trees in a minimum graphoidal tree cover. The following standard notation is used for the partite sets of $K_{m,n}$ with $m \leq n$: $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$.

Lemma 3.2 *If a minimum graphoidal tree cover $\mathscr{J}$ of $K_{m,n}$ contains a tree with a path of length≥ 5, then it also contains a tree with exactly one edge.*

Proof Let $T \in \mathscr{J}$ contain a path $P = (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$ where $x_i \in X$ and $y_j \in Y$. Since y_1 and x_3 are internal in T, these cannot be internal in any other member of $\mathscr{J}$. Therefore $T_1 = \{(y_1x_3)\} \in \mathscr{J}$. □

Lemma 3.3 *If $m \leq n \leq 2mC\ 3$, $\gamma_T(K_{m,n}) \geq \lceil \frac{m+n}{3} \rceil$.*

Proof Suppose $\gamma_T(K_{m,n}) = r$ with $r < \lceil \frac{m+n}{3} \rceil$. Let $\mathscr{J}$ be a minimum graphoidal tree cover of $K_{m,n}$. Since $\delta(G) = m > \frac{m+n}{3} > r$ (as $n \leq 2mC\ 3$), by Observation 1.3, we have every vertex is an internal vertex of a tree in $\mathscr{J}$.

Claim 1. No tree in $\mathscr{J}$ can have more than two internal vertices from X with a common neighbor from Y. Suppose $x_1, x_2, \cdots, x_k$ $(k \geq 3)$ are all adjacent to y_1 in T_1 of $\mathscr{J}$. Then the sum of degrees of $x_1, x_2, \cdots, x_k$ in T_1 is at most $n + k - 1$. But each x_i $(i = 1, 2, \cdots, k)$ is an end vertex in at most $r - 1$ other members of $\mathscr{J}$. So they have at most $n + k - 1 + k(r-1)$ total adjacencies in $\mathscr{J}$. Since $r < \frac{m+n}{3}$, $n + k - 1 + k(r-1) < \frac{3n+3k-3+k(m+n)-3k}{3} \leq \frac{n(2k+3)}{3} - 1$ $(m \leq n)$ $= nk - (\frac{n(k-3)}{3} + 1) < nk$ $(k \geq 3)$, a contradiction. Hence we have Claim 1.

Claim 2 There exists a minimum graphoidal tree cover $\mathscr{J}'$ such that no tree in $\mathscr{J}'$ has a path of length≥ 5.

Suppose $T_1 \in \mathscr{J}$ has a path $(x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$. Then by the previous lemma, a tree T_2 in $\mathscr{J}$ has just the single edge y_1x_3. Let T_1' be the tree containing x_2 obtained by removing the edge y_1x_2 from T_1. Let T_2' be the tree $(T_1 - T_1') \cup T_2$. Let $\mathscr{J}_1'$ be the graphoidal tree cover obtained from $\mathscr{J}$ after replacing T_1, T_2 by T_1', T_2' respectively. If there is a tree in $\mathscr{J}_1'$ again contains a path of length≥ 5 we repeat this process for $\mathscr{J}_1'$ and so on. Finally we get the required minimum graphoidal tree cover $\mathscr{J}'$. Hence we get Claim 2.

Now we can assume that no tree in $\mathscr{J}$ has a path of length ≥ 5.

Claim 3 No tree in $\mathscr{J}$ can have more than two internal vertices from Y with a common neighbor from X.

