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The Characterization of Symmetric Primitive Matrices with Exponent $n-2$

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Abstract: In this paper the symmetric primitive matrices of order n with exponent $n-2$ are completely characterized by applying a combinatorial approach, i.e., mathematical combinatorics ([7]).

Key words: primitive matrix, primitive exponent, graph.

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§1. Introduction

An $n \times n$ nonnegative matrix $\boldsymbol{A} = (a_{ij})$ is said to be *primitive* if $\boldsymbol{A}^k > 0$ for some positive integer k. The least such k is called the *exponent* of the matrix $\boldsymbol{A}$ and is denoted by $\gamma(\boldsymbol{A})$.

Suppose that $SE_n = \{\gamma(\boldsymbol{A}) : \boldsymbol{A}$ is a symmetric and primitive $n \times n$ matrix$\}$ be the exponent set of $n \times n$ symmetric primitive matrices. In 1986, J.Y.Shao[1] proved $SE_n = \{1, 2, \cdots, 2n-2\} \backslash S$, where S is the set of all odd numbers among $\{n, n+1, \cdots, 2n-2\}$ and gave the characterization of the matrix with exponent $2n-2$. In 1990, B.L.Liu et al[2] gave the characterization of the matrix with exponent $2n-4$. In 1991, G.R.Li et al[3] obtained the characterization with exponent $2n-6$. In 1995, J.L.Cai et al[4] derived the complete characterization of symmetric primitive matrices with exponent $2n-2r(\geqslant n)$ which is a generalization of the results in $[1,2,3]$, where $r = 1, 2, 3$, respectively. In 2003, J.L.Cai et al[5] derived the complete characterization of symmetric primitive matrices with exponent $n-1$. However, there are no results regarding the characterization of symmetric primitive matrices of exponent $n-1$. The purpose of this paper is to go further into the problem and give the complete characterization of symmetric primitive matrices with exponent $n-2$ by applying a combinatorial approach, i.e., mathematical combinatorics ([7]).

The associated graph of *symmetric matrix* $\boldsymbol{A}$, denoted by $G(\boldsymbol{A})$, is a graph with a vertex set $V(G(\boldsymbol{A})) = \{1, 2, \cdots, n\}$ such that there is an edge from i to j in graph $G(\boldsymbol{A})$ if and only if $a_{ij} > 0$. Hence $G(\boldsymbol{A})$ may contain loops if $a_{ii} > 0$ for some i. A graph G is called to be

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primitive if there exists an integer $k > 0$ such that for all ordered pairs of vertices $i, j \in V(G)$ (not necessarily distinct), there is a walk from i to j with length k. The least such k is called the *exponent* of G, denoted by $\gamma(G)$. Clearly, a symmetric matrix $\boldsymbol{A}$ is primitive if and only if its associated graph $G(\boldsymbol{A})$ is primitive. And in this case, we have $\gamma(\boldsymbol{A}) = \gamma(G(\boldsymbol{A}))$. By this reason as above, we shall employ graph theory as a major tool and consider $\gamma(G(\boldsymbol{A}))$ to prove our results.

Terminologies and notations not explained in this paper are referred to the reference [6].

§2. Some Lemmas

In the following, we need the conception of the local exponent, i.e., the exponent from vertex u to vertex v, denoted by $\gamma(u, v)$, is the least integer k such that there exists a walk of length m from u to v for all $m \geqslant k$. We denote $\gamma(u, u)$ by $\gamma(u)$ for convenience.

Lemma 2.1([1]) *A undirected graph G is primitive if and only if G is connected and has odd cycles.*

Lemma 2.2([1]) *If G is a primitive graph, then*

$$\gamma(G) = \max_{u,v \in V(G)} \gamma(u, v).$$

Lemma 2.3([2]) *Let G be a primitive graph, and let $u, v \in V(G)$. If there are two walks from u to v with lengths k_1 and k_2, respectively, where $k_1 + k_2 \equiv 1 (\mathrm{mod}\ 2)$, then*

$$\gamma(u, v) \leqslant \max\{k_1, k_2\} - 1.$$

Suppose that $P_{\min}(u, v)$ is a shortest path between u and v in G with the length $d_G(u, v) = |P_{\min}(u, v)|$, called the distance between u and v in G. The *diameter* of G is defined as

$$\mathrm{diam}(G) = \max_{u,v \in V(G)} d_G(u, v).$$

Suppose that $P_{\min}(G_1, G_2)$ is a shortest path between subgraphs G_1 and G_2 of G with the length $d_G(G_1, G_2) = |P_{\min}(G_1, G_2)|$, called the distance between G_1 and G_2 in G. It is obvious that

$$d_G(G_1, G_2) = |P_{\min}(G_1, G_2)| = \min\{|P_{\min}(u, v)| \mid u \in V(G_1), \quad v \in V(G_2)\}.$$

Let u and v be two vertices in G,an (u, v)-walk is said to be a *different walk* if the length of the walk and the distance between u and v have different parity. A shortest different walk is said to be a *primitive walk* , denoted by $W_{\mathrm{rim}}(u, v)$ and its length by $b_G(u, v)$ or simply by $b(u, v)$.

Clearly, for any two vertices u and v in G, we have

$$d_G(u, v) < b_G(u, v), \quad d_G(u, v) + b_G(u, v) \equiv 1 (\mathrm{mod}\ 2).$$

Lemma 2.4([5]) *Suppose that G is a primitive graph and $u, v \in V(G)$,then we have*

(a) $\gamma(u,v) \geqslant d_G(u,v)$;

(b) $\gamma(u,v) \equiv d_G(u,v) (\text{mod } 2)$;

(c) $\gamma(G) \geqslant \text{diam}(G), \quad \gamma(G) \equiv \text{diam}(G) (\text{mod } 2)$.

Lemma 2.5([5]) *Suppose that G is a primitive graph with order n. If there are $u, v \in V(G)$ such that $\gamma(u,v) = \gamma(G) \leqslant n$, then for any odd cycle C in G we have*

$$|V(P_{\min}(u,v)) \cap V(C)| \leqslant n - \gamma(G),$$

where $P_{\min}(u,v)$ is the shortest path from u to v in G.

Lemma 2.6 *Suppose that G is a primitive graph, $u, v \in V(G)$, then*

$$\gamma(u,v) = b_G(u,v) - 1.$$

Thus

$$\gamma(G) = \max_{u,v \in V(G)} b_G(u,v) - 1.$$

Proof Considering the definitions of $\gamma(u,v)$ and $b_G(u,v)$, there is no any (u,v)-walk with the length of $b_G(u,v) - 2$. So $\gamma(u,v) \geqslant b_G(u,v) - 1$.

On the other hand, for any natural number $k \geqslant b_G(u,v) - 1$, from the shortest path $P_{\min}(u,v)$ we can make a walk of the length k between u and v when $d_G(u,v) - k \equiv 0 (\text{mod } 2)$; from the primitive walk $W_{\text{rim}}(u,v)$ we can make a walk of the length k between u and v when $d_G(u,v) - k \equiv 1 (\text{mod } 2)$. So $\gamma(u,v) \leqslant b_G(u,v) - 1$.

