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# On the Crossing Number of the Join of Some 5-Vertex Graphs and $P_n$

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**Abstract**: In this paper, we study the join of graphs, and give the values of crossing numbers for join products $G_i \vee P_n$ for some graphs $G_i (i = 2, 5, 6, 9)$ of order five, which is related with parallel bundles on planar map geometries ([10]), a kind of planar Smarandache geometries.

**Key Words**: graph, crossing number, join, drawing, path.

**AMS(2000)**: 05C10, 05C62

## §1. Introduction

A *drawing* $D$ of a graph $G$ on a surface $S$ consists of an immersion of $G$ in $S$ such that no edge has a vertex as an interior point and no point is an interior point of three edges. We say a drawing of $G$ is a good drawing if the following conditions holds.

($i$) no edge has a self-intersection;

($ii$) no two adjacent edges intersect;

($iii$) no two edges intersect each other more than once;

($iv$) each intersection of edges is a crossing rather than tangential.

Let $G$ be a simple graph with vertex set $V$ and edge set $E$. The *crossing number* $cr(G)$ of a graph $G$ is the smallest number of pairs of nonadjacent edges that intersect in a drawing of $G$ in the plane. An *optimal drawing* of a graph $G$ is a drawing whose number of crossings equals $cr(G)$. Let $A$ and $B$ be disjoint edge subsets of $G$. We denote by $cr_D(A, B)$ the number of crossings between edges of $A$ and $B$, and by $cr_D(A)$ the number of crossings whose two crossed edges are both in $A$. Let $H$ be a subgraph of $G$, the restricted drawing $D|_H$ is said to be a *subdrawing* of $H$. As for more on the theory of crossing numbers, we refer readers to [1] and [2]. In this paper, we shall often use the term *region* also in non-planar drawings. In this case, crossing are considered to be vertices of the map.

Let $G$ and $H$ be two disjoint graphs. The *union* of $G$ and $H$, denoted by $G+H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. And the *join* of $G$ and $H$ is obtained by adjoining every vertex of $G$ to every vertex of $H$ in $G + H$ which is denoted by $G \vee H$ (see [3]).

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Let $K_{m,n}$ denote the complete bipartite graph on sets of $m$ and $n$ vertices, that is, the graph whose edges join exactly those pairs of vertices which belong one to each set. Let $P_n$ be the path with n vertices.

From the definitions, following results are easy.

**Proposition** 1.1 *Let $G$ be a graph homeomorphic to $H$ (for the definition of homeomorphic, readers are referred to [2]), then $cr(G) = cr(H)$.*

**Proposition** 1.2 *If $G$ is a subgraph of $H$, then $cr(G) \leq cr(H)$.*

**Proposition** 1.3 *If $D$ is a good drawing of a graph $G$, $A$, $B$ and $C$ are three mutually disjoint edge subsets of $G$, then we have*

(1) $cr_D(A \cup B) = cr_D(A) + cr_D(A, B) + cr_D(B)$;

(2) $cr_D(A \cup B, C) = cr_D(A, C) + cr_D(B, C)$.

The investigation on the crossing number of graphs is a classical and however very difficult problem. The exact value of the crossing number is known only for few specific families of graphs. The Cartesian product is one of few graph classes, for which exact crossing number results are known. It has long conjectured in [4] that the crossing number $cr(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ equals the Zarankiewicz's Number $Z(m, n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. (For any real $x$,$\lfloor x \rfloor$ denotes the maximum integer not greater than $x$). This conjecture has been verified by Kleitman for $\min\{m, n\} \leq 6$, see [5]. The table in [6] shows the summary of known crossing numbers for Cartesian products of path, cycle and star with connected graphs of order five.

Kulli and Muddebihal [7] gave the Characterization of all pairs of graphs which join is planar graph. In [8] Bogdan Oporowski proved $cr(C_3 \vee C_5) = 6$. In [9] Ling Tang et al. gave the crossing number of the join of $C_m$ and $P_n$. It thus seems natural to inquire about crossing numbers of join product of graphs. In this paper, we give exact values of crossing numbers for join products $G_i \vee P_n$ for some graphs $G_i (i = 2, 5, 6, 9)$see Fig.1 of order five in table [6], which is related with parallel bundles on planar map geometries ([10]), a kind of planar Smarandache geometries.

