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On the Basis Number of the Direct Product of Theta Graphs with Cycles

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Abstract: The basis number of a graph G is defined to be the least integer d such that there is a cycle basis, $\mathcal{B}$, of the cycle space of G such that each edge of G is contained in at most d members of $\mathcal{B}$. MacLane [11] proved that a graph G is planar if and only if $b(G) \leq 2$. Jaradat [5] proved that the basis number of the direct product of a bipartite graph H with a cycle C is bounded above by $3 + b(H)$. In this work, we show that the basis number of the direct product of a theta graph with a cycle is 3 or 4. Our result, improves Jaradat's upper bound in the case that H is a theta graph containing no odd cycle by a combinatorial approach.

Key Words: cycle space; basis number; cycle basis; direct product.

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§1. Introduction

In graph theory, there are many numbers that give rise to a better understanding and interpretation of the geometric properties of a given graph such as the crossing number, the thickness, the genus, the basis number, etc.. The basis number of a graph is of a particular importance because MacLane, in [11], made a connection between the basis number and the planarity of a graph, which is related with parallel bundles on planar map geometries, a kind of Smarandache geometries; in fact, he proved that a graph is planar if and only if its basis number is at most 2.

In general, required cycle bases is not very well behaved under graph operations. That is the basis number $b(G)$ of a graph G is not monotonic (see [2] and [11]). Hence, there does not seem to be a general way of extending required cycle bases of a certain collection of partial graphs of G to a required cycle basis of G, respectively. Global upper bound $b(G) \leq 2\gamma(G) + 2$ where $\gamma(G)$ is the genus of G is proven in [12].

In this paper, we investigate the basis number for the direct product of a theta graphs with cycles.

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§2. Introduction

Unless otherwise specified, the graphs considered in this paper are finite, undirected, simple and connected. For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$.

Cycle Bases

For a given graph G, the set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X\backslash Y) \cup (Y\backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in \mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of G. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that for a connected graph G the dimension of the cycle space is the *cyclomatic number* or the *first Betti number*

$$\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r \tag{1}$$

where r is the number of components in G.

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a d-fold if each edge of G occurs in at most d of the cycles in $\mathcal{B}$. *The basis number,* $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a *d-fold* basis. The following result will be used frequently in the sequel.

Theorem 2.1.1.(MacLane). *The graph G is planar if and only if $b(G) \leq 2$.*

The following theorem due to Schmeichel, which proves the existence of graphs that have arbitrary large basis number.

Theorem 2.1.2. (Schmeichel) *For any positive integer r, there exists a graph G with $b(G) \geq r$.*

Products

Many authors studied the basis number of graph products. The Cartesian product, $\square$, was studied by Ali and Marougi [3] and Alsardary and Wojciechowski [4].

Theorem 2.2.1. (Ali and Marougi) *If G and H are two connected disjoint graphs, then* $\mathrm{b}(G\square H) \leq \max\{\mathrm{b}(G) + \triangle(T_H), \ \mathrm{b}(H) + \triangle(T_G)\}$ *where T_H and T_G are spanning trees of H and G, respectively, such that the maximum degrees $\triangle(T_H)$ and $\Delta(T_G)$ are minimum with respect to all spanning trees of H and G.*

Theorem 2.2.2.(Alsardary and Wojciechowski) *For every $d \geq 1$ and $n \geq 2$, we have $b(K_n^d) \leq 9$ where K_n^d is a d times Cartesian product of the complete graph K_n.*

An upper bound on the strong product $\boxtimes$ was obtained by Jaradat [9] when he gave the following result:

Theorem 2.2.3.(Jaradat) *Let G be a bipartite graph and H be a graph. Then* $b(G \boxtimes H) \leq \max\left\{b(H)+1, 2\Delta(H)+b(G)-1, \left\lfloor \frac{3\Delta(T_G)+1}{2} \right\rfloor, b(G)+2\right\}$.

The lexicographic product, $G[H]$, was studied by Jaradat and Al-zoubi [8] and Jaradat [10]. They obtained the following result results:

Theorem 2.2.4.(Jaradat and Al-Zoubi) *For each two connected graphs G and H,* $b(G[H]) \leq \mathrm{Max}\{4, 2\Delta(G) \ +b(H), 2+b(G)\}$.

Theorem 2.2.5.(Jaradat) *Let G, T_1* and *T_2 be a graph, a spanning tree of G and a tree, respectively. Then,* $b(G[T_2]) \leq b(G[H]) \leq \max\{5, 4+ \ 2\Delta(T^G_{\min})+b(H), 2+b(G)\}$ *where T^G stands for the complement graph of a spanning tree T in G and $T_{\min}$ stands for a spanning tree for G such that $\Delta(T^G_{\min}) = \min\{\ \Delta(T^G) | T$ is a spanning tree of $G\}$.*

Schmeichel [12], Ali [1], [2] and Jaradat [5] gave an upper bound for the basis number on the semi-strong product $\bullet$ and the direct product, $\times$, of some special graphs. They proved the following results:

Theorem 2.2.6. (Schmeichel) *For each $n \geq 7$, $b(K_n \bullet P_2) = 4$.*

Theorem 2.2.7.(Ali) *For each integers n, m, $b(K_m \bullet K_n) \leq 9$.*

Theorem 2.2.8. (Ali) *For any two cycles C_n and C_m* with $n, m \geq 3$, $b(C_n \times C_m) = 3$.

Theorem 2.2.9.(Jaradat) *For each bipartite graphs G and H, $b(G \times H) \leq 5 + b(G) + b(H)$.*

Theorem 2.2.10. (Jaradat) *For each bipartite graph G and cycle C, $b(G \times C) \leq 3 + b(G)$.*

We remark that knowing the number of components in a graph is very important to find the dimension of the cycle space as in (1), so we need the following result.

Theorem 2.2.11.([5]) *Let G and H be two connected graphs. Then $G \times H$ is connected if and only if at least one of them contains an odd cycle. Moreover, If both of them are bipartite graphs, then $G \times H$ consists of two components.*

For completeness, we recall that for two graphs G and H, the direct product $G \times H$ is the graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$.

In the rest of this paper, $f_B(e)$ stand for the number of elements of B containing the edge e where $B \subseteq \mathcal{C}(G)$.

§3. The Basis number of $\theta_n \times C_m$.

By specializing bipartite graph G in Theorem 2.2.10 into a theta graph θ_n containing no odd cycles, we have that $b(\theta_n \times C_m) \leq 5$. In this paper, we reduce the upper bound to 4. In fact, we prove that the basis number of the direct product of a theta graph with a cycle is either

3 or 4. Throughout this work we assume that $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, m\}$ to be the vertex sets of θ_n and C_m, respectively.

Definition 3.1. *A theta graph θ_n is defined to be a cycle C_n to which we add a new edge that joins two non-adjacent vertices. We may assume 1 and δ are the two vertices of θ_n of degree 3.*

The following result follows from Theorem 2.2.11 and noting that at least one of θ_n and C_m contains an odd cycle if and only if at least one of n, m, and δ is odd.