Thus, we have $\gamma(u,v) = b_G(u,v) - 1$. The last result is true by Lemma 2.2. □

According to what is mentioned as above, for arbitrary $u, v \in V(G)$, a different walk of two vertices u and v, denoted by $W(u,v)$, must relate to a cycle C of G. In fact, the symmetric difference $P_{\min}(u,v) \Delta W(u,v)$ of $P_{\min}(u,v)$ and $W(u,v)$ must contain an odd cycle. Conversely, any odd cycle C in G can make a different walk $W(u,v)$ between u and v because of the connectivity of G. So we often write $W(u,v) = W(u,v,C)$. It is clear that for any $u, v \in V(G)$ there must be an odd cycle C' in G such that

$$b_G(u,v) = b_G(u,v,C') = |W_{\text{rim}}(u,v,C')|,$$

then from Lemma 2.6 we have $\gamma(u,v) = \gamma(u,v,C') = b_G(u,v,C') - 1$. The primitive walk can be write as

$$W_{\text{rim}}(u,v) = W_{\text{rim}}(u,v,C') = P_{\min}(u,C') \cup P(C') \cup P_{\min}(C',v),$$

where $P(C')$ is a corresponding segment of the odd cycle C' and

$$\gamma(u,v) = \gamma(u,v,C') = d_G(u,C') + |P(C')| + d_G(C',v) - 1.$$

Moreover, for any odd cycle C in G we have $b_G(u,v) = b_G(u,v,C') \leqslant b_G(u,v,C)$ and $\gamma(u,v) = \gamma(u,v,C') \leqslant \gamma(u,v,C)$. And if there is a vertex $w \in V(C')$ such that $P_{\min}(u,C') =$

$P_{\min}(u,w)$ and $P_{\min}(C',v)=P_{\min}(w,v)$ (i.e., $|P_{\min}(u,C')\cap P_{\min}(C',v)\cap V(C')|$=1), then the odd C' is called a *primitive cycle* between u and v. In this time we have

$$\gamma(u,v)=\gamma(u,v,C')=d_G(u,w)+d_G(w,v)+|C'|-1.$$

Particularly, we put $b(u,C)=b(u,u,C)$, $b(u)=b(u,u)$; $\gamma(u,C)=\gamma(u,u,C)$, $\gamma(u)=\gamma(u,u)$ for convenience.

§3. Constructions of Graphs

Firstly, we define two classes of graphs $\mathcal{M}_{n-2}$ and $\mathcal{N}_{n-2}$ as follows.

(3.1) The set of graphs(*these dashed lines denote possible edges in a graph*)

$$\mathcal{M}_{n-2}=\mathcal{M}^{(1)}_{n-2}\cup\mathcal{M}^{(2)}_{n-2}\cup\mathcal{M}^{(3)}_{n-2}\cup\mathcal{M}^{(4)}_{n-2},$$

where

$\mathcal{M}^{(1)}_{n-2}$: $n=m+2t+2,(t\geqslant 1)$, $0\leqslant i<\frac{m}{2}<j\leqslant m$.

If $\{x_ay_a \mid 1\leqslant a\leqslant t-1\}\cap E(G)\neq\emptyset$, then $j=i+1$ and $m\equiv 1(\bmod 2)$. Otherwise, $j-i<m$

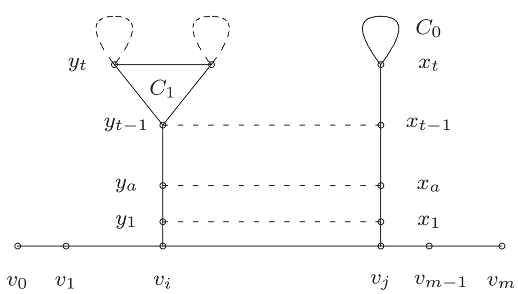

Fig.(1) $\mathcal{M}^{(1)}_{n-2}$

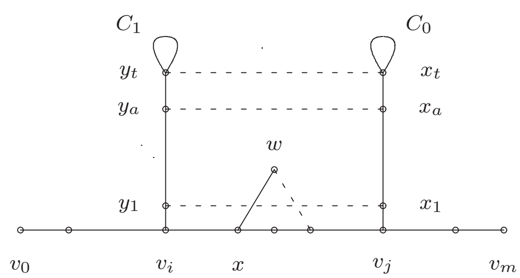

Fig.(2) $\mathcal{M}^{(2)}_{n-2}$

and $m\equiv 0(\bmod 2)$. See Fig.(1).

$\mathcal{M}^{(2)}_{n-2}$: $n=m+2t+2,(t\geqslant 0)$, $0\leqslant i<\frac{m}{2}<j\leqslant m$.

Let $t\geqslant 1$: If $\{x_ky_k \mid 1\leqslant k\leqslant t\}\cap E(G)\neq\emptyset$, then $j=i+1$; $|a-b|=1$ when $\{wx_a,wy_b\}\subseteq E(G)$ $(0\leqslant a,b\leqslant t)$, there may be a loop at w when $a=t$ or $b=t$. If $N_G(w)=\{x,y\}$ but not the case as above, $d(x,y)=2$. If $N_G(w)=\{x\}$ and $d_G(w,P_{uv})\geqslant t$, there may be a loop at w; Otherwise, if $\{wx_a,wy_b\}\subseteq E(G)$ $(0\leqslant a,b\leqslant t)$,then $j=i+1$, $|a-b|=1$ or $j=i+2$, $a=b(m\neq 2)$ and there may be a loop at w when $a=t$ or $b=t$. If $N_G(w)=\{x,y\}$ but not the case as above, $d(x,y)=2$. If $N_G(w)=\{x\}$ and $d_G(w,P_{uv})\geqslant t$, there may be a loop at w. If there is not any loop at w and $x=v_s$, then $i>1$ or $j<m$ when $s=\frac{m}{2}$.

Let $t=0$: There are loops at v_i and v_j $(0\leqslant i<\frac{m}{2}<j\leqslant m)$, respectively, and no loop at w but there is a loop C such that $d_G(w,C)<\frac{m}{2}$. There may be loops at the other vertices. See Fig.(2).

$\mathcal{M}^{(3)}_{n-2}$: $n=m+2t+2,(t\geqslant 1)$,$1\leqslant i+1<\frac{m}{2}<j\leqslant m$.

$j-i<m$ when m is an odd number; $j-i<m-1$ when m is an even number; $j=i+2$ when $\{x_a y_{a+1} \mid 1 \leqslant a \leqslant t\} \cap E(G) \neq \emptyset$. See Fig.(3).