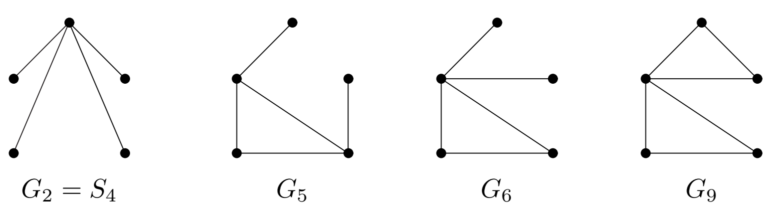
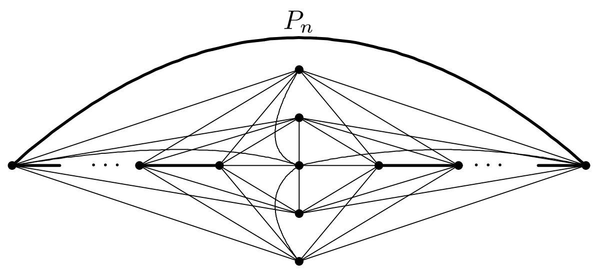



Fig.1

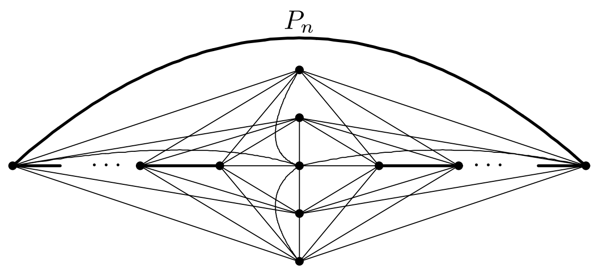
## §2. The Crossing Number of $G_2 \vee P_n$, $G_6 \vee P_n$ and $G_9 \vee P_n$

One of good drawings for graphs $G_2 \vee P_n$, $G_6 \vee P_n$ and $G_9 \vee P_n$ are shown in Fig.2-Fig.4 following.



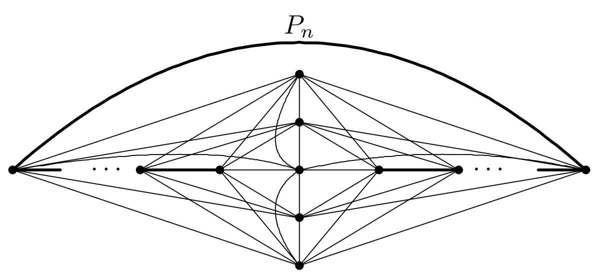

A good drawing of $G_2 \vee P_n$

Fig.2




A good drawing of $G_6 \vee P_n$

Fig.3




A good drawing of $G_9 \vee P_n$

Fig.4

**Theorem 2.1** *$cr(G_i \vee P_n) = n(n-1)(i = 2, 6, 9)$, for $n \geq 1$.*

*Proof* The drawing in Fig.2, Fig.3, Fig.4 following shows that $cr(G_i \vee P_n) \leq Z(5, n) + 2\lfloor \frac{n}{2} \rfloor = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor = n(n-1)(i = 2, 6, 9)$ (see Fig.2). As $G_i$ contains a subgraph homeomorphic to $K_{1,4,n}$, whose crossing number is $n(n-1)$ (see [11]). So we have $cr(G_i \vee P_n) \geq cr(K_{1,4,n}) = n(n-1)(i = 2, 6, 9)$. This complete the proof. □

## §3. The Crossing Number of $G_5 \vee P_n$

Firstly, let us denote by $H_n$ the graph obtained by adding six edges to the graph $K_{5,n}$, containing

