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The Upper and Forcing Vertex Detour Numbers of a Graph

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Abstract: For any vertex x in a connected graph G of order $p \geq 2$, a set $S \subseteq V(G)$ is an x-detour set of G if each vertex $v \in V(G)$ lies on an $x - y$ detour for some element y in S. The minimum cardinality of an x-detour set of G is defined as the x-detour number of G, denoted by $d_x(G)$. An x-detour set of cardinality $d_x(G)$ is called a d_x-set of G. An x-detour set S_x is called a minimal x-detour set if no proper subset of S_x is an x-detour set. The upper x-detour number, denoted by $d_x^+(G)$, is defined as the maximum cardinality of a minimal x-detour set of G. We determine bounds for it and find the same for some special classes of graphs. For any three positive integers a, b and n with $a \geq 2$ and $a \leq n \leq b$, there exists a connected graph G with $d_x(G) = a$, $d_x^+(G) = b$ and a minimal x-detour set of cardinality n. A subset T of a minimum x-detour set S_x of G is an x-forcing subset for S_x if S_x is the unique minimum x-detour set containing T. An x-forcing subset for S_x of minimum cardinality is a minimum x-forcing subset of S_x. The forcing x-detour number of S_x, denoted by $f_{dx}(S_x)$, is the cardinality of a minimum x-forcing subset for S_x. The forcing x-detour number of G is $f_{dx}(G) = \min \{f_{dx}(S_x)\}$, where the minimum is taken over all minimum x-detour sets S_x in G. It is shown that for any three positive integers a, b and c with $2 \leq a \leq b \leq c$, there exists a connected graph G with $f_{dx}(G) = a$, $d_x(G) = b$ and $d_x^+(G) = c$ for some vertex x in G.

Key Words: detour, vertex detour number, upper vertex detour number, forcing vertex detour number.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic

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terminology we refer to Harary [6]. For vertices x and y in a connected graph G, the *distance* $d(x,y)$ is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x,y)$ is called an $x-y$ *geodesic*. The *closed interval* $I[x,y]$ consists of all vertices lying on some $x-y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y\in S} I[x,y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a $g-set$. The geodetic number of a graph was introduced in [1,7] and further studied in [3].

The concept of vertex geodomination number was introduced by Santhakumaran and Titus in [8] and further studied in [9]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-geodominating set* of G if each vertex v of G lies on an $x-y$ geodesic in G for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a *g_x-set*. The connected vertex geodomination number was introduced and studied by Santhakumaran and Titus in [11]. A *connected x-geodominating set* of G is an x-geodominating set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-geodominating set of G is the *connected x-geodomination number* of G and is denoted by $cg_x(G)$. A connected x-geodominating set of cardinality $cg_x(G)$ is called a *cg_x-set* of G.

For vertices x and y in a connected graph G, the *detour distance* $D(x,y)$ is the length of a longest $x-y$ path in G. An $x-y$ path of length $D(x,y)$ is called an $x-y$ *detour*. The closed interval $I_D[x,y]$ consists of all vertices lying on some $x-y$ detour of G, while for $S \subseteq V$, $I_D[S] = \bigcup_{x,y\in S} I_D[x,y]$. A set S of vertices is a *detour set* if $I_D[S] = V$, and the minimum cardinality of a detour set is the *detour number* $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set*. The detour number of a graph was introduced in [4] and further studied in [5].

The concept of vertex detour number was introduced by Santhakumaran and Titus in [10]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-detour set* if each vertex v of G lies on an $x-y$ detour in G for some element y in S. The minimum cardinality of an x-detour set of G is defined as the *x-detour number* of G and is denoted by $d_x(G)$. An x-detour set of cardinality $d_x(G)$ is called a *d_x-set* of G. A vertex v in a graph G is an *x-detour vertex* if v belongs to every minimum x-detour set of G. The connected x-detour number was introduced and studied by Santhakumaran and Titus in [12]. A connected x-detour set of G is an x-detour set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-detour set of G is the *connected x-detour number* of G and is denoted by $cd_x(G)$. A connected x-detour set of cardinality $cd_x(G)$ is called a *cd_x-set* of G.

For the graph G given in Fig.1.1, the minimum vertex detour sets, the vertex detour numbers, the minimum connected vertex detour sets and the connected vertex detour numbers are given in Table 1.1. An elaborate study of results in vertex detour number with several interesting applications is given in [10].

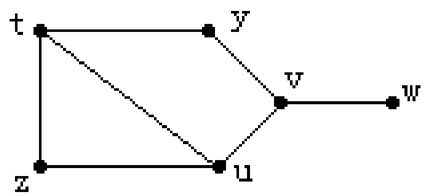

Fig.1.1

Vertex x	d_x-sets	$d_x(G)$	cd_x-sets	$cd_x(G)$
t	$\{y,w\},\{z,w\},\{u,w\}$	2	$\{y,v,w\},\{u,v,w\}$	3
y	$\{w\}$	1	$\{w\}$	1
z	$\{w\}$	1	$\{w\}$	1
u	$\{w\}$	1	$\{w\}$	1
v	$\{y,w\},\{z,w\},\{u,w\}$	2	$\{y,v,w\},\{u,v,w\}$	3
w	$\{y\},\{z\},\{u\}$	1	$\{y\},\{z\},\{u\}$	1

Table 1.1

The following theorems will be used in the sequel.