Lemma 3.2. *Let θ_n be a theta graph and C_m be a cycle. $\theta_n \times C_m$ is connected if and only if at least one of n, m, and δ is odd, otherwise it consists of two components.*

Note that $|E(\theta_n \times C_m)| = 2nm + 2m$ and $|V(\theta_n \times C_m)| = nm$. Hence, by the above lemma and equation (1), we have

$$\dim \mathcal{C}(\theta_n \times C_m) = nm + 2m + s,$$

where

$$s = \begin{cases} 1, & \text{if } \theta_n \times C_m \text{ is connected,} \\ 2, & \text{if } \theta_n \times C_m \text{ is disconnected.} \end{cases}$$

Lemma 3.3. *Let θ_n be a theta graph and C_m be a cycle. Then $b(\theta_n \times C_m) \geq 3$.*

Proof Note that $\theta_n \times C_m$ contains at most 4 cycles of length 3 and the other cycles are of length at least 4. Assume that $\theta_n \times C_m$ has a 2-fold basis $\mathcal{B}$. Then

$$\begin{aligned} 2(|E(\theta_n \times C_m)|) &\geq \sum_{C \in \mathcal{B}} |C| \\ &\geq 4(\dim \mathcal{C}(\theta_n \times C_m) - 4) + 3(4) \\ &\geq 4(\dim \mathcal{C}(\theta_n \times C_m) - 1), \end{aligned}$$

and so,

$$\begin{aligned} \frac{2(2nm + 2m)}{4} &\geq nm + 2m + s - 1 \\ nm + m &\geq nm + 2m + s - 1, \end{aligned}$$

where s is as above. Thus,

$$1 \geq m + s.$$

This is a contradiction. $\square$

Lemma 3.4. *For any graph θ_n of order $n \geq 4$ and cycle C_m of order $m \geq 3$, we have $b(\theta_n \times C_m) \leq 4$.*

Proof To prove the lemma, it is sufficient to exhibit a 4-fold basis, $\mathcal{B}$, for $\mathcal{C}(\theta_n \times C_m)$. According to the parity of m, n and δ (odd or even), we consider the following eight cases:

Case 1. m and n are even and δ is odd. Then, for each $j = 1, 2, \ldots, m-2$, we consider the following sets of cycles:

$$\begin{aligned}
A_1^{(j)} &= \{(i,j)\,(i+1,j+1)\,(i,j+2)\,(i-1,j+1)\,(i,j) : i = 2,3,\ldots n-1\} \\
&\quad \cup \{(1,j)\,(2,j+1)\,(1,j+2)\,(n,j+1)\,(1,j)\} \\
&\quad \cup \{(n,j)\,(n-1,j+1)\,(n,j+2)\,(1,j+1)\,(n,j)\},
\end{aligned}$$

$$\begin{aligned}
A_2^{(j)} &= \{(1,j)\,(2,j+1)\,(1,j+2)\,(\delta,j+1)\,(1,j)\}, \\
A_3^{(j)} &= \{(\delta,j)\,(\delta-1,j+1)\,(\delta,j+2)\,(1,j+1)\,(\delta,j)\}.
\end{aligned}$$

Also, we define the following cycles:

$$\begin{aligned}
c_1 &= (1,1)\,(2,2)\,(3,1)\ldots(n,2)\,(1,1), \\
c_2 &= (1,2)\,(2,1)\,(3,2)\ldots(n,1)\,(1,2), \\
c_3 &= (1,m)\,(2,m-1)\,(3,m)\ldots(\delta,m)\,(1,m-1) \\
&\quad\ (2,m)\ldots(\delta,m-1)\,(1,m).
\end{aligned}$$

Note that, the cycles of $A_1^{(j)}$ are edge pairwise disjoint for each $j = 1, 2, 3, \ldots, m-2$. Thus, $A_1^{(j)}$ is linearly independent and of 1-fold. Let $A_1 = \bigcup_{j=1}^{m-2} A_1^{(j)}$. Note that, each cycle of $A_1^{(j)}$ contains an edge of the form $(i+1, j+1)\,(i, j+2)$ or $(n-1, j+1)\,(n, j+2)$ which is not in $\mathrm{A}_1^{(j-1)}$. In addition, each cycle of $A_1^{(j-1)}$ contains an edge of the form $(i, j-1)\,(i+1, j)$ or $(n, j)\,(n-1, j+1)$ which is not in $A_1^{(j)}$. Therefore, A_1 is linearly independent. Let $V_1^{'} = \{(i,j) : i+j = \text{even}\}$, and $V_2^{'} = \{(i,j) : i+j = \text{odd}\}$. Let H_k be the induced subgraph of $V_k^{'}$ where $k = 1, 2$. For each $j = 1, 2, ..., m-2$, set

$$\begin{aligned}
B_1^{(j)} &= \{(i,j)\,(i+1,j+1)\,(i,j+2)\,(i-1,j+1)\,(i,j) \mid 2 \le i \le n-1 \text{ and} \\
i+j &= \text{even}\} \cup \{(1,j)\,(2,j+1)\,(1,j+2)\,(n,j+1)\,(1,j) : 1+j = \text{even}\} \\
&\quad \cup \{(n,j)\,(n-1,j+1)\,(n,j+2)\,(1,j+1)\,(n,j) :\ n+j = \text{even}\}, \\
B_2^{(j)} &= \{(i,j)\,(i+1,j+1)\,(i,j+2)\,(i-1,j+1)\,(i,j) \mid 2 \le i \le n-1 \text{ and} \\
i+j &= \text{odd}\} \cup \{(1,j)\,(2,j+1)\,(1,j+2)\,(n,j+1)\,(1,j) :\ 1+j = \text{odd}\} \\
&\quad \cup \{(n,j)\,(n-1,j+1)\,(n,j+2)\,(1,j+1)\,(n,j) :\ n+j = \text{odd}\}.
\end{aligned}$$

Let $F^{(k)} = \bigcup_{j=1}^{m-2} B_k^{(j)}$ where $k = 1, 2$. We prove that c_k is independent from the cycles of $F^{(k)}$. Let $E_j^{(k)} = E\,(C_n \times j\,(j+1)) \cap E\,(H_k)$ where C_n is the cycle in θ_n obtained by deleting the edge 1δ from θ_n. Then it is an easy matter to verify that $\left\{E_1^{(k)}, E_2^{(k)}, \ldots, E_{m-1}^{(k)}\right\}$ is a partition of $E\,(C_n \times P_m) \cap E\,(H_k)$ where P_m is the path of C_m obtained by deleting the edge $1m$. Moreover, it is clear that $E_1^{(k)} = E\,(c_k)$ and $E_1^{(k)} \cup E_2^{(k)} = E\left(B_k^{(1)}\right)$. Thus, if c_k is a sum modulo 2 of some cycles of $F^{(k)}$, say $\{T_1, T_2, \ldots, T_r\}$, then $B_k^{(1)} \subseteq \{T_1, T_2, \ldots, T_r\}$.

Since no edges in $E_2^{(k)}$ belongs to $E(c_k)$ and $E_2^{(k)} \cup E_3^{(k)} = E\left(B_k^{(2)}\right), B_k^{(2)} \subseteq \{T_1, T_2, \ldots, T_r\}$. By continuing in this way, it implies that $B_k^{(m-2)} \subseteq \{T_1, T_2, \ldots, T_r\}$. Note that $E_{m-2}^{(k)} \cup E_{m-1}^{(k)} = E\left(B_k^{(m-2)}\right)$ and each edge of $E_{m-1}^{(k)}$ appears in one and only one cycle of $F^{(k)}$. It follows that $E_{m-1}^{(k)} \subseteq E(c_k)$. This is a contradiction. Therefore, $F^{(k)} \cup \{c_k\}$ is linearly independent for $k = 1, 2$. And since $E\left(F^{(1)} \cup \{c_1\}\right) \cap E\left(F^{(2)} \cup \{c_2\}\right) = \phi$, we have $F^{(1)} \cup F^{(2)} \cup \{c_1, c_2\} = A_1 \cup \{c_1, c_2\}$ is linearly independent. Let $A_2 = \overset{m-2}{\underset{j=1}{\cup}} A_2^{(j)}$ and $A_3 = \overset{m-2}{\underset{j=1}{\cup}} A_3^{(j)}$. It is easy to see that the cycles of A_i are edge pairwise disjoint for $i = 2, 3$ and each cycle of A_3 contains at least one edge of the form $(\delta, j)(\delta - 1, j + 1)$ and $(\delta, j)(\delta - 1, j - 1)$ which is not in A_2. And so $A_2 \cup A_3$ is linearly independent. Clearly, c_3 can not be written as a linear combination of cycles of $A_2 \cup A_3$. Therefore, $A_2 \cup A_3 \cup \{c_3\}$ is linearly independent. Let $B_1 = A_1 \cup A_2 \cup A_3 \cup \{c_1, c_2, c_3\}$. We now prove that B_1 is a linearly independent set. Note that $E(A_2 \cup A_3 \cup \{c_3\}) - \{(1, j)(\delta, j + 1), (1, j + 1)(\delta, j) | 1 \leq j \leq m - 1\}$ forms an edge set of a forest. Thus, any linear combinations of cycles of $A_2 \cup A_3 \cup \{c_3\}$ must contains at least one edge of the form $(1, j)(\delta, j + 1)$ and $(1, j + 1)(\delta, j)$ for some $j \leq m - 1$ because any linear combination of a linearly independent set of cycles is a cycle or an edge disjoint union of cycles. Now, Suppose that there are two sets of cycles say $\{d_1, d_2, \ldots, d_{\gamma_1}\} \subseteq A_1 \cup \{c_1, c_2\}$ and $\{f_1, f_2, \ldots, f_{\gamma_2}\} \subseteq A_2 \cup A_3 \cup \{c_3\}$ such that $\sum_{i=1}^{\gamma_1} d_i = \sum_{i=1}^{\gamma_2} f_i \pmod 2$. Consequently, $E(d_1 \oplus d_2 \oplus \cdots \oplus d_{\gamma_1}) = E(f_1 \oplus f_2 \oplus \cdots \oplus f_{\gamma_2})$ and so $d_1 \oplus d_2 \oplus \cdots \oplus d_{\gamma_1}$ contains at least one edge of the form $(1, j)(\delta, j + 1)$ and $(1, j + 1)(\delta, j)$ for some $j \leq m - 1$, which contradicts the fact that no cycle of $A_1 \cup \{c_1, c_2\}$ contains such edges. We now define the following sets of cycles