$n$ vertices of degree 5 and two vertices of degree $n+1$, one vertices of degree $n+2$, two vertices of degree $n+3$, and $5n+6$ edges (see Fig.5). Consider now the graph $G_5$ in Fig.1. It is easy to see that $H_n = G_5 \cup K_{5,n}$, where the five vertices of degree $n$ in $K_{5,n}$, and the vertices of $G_5$ are the same. Let, for $i = 1, 2, \cdots, n$, $T^i$ denote the subgraph of $K_{5,n}$ which consists of the five edges incident with a vertex of degree five in $K_{5,n}$ (see Fig.6). Thus, we have

$$H_n = G_5 \cup K_{5,n} = G_5 \cup (\bigcup_{i=1}^{n} T^i). \tag{1}$$

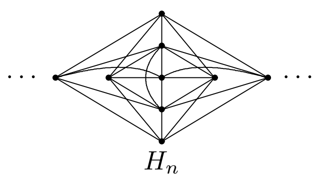



Fig.5

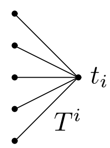
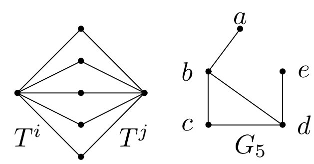



Fig.6




(a) (b)

Fig.7

**Lemma** 3.1 *Let $\phi$ be a good drawing of $H_n$, if there exist $1 \leqslant i \neq j \leqslant n$, such that $cr_\phi(T^i, T^j) = 0$, then*

$$cr_\phi(G_5, T^i \cup T^j) \geqslant 1.$$

*Proof* Let $H$ be the subgraph of $H_n$ induced by the edges of $T^i \cup T^j$. Since $cr_\phi(T^i, T^j) = 0$, and in good drawing two edges incident with the same vertex cannot cross, the subdrawing of $T^i \cup T^j$ induced by $\phi$ induces the map in the plane without crossing, as shown in Fig.7(a). Let $a, b, c, d, e$ denote the five vertices of the subgraph $G_5$ (see Fig.7(b)). Clearly, for any $x \in V(G_5)$, there are exactly two other vertices of $G_5$ on the boundary of common region with $x$. By $d_{G_5}(b) = 3$, at the edges incident with $b$, there are at least one crossing with edges of $H$. Similarly, at the edges incident with $d$, there are at least one crossing with edges of $H$. If the two crossings are different, this completes the proof, otherwise, the same crossing can find at edge $bd$, there are also at least one crossing with edges of $H$. The proof also holds. Therefore, we complete the proof. □

**Theorem** 3.2 $cr(H_n) = Z(5, n) + \lfloor \frac{n}{2} \rfloor$, $n \geq 1$.

*Proof* The drawing in Fig.5 shows that

$$cr(H_n) \leqslant cr(K_{5,n}) + \lfloor \frac{n}{2} \rfloor = Z(5, n) + \lfloor \frac{n}{2} \rfloor.$$

Thus, in order to prove theorem, we need only to prove that $cr_{\phi'}(H_n) \geqslant Z(5, n) + \lfloor \frac{n}{2} \rfloor$ for any drawing $\phi'$ of $H_n$. We prove the reverse inequality by induction on $n$. The case $n = 1$ is trivial, and the inequality also holds when $n = 2$ since $H_2$ contains a subgraph homeomorphic to $K_{3,3}$, whose crossing number is 1. Now suppose that for $n \geqslant 3$,

$$cr(H_{n-2}) \geqslant Z(5, n-2) + \lfloor \frac{n-2}{2} \rfloor \tag{2}$$

and consider such a drawing $\phi$ of $H_n$ that

$$cr_\phi(H_n) < Z(5,n) + \lfloor \frac{n}{2} \rfloor \tag{3}$$

Our next analysis depends on whether or not there are different subgraph $T^i$ and $T^j$ that do not cross each other in $\phi$.

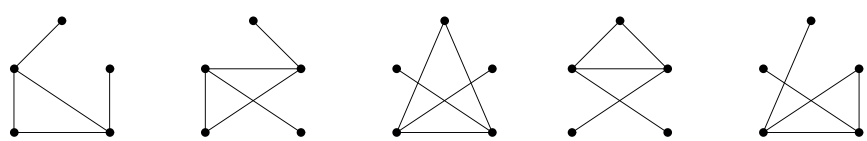
**Case 1** Suppose that $cr_\phi(T_i, T_j) \geq 1$ for any two different subgraphs $T^i$ and $T^j$, $1 \leq i \neq j \leq n$. By Proposition 1.3, using (1), we have

$$cr_\phi(H_n) = cr_\phi(K_{5,n}) + cr_\phi(G_5) + cr_\phi(K_{5,n}, G_5) \geq Z(5,n) + cr_\phi(G_5) + \sum_{i=1}^{n} cr_\phi(G_5, T^i)$$

