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Identities by L-summing Method (II)

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Abstract: In this paper, we introduce 3-dimensional L-summing method by combinatorial speculation ([8]), which is a more complicated version of the usual technique of "changing the order of summation". Applying this method on some special arrays we obtain identities concerning some special functions, and we get more identities by using a Maple program for this method. Finally, we introduce higher dimensional versions of L-summing method.

Key Words: L-summing method, identity, special function.

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§1. Introduction

An identity is a mathematical sentence that has "=" in its middle; Zeilberger [7]. An ancient and well-known proof for the identity

$$\sum_{k=1}^{n}(2k-1)=n^2$$

considers an $n\times n$ array of bullets (the total number of which is abviously n^2) as the following figure

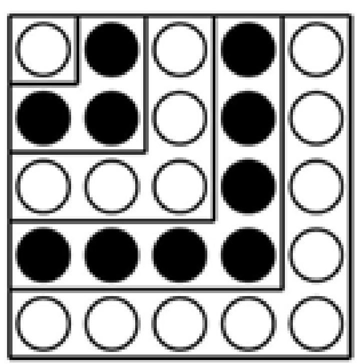

Fig.1

and divides it into n L-shaped zones containing $1,3,\cdots,2n-1$ bullets. In Hassani [3] we have generalized this process to all arrays of numbers with two dimension; to explain briefly, we

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consider the following $n \times n$ multiplication table

1	2	$\cdots$	k	$\cdots$	n
2	4	$\cdots$	$2k$	$\cdots$	$2n$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
k	$2k$	$\cdots$	k^2	$\cdots$	kn
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$\cdots$	kn	$\cdots$	n^2

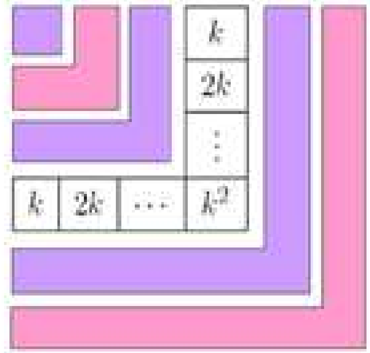

Fig.2

and we set $\Sigma(n)$ for the sum of all numbers in it. By summing line by line and using the identity $1+2+\cdots+n = n(n+1)/2$ we have $\Sigma(n) = (n(n+1)/2)^2$. On the other hand, letting L_k be the sum of numbers in the rotated L in above table (right part of FIGURE 2), we have

$$L_k = k + 2k + \cdots + k^2 + \cdots + 2k + k = 2k(1+2+\cdots+k) - k^2 = k^3,$$

which gives $\Sigma(n) = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} k^3$, and therefore $\sum_{k=1}^{n} k^3 = (n(n+1)/2)^2$. We call L_k, L-summing element and above process is 2-dimensional L-summing method (applied on the array $A_{ab} = ab$). In general, this method is

$$\sum (L - \text{Summing Elements}) = \Sigma. \qquad (1)$$

More precisely, the L-summing method of elements of $n \times n$ array A_{ab} with $1 \le a, b \le n$, is the following rearrangement

$$\sum_{k=1}^{n} \Big\{ \sum_{a=1}^{k} A_{ak} + \sum_{b=1}^{k} A_{kb} - A_{kk} \Big\} = \sum_{1 \le a,b \le n} A_{ab}.$$

This method allows us to obtain easily some classical algebraic identities and also, with help of Maple, some new compact formulas for sums related with the Riemann zeta function, the gamma function and the digamma function, Gilewicz [2] and Hassani [3].

In this paper we introduce a 3-dimensional version of L-summing method for $n \times n \times n$ arrays and we apply it on some special arrays. Also, we give a Maple program for this method and using it we generate and then prove more identities. Finally, we introduce a further generalization of L-summing method in higher dimension spaces. All of these are applications of the combinatorial speculation. The readers can see in [8] for details.

§2. L-Summing Method in $\mathbb{R}^3$

Consider a three dimensional array A_{abc} with $1 \le a, b, c \le n$ and n is a positive integer. We find an explicit version of the general formulation (1) for this array. The sum of all entries is $\Sigma(n) = \sum_{1 \le a,b,c \le n} A_{abc}$. The L-summing elements in this array have the form pictured in Fig.3.