Suppose there is a tree T_1 in $\mathscr{J}$ containing k internal vertices $y_1, y_2, \cdots, y_k$ $(k \geq 3)$ with a common neighbor x_1. Since $m > \gamma_T(G)$ and every vertex is an internal vertex of a tree in $\mathscr{J}$, there is a tree in $\mathscr{J}$, say, T_2 containing at least two vertices from X as internal vertices. By Claim 2, the internal vertices of a tree in $\mathscr{J}$ form a star and so the internal vertices from X in T_2 have a common neighbor from Y. By Claim 1, T_2 has exactly two internal vertices x_2 and x_3 from X with a common neighbor y_s from Y. Between x_2, x_3 and $y_1, y_2, \cdots, y_k$ there are $2k$ edges in $K_{m,n}$. Clearly x_2 and x_3 can be made adjacent with two y's in T_1. Let it be y_1 and y_2. Now $y_1, y_2, \cdots, y_k$ can be made adjacent with x's in T_2. But it will cover exactly $k+2$ edges (out of $2k$ edges) and so by the definition of graphoidal tree cover, each uncovered edge is a tree in $\mathscr{J}$. Without loss of generality let $T_3, T_4, \cdots, T_k$ be the trees with edges $(y_3, x_{l_3}), (y_4, x_{l_4}), \cdots, (y_k, x_{l_k})$ respectively, where $l_i \in \{2,3\}$, $3 \le i \le k$. By Claim 2 the internal vertices of T_1 form a star. Removing all the edges incident with y_i from T_1 to form the tree T_i' $(3 \le i \le k)$. Let T_1' be the tree formed by the remaining edges of T_1 after the removal. Now each T_i in $\mathscr{J}$ is replaced by $T_i' \cup T_i$ for $3 \le i \le k$. Also replace T_1 in $\mathscr{J}$ by T_1'. If $\mathscr{J}$ again contains a tree having more than two internal vertices from Y with a common neighbor from X. We repeat the above process and so on. Hence we have Claim 3. From Claims 1, 2 and 3, it follows that no tree in $\mathscr{J}$ has more than three internal vertices. Since every vertex of $K_{m,n}$ must be an internal vertex of a tree in $\mathscr{J}$ and $\gamma_T(K_{m,n}) = r$, we have only $3r$ ($< m+n$) internal vertices in $\mathscr{J}$. This is a contradiction. Hence $\gamma_T(K_{m,n}) \ge \lceil \frac{m+n}{3} \rceil$. □

Theorem 3.4 *If $m \le n \le 2mC\ 3$, then $\gamma_T(K_{m,n}) = \lceil \frac{m+n}{3} \rceil$. Furthermore, if $n > 2m-3$, then $\gamma_T(K_{m,n}) = m$.*

Proof By Lemma 3.3, $\gamma_T(K_{m,n}) \ge \lceil \frac{m+n}{3} \rceil$. Next we proceed to prove $\gamma_T(K_{m,n}) \le \lceil \frac{m+n}{3} \rceil$, where $3 \le m \le n \le 2m-3$. Let $r = \lfloor \frac{2mn}{3} \rfloor = \frac{2m-n+k}{3}$ where k is $0, 1$ or 2.

Define for $1 \le i \le r$

$P_i = \{(x_i, y_i x_{r+i})\} \cup \{(y_i, x_j) : j \ne i, r+i; 1 \le j \le m-k\} \cup \{(x_i, y_j) : r < j \le m-r-k\} \cup \{(x_{r+i}, y_j) : m-r-k < j \le n-k\}$.

For $1 \le i \le m-2r-k$ we define

$P_{i+r} = \{(y_{r+i}, x_{2r+i}, y_{m-r-k+i})\} \cup \{(x_{2r+i}, y_j); j \ne r+i, m-r-k+i, r < j \le n-k\} \cup \{(y_{r+i}, x_j) : r+1 \le j \le 2r\} \cup \{(y_{m-r-k+i}, x_j) : 1 \le j \le r\}$.

For $k = 1$

$$P_{m-r} = \{(x_m, y_n)\} \cup \{(x_m, y_j) : 1 \le j \le n-1\} \cup \{(y_n, x_j) : 1 \le j \le m-1\}.$$

For $k = 2$

$$P_{m-r-1} = \{(x_{m-1}, y_{n-1})\} \cup \{(x_{m-1}, y_j) : 1 \le j \le nC\ 2\} \cup \{(y_{n-1}, x_j) : 1 \le j \le mC\ 2\},$$

$$P_{m-r} = \{(x_m, y_n)\} \cup \{(x_m, y_j) : 1 \le j \le n-1\} \cup \{(y_n, x_j) : 1 \le j \le m-1\}.$$

Clearly $\mathscr{J} = \{P_1, P_2, \cdots, P_{m-r}\}$ is a graphoidal tree cover for $K_{m,n}$. Therefore $\gamma_T(K_{m,n}) \le m - r = \frac{m+n+k}{3} \le \lceil \frac{m+n}{3} \rceil$.

Let $n = 2m-2+k$, $k \ge 0$. Suppose that $\gamma_T(K_{m,n}) \ne m$. Then there exists a graphoidal tree cover $\mathscr{J}$ with at most $m-1$ trees. Since $\delta(G) > m-1$, it follows that every vertex is an internal vertex of a tree in $\mathscr{J}$. If x_i is an internal vertex of a tree T in $\mathscr{J}$ then $\deg_T(x_i) \ge 2m-2+k-(m-2) = m+k$. This implies that in any minimum graphoidal tree cover exactly one vertex of X should be internal in a tree. But there are m vertices and $|\mathscr{J}| \le m-1$. This leads to a contradiction. Hence $\gamma_T(K_{m,n}) \ge m$. Clearly, $\gamma_T(K_{m,n}) \le m$ and so

$$\gamma_T(K_{m,n}) = m.$$ □

The following examples illustrate $\gamma_T(K_{m,n}) = \lceil \frac{m+n}{3} \rceil$.