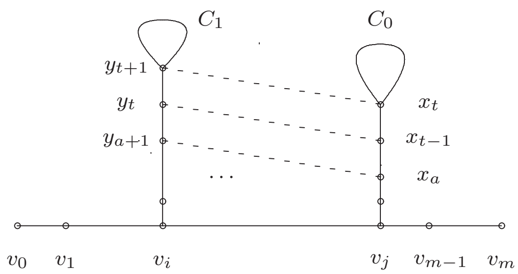

Fig.(3) $\mathcal{M}_{n-2}^{(3)}$

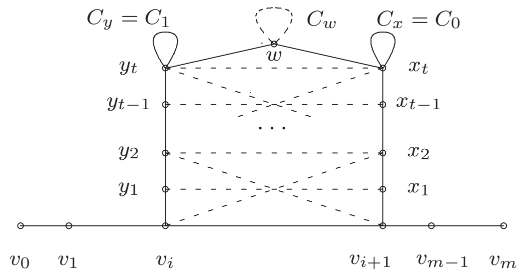

Fig.(4) $\mathcal{M}_{n-2}^{(4)}$

$\mathcal{M}_{n-2}^{(4)}$: $n=m+2t+2,(t \geqslant 0)$.

$t \geqslant 1$: $i=\frac{1}{2}(m-1)$. The set of possible chord edges in $C_{ab}=y_b y_{b+1} \cdots y_t w x_t \cdots x_{a+1} x_a y_b$ is $\{x_a y_b \mid 0 \leqslant a, b \leqslant t, a=b(\neq 0) \text{ or } |a-b|=2\}$. There may be a loop at w and $\{C_{ab} \mid a=b+2\} \cup \{C_y\} \neq \emptyset$, $\{C_{ab} \mid a=b-2\} \cup \{C_x\} \neq \emptyset$.

$t=0$: If $i<\frac{1}{2}(m-1)$, then there are loops at $v_y(\frac{1}{2}(m-1)<y \leqslant m)$. If $i=\frac{1}{2}(m-1)$, then there are loops at v_x and v_y $(0 \leqslant x \leqslant i<y \leqslant m)$. If $i>\frac{1}{2}(m-1)$, then there are loops at $v_x(0 \leqslant x \leqslant \frac{1}{2}(m-1))$, there are loops at the other vertices.Fig.(4).

(3.2) The set of graphs

$$\mathcal{N}_{n-2}=\mathcal{N}_{n-2}^{(1)} \cup \mathcal{N}_{n-2}^{(3)} \cup \cdots \cup \mathcal{N}_{n-2}^{(n-1)}, \quad n \equiv 0(\bmod 2),$$

where the set of subgraphs

$$\mathcal{N}_{n-2}^{(d)}, \quad 1 \leqslant d \leqslant n-1, \quad d \equiv 1(\bmod 2), \quad n \equiv 0(\bmod 2)$$

is constructed in the following.

(1) Let $n=2r+2$,take the copies $K_{r+2}^{c(0)}$, $K_{r+2}^{c(1)}$,$\cdots$, $K_{r+2}^{c(r-1)}$ of r graphs K_{r+2}^{c} of order $r+2$ (The complement of complete graph K_{r+2}) and a complete graph $K_{r+2}^{*(r)}$ with loop at each vertex. Make the *join graph* (the definition of join graph and the complement of graph are referred to [6]): $K_{r+2}^{c(i)} \vee K_{r+2}^{c(i+1)}$, $i=0,1,\cdots,r-2$ and $K_{r+2}^{c(r-1)} \vee K_{r+2}^{*(r)}$. Constructing a new graph K as follows:

$$\begin{aligned} K &= \bigcup_{i=0}^{r-2}(K_{r+2}^{c(i)} \vee K_{r+2}^{c(i+1)}) \bigcup (K_{r+2}^{c(r-1)} \vee K_{r+2}^{*(r)}) \\ &= \underbrace{K_{r+2}^{c(0)} \vee K_{r+2}^{c(1)} \vee \cdots \vee K_{r+2}^{c(r-1)}}_{r\ K_{r+2}^{c}\text{'s}} \vee K_{r+2}^{*(r)}. \end{aligned}$$

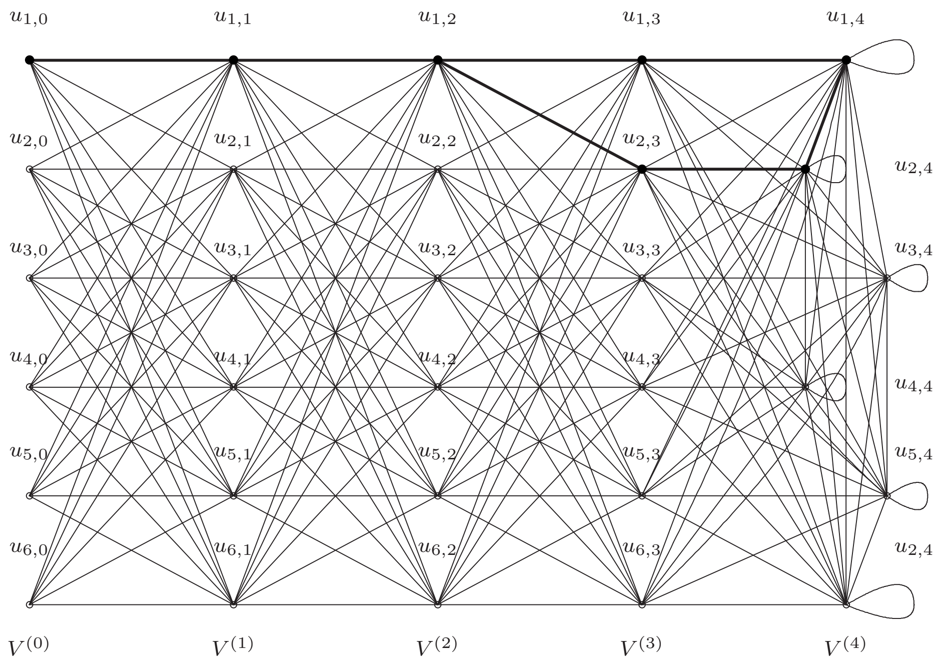

Fig.(5) The graph K with $r = 4$

Suppose that the vertex sets of the graphs $K_{r+2}^{c(0)}$, $K_{r+2}^{c(1)}, \cdots$, $K_{r+2}^{c(r-1)}$ and $K_{r+2}^{*(r)}$ in order are

$$V^{(j)} = \{u_{i,j} \mid i = 1, 2, \cdots, r+2\}, \quad j = 0, 1, \cdots, r,$$

then

$$V(K) = V^{(0)} \cup V^{(1)} \cup \cdots \cup V^{(r-1)} \cup V^{(r)}.$$

Fig.(5) shows a graph K with $r = 4$.

For d: $1 \leqslant d \leqslant 2r+1, d \equiv 1 (\mathrm{mod}\ 2)$ given,take a path $P_t = u_{1,0}u_{1,1}\cdots u_{1,t}$ of length $t = r - \frac{1}{2}(d-1)$ in K and an odd cycle $C_d = u_{1,t}u_{1,t+1}\cdots u_{1,r-1}u_{1,r}u_{2,r}u_{2,r-1}\cdots\ u_{2,t+1}u_{1,t}$ of length d. Constructing a subgraph $K_{(d)}$ of K as follows

$$K_{(d)} = P_t \cup C_d, \quad 1 \leqslant d \leqslant 2r+1, \quad d \equiv 1 (\mathrm{mod}\ 2),$$

which is called a *structure subgraph*. The subgraph in black lines in Fig.(5) shows $K_{(5)}$ $(r = 4)$.