Theorem 1.1([10]) *Let x be any vertex of a connected graph G.*

(i) Every end-vertex of G other than the vertex x (whether x is end-vertex or not) belongs to every x-detour set.

(ii) No cutvertex of G belongs to any d_x-set.

Theorem 1.2([10]) *Let G be a connected graph with cut vertices and let S_x be an x-detour set of G. Then every branch of G contains an element of $S_x \bigcup \{x\}$.*

Theorem 1.3([10]) *If G is a connected graph with k end-blocks, then $d_x(G) \geq k-1$ for every vertex x in G. In particular, if x is a cut vertex of G, then $d_x(G) \geq k$.*

Theorem 1.4([10]) *Let T be a tree with number of end-vertices t. Then $d_x(T) = t-1$ or $d_x(T) = t$ according as x is an end-vertex or not. In fact, if W is the set of all end-vertices of T, then $W - \{x\}$ is the unique d_x-set of T.*

Theorem 1.5([10]) *If G is the complete graph $K_n (n \geq 2)$, the n-cube $Q_n (n \geq 2)$, the cycle $C_n (n \geq 3)$, the wheel $W_n = K_1 + C_{n-1} (n \geq 4)$ or the complete bipartite graph $K_{m,n} (m, n \geq 2)$, then $d_x(G) = 1$ for every vertex x in G.*

Throughout the following G denotes a connected graph with at least two vertices.

§2. Minimal Vertex Detour Sets in a Graph

Definition 2.1 *Let x be any vertex of a connected graph G. An x-detour set S_x is called a minimal x-detour set if no proper subset of S_x is an x-detour set. The upper x-detour number, denoted by $d_x^+(G)$, is defined as the maximum cardinality of a minimal x-detour set of G.*

It is clear from the definition that for any vertex x in G, x does not belong to any minimal x-detour set of G.

Example 2.2 *For the graph G given in Fig.2.1, the minimum vertex detour sets, the minimum vertex detour numbers, the minimal vertex detour sets and the upper vertex detour numbers are given in Table 2.1.*

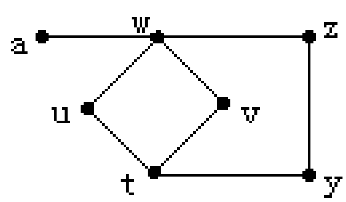

Fig.2.1

Vertex x	Minimum x-detour sets	$d_x(G)$	Minimal x-detour sets	$d_x^+(G)$
t	$\{a, y\}, \{a, z\}$	2	$\{a, u, v\}, \{a, y\}, \{a, z\}$	3
y	$\{a, t\}, \{a, z\}, \{a, u\}, \{a, v\}$	2	$\{a, t\}, \{a, z\}, \{a, u\}, \{a, v\}$	2
z	$\{a\}$	1	$\{a\}$	1
u	$\{a, y\}, \{a, z\}, \{a, v\}$	2	$\{a, y\}, \{a, z\}, \{a, v\}$	2
v	$\{a, y\}, \{a, z\}, \{a, u\}$	2	$\{a, y\}, \{a, z\}, \{a, u\}$	2
w	$\{a, z\}$	2	$\{a, z\}, \{a, t, y\}, \{a, y, u\}, \{a, y, v\}, \{a, u, v\}$	3
a	$\{z\}$	1	$\{z\}, \{t, y\}, \{y, u\}, \{y, v\}, \{u, v\}$	2

Table 2.1

Note 2.3 For any vertex x in a connected graph G, every minimum x-detour set is a minimal x-detour set, but the converse is not true. For the graph G given in Figure 2.1, $\{a, u, v\}$ is a minimal t-detour set but it is not a minimum t-detour set of G.

Theorem 2.4 *Let x be any vertex of a connected graph G.*

(i) Every end-vertex of G other than the vertex x (whether x is end-vertex or not) belongs to every minimal x-detour set.

(ii) No cut vertex of G belongs to any minimal x-detour set.

Proof (i) Let x be any vertex of G. Since x does not belong to any minimal x-detour set, let $v \neq x$ be an end-vertex of G. Then v is the terminal vertex of an $x-v$ detour and v is not an internal vertex of any detour so that v belongs to every minimal x-detour set of G.

