*International J.Math. Combin. Vol.2 (2008), 23-45*

# Combinatorially Riemannian Submanifolds

Linfan MAO

(Chinese Academy of Mathematics and System Science, Beijing 100080, P.R.China)

E-mail: maolinfan@163.com

**Abstract**: Submanifolds are important objectives in classical Riemannian geometry, particularly their embedding or immersion in Euclidean spaces. These similar problems can be also considered for combinatorial manifolds. Serval criterions and fundamental equations for characterizing combinatorially Riemannian submanifolds of a combinatorially Riemannian manifold are found, and the isometry embedding of a combinatorially Riemannian manifold in an Euclidean space is considered by a combinatorial manner in this paper.

**Key Words**: combinatorially Riemannian manifold, combinatorially Riemannian submanifold, criterion, fundamental equation of combinatorially Riemannian submanifolds, embedding in an Euclidean space.

**AMS(2000)**: 51M15, 53B15, 53B40, 57N16, 83C05, 83F05.

## §1. Introduction

Combinatorial manifolds were introduced in [9] by a combinatorial speculation on classical *Riemannian* manifolds, also an application of Smarandache multi-spaces in mathematics (see [12] – [13] for details), which can be used both in theoretical physics for generalizing classical spacetimes to multiple one, also enables one to realize those of non-uniform spaces and multilateral properties of objectives.

For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a *combinatorial manifold* $\widetilde{M}$ is defined to be a *Hausdorff* space such that for any point $p \in \widetilde{M}$, there is a local chart $(U_p, \varphi_p)$ of $p$, i.e., an open neighborhood $U_p$ of $p$ in $\widetilde{M}$ and a homoeomorphism

$$\varphi_p : U_p \to \bigcup_{i=1}^{s} B_i^{n_i},$$

where $B_1^{n_1}, B_2^{n_2}, \cdots, B_s^{n_s}$ are unit balls with $\bigcap\limits_{i=1}^{s} B_i^{n_i} \neq \emptyset$ and $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$ and $\bigcup\limits_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$. Denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context.

Let $\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$ be an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. The maximum value of $s(p)$ and the dimension of $\bigcap\limits_{i=1}^{s(p)} B_i^{n_i}$ are called the dimension and the intersec-

[1]Received January 6, 2008. Accepted March 25, 2008.

tional dimensional of $\widetilde{M}(n_1,n_2,\cdots,n_m)$ at the point $p$, denoted by $d_{\widetilde{M}}(p)$ and $\widehat{d}_{\widetilde{M}}(p)$, respectively. A combinatorial manifold $\widetilde{M}$ is called *finite* if it is just combined by finite manifolds without one manifold contained in the union of others, called *smooth* if it is finite endowed with a $C^\infty$ differential structure. For a smoothly combinatorial manifold $\widetilde{M}$ and a point $p \in \widetilde{M}$, it has been shown in [7] that $\dim T_p\widetilde{M}(n_1,n_2,\cdots,n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)}(n_i - \widehat{s}(p))$ and $\dim T_p^*\widetilde{M}(n_1,n_2,\cdots,n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)}(n_i - \widehat{s}(p))$ with a basis

$$\{\frac{\partial}{\partial x^{i_0 j}}|_p | 1 \le j \le \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \{\frac{\partial}{\partial x^{ij}}|_p \mid \widehat{s}(p)+1 \le j \le n_i\})$$

or

$$\{dx^{i_0 j}|_p|\} 1 \le j \le \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \{dx^{ij}|_p \mid \widehat{s}(p)+1 \le j \le n_i\}$$

for any integer $i_0, 1 \le i_0 \le s(p)$. Let $\widetilde{M}$ be a smoothly combinatorial manifold and

$$g \in A^2(\widetilde{M}) = \bigcup_{p \in \widetilde{M}} T_2^0(p, \widetilde{M}).$$

If $g$ is symmetrical and positive, then $\widetilde{M}$ is called a combinatorially Riemannian manifold, denoted by $(\widetilde{M}, g)$. In this case, if there is a connection $\widetilde{D}$ on $(\widetilde{M}, g)$ with equality following hold

$$Z(g(X,Y)) = g(\widetilde{D}_Z, Y) + g(X, \widetilde{D}_Z Y)$$

then $\widetilde{M}$ is called a combinatorially Riemannian geometry, denoted by $(\widetilde{M}, g, \widetilde{D})$. It has been showed that there exists a unique connection $\widetilde{D}$ on $(\widetilde{M}, g)$ such that $(\widetilde{M}, g, \widetilde{D})$ is a combinatorially Riemannian geometry([7] − [8]).

A subset $\widetilde{S}$ of a combinatorial manifold or a combinatorially Riemannian manifold $\widetilde{M}$ is called a *combinatorial submanifold* or *combinatorially Riemannian submanifold* if it is a combinatorial manifold or a combinatorially Riemannian manifold itself. In classical Riemannian geometry, submanifolds are very important objectives in research, particularly their embedding or immersion in Euclidean spaces. These similar problems should be also considered on combinatorial submanifolds for characterizing combinatorial manifolds, such as those of *what condition ensures a subset of a combinatorial manifold or a combinatorially Riemannian manifold to be a combinatorial submanifold or a combinatorially Riemannian submanifold in topology or in geometry?* Notice that there are no doubts for the existence of submanifolds of a given manifold in classical Riemannian geometry. Thereby one can got various fundamental equations, such as those of the *Gauss*'s, the *Codazzi*'s and the *Ricci*'s for handling the behavior of submanifolds of a Riemannian manifold. But for a combinatorially Riemannian manifold the situation is more complex for it being provided with a combinatorial structure. Therefore, problems without consideration in classical Riemannian geometry should be researched thoroughly in this time. For example, for a given subgraph $\Gamma$ of $G[\widetilde{M}]$ underlying $\widetilde{M}$, *whether is* *there a combinatorial submanifold or a combinatorially Riemannian submanifold underlying* $\Gamma$*? Are those of fundamental equations, i.e., the Gauss's, the Codazzi's or the Ricci's still true for combinatorially Riemannian submanifolds?* If not, *what are their right forms?* All these problems should be answered in this paper.