$$\begin{aligned} A_4 &= \{A_4^{(i)} = (i+1, 1)(i+2, 2)(i+1, 3) \ldots (i+2, m)(i+1, 1) : i = 0, 1, \\ &\quad \ldots, n-2\}, \\ A_5 &= \left\{A_5^{(i)} = (i+1, 1)(i, 2)(i+1, 3) \ldots (i, m)(i+1, 1) : i = 1, 2, \ldots, n-1\right\}, \end{aligned}$$

and

$$\begin{aligned} c_1' &= (\delta, 1)(\delta+1, 2)(\delta, 3)(\delta+1, 4) \ldots (\delta, m-1)(1, m)(\delta, 1), \\ c_2' &= (1, 1)(2, m)(3, 1)(4, m) \ldots (\delta, 1)(1, m)(2, 1) \ldots (\delta, m)(1, 1), \\ c_3' &= (1, 1)(n, 2)(1, 3)(n, 4) \ldots (n, m)(1, 1), \\ c_4' &= (n, 1)(1, 2)(n, 3)(1, 4) \ldots (1, m)(n, 1). \end{aligned}$$

Let $D = A_4 \cup A_5 \cup \left\{c_1', c_2', c_3', c_4'\right\}$. Each cycle $A_4^{(i)}$ of A_4 contains the edge $(i+2, m)(i+1, 1)$ which belongs to no other cycles of $B_1 \cup A_4$. Thus $B_1 \cup A_4$ is linearly independent. Similarly, each cycle $A_5^{(i)}$ of A_5 contains the edge $(i, m)(i+1, 1)$ which belongs to no other cycles of $B_1 \cup A_4 \cup A_5$. Hence $B_1 \cup A_4 \cup A_5$ is linearly independent. Now c_1' is the only cycle of $B_1 \cup A_4 \cup A_5 \cup \left\{c_1'\right\}$ which contains the edge $(1, m)(\delta, 1)$. Hence $B_1 \cup A_4 \cup A_5 \cup \left\{c_1'\right\}$ is linearly independent. Similarly, c_2' is the only cycle of $B_1 \cup A_4 \cup A_5 \cup \left\{c_1', c_2'\right\}$ which contains the edge $(\delta, m)(1, 1)$. Thus $B_1 \cup A_4 \cup A_5 \cup \left\{c_1', c_2'\right\}$ is linearly independent. Now, c_3' and c_4' contain

$(1,1)(n,m)$ and $(1,m)(n,1)$, respectively, which appear in no cycle of $B_1 \cup A_4 \cup A_5 \cup \left\{c_1^{'}, c_2^{'}\right\}$. Therefor, $\mathcal{B} = B_1 \cup D$ is linearly independent. Now,

$$\begin{aligned}
|\mathcal{B}| &= \sum_{i=1}^{5} |A_i| + \sum_{i=1}^{3} |c_i| + \sum_{i=1}^{4} |c_i^{'}| \\
&= n(m-2) + (m-2) + (m-2) + (n-1) + (n-1) + 3 + 4 \\
&= nm + 2m + 1 \\
&= \dim \mathcal{C}(\theta_n \times C_m).
\end{aligned}$$

Hence, $\mathcal{B}$ is a basis of $\theta_n \times C_m$. To complete the proof of this case, we only need to prove that $\mathcal{B}$ is a 4-fold basis. For simplicity, set $Q = \cup_{i=1}^{3} \{c_i\}$. Let $e \in E(\theta_n \times C_m)$. Then

(1) If $e = (i,j)(i+1,j+1)$ or $(n,j)(1,j+1)$ where $1 \le i \le n-1$, and $2 \le j \le m-2$, then $f_{A_1}(e) = 2$, $f_{A_2 \cup A_3}(e) \le 1$, $f_D(e) \le 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 4$.

(2) If $e = (i,j)(i+1,j-1)$ or $(n,j)(1,j-1)$ where $1 \le i \le n-1$, and $3 \le j \le m-1$, then $f_{A_1}(e) = 2$, $f_{A_2 \cup A_3}(e) \le 1$, $f_D(e) \le 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 4$.

(3) If $e = (i,1)(i+1,2)$ or $(1,1)(n,2)$, where $1 \le i \le n-1$, then $f_{A_1}(e) = 1$, $f_{A_2 \cup A_3}(e) \le 1$, $f_D(e) \le 1$, and $f_Q(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 4$.

(4) If $e = (i,2)(i+1,1)$ or $(1,2)(n,1)$ where $1 \le i \le n-1$, then $f_{A_1}(e) = 1$, $f_{A_2 \cup A_3}(e) \le 1$, $f_D(e) \le 1$, and $f_Q(e) = 1$, and so $f_{\mathcal{B}}(e) \le 4$.

(5) If $e = (1,j)(\delta,j+1)$ where $1 \le j \le m-2$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) \le 2$, $f_D(e) = 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 3$.

(6) If $e = (1,j)(\delta,j-1)$ where $2 \le j \le m-2$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) \le 2$, $f_D(e) = 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 3$.

(7) If $e = (i,m-1)(i+1,m)$ or $(i,m)\ (i+1,m-1)$ or $(1,m)\ (n,m-1)$ where $1 \le i \le n-1$, then $f_{A_1}(e) = 1$, $f_{A_2 \cup A_3}(e) \le 1$, $f_D(e) \le 1$, and $f_Q(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 4$.

(8) If $e = (1,m)(\delta,m-1)$ or $(1,m-1)\ (\delta,m)$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) \le 1$, $f_D(e) \le 1$, and $f_Q(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 3$.

(9) If $e = (i+1,1)\ (i,m)$ or $(i,1)(i+1,m)$, where $1 \le i \le n-1$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_D(e) \le 2$, and $f_Q(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 3$.

(10) If $e = (1,1)(\delta,m)$ or $(1,m)\ (\delta,1)$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_D(e) \le 2$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 2$.

(11) If $e = (1,1)(n,m)$ or $(n,1)\ (1,m)$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_D(e) \le 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 1$. Therefore $\mathcal{B}$ is a 4-fold basis. The proof of this case is complete.