(ii) Let $y \neq x$ be a cut vertex of G. Let U and W be two components of $G-\{y\}$. For any vertex x in G, let S_x be a minimal x-detour set of G. Suppose that $x \in U$. Now, suppose that $S_x \bigcap W = \emptyset$. Let $w_1 \in W$. Then $w_1 \notin S_x$. Since S_x is an x-detour set, there exists an element z in S_x such that w_1 lies in some $x-z$ detour $P: x = z_0, z_1, ..., w_1, ..., z_n = z$ in G. Since $S_x \bigcap W = \emptyset$ and y is a cut vertex of G, it follows that the $x-w_1$ subpath of P and the w_1-z subpath of P both contain y so that P is not a path in G. Hence $S_x \bigcap W \neq \emptyset$. Let $w_2 \in S_x \bigcap W$. Then $w_2 \neq y$ so that y is an internal vertex of an $x-w_2$ detour. If $y \in S_x$, let $S = S_x - \{y\}$. It is clear that every vertex that lies on an $x-y$ detour also lies on an $x-w_2$ detour. Hence it follows that S is an x-detour set of G, which is a contradiction to S_x a minimal x-detour set of G. Thus y does not belong to any minimal x-detour set of G. Similarly if $x \in W$, then y does not belong to any minimal x-detour set of G. □

The following theorem is an easy consequence of the definitions of the minimum vertex detour number and the upper vertex detour number of a graph.

Theorem 2.5 *For any non-trivial tree T with k end vertices, $d_x(T) = d_x^+(T) = k$ or $k-1$ according as x is a cut vertex or not.*

(*ii*) *For any vertex x in the complete graph K_p, $d_x(K_p) = d_x^+(K_p) = 1$.*

(*iii*) *For any vertex x in the complete bipartite graph $K_{m,n}$, $d_x(K_{m,n}) = d_x^+(K_{m,n}) = 1$ if $m, n \geq 2$.*

(*iv*) *For any vertex x in the wheel W_p, $d_x(W_p) = d_x^+(W_p) = 1$.* □

Theorem 2.6 *For any vertex x in G, $1 \leq d_x(G) \leq d_x^+(G) \leq p-1$.*

Proof It is clear from the definition of minimum x-detour set that $d_x(G) \geq 1$. Since every minimum x-detour set is a minimal x-detour set, $d_x(G) \leq d_x^+(G)$. Also, since the vertex x does not belong to any minimal x-detour set, it follows that $d_x^+(G) \leq p-1$. □

Remark 2.7 For the complete graph K_p, $d_x(K_p) = 1$ for every vertex x in K_p. For the graph G given in Figure 2.1, $d_y(G) = d_y^+(G)$. Also, for the graph $K_2, d_x^+(K_2) = p-1$ for every vertex x in K_2. All the inequalities in Theorem 2.6 can be strict. For the graph G given in Figure 2.1, $d_w(G) = 2$, $d_w^+(G) = 3$ and $p = 7$ so that $1 < d_w(G) < d_w^+(G) < p-1$.

Theorem 2.8 *For every pair a, b of integers with $1 \leq a \leq b$, there is a connected graph G with $d_x(G) = a$ and $d_x^+(G) = b$ for some vertex x in G.*

Proof For $a = b = 1$, $K_p(p \geq 2)$ has the desired properties. For $a = b$ with $b \geq 2$, let G be any tree of order $p \geq 3$ with b end-vertices. Then by Theorem 2.5(i), $d_x(G) = d_x^+(G) = b$ for any cut vertex x in G. Assume that $1 \leq a < b$. Let $F = (K_2 \bigcup (b-a+2)K_1)) + \overline{K}_2$, where let $Z = V(K_2) = \{z_1, z_2\}$, $Y = V((b-a+2)K_1) = \{x, y_1, y_2, ..., y_{b-a+1}\}$ and $U = V(\overline{K}_2) = \{u_1, u_2\}$. Let G be the graph obtained from F by adding $a-1$ new vertices $w_1, w_2, ..., w_{a-1}$ and joining each w_i to x. The graph G is shown in Fig.2.2. Let $W = \{w_1, w_2, ..., w_{a-1}\}$ be the set of end vertices of G.

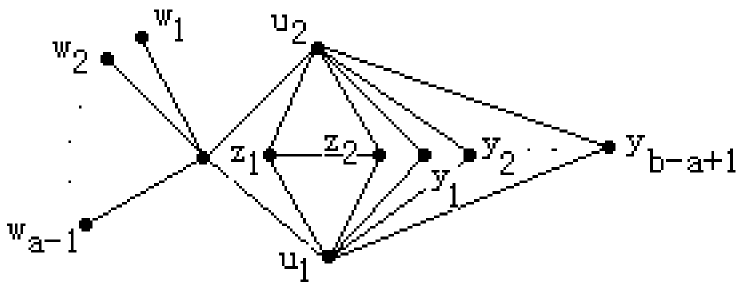

Fig.2.2

First, we show that $d_x(G) = a$ for the vertex x in G. By Theorem 1.3, $d_x(G) \geq a$. On the other hand, let $S = \{w_1, w_2, ..., w_{a-1}, z_1\}$. Then $D(x, z_1) = 5$ and each vertex of F lies on an $x - z_1$ detour. Hence S is an x-detour set of G and so $d_x(G) \leq |S| = a$. Therefore, $d_x(G) = a$. Also, we observe that a minimum x-detour set of G is formed by taking all the end vertices and exactly one vertex from Z.