Now let $\widetilde{M}$, $\widetilde{N}$ be two combinatorial manifolds, $F:\widetilde{M}\rightarrow\widetilde{N}$ a smooth mapping and $p\in\widetilde{M}$. For $\forall v\in T_p\widetilde{M}$, define a tangent vector $F_*(v)\in T_{F(p)}\widetilde{N}$ by

$$F_*(v)=v(f\circ F),\quad \forall f\in C^\infty_{F(p)},$$

called the differentiation of $F$ at the point $p$. Its dual $F^*:T^*_{F(p)}\widetilde{N}\rightarrow T^*_p\widetilde{M}$ determined by

$$(F^*\omega)(v)=\omega(F_*(v))\text{ for }\forall\omega\in T^*_{F(p)}\widetilde{N}\text{ and }\forall v\in T_p\widetilde{M}$$

is called a *pull-back* mapping. We know that mappings $F_*$ and $F^*$ are linear.

For a smooth mapping $F:\widetilde{M}\rightarrow\widetilde{N}$ and $p\in\widetilde{M}$, if $F_{*p}:T_p\widetilde{M}\rightarrow T_{F(p)}\widetilde{N}$ is one-to-one, we call it an *immersion mapping*. Besides, if $F_{*p}$ is onto and $F:\widetilde{M}\rightarrow F(\widetilde{M})$ is a homoeomorphism with the relative topology of $\widetilde{N}$, then we call it an *embedding mapping* and $(F,\widetilde{M})$ a *combinatorially embedded submanifold*. Usually, we replace the mapping $F$ by an inclusion mapping $\widetilde{i}:\widetilde{M}\rightarrow\widetilde{N}$ and denoted by $(\widetilde{i},\widetilde{M})$ a combinatorial submanifold of $\widetilde{N}$.

Terminology and notations used in this paper are standard and can be found in [1]−[2], [14] for manifolds and submanifolds, [3] − [5] for Smarandache multi-spaces and graphs, [7] − [10] for combinatorial manifolds and [11] for topology, respectively.

## §2. Topological Criterions

Let $\widetilde{M}=\widetilde{M}(n_1,n_2,\cdots,n_m)$, $\widetilde{N}=\widetilde{N}(k_1,k_2,\cdots,k_l)$ be two finitely combinatorial manifolds and $F:\widetilde{M}\rightarrow\widetilde{N}$ a smooth mapping. For $\forall p\in\widetilde{M}$, let $(U_p,\varphi_p)$ and $(V_{F(p)},\psi_{F(p)})$ be local charts of $p$ in $\widetilde{M}$ and $F(p)$ in $\widetilde{N}$, respectively. Denoted by

$$J_{X;Y}(F)(p)=[\frac{\partial F^{\kappa\lambda}}{\partial x^{\mu\nu}}]$$

the *Jacobi matrix* of $F$ at $p$. Then we find that

**Theorem** 2.1 *Let $F:\widetilde{M}\rightarrow\widetilde{N}$ be a smooth mapping from $\widetilde{M}$ to $\widetilde{N}$. Then $F$ is an immersion mapping if and only if*

$$rank(J_{X;Y}(F)(p))=d_{\widetilde{M}}(p)$$

*for $\forall p\in\widetilde{M}$.*

*Proof* Assume the coordinate matrixes of points $p\in\widetilde{M}$ and $F(p)\in\widetilde{N}$ are $[x^{ij}]_{s(p)\times n_{s(p)}}$ and $[y^{ij}]_{s(F(p))\times n_{s(F(p))}}$, respectively. Notice that

$$T_p\widetilde{M}=\left\langle\frac{\partial}{\partial x^{i_0j_1}}|_p,\ \frac{\partial}{\partial x^{ij_2}}|_p\ |1\leq i\leq s(p),1\leq j_1\leq\widehat{s}(p),\widehat{s}(p)+1\leq j_2\leq n_i\right\rangle$$

and

$$T_{F(p)}\widetilde{N} = \left\langle \{\frac{\partial}{\partial y^{i_0 j_1}}|_{F(p)}, 1 \leq j_1 \leq \widehat{s}(F(p))\} \bigcup_{i=1}^{s(F(p))} \{\frac{\partial}{\partial y^{i j_2}}|_{F(p)}, \widehat{s}(F(p)) + 1 \leq j_2 \leq k_i\} \right\rangle$$

for any integer $i_0, 1 \leq i_0 \leq \min\{s(p), s(F(p))\}$. By definition, $F_{*p}$ is a linear mapping. We only need to prove that $F_{*p} : T_p\widetilde{M} \to T_p\widetilde{N}$ is an injection for $\forall p \in \widetilde{M}$. For $\forall f \in \mathscr{X}_p$, calculation shows that

$$\begin{aligned} F_{*p}(\frac{\partial}{\partial x^{ij}})(f) &= \frac{\partial(f \circ F)}{\partial x^{ij}} \\ &= \sum_{\mu,\nu} \frac{\partial F^{\mu\nu}}{\partial x^{ij}} \frac{\partial f}{\partial y^{\mu\nu}}. \end{aligned}$$