Example 3.5 (i) Consider $K_{4,5}$. Clearly $k = 0$ and $r = 1$.

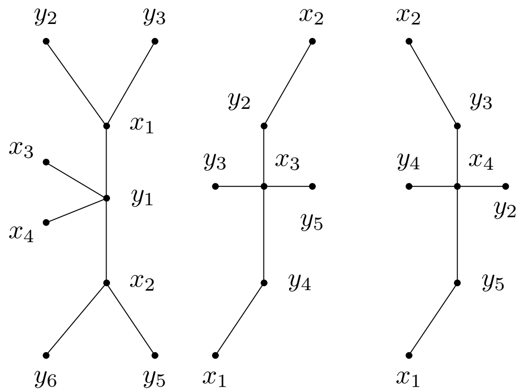


Fig.5

(ii) Consider $K_{8,9}$. Clearly $r = 2$ and $k = 1$.

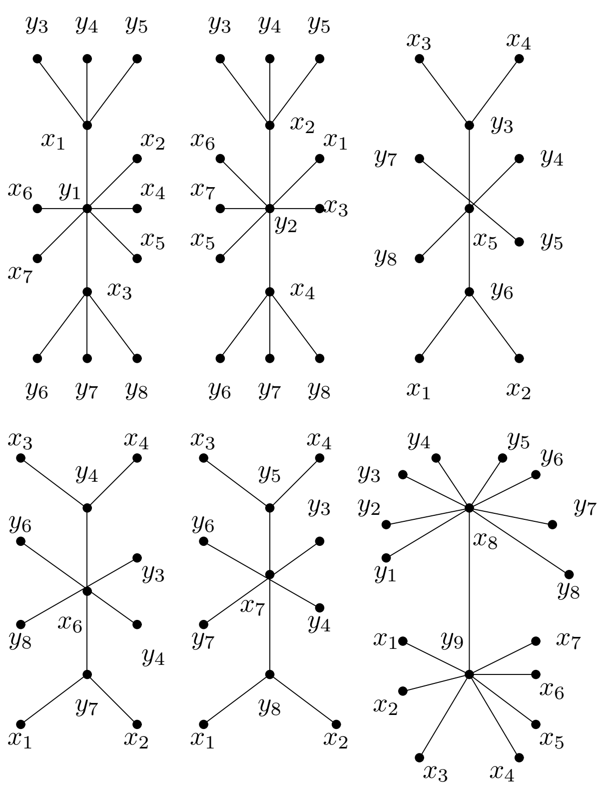


Fig.6

(iii) Consider $K_{12,13}$. Clearly $r = 3$ and $k = 2$.