Case 2. m and δ are even and n is odd. Now, consider the following sets of cycles: A_1, A_2 and A_3 are as in Case 1 and

$$\begin{aligned} c_1 &= (1,m)\,(2,m-1)\,(3,m)\ldots(\delta,m-1)\,(1,m)\,, \\ c_2 &= (1,m-1)\,(2,m)\,(3,m-1)\ldots(\delta,m)\,(1,m-1)\,, \\ c_3 &= (1,1)\,(2,2)\,(3,1)\,...\,(n,1)\,(1,2)\,(2,1)\ldots(n,2)\,(1,1)\,. \end{aligned}$$

Let $B_1 = \left(\cup_{i=1}^{3} A_i\right) \cup \left(\cup_{i=1}^{3} \{c_i\}\right)$. Since $E(c_1) \cap E(c_2) = \varnothing$, $\{c_1, c_2\}$ is linearly independent. Since $\delta \geq 4$, c_1 contains an edge of the form $(2, m-1)\,(3, m)$ and c_2 contains an edge of the form $(2, m)\,(3, m-1)$ and each of which does not appear in any cycles of $A_2 \cup A_3$. Thus $A_2 \cup A_3 \cup \{c_1, c_2\}$ is linearly independent. Next, we show that $A_1 \bigcup \{c_3\}$ is linearly independent. Let $R_i = E(C_n \times\ i\,(i+1))$ where C_n is as in Case 1. Note that $\{R_1, R_2, \ldots, R_{m-1}\}$ is a partition of $E(C_n \times P_m)$ where P_m is as in Case 1. Also, $E(c_3) = R_1$ and $R_1 \cup R_2 = E\left(A_1^{(1)}\right)$. Thus, if c_3 can be written as linear combination of some cycles of A_1, say $\{K_1, K_2, \ldots, K_r\}$, then $A_1^{(1)} \subseteq \{K_1, K_2, \ldots, K_r\}$. Since $R_2 \cup R_3 = E\left(A_1^{(2)}\right)$ and no edges of R_2 belongs to $E(c_3)$, $A_1^{(2)} \subseteq \{K_1, K_2, \ldots, K_r\}$, and so on. This implies that $A_1^{(m-2)} \subseteq \{K_1, K_2, \ldots, K_r\}$. Note that $R_{m-1} \subseteq E\left(A_1^{(m-2)}\right)$ and each edge of R_{m-1} appears only in one cycle of A_1. Thus $R_{m-1} \subseteq E(c_3)$. This is a contradiction. Hence $A_1 \cup \{c_3\}$ is linearly independent. Let $B_1 = A_1 \cup A_2 \cup A_3 \cup \{c_1, c_2, c_3\}$. To show that B_1 is a linearly independent set, we assume that there are two set of cycles say $\{d_1, d_2, \ldots, d_{\gamma_1}\} \subseteq A_1 \cup \{c_3\}$ and $\{f_1, f_2, \ldots, f_{\gamma_2}\} \subseteq A_2 \cup A_3 \cup \{c_1, c_2\}$ such that $\sum_{i=1}^{\gamma_1} d_i = \sum_{i=1}^{\gamma_2} f_i \pmod 2$. By using the same argument as in Case 1, we have that $d_1 \oplus d_2 \oplus \cdots \oplus d_{\gamma_1}$ contains at least one edge of the form $(1, j)\,(\delta, j+1)$ and $(1, j+1)\,(\delta, j)$ for some $j \leq m-1$. Which contradicts the fact that no cycle of $A_1 \cup\ \{c_1\}$ contains such edges. Now, let A_4, $A_5, c_3^{'}$ and $c_4^{'}$ are as defined in Case 1, and define the following cycles:

$$\begin{aligned} c_1^{'} &= (1,1)\,(2,m)\,(3,1)\ldots(\delta-1,1)\,(\delta,m)\,(1,1)\,, \\ c_2^{'} &= (1,m)\,(2,1)\,(3,m)\ldots(\delta,1)\,(1,m)\,. \end{aligned}$$

Let $D = A_4 \cup A_5 \cup \left\{c_1^{'}, c_2^{'}, c_3^{'}, c_4^{'}\right\}$. By following the same arguments as in Case 1, we can prove that $\mathcal{B} = B_1 \cup D$ is a 4-fold basis for $C(\theta_n \times C_m)$. The proof of this case is complete.

Case 3. m, n, and δ are even. Consider the following sets of cycles: A_1, A_2, A_3, A_4, A_5 and $\{c_1, c_2\}$ are as in Case 1. Also, consider $c_3 = c_1$ and $c_4 = c_2$ where c_1 and c_2 are as defined in Case 2. Moreover, $c_1^{'}$ and $c_2^{'}$ are as in Case 2. Define the following two cycles:

$$\begin{aligned} c_3^{'} &= (1,1)\,(2,m)\,(3,1)\ldots(n-1,1)\,(n,m)\,(1,1)\,, \\ c_4^{'} &= (1,m)\,(2,1)\,(3,m)\ldots(n-1,m)\,(n,1)\,(1,m)\,. \end{aligned}$$

By using the same arguments as in Case 1 and Case 2, we can show that

$$\mathcal{B} = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup \left\{c_1, c_2, c_3, c_4, c_1^{'}, c_2^{'}, c_3^{'}, c_4^{'}\right\},$$

is linearly independent. Since

$$\begin{aligned}|\mathcal{B}| &= \sum_{i=1}^{5}|A_i|+8\\ &= n(m-2)+(m-2)+(m-2)+(n-1)+(n-1)+8\\ &= nm+2m+2\\ &= \dim\mathcal{C}(\theta_n\times C_m),\end{aligned}$$

$\mathcal{B}$ is a basis of $\mathcal{C}(\theta_n\times C_m)$. To show that $\mathcal{B}$ is a 4-fold basis, we follow, word by word, (1) to (11) of Case 1. The proof of this case is complete.

Case 4. m is even, and δ and n are odd. By relabeling the vertices of θ_n in the opposite direction, we get a similar case to Case 2. The proof of this case is complete.

Case 5. m is odd, and n and δ are even. Consider the following sets of cycles: A_1, A_2, A_3 and $\{c_1, c_2\}$ are as in Case 1. In addition, $c_3 = c_1$ and $c_4 = c_2$ where c_1 and c_2 are as in Case 2. Using the same arguments as in Case 1 and Case 2, we can show that each of $A_1\cup\{c_1,c_2\}$ and $A_2\cup A_3\cup\{c_3,c_4\}$ are linearly independent. Also, then we show that $A_1\cup A_2\cup A_3\cup\{c_1,c_2,c_3,c_4\}$ is linearly independent. Now, we define the following set of cycles:

$$A_4=\left\{a_4^{(i)}=(i,m)(i+1,m-1)(i+2,m)(i+1,1)(i,m):1\le i\le n-2\right\},$$

and

$$A_5=\left\{a_5^{(i)}=(i,1)(i+1,m)(i+2,1)(i+1,2)(i,1):1\le i\le n-2\right\}.$$

Also, define the following cycle:

$$\begin{aligned}c_5^{'} &= (n-1,1)(n,m)(n-1,m-1)(n,m-2)\ldots(n,1)(n-1,m)(n,m-1)\\ &\quad (n-1,\ m-2)\ldots(n,2)(n-1,1).\end{aligned}$$

Note that, $c_5^{'}$ contains the edge $(n-1,m)(n,1)$ which does not occur in any cycle of $B_1 = A_1\cup A_2\cup A_3\cup\{c_1,c_2,c_3,c_4\}$. Thus, $B_1\cup\left\{c_5^{'}\right\}$ is linearly independent. For simplicity, we set $D=\{D_k\}_{k=1}^{n-2}$, where $D_k=\left\{a_4^{(k)},\ a_5^{(k)}\right\}$. We now, use induction on n to show that the cycles of D are linearly independent. If $n=3$, then $D=D_1=\left\{a_4^{(1)},a_5^{(1)}\right\}$. $a_4^{(1)}$ contains the edge $(2,1)(3,m)$ which does not occur in the cycle $a_5^{(1)}$. Hence D is linearly independent. Assume $n>3$ and it is true for less than n. Note that $D=\{D_k\}_{k=1}^{n-3}\cup\left\{a_4^{(n-2)},a_5^{(n-2)}\right\}$. By the inductive step $\{D_k\}_{k=1}^{n-3}$ is linearly independent. Now, the cycle $a_4^{(n-2)}$ contains the edge $(n-1,1)(n,m)$ which does not occur in any cycle of $\{D_k\}_{k=1}^{2n-3}$, similarly the cycle $a_5^{(n-2)}$ contains the edge $(n,1)(n-1,m)$ which does not occur in any cycle of $\{D_k\}_{k=1}^{n-3}\cup\left\{a_4^{(n-2)}\right\}$. Therefore, D is linearly independent. Note that $E(D)-\{(i+1,1)(i,m),(i,1)(i+1,m)|1\le i\le n-2\}$ forms an edge set of a forest. Thus, any linear combination of cycles of D must contain an edge of the form $(i+1,1)(i,m)$ or $(i,1)(i+1,m)$ for some $1\le i\le n-2$ which does not occur in any cycle of $B_1\cup\left\{c_5^{'}\right\}$. Therefore, $B_1\cup\left\{c_5^{'}\right\}\cup D$ is linearly independent.