Whence, we find that

$$F_{*p}(\frac{\partial}{\partial x^{ij}}) = \sum_{\mu,\nu} \frac{\partial F^{\mu\nu}}{\partial x^{ij}} \frac{\partial}{\partial y^{\mu\nu}}. \qquad (2.1)$$

According to a fundamental result on linear equation systems, these exist solutions in the equation system (2.1) if and only if

$$rank(J_{X;Y}(F)(p)) = rank(J^*_{X;Y}(F)(p)),$$

where

$$J^*_{X;Y}(F)(p) = \begin{bmatrix} \cdots & F_{*p}(\frac{\partial}{\partial x^{11}}) \\ \cdots & \cdots \\ \cdots & F_{*p}(\frac{\partial}{\partial x^{1n_1}}) \\ J_{X;Y}(F)(p) & \cdots \\ \cdots & F_{*p}(\frac{\partial}{\partial x^{s(p)1}}) \\ \cdots & \cdots \\ \cdots & F_{*p}(\frac{\partial}{\partial x^{s(p)n_{s(p)}}}) \end{bmatrix}.$$

We have known that

$$rank(J^*_{X;Y}(F)(p)) = d_{\widetilde{M}}(p).$$

Therefore, $F$ is an immersion mapping if and only if

$$rank(J_{X;Y}(F)(p)) = d_{\widetilde{M}}(p)$$

for $\forall p \in \widetilde{M}$. □

For finding some simple criterions for combinatorial submanifolds, we consider the case that $F : \widetilde{M} \to \widetilde{N}$ maps each manifold of $\widetilde{M}$ to a manifold of $\widetilde{N}$, denoted by $F : \widetilde{M}\ _1 \to_1 \widetilde{N}$, which can be characterized by a purely combinatorial manner. In this case, $\widetilde{M}$ is called a *combinatorial in-submanifold of* $\widetilde{N}$.

Let $G$ be a connected graph. A *vertex-edge labeled graph* $G^L$ defined on $G$ is a triple $(G;\tau_1,\tau_2)$, where $\tau_1 : V(G)\to\{1,2,\cdots,k\}$ and $\tau_2 : E(G)\to\{1,2,\cdots,l\}$ for positive integers $k$ and $l$.

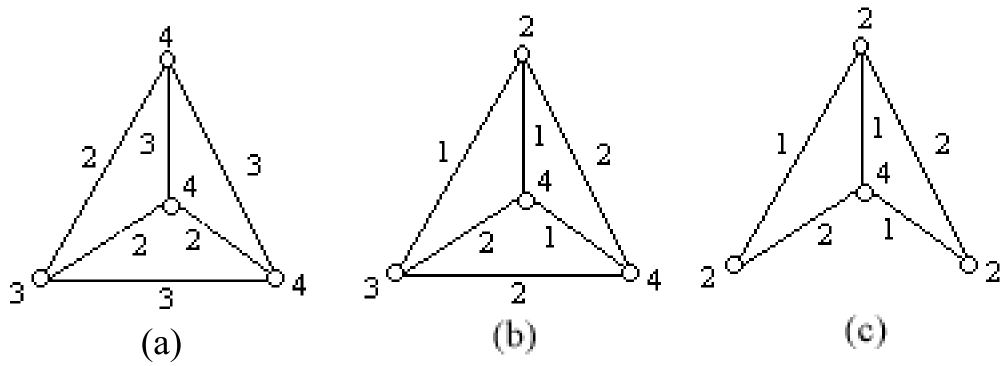


**Fig.**2.1

For a given vertex-edge labeled graph $G^L=(V^L,E^L)$ on a graph $G=(V,E)$, its a subgraph is defined to be a connected subgraph $\Gamma\prec G$ with labels $\tau_1|_\Gamma(u)\leq\tau_1|_G(u)$ for $\forall u\in V(\Gamma)$ and $\tau_2|_\Gamma(u,v)\leq\tau_2|_G(u,v)$ for $\forall(u,v)\in E(\Gamma)$, denoted by $\Gamma^L\prec G^L$. For example, two vertex-edge labeled graphs with an underlying graph $K_4$ are shown in Fig.2.1, in which the vertex-edge labeled graphs (b) and (c) are subgraphs of that (a).

For a finitely combinatorial manifold $\widetilde{M}(n_1,n_2,\cdots,n_m)$ with $1\leq n_1<n_2<\cdots<n_m, m\geq 1$, we can naturally construct a vertex-edge labeled graph $G^L[\widetilde{M}(n_1,n_2,\cdots,n_m)]=(V^L,E^L)$ by defining

$$V^L=\{n_i-\text{manifolds } M^{n_i} \text{ in } \widetilde{M}(n_1,n_2,\cdots,n_m)|1\leq i\leq m\}$$

with a label $\tau_1(M^{n_i})=n_i$ for each vertex $M^{n_i}, 1\leq i\leq m$ and

$$E^L=\{(M^{n_i},M^{n_j})|M^{n_i}\bigcap M^{n_j}\neq\emptyset, 1\leq i,j\leq m\}$$

with a label $\tau_2(M^{n_i},M^{n_j})=\dim(M^{n_i}\bigcap M^{n_j})$ for each edge $(M^{n_i},M^{n_j}), 1\leq i,j\leq m$. This construction then enables us to get a topological criterion for combinatorial submanifolds of a finitely combinatorial manifold by subgraphs in a vertex-edge labeled graph. For this objective, we introduce the *feasibly vertex-edge labeled subgraphs of* $G^L[\widetilde{M}]$ on a finitely combinatorial manifold $\widetilde{M}$ following.

Applying these vertex-edge labeled graphs correspondent to finitely combinatorial manifolds, we get some criterions for combinatorial submanifolds. Firstly, we establish a decomposition result on unit for smoothly combinatorial manifolds.