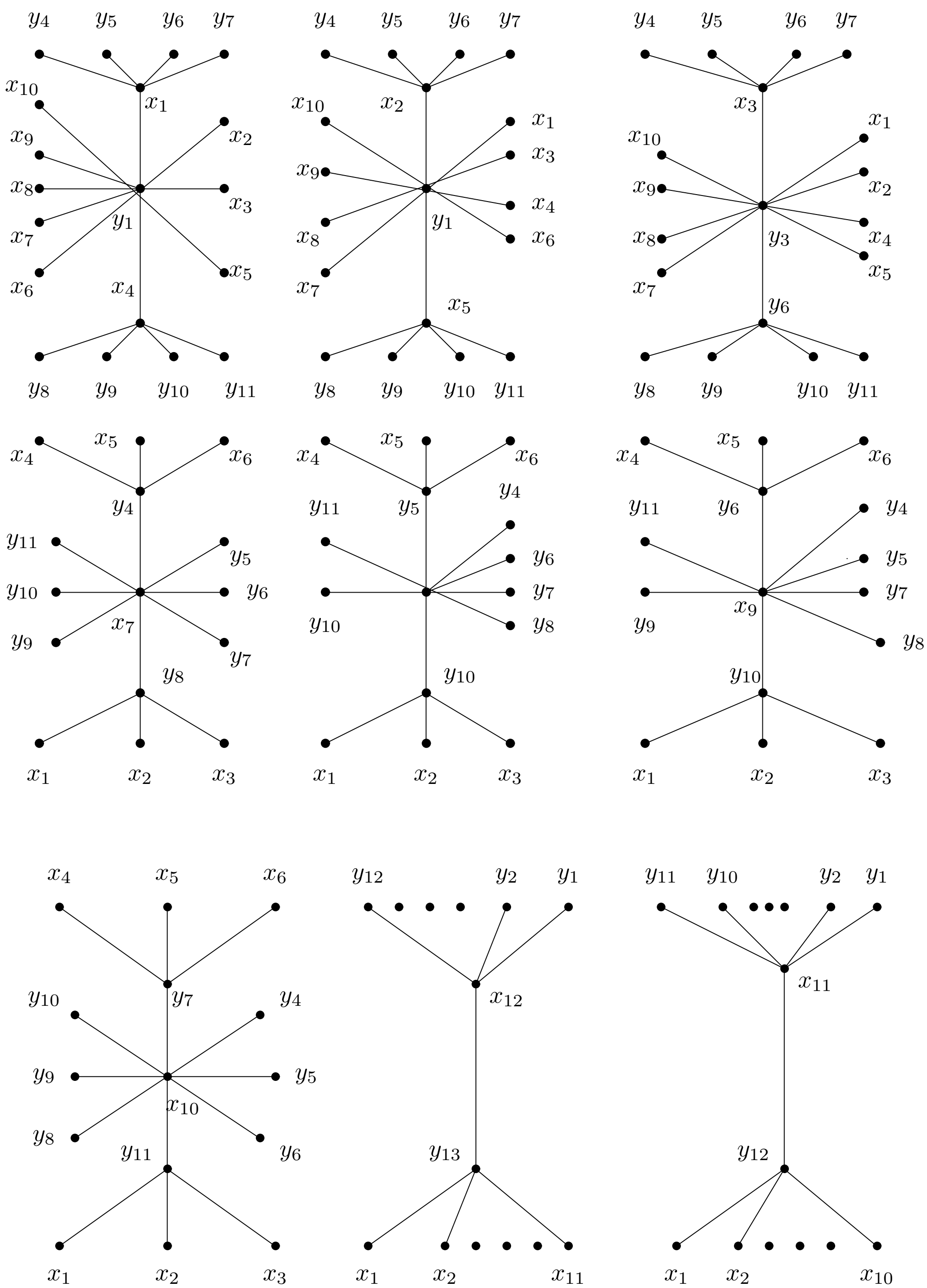


Fig.7

§4. Product of paths and cycles

Theorem 4.1 *(a)* $\gamma_T(P_m \times P_n) = 2$ *for integers* $m, n \geq 2$*;*

(b) $\gamma_T(P_n \times C_m) = 2$ *for integers* $m \geq 3$*,* $n \geq 2$*;*

(c) $\gamma_T(C_m \times C_n) = 3$ *for integers* $m, n \geq 3$*.*

Proof The fact that $P_m \times C_{2r}$ and $P_m \times C_{2r+1}$ can be decomposed into graphoidal tree covers of order 2 clearly follows from Fig.11 and Fig.12. Hence (b) follows. It is easily seen that deleting the edges (W_{i1}, W_{in}), $i = 1, 2, 3, \cdots, m$ (from Fig.8, Fig.9 and Fig.10) produces a graphoidal tree cover for $P_m \times P_n$ and so (a) follows. Since $C_m \times C_n$ is 4-regular, $\gamma_T(C_m \times C_n) > 2$. Now consider the minimum graphoidal tree cover $\{T_1, T_2\}$ of $P_{m-1} \times C_n$.