(2) Let the set of vertex-induced subgraph of order n containing $K_{(d)}$ of K be $\mathcal{K}^{(d)}$ where $1 \leqslant d \leqslant 2r+1, d \equiv 1 (\mathrm{mod}\ 2)$, and for any graph $N \in \mathcal{K}^{(d)}$ let the spanning subgraph containing $K_{(d)}$ of N be $N_{(d)}$, now we construct the set of graphs $\mathcal{N}^{(d)}$ as follows:

$$\mathcal{N}^{(d)} = \{N_{(d)} \mid N \in \mathcal{K}^{(d)}\}, \quad 1 \leqslant d \leqslant 2r+1, \quad d \equiv 1 (\mathrm{mod}\ 2).$$

(3) Let the set of the graph $N_{(d)} \in \mathcal{N}^{(d)}$ satisfying the following conditions be

$$\mathcal{N}_{n-2}^{(d)}, \quad 1 \leqslant d \leqslant n-1, \quad d \equiv 1 (\mathrm{mod}\ 2), \quad n = 2r+2:$$

(i) $\operatorname{diam}(N_{(d)}) \leqslant n-2$;

(ii) For $d' > d$,$N_{(d)}$ has no the subgraph $K_{(d')}$ to be the structure subgraph of $N_{(d)}$;

(iii) For the vertex $x \in N = N_{(d)}$ with $d_N(x, C_d) > t$,there must be odd cycle C such that $2d_N(x,C) + |C| \leqslant n-1$.

§4. Main Results

Theorem 4.1 *G is a primitive graph with order n and for any vertex $w \in V(G)$, $\gamma(w) < \gamma(G) = n-2$ if and only if $G \in \mathcal{M}_{n-2}$.*

Proof We prove the Sufficiency first. Suppose that $G \in \mathcal{M}_{n-2}$, then G is a primitive graph with order n by the construction of $\mathcal{M}_{n-2}$. For any vertex $w \in V(G)$ we have

$$\begin{aligned}\gamma(w) &\leqslant \max\{\gamma(v_0), \gamma(v_m)\} = \max\{\gamma(v_0, C_1), \gamma(v_m, C_0)\}\\ &< 2t + m = n-2 = \gamma(G),\end{aligned}$$

and for any vertices $u, v \in V(G)$ we have

$$\gamma(u,v) \leqslant \gamma(v_0, v_m) = \gamma(v_0, v_m, C_0) = n-2.$$

That is $\gamma(G) = \max\limits_{u,v \in V(G)} \gamma(u,v) = \gamma(v_0, v_m) = n-2$. See Fig.(3-1))$\sim$ (3-4).

For the necessity, suppose that G is a primitive graph with order n and

$$\gamma(w) < \gamma(G) = n-2 \tag{4.1}$$

for any vertex $w \in V(G)$. Without loss of generality, let $u, v \in V(G)$ such that

$$\gamma(u,v) = \max_{x,y \in V(G)} \gamma(x,y) = \gamma(G) = n-2.$$

According to the discussion in § 2, there must be an odd cycle C_0 such that
$\gamma(u,v) = \gamma(u,v,C_0) = \gamma(G) = n-2$. Let

$$P_{uv} = P_{\min}(u,v) = v_0 v_1 \cdots v_i \cdots v_j \cdots v_m,$$

where $v_0 = u$, $v_m = v$,then we know that $n \equiv m \pmod 2$ by Lemma 2.4.

Suppose that C is an odd cycle in G, then by Lemma 2.4 we have

$$|V(P_{uv}) \cap V(C)| \leqslant n - \gamma(G) = n-(n-2) = 2. \tag{4.2}$$

According to (4.2) the following discussion can be partitioned into three cases:

4.1. Suppose that for any odd cycle C in G,

$$V(P_{uv}) \cap V(C) = \emptyset, \tag{4.3}$$

then $t_0 = d_G(P_{uv}, C_0) \geqslant 1$,$d_0 = |C_0| \equiv 1 (\mathrm{mod}\ 2)$. Now chose such odd cycle C_0 and the shortest (u,v)-path P_{uv} in G such that $2t_0 + d_0$ is as small as possible.

Let

$$\begin{cases} P_0 = P_{\min}(P_{uv}, C_0) = x_0 x_1 \cdots x_{t_0}; \\ V_1 = V(P_{uv} \cup P_0 \cup C_0), \quad V_2 = V(G) \setminus V_1. \end{cases}$$

where $x_0 = v_j$, $x_{t_0} \in V(C_0)$,then $n_1 = |V_1| = m + t_0 + d_0$. Since $n-2 = \gamma(u,v) = \gamma(u,v,C_0) \leqslant m + 2t_0 + d_0 - 1$,

$$n_2 = |V_2| = n - (m + t_0 + d_0) \leqslant n - (n - 1 - t_0) = t_0 + 1. \qquad (4.4)$$

4.1.1. Suppose that the odd cycle C' satisfies $\gamma(v) = \gamma(v, C')$ and $P_{\min}(v, C') \cap P_0 \neq \emptyset$,then, by the choice of P_{uv}, C_0, P_0 and the definition of $\gamma(v)$, we know that $\gamma(v, C') = \gamma(v, C_0)$. By (4.1) we have

$$\gamma(v) = \gamma(v, C_0) = 2d(v, C_0) + d_0 - 1 = 2d(v, x_{t_0}) + d_0 - 1 < n - 2.$$

In this time, if there is an odd cycle C_1 such that $\gamma(u) = \gamma(u, C_1)$ and $P_{\min}(u, C_1) \cap P_0 \neq \emptyset$, we can obtain in the same way that

$$\gamma(u) = \gamma(u, C_0) = 2d(u, C_0) + d_0 - 1 = 2d(u, x_{t_0}) + d_0 - 1 < n - 2.$$

Thus, we have

$$\gamma(G) = \gamma(u, v) = \gamma(u, v, C_0) = d(u, x_{t_0}) + d(v, x_{t_0}) + d_0 - 1 < n - 2 = \gamma(G),$$

a contradiction. So we must have

$$P_{\min}(u, C_1) \cap P_0 = \emptyset. \qquad (4.5)$$

Let v_i be the vertex with the maximum suffix in $P_{\min}(u, C_1) \cap P_{uv}$ and

$$d_1 = |C_1|, \quad t_1 = d(v_i, C_1), \quad P_1 = P_{\min}(v_i, C_1) = y_0 y_1 \cdots y_{t_1},$$

where $y_0 = v_i$, $y_{t_1} \in V(C_1)$. By (4.4) and (4.5), we have $P_0 \cap P_1 = \emptyset$,$i < j$ and

$$1 \leqslant t_1 \leqslant t_1 + d_1 - 1 \leqslant |V_2| \leqslant t_0 + 1. \qquad (4.6)$$

By the choice of P_{uv},C_0 and P_0 we also have

$$2t_0 + d_0 \leqslant 2t_1 + d_1. \qquad (4.7)$$

From (4.6) and (4.7) we get

$$2t_1 + 2d_1 - 4 + d_0 \leqslant 2t_0 + d_0 \leqslant 2t_1 + d_1 \leqslant 2t_0 + 3,$$

thus $d_0 \leqslant 3$,$d_0 + d_1 \leqslant 4$,$|t_1 - t_0| \leqslant 1$.