**Lemma** 2.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifolds with the second axiom of countability hold. For $\forall p\in\widetilde{M}$, let $U_p$ be the intersection of $\widehat{s}(p)$ manifolds $M_1,M_2,\cdots,M_{\widehat{s}(p)}$. Then there are functions $f_{M_i}$, $1\leq i\leq\widehat{s}(p)$ in a local chart $(V_p,[\varphi_p]), V_p\subset U_p$ in $\widetilde{M}$ such that*

$$f_{M_i} = \begin{cases} 1 & on\ V_p \bigcap M_i, \\ 0 & otherwise. \end{cases}$$

*Proof* By definition, each manifold $M_i$ is also smooth with the second axiom of countability hold since

$$\mathscr{A}_i = \{(U_p, [\varphi_p])|_{V_p \cap M_i} | p \in M\}$$

is a $C^\infty$ differential structure on $M_i$ for any integer $i, 1 \leq i \leq \widehat{s}(p)$. According to the decomposition theorem of unit on manifolds with the second axiom of countability hold, there is a finite cover

$$\Sigma_{M_i} = \{W_\alpha^i, \alpha \in \mathbb{N}\}$$

on each $M_i$, where $\mathbb{N}$ is a natural number set such that there exists a family function $f_\alpha \in C^\infty(M_i)$ with $f_\alpha|_{W_\alpha^i} \equiv 1$ but $f_\alpha|_{N_i \setminus W_\alpha^i} \equiv 0$.

Not loss of generality, we assume that $p \in W_{\alpha_0}^i$ for any integer $i$. Let

$$V_p = \bigcap_{i=1}^{\widehat{s}(p)} W_{\alpha_0}^i$$

and define

$$f_{M_i}(q) = \begin{cases} f_{\alpha_0}|_{W_{\alpha_0}^i} & if\ q \in W_{\alpha_0}^i, \\ 0 & otherwise. \end{cases}$$

Then we get these functions $f_{M_i}, 1 \leq i \leq \widehat{s}(p)$ satisfied with our desired. □

**Theorem** 2.2 *Let $\widetilde{M}$ be a smoothly combinatorial manifold and $N$ a manifold. If for $\forall M \in V(G[\widetilde{M}])$, there exists an embedding $F_M : M \to N$, then $\widetilde{M}$ can be embedded into $N$.*

*Proof* By assumption, there exists an embedding $F_M : M \to N$ for $\forall M \in V(G[\widetilde{M}])$. For $p \in \widetilde{M}$, let $V_p$ be the intersection of $\widehat{s}(p)$ manifolds $M_1, M_2, \cdots, M_{\widehat{s}(p)}$ with functions $f_{M_i}$, $1 \leq i \leq \widehat{s}(p)$ in Lemma 2.1 existed. Define a mapping $\widetilde{F} : \widetilde{M} \to N$ at $p$ by

$$\widetilde{F}(p) = \sum_{i=1}^{\widehat{s}(p)} f_{M_i} F_{M_i}.$$

Then $\widetilde{F}$ is smooth at each point in $\widetilde{M}$ for the smooth of each $F_{M_i}$ and $\widetilde{F}_{*p} : T_p\widetilde{M} \to T_pN$ is one-to-one since each $(F_{M_i})_{*p}$ is one-to-one at the point $p$. Whence, $\widetilde{M}$ can be embedded into the manifold $N$. □

**Theorem** 2.3 *Let $\widetilde{M}$ and $\widetilde{N}$ be smoothly combinatorial manifolds. If for $\forall M \in V(G[\widetilde{M}])$, there exists an embedding $F_M : M \to \widetilde{N}$, then $\widetilde{M}$ can be embedded into $\widetilde{N}$.*

*Proof* Applying Lemma 2.1, we can get a mapping $\widetilde{F} : \widetilde{M} \to \widetilde{N}$ defined by

$$\widetilde{F}(p) = \sum_{i=1}^{\widehat{s}(p)} f_{M_i} F_{M_i}$$

at $\forall p \in \widetilde{M}$. Similar to the proof of Theorem 2.2, we know that $\widetilde{F}$ is smooth and $\widetilde{F}_{*p} : T_p\widetilde{M} \to T_p\widetilde{N}$ is one-to-one. Whence, $\widetilde{M}$ can be embedded into $\widetilde{N}$. □

Now we introduce conceptions of feasibly vertex-edge labeled subgraphs and labeled quotient graphs in the following.

**Definition** 2.1 *Let $\widetilde{M}$ be a finitely combinatorial manifold with an underlying graph $G^L[\widetilde{M}]$. For $\forall M \in V(G^L[\widetilde{M}])$ and $U^L \subset N_{G^L[\widetilde{M}]}(M)$ with new labels $\tau_2(M, M_i) \leq \tau_2|_{G^L[\widetilde{M}]}(M, M_i)$ for $\forall M_i \in U^L$, let $J(M_i) = \{M_i' | \dim(M \cap M_i') = \tau_2(M, M_i), M_i' \subset M_i\}$ and denotes all these distinct representatives of $J(M_i), M_i \in U^L$ by $\mathscr{T}$. Define the index $o_{\widetilde{M}}(M : U^L)$ of $M$ relative to $U^L$ by*

$$o_{\widetilde{M}}(M : U^L) = \min_{J \in \mathscr{T}} \{\dim(\bigcup_{M' \in J} (M \bigcap M'))\}.$$