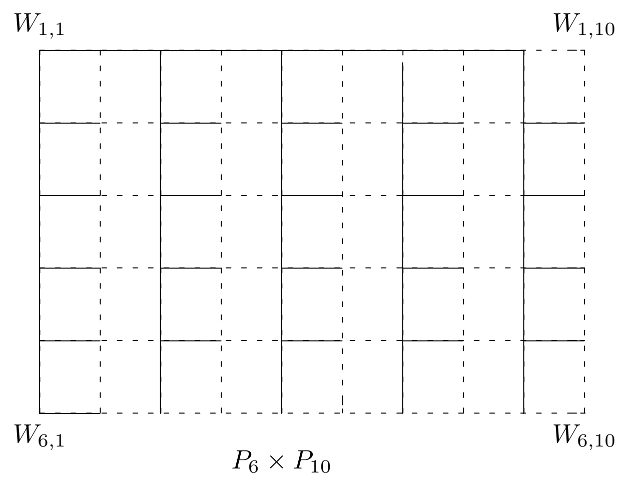


Fig.8

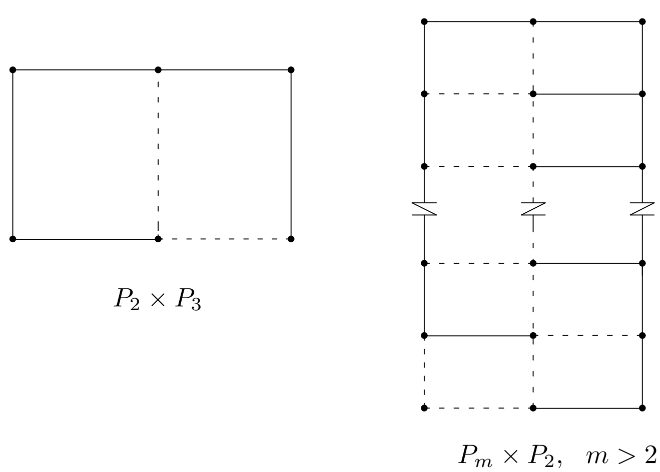
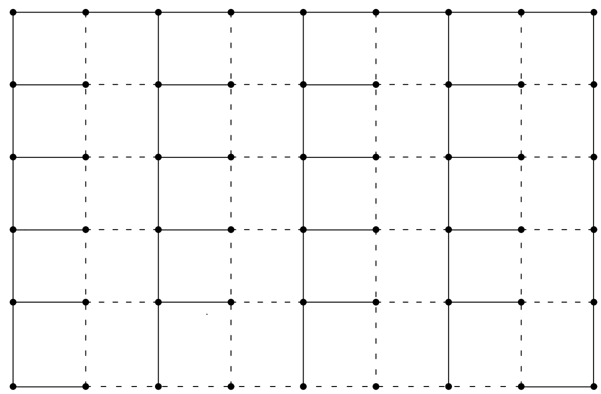


Fig.9



$P_6 \times P_9$

Fig.10

Here in Fig.11 and Fig.12 thick lines form the tree T_1 and dotted lines form the tree T_2.

Case (i) *n is even.*

To the tree T_2, add vertices $w_{m,2}$ and $w_{m,4}$. Then add a vertex $w_{m,3}$ adjacent to $w_{1,3}$, $w_{m-1,3}$, $w_{m,2}$ and $w_{m,4}$. The only additional internal vertex this creates is $w_{m,3}$.

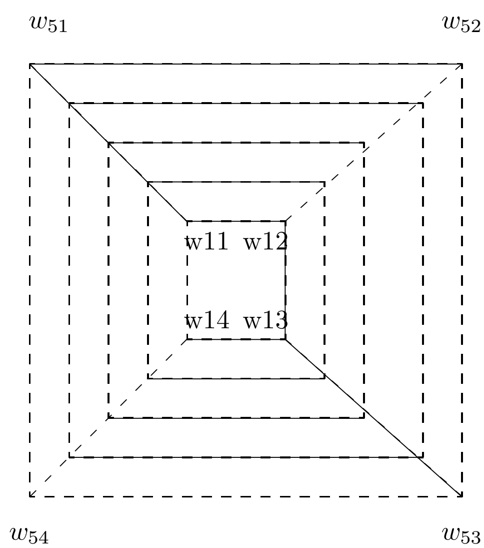
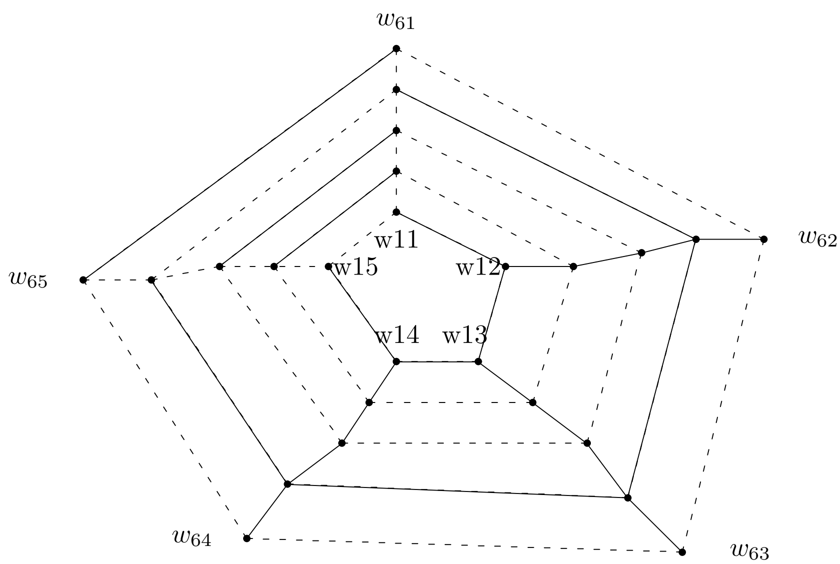


Fig.11

Now take a third tree as $T_3 = \{(w_{m,i-1}, w_{m,i}) : 2 \leq i \leq n; i \neq 3, 4\} \cup \{(w_{m,1}, w_{m,n})\} \cup \{(w_{m,i}, w_{1,i}), (w_{m,i}, w_{m-1,i}) : 1 \leq i \leq n; i \neq 3\}$.