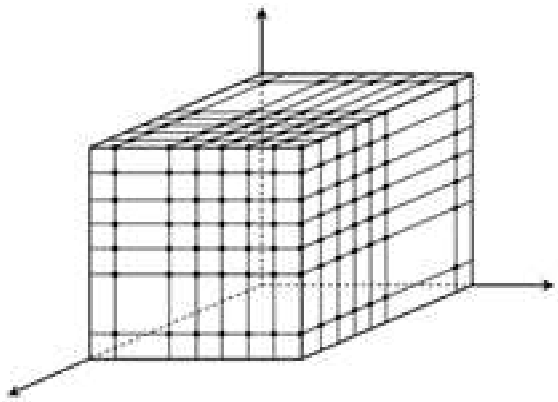

Fig.3

So, we have $L_k = \Sigma_2 - \Sigma_1 + \Sigma_0$, with

$$\begin{aligned}\Sigma_2 &= \sum_{b,c=1}^{k} A_{kbc} + \sum_{a,c=1}^{k} A_{akc} + \sum_{a,b=1}^{k} A_{abk},\\ \Sigma_1 &= \sum_{a=1}^{k} A_{akk} + \sum_{b=1}^{k} A_{kbk} + \sum_{c=1}^{k} A_{kkc},\\ \Sigma_0 &= A_{kkk}.\end{aligned}$$

Note that Σ_2 is the sum of entries in three faces, Σ_1 is the sum of entries in three intersected edges and Σ_0 is the end point of all faces and edges. Therefore, L-summing method in $\mathbb{R}^3$ takes the following formulation

$$\sum_{k=1}^{n}\{\Sigma_2 - \Sigma_1 + \Sigma_0\} = \Sigma(n). \qquad (2)$$

Above equation and its generalization in the last section, rely on the so-called "Inclusion - Exclusion principal".

If the array A_{abc} is symmetric, that is for each permutation $\sigma \in S_3$ it satisfies $A_{abc} = A_{\sigma_a\sigma_b\sigma_c}$, then L-summing elements in $\mathbb{R}^3$ take the following easier form

$$L_k = 3\sum_{b,c=1}^{k} A_{kbc} - 3\sum_{a=1}^{k} A_{akk} + A_{kkk}. \qquad (3)$$

As examples, we apply his method on two special symmetric arrays, related by the Riemann zeta function and digamma function.

The Riemann zeta function Suppose $s \in \mathbb{C}$ and let $A_{abc} = (abc)^{-s}$. Setting $\zeta_n(s) = \sum_{k=1}^{n} k^{-s}$, it is clear that

$$\Sigma(n) = \sum_{1\le a,b,c\le n} (abc)^{-s} = \zeta_n^3(s).$$

Since this array is symmetric, considering (3), we have

$$L_k = 3\frac{\zeta_k^2(s)}{k^s} - 3\frac{\zeta_k(s)}{k^{2s}} + \frac{1}{k^{3s}}.$$

Using (2) and an easy simplification, yield that

$$\sum_{k=1}^{n} \frac{\zeta_k^2(s)}{k^s} - \frac{\zeta_k(s)}{k^{2s}} = \frac{\zeta_n^3(s) - \zeta_n(3s)}{3}. \tag{4}$$

Note that if $\Re(s) > 1$, then $\lim_{n\to\infty} \zeta_n(s) = \zeta(s)$, where $\zeta(s) = \sum_{k=1}^{\infty} n^{-s}$ is the well-known Riemann zeta function defined for complex values of s with $\Re(s) > 1$ and admits a meromorphic continuation to the whole complex plan, Ivić [5]. So, for $\Re(s) > 1$ we have

$$\sum_{k=1}^{\infty} \frac{\zeta_k^2(s)}{k^s} - \frac{\zeta_k(s)}{k^{2s}} = \frac{\zeta^3(s) - \zeta(3s)}{3},$$

which also is true for other values of s by meromorphic continuation, except $s = 1$ and $s = \frac{1}{3}$.

Digamma function Setting $s = 1$ in (**??**) (or equivalently taking $A_{abc} = \frac{1}{abc}$) and considering $\zeta_n(1) = H_n = \sum_{k=1}^{n} \frac{1}{k}$, we obtain

$$\sum_{k=1}^{n} \left\{ \frac{H_k^2}{k} - \frac{H_k}{k^2} \right\} = \frac{H_n^3 - \zeta_n(3)}{3}.$$

We can state this identity in terms of digamma function $\Psi(x) = \frac{d}{dx} \ln \Gamma(x)$, where $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$ is the well-known gamma function. To do this, we use

$$\Psi(n+1) + \gamma = H_n, \tag{5}$$

in which $\gamma = 0.57721\ldots$ is the Euler constant; Abramowitz and Stegun [1]. Therefore, we obtain

$$\sum_{k=1}^{n} \left\{ \frac{(\Psi(k+1)+\gamma)^2}{k} - \frac{\Psi(k+1)+\gamma}{k^2} \right\} = \frac{(\Psi(n+1)+\gamma)^3 - \zeta_n(3)}{3}. \tag{6}$$