In all as above we have the following four cases

$$(d_0, d_1) = \begin{cases} (1,3), & t_1 = t_0 - 1; \\ (3,1), & t_0 = t_1 - 1; \\ (1,1), & t_1 = t_0; \\ (1,1), & t_1 = t_0 + 1, \end{cases} \qquad (4.8)$$

and

$$|V(P_{uv}\cup P_0\cup C_0\cup P_1\cup C_1)| = m+t_0+d_0+t_1+d_1-1 \leqslant n. \tag{4.9}$$

Thus

$$\begin{cases} n-2=\gamma(u,v)=\gamma(u,v,C_0)\leqslant m+2t_0+d_0-1; \\ n-2=\gamma(u,v)\leqslant\gamma(u,v,C_1)\leqslant m+2t_1+d_1-1. \end{cases} \tag{4.10}$$

So it follows from (4.9) and (4.10) we have

$$n-2\leqslant m+t_0+t_1+\frac{1}{2}(d_0+d_1)-1\leqslant n-\frac{1}{2}(d_0+d_1). \tag{4.11}$$

By (4.8) we have four subcases for discussions:

(i) $(d_0,d_1)=(1,3)$,$t_1=t_0-1$,thus $t_1\geqslant 1$,$t_0\geqslant 2$. By (4.10) and (4.11) we have

$$n-2=\gamma(u,v,C_0)=m+2t_0+d_0-1=m+2t_1+d_1-1=\gamma(u,v,C_1)=m+2t_0,$$

i.e., $n=m+2t_0+2$, therefore by (4.9)

$$|V(P_{uv}\cup P_0\cup C_0\cup P_1\cup C_1)|=n.$$

Suppose that $V(C_1)=\{y_{t_0-1},y_{t_0},z\}$,by (4.1) and (4.2) we get

$$v_\lambda x_\alpha\notin E(G),\quad v_\lambda y_\beta\notin E(G),\quad \lambda\neq i,j,\quad 0<\alpha\leqslant t_0,\quad 0<\beta\leqslant t_0.$$

Note that $\gamma(u)=\gamma(u,C_1)<\gamma(G)$ and $\gamma(v)=\gamma(v,C_0)<\gamma(G)$ we have

$$0\leqslant i<\frac{m}{2}<j\leqslant m. \tag{4.12}$$

If $x_ay_b\in E(G),\quad 0\leqslant a\leqslant t_0,\quad 0\leqslant b\leqslant t_1$,then $a+b+1>j-i$,$i+j+a+b\equiv 1 (\mathrm{mod}\ 2)$ and

$$\begin{cases} n-2=\gamma(u,v,C_0)\leqslant i+b+1+a+m-j+2(t_0-a)+d_0-1, \\ n-2=\gamma(u,v,C_1)\leqslant i+a+1+b+m-j+2(t_1-b)+d_1-1. \end{cases} \tag{4.13}$$

So $j-i\leqslant b-a+1$ and $j-i\leqslant a-b+1$. From this we have $j=i+1$ and $1\leqslant a=b\leqslant t_0-1$. By (4.12) m is an odd. Otherwise, since m being an even , $\gamma(v_{\frac{m}{2}})<\gamma(G)$,$j-i<m$.

Additionally, it is clever that there may be loops at the vertices y_{t_0} and z,otherwise no any loop except for at x_{t_0}. So $G\in\mathcal{M}^{(1)}_{n-2}(t\geqslant 2)$. See Fig.(3-1).

(ii) $(d_0,d_1)=(3,1)$,$t_0=t_1-1$,thus $t_0\geqslant 1$,$t_1\geqslant 2$. The discussions of these graphs, which we omit, is analogous to that in (i), and we know that it is must be in $\mathcal{M}^{(1)}_{n-2}(t\geqslant 2)$.

(iii) $(d_0,d_1)=(1,1)$,$t_0=t_1\geqslant 1$. It is analogous to (i) that

$$n-2=\gamma(u,v,C_0)=\gamma(u,v,C_1)=m+2t_0,$$

i.e., $n=m+2t_0+2$,

$$v_\lambda x_\alpha\notin E(G),\quad v_\lambda y_\beta\notin E(G),\quad \lambda\neq i,j,\quad 0<\alpha\leqslant t_0,\quad 0<\beta\leqslant t_0,$$

and

$$0 \leqslant i < \frac{m}{2} < j \leqslant m.$$

Thus, by (4.9) we have

$$|V(P_{uv} \cup P_0 \cup P_1)| = n - 1.$$

It is easy to see that the graph G has also another vertex, denoted by w and $1 \leqslant N_G(w) \leqslant 2$. If $x_k y_l \in E(G),\ 0 \leqslant k, l \leqslant t_0$, then $j = i+1$,$1 \leqslant k = l \leqslant t_0$ and

(a) When $\{wx_a, wy_b\} \subseteq E(G)$, $(0 \leqslant a, b \leqslant t_0)$, similar to (4.13) we have

$$a + b \equiv 1 (\text{mod } 2), \quad 1 = j - i \leqslant \min\{b - a + 2, a - b + 2\}.$$

That is $|a - b| = 1$. It is clear that as $a = t_0$ or $b = t_0$, there may add a loop at vertex w;

(b) When $N_G(w) = \{x, y\}$ but not the case (a),$d(x, y) = 2$;

(c) When $N_G(w) = \{x\}$ and $d_G(w, P_{uv}) \geqslant t_0$, there may add a loop at vertex w.

If there is not $x_k y_l \in E(G), \quad 0 \leqslant k, l \leqslant t_0$, then we have by similar discussions:

(a′) When $\{wx_a, wy_b\} \subseteq E(G)$, $(0 \leqslant a, b \leqslant t_0)$,we have $j = i + 1$, $|a - b| = 1$ or $j = i + 2$, $a = b$, but $m \neq 2$ and as $a = t_0$ or $b = t_0$, there may add a loop at vertex w;

(b′) When $N_G(w) = \{x, y\}$ but not the case (a′), $d(x, y) = 2$;

(c′) When $N_G(w) = \{x\}$ and $d_G(w, P_{uv}) \geqslant t_0$, there may add a loop at vertex w. If there is not any loop at vertex w and $x = v_s$, then by $\gamma(w) < \gamma(G)$,$i > 1$ or $j < m$ as $s = \frac{m}{2}$.

To sum up we have $G \in \mathcal{M}^{(2)}_{n-2}(t \geqslant 1)$(See Fig.(3-2)).