Next, we show that $d_x^+(G) = b$. Let $M = \{w_1, w_2, ..., w_{a-1}, y_1, y_2, ..., y_{b-a+1}\}$. It is clear that M is an x-detour set of G. We claim that M is a minimal x-detour set of G. Assume, to the contrary, that M is not a minimal x-detour set. Then there is a proper subset T of M such that T is an x-detour set of G. Let $s \in M$ and $s \notin T$. By Theorem 1.1(i), clearly $s = y_i$, for some $i = 1, 2, ..., b - a + 1$. For convenience, let $s = y_1$. Since y_1 does not lie on any $x - y_j$ detour where $j = 2, 3, ..., b - a + 1$, it follows that T is not an x-detour set of G, which is a contradiction. Thus M is a minimal x-detour set of G and so $d_x^+(G) \geq |M| = b$.

Now we prove that $d_x^+(G) = b$. Suppose that $d_x^+(G) > b$. Let N be a minimal x-detour set of G with $|N| > b$. Then there exists at least one vertex, say $v \in N$ such that $v \notin M$. Thus $v \in \{u_1, u_2, z_1, z_2\}$.

Case 1. $v \in \{z_1, z_2\}$, say $v = z_1$. Clearly $W \bigcup \{z_1\}$ is an x-detour set of G and also it is a proper subset of N, which is a contradiction to N a minimal x-detour set of G.

Case 2. $v \in \{u_1, u_2\}$, say $v = u_1$. Suppose $u_2 \notin N$. Then there is at least one y in Y such that $y \in N$. Clearly, $D(x, u_1) = 4$ and the only vertices of any $x - u_1$ detour are x, z_1, z_2, u_1 and u_2. Also x, u_2, z_1, z_2, u_1, y is an $x - y$ detour and hence $N - \{u_1\}$ is an x-detour set, which is a contradiction to N a minimal x-detour set of G. Suppose $u_2 \in N$. It is clear that the only vertices of any $x - u_1$ or $x - u_2$ detour are x, u_1, u_2, z_1 and z_2. Since $u_1, u_2 \in N$, it follows that both $N - \{u_1\}$ and $N - \{u_2\}$ are x-detour sets, which is a contradiction to N a minimal x-detour set of G.

Thus there is no minimal x-detour set N of G with $|N| > b$. Hence $d_x^+(G) = b$. □

Remark 2.9 The graph G of Figure 2.2 contains exactly three minimal x-detour sets, namely $W \bigcup \{z_1\}$, $W \bigcup \{z_2\}$ and $W \bigcup (Y - \{x\})$. This example shows that there is no "Intermediate Value Theorem" for minimal x-detour sets, that is, if n is an integer such that $d_x(G) < n < d_x^+(G)$, then there need not exist a minimal x-detour set of cardinality n in G.

Theorem 2.10 *For any three positive integers a, b and n with $a \geq 2$ and $a \leq n \leq b$, there exists a connected graph G with $d_x(G) = a$, $d_x^+(G) = b$ and a minimal x-detour set of cardinality n.*

Proof We consider four cases.

Case 1. Suppose $a = n = b$.

Let G be any tree of order $p \geq 3$ with a end vertices. Then by Theorem 2.5(i), $d_x(G) = d_x^+(G) = a$ for any cut vertex x in G and the set of all end vertices in G is a minimal x-detour set with cardinality n by Theorem 2.4.

Case 2. Suppose $a = n < b$. For the graph G given in Figure 2.2 of Theorem 2.8, it is proved that $d_x(G) = a$, $d_x^+(G) = b$ and $S = \{w_1, w_2, ..., w_{a-1}, z_1\}$ is a minimal x-detour set of cardinality n.

Case 3. Suppose $a < n = b$. For the graph G given in Figure 2.2 of Theorem 2.8, it is proved that $d_x(G) = a$, $d_x^+(G) = b$ and $S = \{w_1, w_2, ..., w_{a-1}, y_1, y_2, ..., y_{b-a+1}\}$ is a minimal x-detour set of cardinality n.

Case 4. Suppose $a < n < b$. Let $l = n - a + 1$ and $m = b - n + 1$.