Letting

$$\S(m,n) = \sum_{k=1}^{n} \frac{\Psi(k)^m}{k},$$

the following identity in Hassani [3] is a result of 2-dimensional L-summing method

$$\S(1,n) = \frac{(\Psi(n+1)+\gamma)^2 + \Psi(1,n+1)}{2} - \frac{\pi^2}{12} - \Psi(n+1)\gamma - \gamma^2, \tag{7}$$

where $\Psi(m,x) = \frac{d^m}{dx^m}\Psi(x)$ is called m^{th} polygamma function; Abramowitz and Stegun [1], and we have

$$\zeta_n(s) = \frac{(-1)^{s-1}}{(s-1)!}\Psi(s-1,n+1) + \zeta(s) \qquad (s \in \mathbb{Z}, s \geq 2). \tag{8}$$

Using (8) in (4) we can get a generalization of (6), however (6) itself is the key of obtaining an analogue of (7) in $\mathbb{R}^3$.

Theorem 1 *For every integer $n \geq 1$, we have*

$$\begin{aligned} \sum_{k=1}^{n} \left\{ \frac{\Psi(k)^2}{k} + \frac{\Psi(k)}{k^2} \right\} &= \frac{(\Psi(n+1)+\gamma)^3}{3} - \frac{\zeta_n(3)}{3} + (\gamma-2)\frac{\pi^2}{6} \\ &- (\gamma-2)\Psi(1,n+1) - \gamma^2\Psi(n+1) - \gamma^3 - 2\S(1,n). \end{aligned}$$

Proof We begin from the left hand side of the identity (6), then we simplify it by using the relations $\Psi(n+1)=\frac{1}{n}+\Psi(n)$, (5) and the relation (8) with $s=2$. This completes the proof.□

Corollary 2 *For every integer $n\geq 1$, we have*

$$\begin{aligned} \S(2,n) &= \frac{(\Psi(n+1)+\gamma)^3}{3}-\frac{\zeta_n(3)}{3}+(\gamma-2)\frac{\pi^2}{6}-(\gamma-2)\Psi(1,n+1) \\ &- \gamma^2\Psi(n+1)-\gamma^3-2\S(1,n)-\sum_{k=1}^{n}\frac{\Psi(k)}{k^2}. \end{aligned}$$

In the above corollary, the main term in the right hand side is $\frac{\Psi(n+1)^3}{3}$. Also, computations show that $\sum_{k=1}^{\infty}\frac{\Psi(k)}{k^2}=0.252\ldots$.

Note and Problem 3 Since $\Psi(x)\sim\ln x$, we obtain

$$\S(m,n)\sim\sum_{k=1}^{n}\frac{\ln^m k}{k}\sim\int_1^n\frac{\ln^m k}{k}dk=\frac{\ln^{m+1}n}{m+1}\sim\frac{\Psi(n+1)^{m+1}}{m+1}.$$

It is interesting to find an explicit recurrence relation for the function $\S(m,n)$. One can attack this problem by considering generalization of L-summing method in higher dimension spaces, considered in the last section of this paper.

§3. An Identity - Generator Machine

Based on the formulation of 3-dimensional L-summing method, we can write a Maple program (see Appendix 1), with input a 3-dimensional array A_{abc}, and out put an identity, which we show it by `LSMI` $< A_{abc} >$. We introduce some examples; the first one is `LSMI` $< \ln(a) >$, which is

$$\sum_{k=1}^{n}\left\{k^2\ln k+2\,k\ln\Gamma(k+1)-2\,k\ln k-\ln\Gamma(k+1)+\ln k\right\}=n^2\ln\Gamma(n+1).$$

To prove this, we consider relations (2) and $\Gamma(n+1)=n!$, and we obtain $\Sigma(n)=n^2\sum_{a=1}^{n}\ln a=n^2\ln\Gamma(n+1)$. Also, $\Sigma_2=k^2\ln k+2k\ln\Gamma(k+1)$, $\Sigma_1=\ln\Gamma(k+1)+2k\ln k$ and $\Sigma_0=\ln k$.