*A vertex-edge labeled subgraph $\Gamma^L$ of $G^L[\widetilde{M}]$ is feasible if for $\forall u \in V(\Gamma^L)$,*

$$\tau_1|_{\Gamma}(u) \geq o_{\widetilde{M}}(u : N_{\Gamma^L}(u)).$$

*Denoted by $\Gamma^L \prec_o G^L[\widetilde{M}]$ a feasibly vertex-edge labeled subgraph $\Gamma^L$ of $G^L[\widetilde{M}]$.*

**Definition** 2.2 *Let $\widetilde{M}$ be a finitely combinatorial manifold, $\mathscr{L}$ a finite set of manifolds and $F_1^1 : \widetilde{M} \to \mathscr{L}$ an injection such that for $\forall M \in V(G[\widetilde{M}])$, there are no two different $N_1, N_2 \in \mathscr{L}$ with $F_1^1(M) \cap N_1 \neq \emptyset$, $F_1^1(M) \cap N_2 \neq \emptyset$ and for different $M_1, M_2 \in V(G[\widetilde{M}])$ with $F_1^1(M_1) \subset N_1, F_1^1(M_2) \subset N_2$, there exist $N_1', N_2' \in \mathscr{L}$ enabling that $N_1 \cap N_1' \neq \emptyset$ and $N_2 \cap N_2' \neq \emptyset$. A vertex-edge labeled quotient graph $G^L[\widetilde{M}]/F_1^1$ is defined by*

$$V(G^L[\widetilde{M}]/F_1^1) = \{N \subset \mathscr{L} | \exists M \in V(G[\widetilde{M}]) \text{ such that } F_1^1(M) \subset N\},$$

$$E(G^L[\widetilde{M}]/F_1^1) = \{(N_1, N_2) | \exists (M_1, M_2) \in E(G[\widetilde{M}]), N_1, N_2 \in \mathscr{L} \text{ such that}$$
$$F_1^1(M_1) \subset N_1, F_1^1(M_2) \subset N_2 \text{ and } F_1^1(M_1) \cap F_1^1(M_2) \neq \emptyset\}$$

*and labeling each vertex $N$ with $dimM$ if $F_1^1(M) \subset N$ and each edge $(N_1, N_2)$ with $dim(M_1 \cap M_2)$ if $F_1(M_1) \subset N_1, F_1^1(M_2) \subset N_2$ and $F_1^1(M_1) \cap F_1^1(M_2) \neq \emptyset$.*

According to Theorems 2.2 and 2.3, we find criterion for combinatorial submanifolds in the following.

**Theorem** 2.4 *Let $\widetilde{M}$ and $\widetilde{N}$ be finitely combinatorial manifolds. Then $\widetilde{M}$ is a combinatorial in-submanifold of $\widetilde{N}$ if and only if there exists an injection $F_1^1$ on $\widetilde{M}$ such that*

$$G^L[\widetilde{M}]/F_1^1 \prec_o \widetilde{N}.$$

*Proof* If $\widetilde{M}$ is a combinatorial in-submanifold of $\widetilde{N}$, by definition, we know that there is an injection $F : \widetilde{M} \rightarrow \widetilde{N}$ such that $F(\widetilde{M}) \in V(G[\widetilde{N}])$ for $\forall M \in V(G[\widetilde{M}])$ and there are no two different $N_1, N_2 \in \mathscr{L}$ with $F_1^1(M) \cap N_1 \neq \emptyset$, $F_1^1(M) \cap N_2 \neq \emptyset$. Choose $F_1^1 = F$. Since $F$ is locally $1-1$ we get that $F(M_1 \cap M_2) = F(M_1) \cap F(M_2)$, i.e., $F(M_1, M_2) \in E(G[\widetilde{N}])$ or $V(G[\widetilde{N}])$ for $\forall (M_1, M_2) \in E(G[\widetilde{M}])$. Whence, $G^L[\widetilde{M}]/F_1^1 \prec G^L[\widetilde{N}]$. Notice that $G^L[\widetilde{M}]$ is correspondent with $\widetilde{M}$. Whence, it is a feasible vertex-edge labeled subgraph of $G^L[\widetilde{N}]$ by definition. Therefore, $G^L[\widetilde{M}]/F_1^1 \prec_o G^L[\widetilde{N}]$.

Now if there exists an injection $F_1^1$ on $\widetilde{M}$, let $\Gamma^L \prec_o G^L[\widetilde{N}]$. Denote by $\overline{\Gamma}$ the graph $G^L[\widetilde{N}] \setminus \Gamma^L$, where $G^L[\widetilde{N}] \setminus \Gamma^L$ denotes the vertex-edge labeled subgraph induced by edges in $G^L[\widetilde{N}] \setminus \Gamma^L$ with non-zero labels in $G[\widetilde{N}]$. We construct a subset $\widetilde{M}^*$ of $\widetilde{N}$ by

$$\widetilde{M}^* = \widetilde{N} \setminus ((\bigcup_{M' \in V(\overline{\Gamma})} M') \bigcup (\bigcup_{(M', M'') \in E(\overline{\Gamma})} (M' \bigcap M'')))$$

and define $\widetilde{M} = F_1^{1-1}(\widetilde{M}^*)$. Notice that any open subset of an $n$-manifold is also a manifold and $F_1^{1-1}(\Gamma^L)$ is connected by definition. It can be shown that $\widetilde{M}$ is a finitely combinatorial submanifold of $\widetilde{N}$ with $G^L[\widetilde{M}]/F_1^1 \cong \Gamma^L$. □

An injection $F_1^1 : \widetilde{M} \rightarrow \mathscr{L}$ is *monotonic* if $N_1 \neq N_2$ if $F_1^1(M_1) \subset N_1$ and $F_1^1(M_2) \subset N_2$ for $\forall M_1, M_2 \in V(G[\widetilde{M}]), M_1 \neq M_2$. In this case, we get a criterion for combinatorial submanifolds of a finite combinatorial manifold.