Let $F_1 = (K_2 \bigcup lK_1) + \overline{K}_2$, where let $Z_1 = V(K_2) = \{z_1, z_2\}$, $Y_1 = V(lK_1) = \{y_1, y_2, ..., y_l\}$ and $U_1 = V(\overline{K}_2) = \{u_1, u_2\}$. Similarly let $F_2 = (K_2 \bigcup mK_1) + \overline{K}_2$, where let $Z_2 = V(K_2) = \{z_3, z_4\}$, $Y_2 = V(mK_1) = \{x_1, x_2, ..., x_m\}$ and $U_2 = V(\overline{K}_2) = \{u_3, v_4\}$. Let $K_{1,a-2}$ be the star at the vertex x and let $W = \{w_1, w_2, ..., w_{a-2}\}$ be the set of end vertices of $K_{1,a-2}$. Let G be the graph obtained from $K_{1,a-2}$, F_1 and F_2 by joining the vertex x of $K_{1,a-2}$ to the elements of U_1 and U_2. The graph G is shown in Fig.2.3. It follows from Theorem 2.4(i) that for the vertex x, W is a subset of every minimal x-detour set of G.

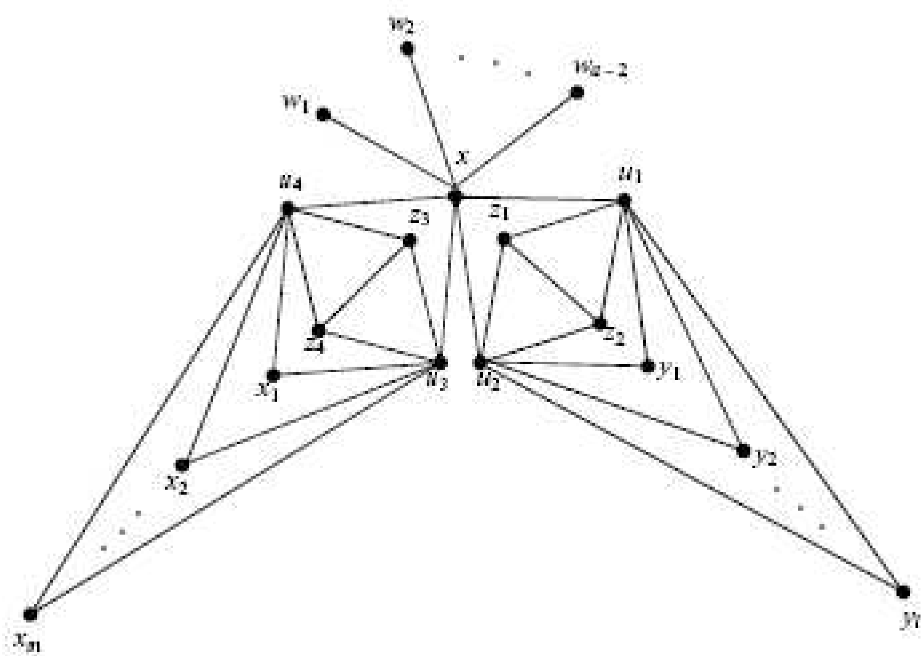

Fig.2.3

First, we show that $d_x(G) = a$ for the vertex x in G. By Theorem 1.3, $d_x(G) \geq a$. On the other hand, let $S = \{w_1, w_2, ..., w_{a-2}, z_1, z_3\}$. Then $D(x, z_1) = 5$ and each vertex of F_1 lies on an $x - z_1$ detour. Similarly, $D(x, z_3) = 5$ and each vertex of F_2 lies on an $x - z_3$ detour. Hence S is an x-detour set of G and so $d_x(G) \leq |S| = a$. Therefore, $d_x(G) = a$.

Next, we show that $d_x^+(G) = b$. Let $M = W \bigcup Y_1 \bigcup Y_2$. It is clear that M is an x-detour set of G. We claim that M is a minimal x-detour set of G. Assume, to the contrary, that M is not a minimal x-detour set. Then there is a proper subset T of M such that T is an x-detour set of G. Let $s \in M$ and $s \notin T$. By Theorem 1.1(i), clearly $s \in Y_1 \bigcup Y_2$. For convenience, let $s = y_1$. Since y_1 does not lie on any $x - y_j$ detour, where $j = 2, 3, ..., l$ and y_1 does not lie on any $x - x_j$ detour, where $j = 1, 2, ..., m$, it follows that T is not an x-detour set of G, which is a contradiction. Thus M is a minimal x-detour set of G and so $d_x^+(G) \geq |M| = b$.

Now, we prove that $d_x^+(G) = b$. Suppose that $d_x^+(G) > b$. Let N be a minimal x-detour set of G with $|N| > b$. Then there exists at least one vertex, say $v \in N$ such that $v \notin M$. Thus, $v \in \{u_1, u_2, u_3, u_4, z_1, z_2, z_3, z_4\}$.

Subcase 1. Suppose $v \in \{z_1, z_2\}$, say $v = z_1$. Clearly, every vertex of F_1 lies on an $x - z_1$ detour and so $(N - V(F_1)) \bigcup \{v\}$ is an x-detour set of G and it is a proper subset of N, which is a contradiction to N a minimal x-detour set of G.

Subcase 2. Suppose $v \in \{z_3, z_4\}$. It is similar to Subcase 1.