(*iv*) $(d_0, d_1) = (1, 1)$, $t_1 = t_0 + 1 \geqslant 2$. From (4.9) we get

$$|V(P_{uv} \cup P_0 \cup P_1)| = n.$$

And from (4.10) we have

$$n - 2 = \gamma(G) = \gamma(u, v, C_0) = m + 2t_0, \quad \gamma(u, v, C_1) \leqslant m + 2t_1 = m + 2t_0 + 2.$$

i.e., $n = m + 2t_0 + 2$,and from (4.1) and (4.2), we have

$$v_\lambda x_\alpha \notin E(G), \quad \lambda \neq j, \quad 0 < \alpha \leqslant t_0; \quad v_\mu y_\beta \notin E(G), \quad 0 \leqslant \mu < i, \quad 0 < \beta \leqslant t_1.$$

Sine $\gamma(u) = \gamma(u, C_1) = 2i + 2t_0 + 2 < m + 2t_0$, $i + 1 < \frac{m}{2}$,thus

$$1 \leqslant i + 1 < \frac{m}{2} < j \leqslant m. \tag{4.14}$$

Now, if $v_\mu y_\beta \notin E(G), \quad \mu > i, \quad 1 \leqslant \beta \leqslant t_0 + 1$, then by (4.1) we have $j - i < m$ as m is odd and $j - i < m - 1$ as m is even.

If $x_a y_b \in E(G)$, $0 \leqslant a \leqslant t_0$,$0 \leqslant b \leqslant t_1$, then $a + b + i + j \equiv 1 (\text{mod } 2)$,

$$\begin{cases} n - 2 = \gamma(u, v, C_0) \leqslant i + b + 1 + a + m - j + 2(t_0 - a), \\ n - 2 = \gamma(u, v, C_1) \leqslant i + b + 1 + a + m - j + 2(t_1 - b). \end{cases}$$

Thus $1 < j-i \leqslant b-a+1$ and $1 < j-i \leqslant a-b+3$. this means that $j=i+2$ and $b=a+1$. So $G \in \mathcal{M}^{(3)}_{n-2}$(See Fig.(3–3)).

If $v_\mu y_\beta \in E(G)$, $(\mu > i)$, then it is must be that $y_1 v_{i+2}$ and $i+2 \leqslant j$ by (4.14). The case similar to (b) in (iii).

4.1.2. Suppose that the odd cycle C' satisfies $\gamma(v)=\gamma(v,C')$ and $P_{\min}(v,C') \cap P_0 = \emptyset$,then there is also an odd cycle C'' such that $\gamma(u)=\gamma(u,C'')$ and $P_{\min}(u,C'') \cap P_0 = \emptyset$. Otherwise, similar to 4.1.1. Let

$$P' = P_{\min}(P_{uv}, C'), \quad P'' = P_{\min}(P_{uv}, C'')$$

and writing $t' = d_G(P_{uv}, C')$, $d' = |C'|$; $t'' = d_G(P_{uv}, C'')$, $d'' = |C''|$, therefore

$$|V(P_{uv} \cup P_0 \cup P' \cup P'' \cup C_0 \cup C' \cup C'')| = m + t_0 + t' + t'' + d_0 + d' + d'' - 2 \leqslant n.$$

Since

$$\begin{cases} n-2 = \gamma(u,v,C_0) \leqslant m + 2t_0 + d_0 - 1, \\ n-2 \leqslant \gamma(u,v,C') \leqslant m + 2t' + d' - 1, \\ n-2 \leqslant \gamma(u,v,C'') \leqslant m + 2t'' + d'' - 1. \end{cases}$$

Thus, we have

$$\begin{aligned} n-2 \quad &\leqslant m + \tfrac{2}{3}(t_0 + t' + t'') + \tfrac{1}{3}(d_0 + d' + d'') - 1 \\ &\leqslant n + 1 - \tfrac{1}{3}(t_0 + t' + t'') - \tfrac{2}{3}(d_0 + d' + d'') \\ &\leqslant n-2. \end{aligned}$$

i.e.,

$$t_0 = t' = t'' = 1, \quad d_0 = d' = d'' = 1.$$

So $G \in \mathcal{M}^{(2)}_{n-2}(t=1)$(See Fig.(3-2)).

4.2. Suppose that there is an odd cycle C in G such that

$$V(P_{uv}) \cap V(C) = \{v_i, v_\lambda\}, \quad (i < \lambda). \tag{4.15}$$

Then from (4.2) we see that $\lambda = i+1$,thus

$$n-2 = \gamma(u,v) \leqslant i + (m-\lambda) + |C| - 2 = m + |C| - 3 = |V(P_{uv} \cup C)| - 2 \leqslant n-2,$$

i.e.,

$$V(P_{uv} \cup C) = V(G), \quad n-2 = \gamma(u,v) = \gamma(G) = m + |C| - 3,$$

or

$$G = P_{uv} \cup C, \quad n = m + |C| - 1. \tag{4.16}$$

In the same time from (4.2), also we see that

$$N[C_0 - \{v_i, v_{i+1}\}] \cap V(P_{uv}) = \{v_i, v_{i+1}\}.$$

By (4.2) we have $v_i v_{i+1} \in C$,and $\gamma(u,v) = \gamma(u,v,C)$,i.e., putting $C = C_0$. Let $C_0 = y_0 y_1 \cdots y_{t_0} w x_{t_0} \cdots x_1 x_0 y_0$ where $y_0 = v_i, x_0 = v_{i+1}, t_0 \geqslant 0$, then $|C_0| = 2t_0 + 3$,that is $n = m + 2t_0 + 2$.

If there is $x_a y_b \in E(G), 0 \leqslant a, b \leqslant t_0$,then by $\gamma(u,v) = \gamma(u,v,C_0)$ we have $|a-b| \equiv 0 (\mathrm{mod}\ 2)$. In this time we have the odd cycle $C_{ab} = y_b y_{b+1} \cdots y_{t_0} w x_{t_0} \cdots x_{a+1} x_a y_b$,and so

$$\begin{cases} n-2 \leqslant \gamma(u,v,C_{ab}) \leqslant m + 2a + 2t_0 - a - b + 2, \\ n-2 \leqslant \gamma(u,v,C_{ab}) \leqslant m + 2b + 2t_0 - a - b + 2. \end{cases}$$

That is $|a-b| \leqslant 2$,or $a = b \neq 0 (t_0 \geqslant 1)$ or $|a-b| = 2 (t_0 \geqslant 2)$.

It is easily seen that the all of odd cycles in G is included in $\mathcal{Z} = \{C_{ab} \mid 0 \leqslant a, b \leqslant t_0, a = b \neq 0 \text{ or } |a-b| = 2\}$ where $C_0 = C_{00}$ except for possible loops C_y at y_{t_0}, C_x at x_{t_0} and C_w at w.