We now consider $c_1^{'}$ and $c_2^{'}$ as in Case 2 and $c_3^{'}$ and $c_4^{'}$ as in Case 1. Note that $c_1^{'}$ and $c_2^{'}$ contain the edges $(1,1)(\delta,m)$ and $(1,m)(\delta,1)$, respectively, which do not appear in any cycle of $B_1 \cup \left\{c_5^{'}\right\} \cup D$. Thus $B_1 \cup D \cup \left\{c_1^{'}, c_2^{'}, c_5^{'}\right\}$ is linearly independent. Similarly $c_3^{'}$ and $c_4^{'}$ contain the edges $(1,1)(n,m)$ and $(1,m)(n,1)$, respectively, which do not appear in any cycle of $B_1 \cup \left\{c_1^{'}, c_2^{'}, c_5^{'}\right\} \cup D$. Thus

$$\mathcal{B} = B_1 \cup D \cup \left\{c_1^{'}, c_2^{'}, c_3^{'}, c_4^{'}, c_5^{'}\right\}$$

is linearly independent. Now,

$$\begin{aligned}
|\mathcal{B}| &= \sum_{i=1}^{5} |A_i| + 9 \\
&= n(m-2) + (m-2) + (m-2) + (n-2) + (n-2) + 9 \\
&= nm + 2m + 1 \\
&= \dim \mathcal{C}(\theta_n \times C_m).
\end{aligned}$$

Hence, B is a basis of $\mathcal{C}(\theta_n \times C_m)$. To complete the proof of this case, we show that B is a 4-fold basis. Let $e \in E(\theta_n \times C_m)$. Then,

(1) If $e = (i+1,1)(i,m)$ or $(i,1)\ (i+1,m)$ where $1 \le i \le n-1$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_{D \cup \{c_i^{'}\}_{i=1}^{5}}(e) \le 3$, and $f_{\cup_{i=1}^{4}\{c_i\}}(e) = 0$, and so $f_{\mathcal{B}}(e) \le 3$.

(2) If e is as in (1) to (11) of Case 1 and not of the above form, then, as in that Case 1, $f_{\mathcal{B}}(e) \le 4$. Therefore $\mathcal{B}$ is a 4-fold basis. The proof of this case is complete.

Case 6. m and δ are odd and n is even. According to the relation between m and δ, we split this case into two subcases.

Subcase 6a. $\delta \le m$. Then consider the following sets of cycles: $A_1, A_2, A_3, c_1, c_2, c_3$ are as in Case 1. In addition, for each $i = 2, 3, \ldots, \delta$, we define the following sets of cycles.

$$\begin{aligned}
F_i = {} & (i,1)(i-1,2)(i-2,3)\ldots(1,i)(\delta,i+1)(\delta-1,i+2) \\
& (\delta-2,i+3)\ldots(i,\delta+1)(i+1,\delta+2)(i,\delta+3)\ldots(i-1,m)(i,1),
\end{aligned}$$

and for each $i = 1, 2, 3, \ldots, \delta - 1$

$$\begin{aligned}
F_i' = {} & (i,1)(i+1,2)(i+2,3)\ldots(\delta,\delta-i+1)(1,\delta-i+2) \\
& (2,\delta-i+3)\ldots(i,\delta+1)(i+1,\delta+2)(i,\delta+3)\ldots(i+1,m)(i,1).
\end{aligned}$$

Also, set

$$\begin{aligned}
F_1 = {} & (1,1)(\delta,2)(\delta-1,3)(\delta-2,4)\ldots(1,\delta+1)(\delta,\delta+2) \\
& (1,\delta+3)\ldots(1,m-1)(\delta,m)(1,1),
\end{aligned}$$

and

$$\begin{aligned} F_{\delta}^{'} &= (\delta,1)\,(1,2)\,(2,3)\,(3,4)\ldots(\delta,\delta+1)\,(1,\delta+2)\,(\delta,\delta+3) \\ &\quad (1,\ \delta+4)\ldots(\delta,m-1)\,(1,m)\,(\delta,1)\,. \end{aligned}$$

Let

$$F=\cup_{i=1}^{\delta}F_i \text{ and } F^{'}=\cup_{i=1}^{\delta}F_i^{'}.$$

By Case 1, $A_1\cup A_2\cup A_3\cup\{c_1,c_2,c_3\}$ is linearly independent. Note that each cycle of F contains an edge of the form $(i-1,m)\,(i,1)$ or $(\delta,m)\,(1,1)$ for some $2\le i\le\delta$ which does not occur in any other cycle of $A_1\cup A_2\cup A_3\cup F\cup\{c_1,c_2,c_3\}$. Thus, $A_1\cup A_2\cup A_3\cup F\cup\{c_1,c_2,c_3\}$ is linearly independent. Similarly, each cycle of $F^{'}$ contains an edge of the form $(i+1,m)\,(i,1)$ or $(1,m)\,(\delta,1)$ for some $1\le i\le\delta-1$ which does not occur in any other cycle of $A_1\cup A_2\cup A_3\cup F\cup F^{'}\cup\{c_1,c_2,c_3\}$. Thus, $A_1\cup A_2\cup A_3\cup F\cup F^{'}\cup\{c_1,c_2,c_3\}$ is linearly independent. Now, define the following sets of cycles:

$$A_4=\left\{a_4^{(i)}=(i,m)\,(i+1,m-1)\,(i+2,m)\,(i+1,1)\,(i,m):\delta-1\le i\le n-2\right\},$$

and

$$A_5=\left\{a_5^{(i)}=(i,1)\,(i+1,m)\,(i+2,1)\,(i+1,2)\,(i,1):\delta-1\le i\le n-2\right\}.$$

Also, set the following cycles:

$$\begin{aligned} c_4 &= (1,1)\,(2,m)\,(3,1)\ldots(n,m)\,(1,1)\,, \\ c_5 &= (1,m)\,(2,1)\,(3,m)\ldots(n,1)\,(1,m)\,. \end{aligned}$$

By using the same arguments as in Case 5, we can show that $A_4\cup A_5$ is linearly independent. Since each linear combination of cycles of $A_4\cup A_5$ contains an edge of the form $(i+1,1)\,(i,m)$ or $(i,1)\,(i+1,m)$ for some $\delta\le i\le n-2$ which does not occurs in any cycle of $A_1\cup A_2\cup A_3\cup F\cup F^{'}\cup\{c_1,c_2,c_3\}$, $A_1\cup A_2\cup A_3\cup A_4\cup A_5\cup F\cup F^{'}\cup\{c_1,c_2,c_3\}$ is linearly independent. Finally, c_4 contains the edge $(n,m)\,(1,1)$ and c_5 contains the edge $(n,1)\,(1,m)$ which do not appear in any cycle of $A_1\cup A_2\cup A_3\cup A_4\cup A_5\cup F\cup F^{'}\cup\{c_1,c_2,c_3\}$. Thus,

$$\mathcal{B}=A_1\cup A_2\cup A_3\cup A_4\cup A_5\cup F\cup F^{'}\cup\{c_1,c_2,c_3,c_4,c_5\}$$

is linearly independent. Since

$$\begin{aligned} |\mathcal{B}| &= \sum_{i=1}^{5}|A_i|+|F|+|F^{'}|+\sum_{i=1}^{5}|c_i| \\ &= n(m-2)+(m-2)+(m-2)+(n-\delta)+ \\ &\quad (n-\delta)+\delta+\delta+5 \\ &= mn+2m+1 \\ &= \dim\mathcal{C}(\theta_n\times C_m), \end{aligned}$$