Subcase 3. Suppose $v \in \{u_1, u_2\}$, say $v = u_1$. Suppose $u_2 \notin N$. Then there is at least one element y in Y_1 such that $y \in N$. Clearly, $D(x, u_1) = 4$ and the only vertices of any $x - u_1$ detour are x, z_1, z_2, u_1 and u_2. Also x, u_2, z_1, z_2, u_1, y is an $x - y$ detour and hence $N - \{u_1\}$ is an x-detour set, which is a contradiction to N a minimal x-detour set of G. Suppose $u_2 \in N$. It is clear that the only vertices of any $x - u_1$ or $x - u_2$ detour are x, u_1, u_2, z_1 and z_2. Since $u_1, u_2 \in N$, it follows that both $N - \{u_1\}$ and $N - \{u_2\}$ are x-detour sets, which is a contradiction to N a minimal x-detour set of G.

Subcase 4. Suppose $v \in \{u_3, u_4\}$. It is similar to Subcase 3.

Thus there is no minimal x-detour set N of G with $|N| > b$. Hence $d_x^+(G) = b$.

Now, we show that there is a minimal x-detour set of cardinality n. Let $S = \{w_1, w_2, ..., w_{a-2}, z_3, y_1, y_2, ..., y_l\}$. It is clear that S is an x-detour set of G. We claim that S is a minimal x-detour set of G. Assume, to the contrary, that S is not a minimal x-detour set. Then there is a proper subset T of S such that T is an x-detour set of G. Let $s \in S$ and $s \notin T$. By Theorem 1.1(i) and Theorem 1.2, clearly $s = y_i$ for some $i = 1, 2, ..., l$. For convenience, let $s = y_1$. Since y_1 does not lie on any $x - y_j$ detour where $j = 2, 3, ..., l$, it follows that T is not an x-detour set of G, which is a contradiction. Thus S is a minimal x-detour set of G with cardinality $|S| = n$. Hence we obtain the theorem. □

§3. Vertex Forcing Subsets in Vertex Detour Sets of a Graph

Let x be any vertex of a connected graph G. Although G contains a minimum x-detour set there are connected graphs which may contain more than one minimum x-detour set. For example the graph G given in Fig. 2.1 contains more than one minimum x-detour set. For each minimum x-detour set S_x in a connected graph G there is always some subset T of S_x that uniquely determines S_x as the minimum x-detour set containing T. Such sets are called "vertex forcing subsets" and we discuss these sets in this section.

Definition 3.1 *Let x be any vertex of a connected graph G and let S_x be a minimum x-detour set of G. A subset $T \subseteq S_x$ is called an x-forcing subset for S_x if S_x is the unique minimum x-detour set containing T. An x-forcing subset for x of minimum cardinality is a minimum x-forcing subset of S_x. The forcing x-detour number of S_x, denoted by $f_{dx}(S_x)$, is the cardinality of a minimum x-forcing subset for S_x. The forcing x-detour number of G is $f_{dx}(G) = min\ \{f_{dx}(S_x)\}$, where the minimum is taken over all minimum x-detour sets S_x in G.*

Example 3.2 For the graph G given in Figure 1.1, the minimum x-detour sets, the x-detour numbers and the forcing x-detour numbers for every vertex x in G are given in Table 3.1.

Vertex x	Minimum x-detour sets	x-detour number	Forcing x-detour number
t	$\{y, w\}, \{z, w\}, \{u, w\}$	2	1
y	$\{w\}$	1	0
z	$\{w\}$	1	0
u	$\{w\}$	1	0
v	$\{y, w\}, \{z, w\}, \{u, w\}$	2	1
w	$\{y\}, \{z\}, \{u\}$	1	1

Table 3.1

Theorem 3.3 *any vertex x in a connected graph G, $0 \leq f_{dx}(G) \leq d_x(G)$.*

Proof Let x be any vertex of G. It is clear from the definition of $f_{dx}(G)$ that $f_{dx}(G) \geq 0$. Let S_x be any minimum x-detour set of G. Since $f_{dx}(S_x) \leq d_x(G)$ and since $f_{dx}(G) =$ min $\{f_{dx}(S_x) : S_x$ is a minimum x-detour set in $G\}$, it follows that $f_{dx}(G) \leq d_x(G)$. Thus $0 \leq f_{dx}(G) \leq d_x(G)$. □

Remark 3.4 The bounds in Theorem 3.3 are sharp. For the graph G given in Figure 1.1, $f_{dy}(G) = 0$ and $f_{dw}(G) = d_w(G) = 1$. Also, the inequality in Theorem 3.3 can be strict. For the same graph G given in Figure 1.1, $0 < f_{dv}(G) < d_v(G)$.

The following theorem characterizes those graphs G having $f_{dx}(G) = 0$, $f_{dx}(G) = 1$ or $f_{dx}(G) = d_x(G)$. Since the proof of this theorem is straight forward, we omit it.

Theorem 3.5 *Let x be any vertex of a graph G. Then*

(i) $f_{dx}(G) = 0$ if and only if G has a unique minimum x-detour set.