**Corollary** 2.1 *For two finitely combinatorial manifolds $\widetilde{M}, \widetilde{N}$, $\widetilde{M}$ is a combinatorially monotonic submanifold of $\widetilde{N}$ if and only if $G^L[\widetilde{M}] \prec_o G^L[\widetilde{N}]$.*

*Proof* Notice that $F_1^1 \equiv \mathbf{1}_1^1$ in the monotonic case. Whence, $G^L[\widetilde{M}]/F_1^1 = G^L[\widetilde{M}]/\mathbf{1}_1^1 = G^L[\widetilde{M}]$. Thereafter, by Theorem 2.4, we know that $\widetilde{M}$ is a combinatorially monotonic submanifold of $\widetilde{N}$ if and only if $G^L[\widetilde{M}] \prec_o G^L[\widetilde{N}]$. □

## §3. Fundamental Formulae

Let $(\widetilde{i}, \widetilde{M})$ be a smoothly combinatorial submanifold of a Riemannian manifold $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$. For $\forall p \in \widetilde{M}$, we can directly decompose the tangent vector space $T_p\widetilde{N}$ into

$$T_p\widetilde{N} = T_p\widetilde{M} \oplus T_p^{\perp}\widetilde{M}$$

on the Riemannian metric $g_{\widetilde{N}}$ at the point $p$, i.e., choice the metric of $T_p\widetilde{M}$ and $T_p^{\perp}\widetilde{M}$ to be $g_{\widetilde{N}}|_{T_p\widetilde{M}}$ or $g_{\widetilde{N}}|_{T_p^{\perp}\widetilde{M}}$, respectively. Then we get a tangent vector space $T_p\widetilde{M}$ and a orthogonal complement $T_p^{\perp}\widetilde{M}$ of $T_p\widetilde{M}$ in $T_p\widetilde{N}$, i.e.,

$$T_p^{\perp}\widetilde{M} = \{v \in T_p\widetilde{N} |\ \langle v, u \rangle = 0 \text{ for } \forall u \in T_p\widetilde{M}\}.$$

We call $T_p\widetilde{M}$, $T_p^{\perp}\widetilde{M}$ the *tangent space* and *normal space* of $(\widetilde{i}, \widetilde{M})$ at the point $p$ in $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$, respectively. They both have the Riemannian structure, particularly, $\widetilde{M}$ is a combinatorially Riemannian manifold under the induced metric $g = \widetilde{i}^* g_{\widetilde{N}}$.

Therefore, a vector $v \in T_p\widetilde{N}$ can be directly decomposed into

$$v = v^{\top} + v^{\perp},$$

where $v^{\top} \in T_p\widetilde{M}, v^{\perp} \in T_p^{\perp}\widetilde{M}$ are the tangent component and the normal component of $v$ at the point $p$ in $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$. All such vectors $v^{\perp}$ in $T\widetilde{N}$ are denoted by $T^{\perp}\widetilde{M}$, i.e.,

$$T^{\perp}\widetilde{M} = \bigcup_{p\in\widetilde{M}} T_p^{\perp}\widetilde{M}.$$

Whence, for $\forall X, Y \in \mathscr{X}(\widetilde{M})$, we know that

$$\widetilde{D}_X Y = \widetilde{D}_X^{\top} Y + \widetilde{D}_X^{\perp} Y,$$

called the *Gauss formula on the combinatorially Riemannian submanifold* $(\widetilde{M}, g)$, where $\widetilde{D}_X^{\top} Y = (\widetilde{D}_X Y)^{\top}$ and $\widetilde{D}_X^{\perp} Y = (\widetilde{D}_X Y)^{\perp}$.

**Theorem** 3.1 *Let $(\widetilde{i}, \widetilde{M})$ be a combinatorially Riemannian submanifold of $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ with an induced metric $g = \widetilde{i}^* g_{\widetilde{N}}$. Then for $\forall X, Y, Z$, $\widetilde{D}^{\top} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ determined by $\widetilde{D}^{\top}(Y, X) = \widetilde{D}_X^{\top} Y$ is a combinatorially Riemannian connection on $(\widetilde{M}, g)$ and $\widetilde{D}^{\perp} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to T^{\perp}(\widetilde{M})$ is a symmetrically coinvariant tensor field of order 2, i.e.,*

(1) $\widetilde{D}_{X+Y}^{\perp} Z = \widetilde{D}_X^{\perp} Z + \widetilde{D}_Y^{\perp} Z$;

(2) $\widetilde{D}_{\lambda X}^{\perp} Y = \lambda \widetilde{D}_X^{\perp} Y$ *for* $\forall \lambda \in C^{\infty}(\widetilde{M})$;

(3) $\widetilde{D}_X^{\perp} Y = \widetilde{D}_Y^{\perp} X$.

*Proof* By definition, there exists an inclusion mapping $\widetilde{i} : \widetilde{M} \to \widetilde{N}$ such that $(\widetilde{i}, \widetilde{M})$ is a combinatorially Riemannian submanifold of $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ with a metric $g = \widetilde{i}^* g_{\widetilde{N}}$.