$\mathcal{B}$ is a basis of $\mathcal{C}(\theta_n \times C_m)$. To complete the proof of the theorem we only need to prove that $\mathcal{B}$ is a 4-fold basis. For simplicity let $Q = \cup_{i=1}^{5}\{c_i\}$. Let $e \in E\left(\theta_n \times C_m\right)$. Then

(1) if $e = (i,j)\,(i+1,j+1)$ or $(n,j)\,(1,j+1)$, where $1 \le i \le n-1$, and $2 \le j \le m-2$, then $f_{A_1}(e) = 2$, $f_{A_2 \cup A_3}(e) \le 1$, $f_{F \cup F'}(e) = 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 4$.

(2) If $e = (i,j)\,(i+1,j-1)$ or $(n,j)\,(1,j-1)$, where $1 \le i \le n-1$, and $3 \le j \le m-1$, then $f_{A_1}(e) = 2$, $f_{A_2 \cup A_3}(e) \le 1$, $f_{F \cup F'}(e) = 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 4$.

(3) If $e = (i,1)\,(i+1,2)$ or $(1,1)\,(n,2)$, where $1 \le i \le n-1$, then $f_{A_1}(e) = 1$, $f_{A_2 \cup A_3}(e) \le 1$, $f_{F \cup F'}(e) = 1$, and $f_Q(e) = 1$, and so $f_{\mathcal{B}}(e) \le 4$.

(4) If $e = (i,2)\,(i+1,1)$ or $(1,2)\,(n,1)$, where $1 \le i \le n-1$, then $f_{A_1}(e) = 1$, $f_{A_2 \cup A_3}(e) \le 1$, $f_{F \cup F'}(e) = 0$, and $f_Q(e) = 1$, and so $f_{\mathcal{B}}(e) \le 3$.

(5) If $e = (1,j)\,(\delta,j+1)$,where $1 \le j \le m-2$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) \le 2$, $f_{F \cup F'}(e) = 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 3$.

(6) If $e = (1,j)\,(\delta,j-1)$,where $2 \le j \le m-2$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) \le 2$, $f_{F \cup F'}(e) = 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 3$.

(7) If $e = (i,m-1)\,(i+1,m)$ or $(i,m)\ (i+1,m-1)$ or $(1,m)\ (n,m-1)$, where $1 \le i \le n-1$, then $f_{A_1}(e) = 1$, $f_{A_2 \cup A_3}(e) \le 1$, $f_{F \cup F'}(e) = 1$, and $f_Q(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 4$.

(8) If $e = (1,m)\,(\delta,m-1)$ or $(1,m{-}1)\ (\delta,m)$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) \le 1$, $f_{F \cup F'}(e) = 1$, and $f_Q(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 3$.

(9) If $e = (i,1)\,(i+1,m)$ or $(i+1,1)\ (i,m)$, where $1 \le i \le n-2$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_{F \cup F'}(e) \le 2$, and $f_Q(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 3$.

(10) If $e = (1,1)\,(\delta,m)$ or $(1,m)\ (\delta,1)$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_{F \cup F'}(e) = 1$, and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \le 1$.

(11) If $e = (1,1)\,(n,m)$ or $(n,1)\ (1,m)$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_{F \cup F'}(e) \le 1$, and $f_Q(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 2$. Therefore $\mathcal{B}$ is a 4-fold basis.

Subcase 6b. $m < \delta$. Then consider the following set of cycles: A_1, c_1, c_2 are as in Case 1 and A_4 and A_5 are as in Case 4, and

$$c_3 = (1,1)(2,2)(1,3)(2,4)\ldots(1,m)(2,1)(1,2)(2,3)\ldots(2,m)(1,1).$$

Using similar arguments to Case 5, we can show that $A_1 \cup A_4 \cup A_5 \cup \{c_1, c_2, c_3\}$ is a linearly independent set. Now, let c_4 and c_5 be the two cycles as in the Subcase 6a. Then c_4 contains the edge $(n,m)(1,1)$ which does not appear in the cycles of $A_1 \cup A_4 \cup A_5 \cup \{c_1, c_2, c_3\}$. Thus, $A_1 \cup A_4 \cup A_5 \cup \{c_1, c_2, c_3, c_4\}$ is linearly independent. Similarly, c_5 contains the edge $(n,1)(1,m)$ which does not appear in any cycle of $A_1 \cup A_4 \cup A_4 \cup \{c_1, c_2, c_3, c_4\}$. Therefore, $A_1 \cup A_4 \cup A_5 \cup \{c_1, c_2, c_3, c_4, c_5\}$ is linearly independent. Now, for $j = 2, 3, \ldots, m$ define the following cycles:

$$\begin{aligned} F_i &= (1,j)(2,j-1)(3,j-2)\ldots(j,1)(j+1,m)(j+2,m-1)(j+3,m-2)\ldots \\ &\quad (m+1,j)(m+2,j-1)(m+3,j)\ldots(\delta,j-1)(1,j), \end{aligned}$$

and for $j = 1, 2, 3, \ldots, m-1$

$$\begin{aligned} F_i^{'} &= (1,j)(2,j+1)(3,j+2)\ldots(m-j+1,m)(m-j+2,1)(m-j+3,2) \\ &\quad (m-j+4,3)\ldots(m+1,j)(m+2,j+1)(m+3,j)\ldots(\delta,j+1)(1,j). \end{aligned}$$

Moreover, define

$$\begin{aligned} F_1 &= (1,1)(2,m)(3,m-1)(4,m-2)\ldots(m+1,1)(m+2,m)(m+3,1)\ldots(\delta,m)(1,1) \\ F_m^{'} &= (1,m)(2,1)(3,2)(4,3)\ldots(m+1,m)(m+2,1)(m+3,2)\ldots(\delta,1)(1,m). \end{aligned}$$

Let

$$F = \cup_{i=1}^{m} F_i \text{ and } F^{'} = \cup_{i=1}^{m} F_i^{'}.$$

Note that each cycle of $F \cup F^{'}$ contains an edge of the form $(\delta, j+1)(1,j)$ or $(\delta, j-1)(1,j)$ which does not appear in other cycles of

$$\mathcal{B} = A_1 \cup A_4 \cup A_5 \cup F \cup F^{'} \cup \{c_1, c_2, c_3, c_4, c_5\}$$

Thus, $\mathcal{B}$ is linearly independent. Since

$$\begin{aligned} |B| &= |A_1| + |A_4| + |A_5| + |F| + |F^{'}| + \sum_{i=1}^{5} |c_i| \\ &= (m-2)n + (n-2) + (n-2) + m + m + 5 \\ &= mn + 2m + 1 \\ &= \dim \mathcal{C}(\theta_n \times C_m), \end{aligned}$$

$\mathcal{B}$ is a basis for $\mathcal{C}(\theta_n \times C_m)$. Now to complete the proof, we show that $\mathcal{B}$ is a 4-fold basis. Let $e \in E(\theta_n \times C_m)$. Then

(1) if $e = (i,j)\,(i+1,j+1)$ where $1 \le i \le \delta - 2$, and $1 \le j \le m-1$ or $(i,j)\,(i-1,j-1)$, where $2 \le i \le \delta - 1$, and $2 \le j \le m$, then $f_{A_1 \cup \{c_i\}_{i=1}^5}(e) \le 2$, $f_{A_4 \cup A_5}(e) \le 1$, $f_{F \cup F'}(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 4$.