Case (ii) *n is odd.*

To the tree T_2, add vertices $w_{m,1}$ and $w_{m,3}$. Then add a vertex $w_{m,2}$ adjacent to $w_{1,2}$, $w_{m-1,2}$, $w_{m,1}$, $w_{m,3}$. The only additional internal vertex this creates is $w_{m,2}$.



$P_6 \times C_5$

Fig.12

Now take a third tree as $T_3 = \{(w_{m,i-1}, w_{m,i}) : 4 \leq i \leq n\} \cup \{(w_{m,1}, w_{m,n})\} \cup \{(w_{m,i}, w_{1,i}), (w_{m,i}, w_{m-1,i}) : 1 \leq i \leq n; i \neq 2\}$. Now $\{T_1, T_2, T_3\}$ is a graphoidal tree cover for $C_m \times C_n$ and so $\gamma_T(C_m \times C_n) = 3$.

Observation 4.2 It is observed that $\tau(G) \leq \gamma_T(G)$. From Theorems 2.1 and 3.1, it follows that (i) $\tau(K_n) = \gamma_T(K_n) = \lceil \frac{n}{2} \rceil$. It is also observed that $\tau(G) = \gamma_T(G)$ for all graphs with maximum degree≤ 3. From Theorems 2.2, 2.3 and 2.4, it follows that

Theorem 4.3 *If G is a 2-connected cubic graph with p vertices, $p \geq 8$, then $\gamma_T(G) \leq \lfloor \frac{p}{4} \rfloor$.*

Theorem 4.4 *If G is a 3-connected cubic graph with p vertices, $p \geq 12$, then $\gamma_T(G) \leq \lfloor \frac{p}{6} \rfloor$.*

Theorem 4.5 *If G is a cyclically 4-edge connected cubic graph with p vertices, $8 \leq p \leq 16$ then $\gamma_T(G) = 2$.*

§5. Relationship between $\tau'(G)$ and $\gamma_T(G)$

In [5] Foregger, M F and Foregger, T.H defined $\tau'(G)$ as the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a tree. In this section we try to find some relationship between $\tau'(G)$ and $\gamma_T(G)$.

Theorem 5.1 *Let G be a graph with vertices $p \geq 4$ and let $\mathscr{J} = \{T_1, T_2, \cdots, T_n\}$ be a minimum graphoidal tree cover of G with $|E(T_j)| = 1$ for some j and $|E(T_i)| > 1$ for all $i \neq j$. Then we can always find a minimum graphoidal tree cover $\mathscr{J}' = \{T_1', T_2', \cdots, T_n'\}$ with $|E(T_i')| > 1$ for all i.*

Proof Let $T_j = \{(xy)\}$. Consider cases following.

Case (i) Suppose at least one of the vertices x and y, say x, is internal in a tree of $\mathscr{J}$. First assume that x is internal in a tree T_i of $\mathscr{J}$. If $y \notin V(T_i)$ then replacing T_i by $T_i \cup T_j$ and removing T_j from $\mathscr{J}$, we get a graphoidal tree cover $\mathscr{J}'$ with $|\mathscr{J}'| < |\mathscr{J}|$. Hence $y \in V(T_i)$. Let $(w, x, z, \cdots, y)$ be a path in T_i. Let C_1 and C_2 be the two components of $T_i - (xz)$ containing x and y respectively. Replace T_i and T_j by $C_2 \cup (xz)$ and $C_1 \cup T_j$ respectively so that both of them have at least two edges. Now $\mathscr{J}$ is still a minimum graphoidal tree cover and $|E(T)| > 1$ for every $T \in \mathscr{J}$.