If there exists $C_{ab}, C_{a'b'} \in \mathcal{Z}$ in G such that $\gamma(u) = \gamma(u, C_{ab}) < n-2$, $\gamma(v) = \gamma(v, C_{a'b'}) < n-2$,then from (4.1) we get

$$\begin{cases} 2i + 2b + 2t_0 + 2 - a - b \leqslant n - 3, \\ 2(m-i-1) + 2a' + 2t_0 + 2 - a' - b' \leqslant n - 3, \end{cases}$$

note that $n = m + 2t_0 + 2$,the formula as above equivalent to

$$\begin{cases} 2i \leqslant m - 3 + a - b, \\ 2(m-i-1) \leqslant m - 3 + b' - a' \end{cases}$$

i.e.,

$$a = b + 2, \quad a' = b' - 2, \quad i = \frac{1}{2}(m-1).$$

Otherwise,we must have $\gamma(u) = \gamma(u, C_y) < n-2$ and $\gamma(v) = \gamma(v, C_x) < n-2$,i.e.,

$$\begin{cases} 2i + 2t_0 \leqslant n - 3, \\ 2(m-i-1) + 2t_0 \leqslant n - 3, \end{cases}$$

From this we can get $i = \frac{1}{2}(m-1)$,and there are loops C_y at y_{t_0} and C_x at x_{t_0}.

To sum up,we obtain that

$$\begin{cases} \{C_{ab} \mid 0 \leqslant a, b \leqslant t_0, a = b + 2\} \cup \{C_y\} \neq \emptyset; \\ \{C_{ab} \mid 0 \leqslant a, b \leqslant t_0, a = b - 2\} \cup \{C_x\} \neq \emptyset. \end{cases}$$

Evidently,this result is the same as the case of no any $x_a y_b \in E(G), 0 \leqslant a, b \leqslant t_0$ but $t_0 \geqslant 1$.

When $t_0 = 0$, i.e., $|C_0| = 3$,$n-2 = \gamma(u,v) = \gamma(u,v,C_0) = \gamma(G) = m$. Set $C_0 = v_i v_{i+1} w v_i$, then there are loops at vertex v_x and v_y as $i = \frac{1}{2}(m-1)$ where $0 \leqslant x \leqslant i = \frac{1}{2}(m-1) < y \leqslant m$, and there may be loops at the rest vertices;There must be a loop at v_y as $i < \frac{1}{2}(m-1)$ where $\frac{1}{2}(m-1) < y \leqslant m$, and there may be loops at the rest vertices; there must be a loop at v_x as $i > \frac{1}{2}(m-1)$ where $0 \leqslant x \leqslant \frac{1}{2}(m-1)$, and there may be loops at the rest vertices. So $G \in \mathcal{M}_{n-2}^{(4)}$(See Fig.(3-4)).

4.3. There is an odd cycle C such that

$$V(P_{uv}) \cap V(C) = \{v_i\}, \tag{4.17}$$

but there is not odd cycle C' such that $|V(P_{uv}) \cap V(C')| \geqslant 2$. Thus,we have

$$n-2 = \gamma(u,v) \leqslant \gamma(u,v,C) = m + |C| - 1 = |V(P_{uv} \cup C)| - 1 \leqslant n-1.$$

Since $n \equiv m(\text{mod } 2)$,

$$n-2 = \gamma(u,v) = \gamma(u,v,C) = m + |C| - 1, \quad |V(P_{uv} \cup C)| = n-1,$$

i.e., $n = m + |C| + 1$. Evidently,there is only one vertex w in G except for the vertices of $V(P_{uv} \cup C)$ and $N(C - \{v_i\}) \cap V(P_{uv}) = \{v_i\}$. This indicates that $C = C_0$ and $|C_0| \leqslant 3$,otherwise,there must have $\gamma(u) \geqslant \gamma(G)$ or $\gamma(v) \geqslant \gamma(G)$, contradicts to (4.1).

When $|C_0| = 3$ we have $\gamma(G) = m+2$ and $n = m+4$. There is no any loop at the vertices on P_{uv}. By (4.1) we have $G \in \mathcal{M}^{(1)}_{n-2}(t=1)$ (See Fig.(3–1)).

When $|C_0| = 1$ we have $\gamma(G) = m$ and $n = m+2$. By (4.1) the set of graphs have the characteristic: there are loops at v_i and v_j as $0 \leqslant i < \frac{m}{2} < j \leqslant m$,there is no any loop at w,and there may be loops at the rest vertices. There exists a loop C such that $d_G(w,C) < \frac{m}{2}$. So $G \in \mathcal{M}^{(2)}_{n-2}(t=0)$(See Fig.(3-2)).

The proof is complete. □

Theorem 4.2 *Suppose that G is a primitive graph with order n, then there exists a vertex $w \in V(G)$ such that $\gamma(w) = \gamma(G) = n-2$ if and only if $G \in \mathcal{N}_{n-2}$.*

Proof For the sufficiency, suppose that $G \in \mathcal{N}_{n-2}$, without loss of generality suppose that $G \in \mathcal{N}^{(d)}_{n-2}$,$1 \leqslant d \leqslant 2r+1$, $d \equiv 1(\text{mod } 2)$. Since diam$(G) \leqslant n-2$,$G$ is connected and it is clear that there is at least an odd cycle C_d in G. By Lemma 2.1 we know that G is a primitive graph and $|V(G)| = n_1 + n_2 = 2t + d + 1 = n$.

In the following, we need only to prove two results:

(1)$\gamma(u_0) = n-2$.

Evidently, $\gamma(u_0, C_d) = 2d_G(u_0, C_d) + |C_d| - 1 = 2t + d - 1 = n-2$.

Suppose that there is any odd cycle C in G such that $\gamma(u_0, C) < n-2 = 2r$,then $2d_G(u_0,C) + |C| - 1 < 2r$,i.e.,

$$d_G(u_0,C) + \frac{1}{2}(|C|-1) < r.$$

This means that there is an odd cycle C in the vertex-induced subgraph $G[U']$ in G where

$$U' = \{u \mid d_G(u_0,u) < r, \quad u \in V(G)\}.$$

This is impossible, since $G[U']$ is the subgraph of the vertex-induced subgraph $K[V']$ in K where

$$V' = \{u \mid d_K(u_0,u) < r, \quad u \in V(K)\},$$

and $K[V']$ is a bipartite graph. So $\gamma(u_0) = \gamma(u_0, C_d) = n-2$.

(2) $\forall u,v \in V(G),\ \gamma(u,v) \leqslant n-2$.

When $u=v$,If $d_G(u,C_d)\leqslant t$, then

$$\gamma(u)\leqslant\gamma(u,C_d)=2t+d-1=2r=n-2;$$

If $d_G(u,C_d)>t$, then by the constructed condition (iii) of G we see that there exists an odd cycle C in G such that $2d_G(u,C)+|C|\leqslant n-1$, that is

$$\gamma(u)\leqslant\gamma(u,C)=2d_G(u,C)+|C|-1\leqslant n-2.$$

Thus, we get $\gamma(u,v)\leqslant n-2$.