(ii) $f_{dx}(G) = 1$ if and only if G has at least two minimum x-detour sets, one of which is a unique minimum x-detour set containing one of its elements.

(iii) $f_{dx}(G) = d_x(G)$ if and only if no minimum x-detour set of G is the unique minimum x-detour set containing any of its proper subsets.

Theorem 3.6 *Let x be any vertex of a connected graph G and let S_x be any minimum x-detour set of G. Then*

(*i*) *no cut vertex of G belongs to any minimum x-forcing subset of S_x.*

(*ii*) *no x-detour vertex of G belongs to any minimum x-forcing subset of S_x.*

Proof Let x be any vertex of a connected graph G and let S_x be any minimum x-detour set of G.

(i) Since any minimum x-forcing subset of S_x is a subset of S_x, the result follows from Theorem 1.1(ii).

(ii) Let v be an x-detour vertex of G. Then v belongs to every minimum x-detour set of G. Let $T \subseteq S_x$ be any minimum x-forcing subset for any minimum x-detour set S_x of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S_x is the unique minimum x-detour set containing T' so that T' is an x-forcing subset for S_x with $|T'| < |T|$, which is a contradiction to T a minimum x-forcing subset for S_x. □

Corollary 3.7 *Let x be any vertex of a connected graph G. If G contains k end-vertices, then $f_{dx}(G) \leq d_x(G) - k + 1$.*

Proof This follows from Theorem 1.1(i) and Theorem 3.6(ii). □

Remark 3.8 The bound for $f_{dx}(G)$ in Corollary 3.7 is sharp. For a non-trivial tree T with k end vertices, $f_{dx}(T) = 0 = d_x(T) - k + 1$ for any end vertex x in T.

Theorem 3.9 (*i*)*If T is a non-trivial tree, then $f_{dx}(T) = 0$ for every vertex x in T.*

(*ii*) *If G is the complete graph K_n ($n \geq 3$), the n-cube Q_n ($n \geq 2$), the cycle C_n ($n \geq 3$), the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 4$) or the complete bipartite graph $K_{m,n}$ ($m, n \geq 2$), then $f_{dx}(G) = d_x(G) = 1$ for every vertex x in G.*

Proof (i) This follows from Theorem 1.4 and Theorem 3.5(i).

(ii) For each of the graphs in (ii) it is easily seen that there is more than one minimum x-detour set for any vertex x. Hence it follows from Theorem 3.5(i) that $f_{dx}(G) \neq 0$ for each of the graphs. Also, by Theorem 3.3, $f_{dx}(G) \leq d_x(G)$. Now it follows from Theorem 1.5 that $f_{dx}(G) = d_x(G) = 1$ for each of the graphs. □

Theorem 3.10 *For any vertex x in a connected graph G, $0 \leq f_{dx}(G) \leq d_x(G) \leq d_x^+(G)$.*

Proof This follows from Theorems 2.6 and 3.3. □

The following theorem gives a realization for the parameters $f_{dx}(G)$, $d_x(G)$ and $d_x^+(G)$.

Theorem 3.11 *For any three positive integers a, b and c with $2 \leq a \leq b \leq c$, there exists a connected graph G with $f_{dx}(G) = a$, $d_x(G) = b$ and $d_x^+(G) = c$ for some vertex x in G.*

Proof For each integer i with $1 \leq i \leq a-1$, let F_i be a copy of K_2, where v_i and v_i' are the vertices of F_i. Let $K_{1,b-a}$ be the star at the vertex x and let $U = \{u_1, u_2, ..., u_{b-a}\}$ be the set of end vertices of $K_{1,b-a}$. Let H be the graph obtained from $K_{1,b-a}$ by joining the vertex x to the vertices of F_i $(1 \le i \le a-1)$. Let $K = (K_2 \bigcup (c-b+1)K_1) + \overline{K}_2$, where $Z = V(K_2) = \{z_1, z_2\}$, $Y = V((c-b+1)K_1) = \{y_1, y_2, ..., y_{c-b+1}\}$ and $X = V(\overline{K}_2) = \{x_1, x_2\}$. Let G be the graph obtained from H and K by joining x with x_1 and x_2. The graph G is shown in Fig.3.1.