Breaking up the statement under the sum obtained by `LSMI` $< \ln(a) >$ into the sum of $(k^2-k)\ln k+2k\ln\Gamma(k+1)$ and $\ln\Gamma\,(k+1)+k\ln k-\ln k$, and considering Proposition 6 of Hassani [3], which states

$$\sum_{k=1}^{n}\{\ln\Gamma\,(k+1)+k\ln k-\ln k\}=n\ln\Gamma(n+1),$$

led us to the following result

$$\sum_{k=1}^{n}\left\{(k^2-k)\ln k+2k\ln\Gamma(k+1)\right\}=(n^2+n)\ln\Gamma(n+1). \qquad (9)$$

This is an important example, because examining Maple code of expressed sum in (9), we see that Maple has no comment for computing it. But, it is obtained by Maple itself and L-summing method. There is another gap in Maple recognized by this method (see Appendix 2).

As we see, Maple program of 3-dimension L-summing method is a machine of generating identities. Many of them are similar and are not interesting, but we can choose some interesting ones. Another easy example is `LSMI` $< \tan(a) >$, which (after simplification) is

$$\sum_{k=1}^{n}\left\{(k-1)^2\tan k+(2k-1)\mathfrak{T}(k)\right\}=n^2\mathfrak{T}(n),$$

where $\mathfrak{T}(n)=\sum_{k=1}^{n}\tan k$. Our last example is an identity concerning hypergeometric functions, denoted in Maple by

$$\texttt{hypergeom([}a_1\ \ a_2\ \cdots\ a_p\texttt{],[}b_1\ \ b_2\ \cdots\ b_q\texttt{],}x\texttt{)}.$$

Standard notation and definition; Petkovšek, Wilf and Zeilberger [6], is as follows

$${}_pF_q\left[\begin{matrix}a_1 & a_2 & \cdots & a_p\\ b_1 & b_2 & \cdots & b_q\end{matrix};x\right]=\sum_{k\geq 0}t_kx^k,$$

where

$$\frac{t_{k+1}}{t_k}=\frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x.$$

Now, setting

$$\mathfrak{H}(\alpha,\beta)={}_2F_0\left[\begin{matrix}\alpha & \beta\\ - & \end{matrix};1\right],$$

after simplification of `LSMI` $< a! >$ we obtain

$$\sum_{k=1}^{n}\left\{(k-1)^2k!+(2k-1)(k+1!)\mathfrak{H}(1,k+2)\right\}=n^2(n+1)!\mathfrak{H}(1,n+2).$$

To prove this, considering definition of hypergeometric functions we have $\mathfrak{H}(1,n+1)=(n+1)\mathfrak{H}(1,n+2)$, which implies $\sum_{a=1}^{n}a!=\mathfrak{H}(1,2)-(n+1)!\mathfrak{H}(1,n+2)=\mathfrak{P}(n)$, say. This gives $\Sigma(n)=n^2\mathfrak{P}(n)$ and in similar way it yields that $L_k=(k-1)^2k!+(2k-1)\left((k+1!)\mathfrak{H}(1,k+2)-\mathfrak{H}(1,2)\right)$.

Above examples are special cases of the array $A_{abc}=f(a)$, for a given function f. In this general case, L-summing method takes the following formulation

$$\sum_{k=1}^{n}\left\{(2k-1)\mathfrak{F}(k)+(k-1)^2f(k)\right\}=n^2\mathfrak{F}(n),$$

where $\mathfrak{F}(n)=\sum_{a=1}^{n}f(a)$.

§4. Futher Generalizations and Comments

L-summing method in $\mathbb{R}^t$ Consider a $t-$dimensional array $A_{x_1x_2\cdots x_t}$ and let $\Sigma(n)=\sum A_{x_1x_2\cdots x_t}$ with $1\leq x_1,x_2,\cdots,x_t\leq n$. L-summing method in $\mathbb{R}^t$ is the rearrangement $\Sigma(n)=\sum L_k$, where

$$L_k=\sum_{m=1}^{t}\left\{(-1)^{m-1}\Sigma_{t-m}\right\},$$

with

$$\Sigma_{t-m} = \sum_{1\leq i_1<i_2<\cdots<i_m\leq t} \left\{ {\sum}' A_{\mathbf{x}_{i_1 i_2\cdots i_m}} \right\}.$$

The inner sum $\sum'$ is over $x_j \in \{x_{i_1}, \cdots, x_{i_m}\}^C = \{x_1, x_2, \cdots, x_t\} - \{x_{i_1}, \cdots, x_{i_m}\}$ with $1 \leq x_j \leq k$, and the index $\mathbf{x}_{i_1 i_2\cdots i_m}$ denotes $x_1 x_2 \cdots x_t$ with $x_{i_1} = x_{i_2} = \cdots = x_{i_m} = k$. One can apply this generalized version to get more general form of relations obtained in previous sections. For example, considering the array $A_{x_1x_2\cdots x_t} = (x_1x_2\cdots x_t)^{-s}$ with $s \in \mathbb{C}$, yields

$$\sum_{k=1}^{n} \left\{ \sum_{m=1}^{t-1} (-1)^{m-1} \binom{t}{m} k^{-ms} \zeta_k(s)^{t-m} \right\} = \zeta_n(s)^t + (-1)^t \zeta_n(ts).$$