(2) If $e = (i,j)\,(i+1,j+1)$ where $\delta \le i \le n-1$, and $1 \le j \le m-1$ or $(i,j)\,(i-1,j-1)$, where $\delta + 1 \le i \le n$, and $2 \le j \le m$, then $f_{A_1 \cup \{c_i\}_{i=1}^5}(e) \le 3$, $f_{A_4 \cup A_5}(e) \le 1$, $f_{F \cup F'}(e) = 0$ and so $f_{\mathcal{B}}(e) \le 4$.

(3) If $e = (i,1)\,(i+1,m)$ or $(i+1,1)\,(i,m)$ or $(1,1)\,(n,m)$ or $(1,m)\,(n,1)$ where $1 \le i \le n-2$, then $f_{A_1 \cup \{c_i\}_{i=1}^5}(e) \le 1$, $f_{A_4 \cup A_5}(e) \le 2$, $f_{F \cup F'}(e) = 0$ and so $f_{\mathcal{B}}(e) \le 3$.

(4) If $e = (1,j)\,(\delta,j+1)$ or $(1,j+1)\,(\delta,j)$ or $(1,1)\,(\delta,m)$ or $(1,m)\,(\delta,1)$ where $1 \le j \le m-1$, then $f_{A_1 \cup \{c_i\}_{i=1}^5}(e) \le 1$, $f_{A_4 \cup A_5}(e) \le 1$, $f_{F \cup F'}(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 3$.

(5) If $e = (1,1)\,(n,m)$ or $(1,m)\,(n,1)$, then $f_{A_1 \cup \{c_i\}_{i=1}^5}(e) \le 1$, $f_{A_4 \cup A_5}(e) = 0$, $f_{F \cup F'}(e) = 0$, so $f_{\mathcal{B}}(e) \le 1$.

Thus, $\mathcal{B}$ is a 4-fold basis. The proof of this case is complete.

Case 7. m and n are odd, and δ is even. According to the relation between m and n, we split this case into two subcases.

Subcase 7a. $m \geq n$. Then consider the following sets of cycles: A_1, A_2 and A_3 are as in Case 1 and $c_1, c_2, c_3, c_1^{'}$ and $c_2^{'}$ are as in Case 2. Also, for $i = 2, 3, \ldots, n$, define the following cycles:

$$\begin{aligned} F_i &= (i,1)\,(i-1,2)\,(i-2,3)\ldots(1,i)\,(n,i+1)\,(n-1,i+2)(n-2,i+3)\ldots(i,n+1) \\ &\quad (i-1,n+2)\,(i,n+3)\ldots(i-1,m)\,(i,1)\,, \end{aligned}$$

and for $i = 1, 2, 3, \ldots, n-1$

$$\begin{aligned} F_i^{'} &= (i,1)\,(i+1,2)\,(i+2,3)\ldots(n,n-i+1)\,(1,m-i+2) \\ &\quad (2,m-i+3)\ldots(i,n+1)(i+1,n+2)(i,n+3)\ldots(i+1,m)\,(i,1)\,. \end{aligned}$$

Moreover, set

$$\begin{aligned} F_1 &= (1,1)\,(n,2)\,(n-1,3)\,(n-2,4)\ldots(1,n+1)\,(n,n+2)\,(1,n+3)\ldots \\ &\quad (1,m-1)\,(n,m)\,(1,1)\,, \end{aligned}$$

and

$$\begin{aligned} F_n^{'} &= (n,1)\,(1,2)\,(2,3)\,(3,4)\ldots(n,n+1)\,(1,n+2)\,(n,n+3) \\ &\quad (1,n+4)\ldots(n,m-1)\,(1,m)\,(n,1)\,. \end{aligned}$$

Let

$$F = \cup_{i=1}^{n} F_i \text{ and } F^{'} = \cup_{i=1}^{n} F_i^{'}.$$

By Case 2, $A_1 \cup A_2 \cup A_3 \cup c_1 \cup c_2 \cup c_3 \cup c_1^{'} \cup c_2^{'}$ is linearly independent. By a similar argument as in Subcase 6a, we can show that

$$\mathcal{B} = A_1 \cup A_2 \cup A_3 \cup F \cup F^{'} \cup c_1 \cup c_2 \cup c_3 \cup c_1^{'} \cup c_2^{'}$$

is a linearly independent set of cycles. Since

$$\begin{aligned} |\mathcal{B}| &= \sum_{i=1}^{3} |A_i| + |F| + |F^{'}| + \sum_{i=1}^{3} |c_i| + |c_1^{'}| + |c_2^{'}| \\ &= (m-2)n + (m-2) + (m-2) + n + n + 5 \\ &= mn + 2m + 1 \\ &= \dim \mathcal{C}(\theta_n \times C_m), \end{aligned}$$

$\mathcal{B}$ is a cycle basis of $\theta_n \times C_m$. For simplicity, set $Q = \cup_{i=1}^{3}\{c_i\}_{i=1}^{3}$. Let $e \in E(\theta_n \times C_m)$. Then

(1) if $e = (i,j)\,(i+1,j+1)$ or $(n,j)\,(1,j+1)$, where $1 \leq i \leq n-1$, and $2 \leq j \leq m-2$, then $f_{A_1}(e) = 2$, $f_{A_2 \cup A_3}(e) \leq 1$, $f_{F \cup F'}(e) = 1$, $f_{c_1^{'} \cup c_2^{'}}(e) = 0$ and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \leq 4$.

(2) If $e = (i, j)(i+1, j-1)$ or $(n, j)(1, j-1)$, where $1 \leq i \leq n-1$, and $3 \leq j \leq m-1$, then $f_{A_1}(e) = 2$, $f_{A_2 \cup A_3}(e) \leq 1$, $f_{F \cup F'}(e) = 1$, $f_{c_1' \cup c_2'}(e) = 0$ and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \leq 4$.

(3) If $e = (i, 1)(i+1, 2)$ or $(1, 1)(n, 2)$, where $1 \leq i \leq n-1$, then $f_{A_1}(e) = 1$, $f_{A_2 \cup A_3}(e) \leq 1$, $f_{F \cup F'}(e) = 1$, $f_{c_1' \cup c_2'}(e) = 0$ and $f_Q(e) = 1$, and so $f_{\mathcal{B}}(e) \leq 4$.

(4) If $e = (i, 2)(i+1, 1)$ or $(1, 2)(n, 1)$, where $1 \leq i \leq n-1$, then $f_{A_1}(e) = 1$, $f_{A_2 \cup A_3}(e) \leq 1$, $f_{F \cup F'}(e) = 0$, $f_{c_1' \cup c_2'}(e) = 0$ and $f_Q(e) = 1$, and so $f_{\mathcal{B}}(e) \leq 3$.

(5) If $e = (1, j)(\delta, j+1)$, where $1 \leq j \leq m-2$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) \leq 2$, $f_{F \cup F'}(e) = 1$, $f_{c_1' \cup c_2'}(e) = 0$ and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \leq 3$.

(6) If $e = (1, j)(\delta, j-1)$, where $2 \leq j \leq m-2$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) \leq 2$, $f_{F \cup F'}(e) = 0$, $f_{c_1' \cup c_2'}(e) = 0$ and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \leq 2$.

(7) If $e = (i, m-1)(i+1, m)$ or $(i, m)(i+1, m-1)$ or $(1, m)(n, m-1)$, where $1 \leq i \leq n-1$, then $f_{A_1}(e) = 1$, $f_{A_2 \cup A_3}(e) \leq 1$, $f_{F \cup F'}(e) = 1$, $f_{c_1' \cup c_2'}(e) = 0$ and $f_Q(e) \leq 1$, and so $f_{\mathcal{B}}(e) \leq 4$.

(8) If $e = (1, m)(\delta, m-1)$ or $(1, m-1)(\delta, m)$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) \leq 1$, $f_{F \cup F'}(e) = 0$, $f_{c_1' \cup c_2'}(e) = 0$ and $f_Q(e) \leq 1$, and so $f_{\mathcal{B}}(e) \leq 2$.