This, together with our assumption (3), implies that

$$cr_\phi(G_5) + \sum_{i=1}^{n} cr_\phi(G_5, T^i) < \lfloor \frac{n}{2} \rfloor$$

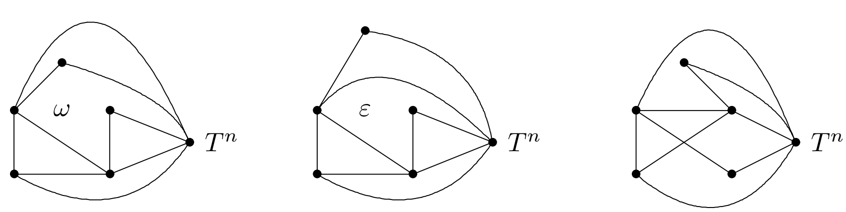
We can see that in $\phi$ there are no more than $\lfloor \frac{n}{2} \rfloor$ subgraphs $T^i$ which cross $G_5$, and at least have $\lceil \frac{n}{2} \rceil$ subgraphs $T^i$ which does not cross $G_5$. Now, we consider $T^i$, which satisfy $cr_\phi(G_5, T^i) = 0$. Without loss of generality, we suppose $cr_\phi(G_5, T^n) = 0$ and let $F$ be the subgraph $G_5 \cup T^n$ of the graph $H_n$.



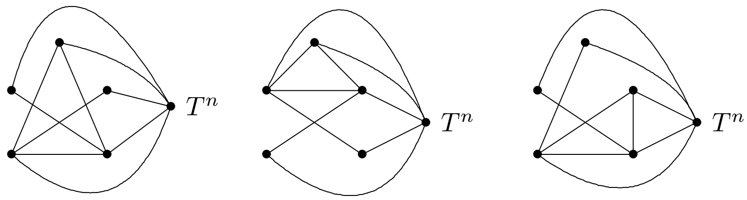
(1) (2) (3) (4) (5)

Fig.8

Consider the subdrawings $\phi^*$ and $\phi^{**}$ of $G_5$ and $F$, respectively, induced by $\phi$. Since $cr_\phi(G_5, T^n) = 0$, the subdrawing $\phi^*$ divides the plane in such a way that all vertices are on the boundary of one region. It is easy to verify that all possibilities of the subdrawing $\phi^*$ are shown in Fig.8. Thus, all possibilities of the subdrawing $\phi**$ are shown in Fig.9.




(1) (2) (3)




(4) (5) (6)

Fig.9

**(a)** The subdrawing $\phi^{**}$ of $\langle G_5 \cup T^n \rangle$ is isomorphic to Figure 9(1). When the vertex $t_i (1 \leq i \leq n-1)$ locates in the region labeled $\omega$, we have $cr_\phi(T^i, G_5 \cup T^n) \geq 1$, using $cr_\phi(T^i, T^j) \geq 1$, we have $cr_\phi(T^i, G_5 \cup T^n) \geq 2$; when the vertex $t_i$ locates in the other regions, we have $cr_\phi(T^i, G_5 \cup T^n) \geq 3$.

We suppose there are $x$ vertices $t_i$ locates in the region labeled $\omega$, and the other $n-1-x$ vertices locates in the other regions. It has been proved that $x$ is no more than $\lfloor \frac{n}{2} \rfloor$, so by Proposition 1.3, we have

$$\begin{aligned} cr_\phi(H_n) &= cr_\phi(G_5 \cup T^n \cup \bigcup_{i=1}^{n-1} T^i) \\ &= cr_\phi(G_5 \cup T^n, \bigcup_{i=1}^{n-1} T^i) + cr_\phi(G_5 \cup T^n) + cr_\phi(\bigcup_{i=1}^{n-1} T^i) \\ &\geq Z(5, n-1) + 2x + 3(n-1-x) \\ &\geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 3n - 3 - \lfloor \frac{n}{2} \rfloor \\ &\geq Z(5, n) + \lfloor \frac{n}{2} \rfloor \end{aligned}$$

**(b)** The subdrawing $\phi^{**}$ of $\langle G_5 \cup T^n \rangle$ is isomorphic to Figure 9(2). When the vertex $t_i (1 \leq i \leq n-1)$ locates in the region labeled $\varepsilon$, we have $cr_\phi(T^i, G_5 \cup T^n) \geq 2$; when the vertex $t_i$ locates in the other regions, we have $cr_\phi(T^i, G_5 \cup T^n) \geq 3$. Using the similar way as Fig.9(1), we can have $cr_\phi(H_n) \geq Z(5, n) + \lfloor \frac{n}{2} \rfloor$.