For $\forall X, Y, Z \in \mathscr{X}(\widetilde{M})$, we know that

$$\begin{aligned}\widetilde{D}_{X+Y} Z &= \widetilde{D}_X Z + \widetilde{D}_Y Z \\ &= (\widetilde{D}_X^{\top} Z + \widetilde{D}_X^{\top} Z) + (\widetilde{D}_X^{\perp} Z + \widetilde{D}_X^{\perp} Z)\end{aligned}$$

by properties of the combinatorially Riemannian connection $\widetilde{D}$. Thereby, we find that

$$\widetilde{D}_{X+Y}^{\top} Z = \widetilde{D}_X^{\top} Z + \widetilde{D}_Y^{\top} Z, \quad \widetilde{D}_{X+Y}^{\perp} Z = \widetilde{D}_X^{\perp} Z + \widetilde{D}_Y^{\perp} Z.$$

Similarly, we also find that

$$\widetilde{D}_X^{\top}(Y + Z) = \widetilde{D}_X^{\top} Y + \widetilde{D}_X^{\top} Z, \quad \widetilde{D}_X^{\perp}(Y + Z) = \widetilde{D}_X^{\perp} Y + \widetilde{D}_X^{\perp} Z.$$

Now for $\forall \lambda \in C^{\infty}(\widetilde{M})$, since

$$\widetilde{D}_{\lambda X} Y = \lambda \widetilde{D}_X Y, \quad \widetilde{D}_X(\lambda Y) = X(\lambda) + \lambda \widetilde{D}_X Y,$$

we find that

$$\widetilde{D}_{\lambda X}^{\top} Y = \lambda \widetilde{D}_X^{\top} Y, \quad \widetilde{D}_X^{\top}(\lambda Y) = X(\lambda) + \lambda \widetilde{D}_X^{\top} Y$$

and

$$\widetilde{D}^{\perp}_X(\lambda Y)=\lambda\widetilde{D}^{\perp}_X Y.$$

Thereafter, the mapping $\widetilde{D}^{\top}:\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\rightarrow\mathscr{X}(\widetilde{M})$ is a combinatorially connection on $(\widetilde{M},g)$ and $\widetilde{D}^{\perp}:\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\rightarrow T^{\perp}(\widetilde{M})$ have properties (1) and (2).

By the torsion-free of the Riemannian connection $\widetilde{D}$, i.e.,

$$\widetilde{D}_X Y-\widetilde{D}_Y X=[X,Y]\in\mathscr{X}(\widetilde{M})$$

for $\forall X,Y\in\mathscr{X}(\widetilde{M})$, we get that

$$\widetilde{D}^{\top}_X Y-\widetilde{D}^{\top}_Y X=(\widetilde{D}_X Y-\widetilde{D}_Y X)^{\top}=[X,Y]$$

and

$$\widetilde{D}^{\perp}_X Y-\widetilde{D}^{\perp}_Y X=(\widetilde{D}_X Y-\widetilde{D}_Y X)^{\perp}=0,$$

i.e., $\widetilde{D}^{\perp}_X Y=\widetilde{D}^{\perp}_Y X$. Whence, $\widetilde{D}^{\top}$ is also torsion-free on $(\widetilde{M},g)$ and the property (3) on $\widetilde{D}^{\perp}$ holds. Applying the compatibility of $\widetilde{D}$ with $g_{\widetilde{N}}$ in $(\widetilde{N},g_{\widetilde{N}},\widetilde{D})$, we finally get that

$$\begin{aligned}Z\langle X,Y\rangle &= \left\langle\widetilde{D}_Z X,Y\right\rangle+\left\langle X,\widetilde{D}_Z Y\right\rangle\\ &= \left\langle\widetilde{D}^{\top}_Z X,Y\right\rangle+\left\langle X,\widetilde{D}^{\top}_Z Y\right\rangle,\end{aligned}$$

which implies that $\widetilde{D}^{\top}$ is also compatible with $(\widetilde{M},g)$, namely $\widetilde{D}^{\top}:\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\rightarrow\mathscr{X}(\widetilde{M})$ is a combinatorially Riemannian connection on $(\widetilde{M},g)$. $\square$

Now for $\forall X\in\mathscr{X}(\widetilde{M})$ and $Y^{\perp}\in T^{\perp}\widetilde{M}$, we know that $\widetilde{D}_X Y^{\perp}\in T\widetilde{N}$. Whence, we can directly decompose it into

$$\widetilde{D}_X Y^{\perp}=\widetilde{D}^{\top}_X Y^{\perp}+\widetilde{D}^{\perp}_X Y^{\perp},$$

called the *Weingarten formula on the combinatorially Riemannian submanifold* $(\widetilde{M},g)$, where $\widetilde{D}^{\top}_X Y^{\perp}=(\widetilde{D}_X Y^{\perp})^{\top}$ and $\widetilde{D}^{\perp}_X Y^{\perp}=(\widetilde{D}_X Y^{\perp})^{\perp}$.

**Theorem** 3.2 *Let $(\widetilde{i},\widetilde{M})$ be a combinatorially Riemannian submanifold of $(\widetilde{N},g_{\widetilde{N}},\widetilde{D})$ with an induced metric $g=\widetilde{i}^{*}g_{\widetilde{N}}$. Then the mapping $\widetilde{D}^{\perp}:T^{\perp}\widetilde{M}\times\mathscr{X}(\widetilde{M})\rightarrow T^{\perp}\widetilde{M}$ determined by $\widetilde{D}(Y^{\perp},X)=\widetilde{D}^{\perp}_X Y^{\perp}$ is a combinatorially Riemannian connection on $T^{\perp}\widetilde{M}$.*