(9) If $e = (i, 1)(i+1, m)$ or $(i+1, 1)(i, m)$, where $1 \leq i \leq n-2$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_{F \cup F'}(e) = 1$, $f_{c_1' \cup c_2'}(e) \leq 1$ and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \leq 2$.

(10) If $e = (1, 1)(\delta, m)$ or $(m, 1)(\delta, 1)$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_{F \cup F'}(e) = 0$, $f_{c_1' \cup c_2'}(e) = 1$ and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \leq 1$.

(11) If $e = (1, 1)(n, m)$ or $(n, 1)(1, m)$, then $f_{A_1}(e) = 0$, $f_{A_2 \cup A_3}(e) = 0$, $f_{F \cup F'}(e) = 1$, $f_{c_1' \cup c_2'}(e) \leq 1$ and $f_Q(e) = 0$, and so $f_{\mathcal{B}}(e) \leq 2$. Therefore $\mathcal{B}$ is a 4-fold basis.

Subcase 7b. $m < n$. Then consider c_1' and c_2' as in Case 2 and c_5', A_4 and A_5 as in Case 5. Moreover, set

$$\begin{aligned} A_1' &= A_1 - \{\{(1, j)(2, j+1)(1, j+2)(n, j+1)(1, j) : j = 1, 2, \ldots, m-2\} \\ &\quad \cup \{(n, j)(n-1, j+1)(n, j+2)(1, j+1)(n, j) : j = 1, 2, \ldots, m-2\}\}, \end{aligned}$$

where A_1 is as in Case 1. Also, set

$$\begin{aligned} A_2' &= \{(1, j)(2, j+1)(3, j)(4, j+1) \ldots (\delta, j+1)(1, j) | j = 1, 2, \ldots m-1\} \\ A_3' &= \{(1, j-1)(2, j)(3, j-1) \ldots (\delta, j)(1, j-1) | j = 2, \ldots m\}\}. \end{aligned}$$

By Case 5 and noting that each cycle of $A_2' \cup A_3'$ contains an edge of the form $(\delta, j+1)(1, j)$ for some $1 \leq j \leq m-1$ or an edge of the form $(\delta, j)(1, j-1)$ for some $2 \leq j \leq m$ which appears in no cycle of $A_1' \cup A_4 \cup A_5 \cup c_1' \cup c_2' \cup c_5'$, we have that $A_1' \cup A_2' \cup A_3' \cup A_4 \cup A_5 \cup c_1' \cup c_2' \cup c_5'$ is linearly independent. Now, for $j = 2, 3, \ldots, m$, consider the following cycles:

$$\begin{aligned} F_i &= (1, j)(2, j-1)(3, j-2) \ldots (j, 1)(j+1, m)(j+2, m-1)(j+3, m-2) \ldots \\ &\quad (m+1, j)(m+2, j-1)(m+3, j) \ldots (n, j-1)(1, j), \end{aligned}$$

and for $j = 1, 2, 3, \ldots, m-1$

$$\begin{aligned} F_i^{'} &= (1,j)(2,j+1)(3,j+2)\ldots(m-j+1,m)(m-j+2,1)(m-j+3,2)(m-j+4,3)\ldots \\ &\quad (m+1,j)(m+2,j+1)(m+3,j)\ldots(n,j+1)(1,j). \end{aligned}$$

Moreover, set

$$\begin{aligned} F_1 &= (1,1)(2,m)(3,m-1)(4,m-2)\ldots(m+1,1)(m+2,m)(m+3,1)\ldots(n,m)(1,1) \\ F_m^{'} &= (1,m)(2,1)(3,2)(4,3)\ldots(m+1,m)(m+2,1)(m+3,2)\ldots(n,1)(m,1), \end{aligned}$$

Let

$$F = \cup_{i=1}^{m} F_i \text{ and } F^{'} = \cup_{i=1}^{m} F_i^{'}.$$

Using a similar arguments as in Subcase 6b, we show that

$$\mathcal{B} = A_1^{'} \cup A_2^{'} \cup A_3^{'} \cup A_4 \cup A_5 \cup F \cup F^{'} \cup c_1^{'} \cup c_2^{'} \cup c_5^{'}$$

is linearly independent. Note that

$$\begin{aligned} |B| &= \sum_{i=1}^{3} |A_i^{'}| + |A_4| + |A_5| + |F| + |F^{'}| + 3 \\ &= (m-2)(n-2) + (m-1) + (m-1) + (n-2) + (n-2) + m + m + 3 \\ &= mn + 2m + 1 \\ &= \dim \mathcal{C}\left(\theta_n \times C_m\right). \end{aligned}$$

Thus, $\mathcal{B}$ is a basis for $\mathcal{C}\left(\theta_n \times C_m\right)$. Now, let $e \in E(\theta_n \times C_m)$. Then

(1) If $e = (i,j)\,(i+1,j+1)$, where $1 \le i \le n-2$, and $1 \le j \le m-1$, then $f_{A_1' \cup A_2' \cup A_3' \cup A_4 \cup A_5}(e) \le 3$, $f_{F \cup F'}(e) = 1$, $f_{\{c_5'\}}(e) = 0$ and $f_{\{c_1', c_2'\}}(e) = 0$, and so $f_{\mathcal{B}}(e) \le 4$.

(2) If $e = (i,j)\,(i-1,j+1)$, where $2 \le i \le n-1$, and $1 \le j \le m-1$, then $f_{A_1' \cup A_2' \cup A_3' \cup A_4 \cup A_5}(e) \le 3$, $f_{F \cup F'}(e) \le 1$, $f_{\{c_5'\}}(e) = 0$ and $f_{\{c_1', c_2'\}}(e) = 0$, and so $f_{\mathcal{B}}(e) \le 4$.

(3) If $e = (n-1,j)\,(n,j+1)$ or $(n-1,j+1)\,(n,j)$or $(n-1,1)(n,m)$ or $(n-1,m)(n,1)$, where $1 \le j \le m-2$, then $f_{A_1' \cup A_2' \cup A_3' \cup A_4 \cup A_5}(e) \le 3$, $f_{F \cup F'}(e) = 0$, $f_{\{c_5'\}}(e) \le 1$ and $f_{\{c_1', c_2'\}}(e) = 0$, and so $f_{\mathcal{B}}(e) \le 4$.

(4) If $e = (1,j)\,(\delta,j+1)$ or $(1,j+1)\,(\delta,j)$or $(1,1)\,(\delta,m)$ or $(1,m)\,(\delta,1)$ where $1 \le j \le m$, then $f_{A_1' \cup A_2' \cup A_3' \cup A_4 \cup A_5}(e) \le 1$, $f_{F \cup F'}(e) \le 1$, $f_{\{c_5'\}}(e) = 0$ and $f_{\{c_1', c_2'\}}(e) \le 1$, and so $f_{\mathcal{B}}(e) \le 3$.

(5) If $e = (1,j)\,(n,j+1)$ or $(1,j+1)\,(n,j+1)$ or$(1,1)\,(n,m)$ or $(1,m)\,(n,1)$ where $1 \le j \le m-1$, then $f_{A_1' \cup A_2' \cup A_3' \cup A_4 \cup A_5}(e) = 0$, $f_{F \cup F'}(e) \le 1$, $f_{\{c_5'\}}(e) = 0$ and $f_{\{c_1', c_2'\}}(e) = 0$, and so $f_{\mathcal{B}}(e) \le 1$. Thus, $\mathcal{B}$ is a 4-fold basis of $\mathcal{C}(\theta_n \times C_m)$. The proof of this case is complete.

Case 8. m, n and δ are odd. By relabeling the vertices of θ_n in the opposite direction, we get a similar case to Case 6. The proof of this case is complete. □

By combining Lemma 3.3 and Lemma 3.4, we have the following result.

Theorem 3.5. *For any graph θ_n of order $n \geq 4$ and cycle C_m of order $m \geq 3$, we have* $3 \leq b(\theta_n \times C_m) \leq 4$.

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