Case (ii) Suppose both x and y are external vertices in $\mathscr{J}$. If $x \in V(T_i)$ and $y \notin V(T_i)$ then as in Case (i), we get a graphoidal tree cover $\mathscr{J}'$ with $|\mathscr{J}'| < |\mathscr{J}|$. Hence either $x, y \in V(T)$ or $x, y \notin V(T)$ for every T in $\mathscr{J}$. Let $x, y \in V(T_r)$, $T_r \in \mathscr{J}$. Suppose $|E(T_r)| > 2$. Let $e = (xz)$ be an edge in T_r. Replace T_r and T_j by $T_r - e$ and $T_j \cup e$ respectively and the result is true in this case. So let us assume that $|E(T_i)| = 2$ for some $T_i \in \mathscr{J}$ and $x, y \in V(T_i)$. Suppose $T_i = (xzy) \in \mathscr{J}$. Then $\deg(z) \geq 3$ in G. For, suppose $\deg(z) = 2$ in G. Since G is connected and $p \geq 4$, we must have at least one of the vertices x, y is of degree≥ 3. Since x or y alone can not be a member of a tree in $\mathscr{J}$ and $x, y \in V(T_i), V(T_j)$ we have $\deg(x) \geq 3$ and $\deg(y) \geq 3$.

Let x and y be external vertices in a tree T_r of $\mathscr{J}$ $(r \neq i, j)$. Replace T_r and T_j by $T_r \cup (xz)$ and $T_j \cup (zy)$ respectively. Now $\{T_1, T_2, \cdots, T_{i-1}, T_{i+1}, \cdots, T_n\}$ is clearly a graphoidal tree cover for G. This is a contradiction to the minimality of $\mathscr{J}$. Hence $\deg(z) \geq 3$ in G. Now, z must be external in some tree T_r of $\mathscr{J}$. Clearly $x, y \in V(T_r)$. Suppose $x, y \notin V(T_r)$. Replace T_r and T_j by $T_r \cup \{(xz)\}$ and $T_j \cup \{(zy)\}$ respectively in $\mathscr{J}$. Now $\{T_1, T_2, \cdots, T_{i-1}, T_{i+1}, \cdots, T_n\}$ is clearly a graphoidal tree cover for G. This is a contradiction to the minimality of $\mathscr{J}$. It shows that $x, y \in V(T_r)$. Since x, y and z are external vertices in T_r we have $|E(T_r)| \geq 3$. Let e be an edge in T_r containing z. Replace T_i and T_j by $\{T_i - (xz)\} \cup \{e\}$ and $T_j \cup \{(xz)\}$ respectively. Now $\mathscr{J}$ is a minimum graphodial tree cover and $|E(T)| > 1$ for every $T \in \mathscr{J}$. □

Proposition 5.2 *If $p \geq 4$, then there exists a minimum graphoidal tree cover of a connected graph G, in which every tree has more than one edge.*

Proof Let $\mathscr{J}$ be a minimum graphoidal tree cover of G and let $\mathscr{J} = \{T_1, T_2, \cdots, T_n\}$. Let us assume that $T_i = \{e_i\}, 1 \leq i \leq k$ and $|E(T_j)| > 1$ for $k+1 \leq j \leq n$. Let $G' = G - \{e_1, e_2, \cdots, e_k\}$. Clearly $\mathscr{J}' = \mathscr{J} - \{T_1, T_2, \cdots, T_k\}$ is a graphoidal tree cover for G'. Suppose G' is a disconnected graph. Then the number of components $\omega(G')$ is greater than one. If $\omega(G' \cup e_i) = \omega(G')$ for every $i \in \{1, 2, \cdots, k\}$ then G is disconnected. Hence we can choose $e_i = (x_i, y_i)$ for some $i \in \{1, 2, \cdots, k\}$ such that $\omega(G' \cup e_i) < \omega(G')$. Let G_1', G_1'' be the components of G' such that $G_1' \cup G_1'' \cup e_i$ is connected. Without loss of generality assume that $x_i \in G_1'$, $y_i \in G_1''$. If at all x_i is internal in a tree of $\mathscr{J}$, let it be in a tree T (of $\mathscr{J}$) in G_1'. Clearly $\mathscr{J}_1 = (\mathscr{J} - \{T, T_i\}) \cup \{T \cup T_i\}$ is a graphoidal tree cover of G