When $u\neq v$, if $d_G(u,C_d)+d_G(v,C_d)\leqslant 2t$, then

$$\gamma(u,v)\leqslant\gamma(u,v,C_d)\leqslant 2t+d-1=n-2;$$

If $d_G(u,C_d)+d_G(v,C_d)>2t$,it might just as well suppose that $d_G(u,C_d)>t$,then by the constructed condition (iii) of G we also see that there exists an odd cycle C in G such that

$$2d_G(u,C)+|C|\leqslant n-1.$$

By considering the shortest path $P_{\min}(u,C)$ from u to C and $P_{\min}(u_0,C_d)$ from u_0 to C_d, if they intersect each other, let w be the first intersect vertex of $P_{\min}(u,C)$ from u to C and $P_{\min}(u_0,C_d)$, then $d_G(u,w)>d_G(u_0,w)$. Thus

$$\begin{aligned}\gamma(u_0)\quad&\leqslant\gamma(u_0,C)\leqslant 2(d_G(u_0,w)+d_G(w,C))+|C|-1\\&<2(d_G(u,w)+d_G(w,C))+|C|-1\\&=2d_G(u,C)+|C|-1\\&\leqslant n-2=\gamma(u_0),\end{aligned}$$

a contradiction. Therefore,there are no any intersect vertex between $P_{\min}(u,C)$ and $P_{\min}(u_0,C_d)$. Thus, by the connectivity of G and the condition $n_2=t+1$, we have $d_G(u,C_d)=t+1$ and $d_G(v,C_d)=t$. This means that $uv\in E(G)$ or $v=u_0$.

If $uv\in E(G)$, then

$$\begin{aligned}\gamma(u,v)\quad&\leqslant\gamma(u,v,C)=d_G(u,C)+d_G(v,C)+|C|-1\\&<2d_G(u,C)+|C|-1\leqslant n-2;\end{aligned}$$

If $v=u_0$, then

$$\begin{aligned}\gamma(u,v)\quad&\leqslant\gamma(u,u_0,C_d)\leqslant d_G(u,C_d)+d_G(u_0,C_d)+|C_d|-2\\&=2t+d-1=n-2.\end{aligned}$$

To sum up, we get $\forall u,v\in V(G),\ \gamma(u,v)\leqslant n-2$.

For the necessity, suppose that G is a primitive graph with order n,then there must be a vertex u_0 and an odd cycle C in G such that $\gamma(u_0)=\gamma(u_0,C)=\gamma(G)=n-2$, choosing such vertex u_0 and odd cycle C that the length $d=|C|$ as large as possible and writing $C=C_d$. By the Lemma 2.4,we have $\gamma(G)=\gamma(u_0)\equiv d_G(u_0,u_0)=0(\mathrm{mod}\ 2)$. So, let $\gamma(G)=2r$, thus $n=2r+2$.

It is clear that C_d is a primitive cycle at vertex u_0, let $t = d_G(u_0, C_d)$, then $\gamma(u_0) = 2t + d - 1 = 2r$. So $t = r - \frac{1}{2}(d-1)$,$1 \leqslant d \leqslant 2r+1$. Suppose that

$$P_t = P_{\min}(u_0, C_d) = u_0 u_1 \cdots u_t, \quad C_d = u_t u_{t+1} \cdots u_{t+d-1} u_t,$$

and write

$$\begin{cases} V_1(t,d) = V(P_t \cup C_d), & V_2(t,d) = V(G) - V_1(t,d); \\ E_1(t,d) = E(P_t \cup C_d), & E_2(t,d) = E(G) - E_1(t,d). \end{cases}$$

Then, we calculate

$$n_1 = |V_1(t,d)| = t + d, \quad n_2 = |V_2(t,d)| = t + 1, \quad n = 2t + d + 1.$$

What is mentioned as above indicates that there must be the structure subgraph $K_{(d)} = P_t \cup C_d$ in G. In order to prove $G \in \mathcal{N}_{n-2}^{(d)} \subseteq \mathcal{N}_{n-2}$,it is suffice to prove that (a) The graph G satisfies the constructed conditions (i),(ii) and (iii) of the set of $\mathcal{N}_{n-2}^{(d)}$;(b)The graph G is a subgraph of K.

(a) By Lemma 2.4 we get $\mathrm{diam}(G) \leqslant \gamma(G) = 2r = n-2$,so the condition (i) holds. By the choice of C_d we know that for $d' > d$ there is not the structure subgraph $K(d')$ in G,so the condition (ii) holds. Suppose that there exists a vertex x in G such that $d_G(x, C_d) > t$,then $\gamma(x, C_d) = 2d_G(x, C_d) + d - 1 > 2t + d - 1 = 2r$. If $2d_G(x, C') + |C'| > n-1$ for all odd cycle C' different from odd C_d in G, then also $\gamma(x, C') = 2d_G(x, C') + |C'| - 1 > n - 2 = 2r$. So $\gamma(G) \geqslant \gamma(x) > 2r = \gamma(G)$, a contradiction. Therefore the condition (iii) holds too.

(b) Suppose that the vertex set $V(G)$ of G is divided into as follows:

$$V(G) = U_0 \cup U_1 \cup \cdots \cup U_{r-1} \cup U_r,$$

in which $U_i = \{u \mid d_G(u_0, u) = i,\ u \in V(G)\}$, $i = 0, 1, 2, \cdots, r-1$, $U_r = \{u \mid d_G(u_0, u) \geqslant r,\ u \in V(G)\}$.

Firstly, we prove that the induced vertex subgraphs $G[U_i]$ all are zero graphs, $i = 0, 1, 2, \cdots, r-1$. Otherwise,there must be odd cycle in the vertex-induced subgraph $G' = G[U_0 \cup U_1 \cup \cdots \cup U_{r-1}]$. Let C be an odd cycle in G',then $d_{G'}(u_0, C) + \frac{1}{2}(|C| - 1) < r$. Thus, $\gamma(u_0) \leqslant \gamma(u_0, C) = 2d_{G'}(u_0, C) + |C| - 1 < 2r = \gamma(u_0)$,a contradiction.

Secondly, we prove that $G[U_r]$ is a subgraph of $K_{r+2}^{(r)}$. By the definition of $K_{r+2}^{(r)}$,it is suffice to prove that $|U_r| \leqslant |K_{r+2}^{(r)}| = r+2$. In fact,when $d = 1$ since $|U_i| \geqslant 1$, $i = 0, 1, \cdots, r-1$,we have $2r + 2 = |V(G)| \geqslant r + |U_r|$, i.e., $|U_r| \leqslant r + 2$. When $d \geqslant 3$ since $|U_i| \geqslant 1$, $i = 0, 1, \cdots, t$,$|U_j| \geqslant 2$, $j = t+1, \cdots, r-1$,we have $2r + 2 = |V(G)| \geqslant t + 1 + 2(r - t - 1) + |U_r|$, i.e., $|U_r| \leqslant t + 3 = r - \frac{1}{2}(d-1) + 3 \leqslant r + 2$.

To sum up, we obtain $G \in \mathcal{N}_{n-2}^{(d)} \subseteq \mathcal{N}_{n-2}$. The theorem is proved completely. □

Theorem 4.3 *Suppose that* $\boldsymbol{A}$ *is a symmetric primitive matrix with order* n*,then* $\gamma(\boldsymbol{A}) = n - 2$ *if and only if* $G(\boldsymbol{A}) \in \mathcal{M}_{n-2} \cup \mathcal{N}_{n-2}$.

Proof According to Theorem 4.1 and Theorem 4.2 the theory holds. □

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