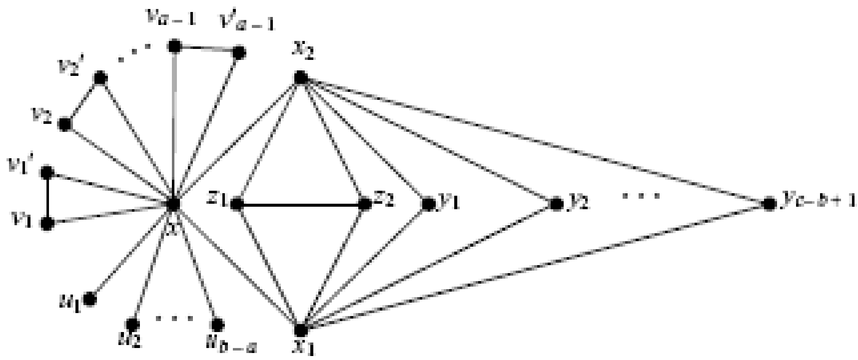

Fig.3.1

Step I. First, we show that $d_x(G) = b$ for the vertex x in G. By Theorem 1.3, $d_x(G) \ge b$. On the other hand, if $c-b+1 > 1$, let $S = \{u_1, u_2, ..., u_{b-a}, v_1, v_2, ..., v_{a-1}, z_1\}$ be the set formed by taking all the end vertices and exactly one vertex from each F_i and Z, and if $c-b+1 = 1$, let $S = \{u_1, u_2, ..., u_{b-a}, v_1, v_2, ..., v_{a-1}, z_1\}$ be the set formed by taking all the end vertices and exactly one vertex from each F_i and $Z \bigcup \{y_1\}$. Then $D(x, z_1) = 5$ and each vertex of K lies on an $x - z_1$ detour and each vertex of F_i lies on an $x - v_i$ detour. Hence S is an x-detour set of G and so $d_x(G) \le |S| = b$. Therefore, $d_x(G) = b$.

Step II. Now, we show that $f_{dx}(G) = a$. Since every minimum x-detour set of G contains U, exactly one vertex from each F_i $(1 \le i \le a-1)$ and one vertex from Z or $Z \bigcup \{y_1\}$ according as $c > b$ or $c = b$ respectively, let $S = \{u_1, u_2, ..., u_{b-a}, v_1, v_2, ..., v_{a-1}, z_1\}$ be a minimum x-detour set of G and let $T \subseteq S$ be any minimum x-forcing subset of S. Then by Theorem 3.6(ii), $T \subseteq S - U$. Therefore, $|T| \le a$. If $|T| < a$, then there is a vertex $y \in S - U$ such that $y \notin T$. Now there are two cases.

Case 1. Let $y \in \{v_1, v_2, ..., v_{a-1}\}$, say $y = v_1$. Let $S' = (S - \{v_1\}) \bigcup \{v_1'\}$, where v_1' be the vertex of F_1 other than v_1. Then $S' \ne S$ and S' is also a minimum x-detour set of G such that it contains T, which is a contraction to T an x-forcing subset of S.

Case 2. Let $y = z_1$. Then exactly similar to Case 1 we see that $|T| < a$ is not possible. Thus $|T| = a$ and so $f_{dx}(G) = a$.

Step III. Next, we show that $d_x^+(G) = c$. Let $M = \{u_1, u_2, ..., u_{b-a}, v_1, v_2, ..., v_{a-1}, y_1, y_2, .., y_{c-b+1}\}$. It is clear that M is an x-detour set of G. We claim that M is a minimal x-detour set of G. Assume, to the contrary, that M is not a minimal x-detour set. Then there is a proper subset T of M such that T is an x-detour set of G. Let $s \in M$ and $s \notin T$. By Theorem 1.2, clearly $s = y_i$ for some $i = 1, 2, ..., c-b+1$. For convenience, let $s = y_1$. Since y_1 does not lie on any $x - y_j$ detour where $j = 2, 3, ..., c-b+1$, it follows that T is not an x-detour set of G, which is a contradiction. Thus M is a minimal x-detour set of G and so $d_x^+(G) \ge |M| = c$. Now suppose $d_x^+(G) > c$. Let N be a minimal x-detour set of G with $|N| > c$. Then at least one vertex $w \in N$ such that $w \notin M$. It is clear that every minimal x-detour set contains exactly one vertex from each F_i. Then by Theorem 2.4(i), $w \in \{x_1, x_2, z_1, z_2\}$.

Case 1. Let $w \in \{z_1, z_2\}$, say $w = z_1$. Since every vertex of K lies on an $x - z_1$ detour we have $(N - V(K)) \bigcup \{z_1\}$ is an x-detour set and it is a proper subset of N, which is a contradiction to N a minimal x-detour set of G.

Case 2. Let $w \in \{x_1, x_2\}$, say $w = x_1$. Suppose $x_2 \notin N$. Then there is at least one y in Y such that $y \in N$. Clearly, $D(x, x_1) = 4$ and the only vertices of any $x - x_1$ detour are x, z_1, z_2, x_1 and x_2. Also x, x_2, z_1, z_2, x_1, y is an $x - y$ detour and hence $N - \{x_1\}$ is an x-detour set, which is a contradiction to N a minimal x-detour set of G. Suppose $x_2 \in N$. It is clear that the only vertices of any $x - x_1$ or $x - x_2$ detour are x, z_1, z_2, x_1 and x_2. Since $x_1, x_2 \in N$, it follows that both $N - \{x_1\}$ and $N - \{x_2\}$ are x-detour sets, which is a contradiction to N a minimal x-detour set of G.

Thus there is no minimal x-detour set N of G with $|N| > c$. Hence $d_x^+(G) = c$. □

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