and $\mathscr{J}_1 < |\mathscr{J}|$. This is a contradiction. Hence G' is connected. Take $G_1 = G' \cup \{e_1\}$. Clearly $\mathscr{J}_1 = \mathscr{J}' \cup \{T_1\}$ is a minimum graphoidal tree cover for G_1 and $|\mathscr{J}_1| = n - k + 1$. For, suppose $\gamma_T(G_1) < n - k + 1$ and let $\mathscr{J}''$ be a minimum graphoidal tree cover for G_1. Then $|\mathscr{J}''| < n - k + 1$. Since $G = G_1 \cup \{e_2, \cdots, e_k\}$, $\mathscr{J}''' = \mathscr{J}'' \cup \{T_2, \cdots, T_k\}$ is a graphoidal tree cover for G and $|\mathscr{J}'''| = |\mathscr{J}''| + k - 1 < n - k + 1 + k - 1 = n$. This is a contradiction to the minimality of $\mathscr{J}$. Hence $\gamma_T(G_1) = n - k + 1$. By Theorem 5.1, there exists a minimum graphoidal tree cover $\mathscr{J}_1'$ of G_1 in which every tree has more than one edge and $|\mathscr{J}_1'| = |\mathscr{J}_1| = n - k + 1$. Let $G_2 = G_1 \cup \{e_2\}$. Proceeding as above, we find a minimum graphoidal tree cover $\mathscr{J}_2$ of G_2 in which every tree has more than one edge. Finally, we get $G = G_n = G_{n-1} \cup \{e_n\}$ and by a similar argument as above, we find a minimum graphoidal tree cover $\mathscr{J}_n$ of $G = G_n$ in which $|E(T)| > 1$ for every $T \in \mathscr{J}_n$. □

Lemma 5.3 *Let $p(G) \geq 4$. Let $\mathscr{J}$ be a graphoidal tree cover of G such that $|E(T)| > 1$ for every tree $T \in \mathscr{J}$. Let $i(T)$ be the set of internal vertices of T. Then $\langle i(T) \rangle$-the subgraph induced by $i(T)$ is a subgraph of T and it is a tree for every $T \in \mathscr{J}$.*

Proof If $|i(T)| = 1$ then clearly the result is true. Let $|i(T)| > 1$. Let $x, y \in i(T)$ and $xy \in E(G)$. Suppose $xy \notin E(T)$. Then there exists T' of $\mathscr{J}$ such that $T' = \{(xy)\}$ by the definition of graphoidal tree cover. By our assumption this is not possible. Hence $\langle i(T) \rangle$ is a subgraph of T and it is a tree. Moreover, it is got by removing all the pendant vertices of T.□

Theorem 5.4 *If G is a (p, q) graph with $p \geq 4$, then $\tau'(G) \leq \gamma_T(G)$.*

Proof By Proposition 5.2, we have known that result (1) following:

there exists a minimum graphoidal tree cover $\mathscr{J}$ such that $|E(T)| > 1$ for all $T \in \mathscr{J}$ and $|\mathscr{J}| = n$.

Let $\mathscr{J} = \{T_1, T_2, \cdots, T_n\}$.

Case (i) If every vertex is an internal vertex of a tree of $\mathscr{J}$, then $V(G) = i(T_1) \cup \cdots \cup i(T_n)$ is clearly a vertex partition of G. By Lemma 5.3, $\langle i(T_j) \rangle$ is a subgraph of T_j and is a tree for $1 \leq j \leq n$. Hence $\tau'(G) \leq n \leq \gamma_T(G)$.

Case (ii) Let x be one of the vertices which is not internal in any tree of $\mathscr{J}$. Let $x \in V(T_k)$ and $v \in i(T_k)$ such that $xv \in E(T_k)$. Since x is not internal in any tree of $\mathscr{J}$ and v is not internal in any tree except T_k, we have $\langle i(T_k) \cup \{x\} \rangle$ is a tree. For, if $xu \in E(G)$ and $xu \notin E(T_k)$ where $u \neq v$ in $i(T_k)$, then by the definition of graphoidal tree cover there exists T' of $\mathscr{J}$ such that $T' = \{(xu)\}$. This is a contradiction to claim in (1).

Let x, y be non-internal vertices in any tree of $\mathscr{J}$. Let $x, y \in V(T_k)$. If $xy \in E(G)$ then there exists T' of $\mathscr{J}$ such that $T' = \{(xy)\}$. This is a contradiction to the claim (1) also. Clearly, in this case $\langle i(T_k) \cup \{x, y\} \rangle$ is a tree. In this way we adjoin every such vertex to an $i(T_k)$. We make sure that each such vertex is adjoined to only one $i(T_k)$. These induced subgraphs give rise to a partition of $V(G)$ and these induced subgraphs form $n = \gamma_T(G)$ trees. Hence $\tau'(G) \leq n = \gamma_T(G)$. From Theorems 3.1 and 4.1 it follows that $\gamma_T(G) = \tau'(G)$ for the following graphs $K_n, P_m \times P_n$ and $P_n \times C_m$. □

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