**(c)** The subdrawing $\phi^{**}$ of $\langle G_5 \cup T^n \rangle$ is isomorphic to Figure 9(3)-9(6). No matter which region $t_i$ locates in, we have $cr_\phi(T^i, G_5 \cup T^n) \geq 3$. Then by Proposition 1.3, we have

$$\begin{aligned} cr_\phi(H_n) &= cr_\phi(G_5 \cup T^n \cup \bigcup_{i=1}^{n-1} T^i) \\ &= cr_\phi(G_5 \cup T^n, \bigcup_{i=1}^{n-1} T^i) + cr_\phi(G_5 \cup T^n) + cr_\phi(\bigcup_{i=1}^{n-1} T^i) \\ &\geq Z(5, n-1) + 3(n-1) \\ &\geq Z(5, n) + \lfloor \frac{n}{2} \rfloor \end{aligned}$$

This contradicts (3).

**Case 2** Suppose that there are at least two different subgraphs $T^i$ and $T^j$ that do not cross each other in $\phi$. Without loss of generality, we may assume that $cr_\phi(T^{n-1}, T^n) = 0$. By Lemma 3.1, $cr_\phi(G_5, T^{n-1} \cup T^n) \geqslant 1$, as $cr(K_{3,5}) = 4$, for all $i = 1, 2, \cdots, n-2$, $cr_\phi(T^i, T^{n-1} \cup T^n) \geqslant 4$. This implies that

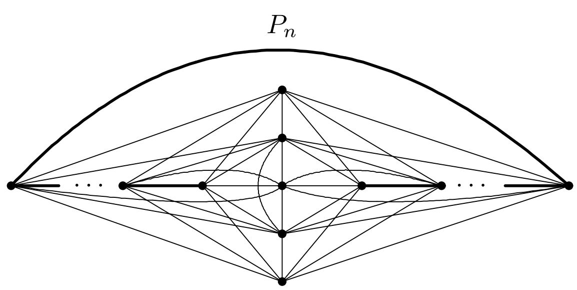
$$cr_\phi(H_{n-2}, T^{n-1} \cup T^n) \geq 4(n-2) + 1 = 4n - 7 \quad (4)$$

As $H_n = H_{n-2} \cup (T^{n-1} \cup T^n)$, using (1),(2) and (4), we have

$$\begin{aligned} cr_\phi(H_n) &= cr_\phi(H_{n-2}) + cr_\phi(T^{n-1} \cup T^n) + cr_\phi(H_{n-2}, T^{n-1} \cup T^n) \\ &\geq 4\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor + \lfloor\frac{n-2}{2}\rfloor + 4n - 7 \\ &= Z(5, n) + \lfloor\frac{n}{2}\rfloor \end{aligned}$$

This contradiction to (3). So the conclusion is held.

This completes the proof of Theorem 3.2. □




A good drawing of $G_5 \vee P_n$

Fig.10

**Theorem** 3.3 *$cr(G_5 \vee P_n) = Z(5, n) + \lfloor\frac{n}{2}\rfloor$, for $n \geq 1$.*

*Proof* The drawing in Fig.10 shows that $cr(G_5 \vee P_n) \leq Z(5, n) + \lfloor\frac{n}{2}\rfloor$. Contrast Fig.10 with Fig.5, it is easy to check that $G_5 \vee P_n$ has a subgraph which is homeomorphic to $H_n$, whose crossing number is $Z(5, n) + \lfloor\frac{n}{2}\rfloor$ in Theorem 3.2. So we have $cr(G_5 \vee P_n) \geq cr(H_n) = Z(5, n) + \lfloor\frac{n}{2}\rfloor$.

This completes the proof of Theorem 3.3. □

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