L-summing method on manifolds. As we told at the beginning, the base of the L-summing method is multiplication table. Above generalization of L-summing method in $\mathbb{R}^t$ is based on the generalized multiplication tables; see Hassani [4]. But, $\mathbb{R}^t$ is a very special t-dimensional manifold, and if we replace it by Γ, an $l-$dimensional manifold with $l \leq t$, then we can define generalized multiplication table on Γ by considering lattice points on it (which of course isn't easy problem). Let

$$L_\Gamma(n) = \big\{(a_1, a_2, \cdots, a_t) \in \Gamma \cap \mathbb{N}^t : 1 \leq a_1, a_2, \cdots, a_t \leq n\big\},$$

and $f : \mathbb{R}^k \longrightarrow \mathbb{C}$ is a function. If $\mathcal{O}_\Gamma$ is a collection of $k-1$ dimension orthogonal manifolds, in which $L_\Gamma(n) = \cup_{\Lambda\in\mathcal{O}_\Gamma} L_\Lambda(n)$ and $L_{\Lambda_i}(n) \cap L_{\Lambda_j}(n) = \phi$ for distinct $\Lambda_i, \Lambda_j \in \mathcal{O}_\Gamma$, then we can formulate L-summing method as follows

$$\sum_{X\in L_\Gamma(n)} f(X) = \sum_{\Lambda\in\mathcal{O}_\Gamma} \Big\{ \sum_{X\in L_\Lambda(n)} f(X) \Big\}.$$

Here L-summing elements are $\sum_{X\in L_\Lambda(n)} f(X)$. This may ends to some interesting identities, provided one applies it on some suitable manifolds.

Stronger form of L-summing method. One can state the relation $\sum L_k = \Sigma(n)$ in the following stronger form

$$L_n = \Sigma(n) - \Sigma(n-1).$$

Specially, this will be useful for those arrays with $\Sigma(n)$ computable explicitly and L_k maybe note. For example, considering the array $A_{x_1x_2\cdots x_t} = (x_1x_2\cdots x_t)^{-s}$ we obtain

$$\sum_{m=1}^{t-1} (-1)^{m-1} \binom{t}{m} n^{-ms} \zeta_n(s)^{t-m} = \zeta_n(s)^t + (-1)^t \zeta_n(ts) - \zeta_{n-1}(s)^t - (-1)^t \zeta_{n-1}(ts).$$

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Appendix 1. Maple program of 3-dimension $L-$summing method for the array $A_{abc} = \frac{1}{abc}$

```
restart:
A[abc]:=1/(a*b*c);
S21:=sum(sum(eval(A[abc],a=k),b=1..k),c=1..k):
S22:=sum(sum(eval(A[abc],b=k),a=1..k),c=1..k):
S23:=sum(sum(eval(A[abc],c=k),a=1..k),b=1..k):
S2:=S21+S22+S23:
S11:=sum(eval(eval(A[abc],a=k),b=k),c=1..k):
S12:=sum(eval(eval(A[abc],a=k),c=k),b=1..k):
S13:=sum(eval(eval(A[abc],b=k),c=k),a=1..k):
S1:=S11+S12+S13:
S0:=eval(eval(eval(A[abc],a=k),b=k),c=k):
L[k]:=simplify(S2-S1+S0):
ST(A):=(simplify(sum(sum(sum(A[abc],a=1..n),b=1..n),c=1..n))):
Sum(L[k],k=1..n)=ST(A);
```

Appendix 2. A note on the operator "is" in Maple

The operator "is" in Maple software verifies the numerical and symbolic identities and inequalities, and it's out put is "*true*", "*false*" or "*FAIL*". We consider the following example, with "*FAIL*" as out put.

```
A:=binomial(2*k+1,k+1)+binomial(2*k+1,k)-binomial(2*k,k):
is(sum(A,k = 1 .. n) = binomial(2*n+2,n+1)-2);
```

This example is verifying the following identity:

$$\sum_{k=1}^{n}\left\{\binom{2\,k+1}{k}+\binom{2\,k+1}{k+1}-\binom{2\,k}{k}\right\}=\binom{2\,n+2}{n+1}-2,$$

which is true by using Maple and L-summing method; Hassani [3].