*Proof* By definition, we have known that there is an inclusion mapping $\widetilde{i}:\widetilde{M}\rightarrow\widetilde{N}$ such that $(\widetilde{i},\widetilde{M})$ is a combinatorially Riemannian submanifold of $(\widetilde{N},g_{\widetilde{N}},\widetilde{D})$ with a metric $g=\widetilde{i}^{*}g_{\widetilde{N}}$. For $\forall X,Y\in\mathscr{X}(\widetilde{M})$ and $\forall Z^{\perp},Z^{\perp}_1,Z^{\perp}_2\in T^{\perp}\widetilde{M}$, we know that

$$\widetilde{D}^{\perp}_{X+Y}Z^{\perp}=\widetilde{D}^{\perp}_X Z^{\perp}+\widetilde{D}^{\perp}_Y Z^{\perp},\quad \widetilde{D}^{\perp}_X(Z^{\perp}_1+Z^{\perp}_2)=\widetilde{D}^{\perp}_X Z^{\perp}_1+\widetilde{D}^{\perp}_X Z^{\perp}_2$$

similar to the proof of Theorem 3.1. For $\forall\lambda\in C^{\infty}(\widetilde{M})$, we know that

$$\widetilde{D}_{\lambda X}Z^{\perp} = \lambda\widetilde{D}_X Z^{\perp}, \quad \widetilde{D}_X(\lambda Z^{\perp}) = X(\lambda)Z^{\perp} + \lambda\widetilde{D}_X Z^{\perp}.$$

Whence, we find that

$$\widetilde{D}^{\perp}_{\lambda X}Z^{\perp} = (\lambda\widetilde{D}_X Z^{\perp})^{\perp} = \lambda(\widetilde{D}_X Z^{\perp})^{\perp} = \lambda\widetilde{D}^{\perp}_X Z^{\perp},$$

$$\widetilde{D}^{\perp}_X(\lambda Z^{\perp}) = X(\lambda)Z^{\perp} + \lambda(\widetilde{D}_X Z^{\perp})^{\perp} = X(\lambda)Z^{\perp} + \lambda\widetilde{D}^{\perp}_X Z^{\perp}.$$

Therefore, the mapping $\widetilde{D}^{\perp} : T^{\perp}\widetilde{M}\times\mathscr{X}(\widetilde{M}) \to T^{\perp}\widetilde{M}$ is a combinatorially connection on $T^{\perp}\widetilde{M}$. Applying the compatibility of $\widetilde{D}$ with $g_{\widetilde{N}}$ in $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$, we finally get that

$$X\left\langle Z_1^{\perp}, Z_2^{\perp}\right\rangle = \left\langle \widetilde{D}_X Z_1^{\perp}, Z_2^{\perp}\right\rangle + \left\langle Z_1^{\perp}, \widetilde{D}_X Z_2^{\perp}\right\rangle = \left\langle \widetilde{D}^{\perp}_X Z_1^{\perp}, Z_2^{\perp}\right\rangle + \left\langle Z_1^{\perp}, \widetilde{D}^{\perp}_X Z_2^{\perp}\right\rangle,$$

which implies that $\widetilde{D}^{\perp} : \mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ is a combinatorially Riemannian connection on $T^{\perp}\widetilde{M}$. $\square$

**Definition** 3.1 *Let $(\widetilde{i}, \widetilde{M})$ be a smoothly combinatorial submanifold of a Riemannian manifold $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$. The two mappings $\widetilde{D}^{\top}$, $\widetilde{D}^{\perp}$ are called the induced Riemannian connection on $\widetilde{M}$ and the normal Riemannian connection on $T^{\perp}\widetilde{M}$, respectively.*

**Theorem** 3.3 *Let $(\widetilde{i}, \widetilde{M})$ be a combinatorially Riemannian submanifold of $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ with an induced metric $g = \widetilde{i}^* g_{\widetilde{N}}$. Then for any chosen $Z^{\perp} \in T^{\perp}\widetilde{M}$, the mapping $D^{\top}_{Z^{\perp}} : \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ determined by $\widetilde{D}^{\top}_{Z^{\perp}}(X) = \widetilde{D}^{\top}_X Z^{\perp}$ for $\forall X \in \mathscr{X}(\widetilde{M})$ is a tensor field of type $(1,1)$. Besides, if $\widetilde{D}^{\top}_{Z^{\perp}}$ is treated as a smoothly linear transformation on $\widetilde{M}$, then $\widetilde{D}^{\top}_{Z^{\perp}} : T_p\widetilde{M} \to T_p\widetilde{M}$ at any point $p \in \widetilde{M}$ is a self-conjugate transformation on $g$ with the equality following hold*

$$\left\langle \widetilde{D}^{\top}_{Z^{\perp}}(X), Y\right\rangle = \left\langle \widetilde{D}^{\perp}_X(Y), Z^{\perp}\right\rangle, \quad \forall X, Y \in T_p\widetilde{M}. \quad (*)$$

*Proof* First, we establish the equality $(*)$. By applying equalities $X\left\langle Z^{\perp}, Y\right\rangle = \left\langle \widetilde{D}_X Z^{\perp}, Y\right\rangle + \left\langle Z^{\perp}, \widetilde{D}_X Y\right\rangle$ and $\left\langle Z^{\perp}, Y\right\rangle = 0$ for $\forall X, Y \in \mathscr{X}(\widetilde{M})$ and $\forall Z^{\perp} \in T^{\perp}\widetilde{M}$, we find that

$$\begin{aligned}\left\langle \widetilde{D}^{\top}_{Z^{\perp}}(X), Y\right\rangle &= \left\langle \widetilde{D}_X Z^{\perp}, Y\right\rangle \\ &= X\left\langle Z^{\perp}, Y\right\rangle - \left\langle Z^{\perp}, \widetilde{D}_X Y\right\rangle = \left\langle \widetilde{D}^{\perp}_X Y, Z^{\perp}\right\rangle.\end{aligned}$$

Thereafter, the equality $(*)$ holds.

Now according to Theorem 3.1, $\widetilde{D}^{\perp}_X Y$ posses tensor properties for $X, Y \in T_p\widetilde{M}$. Combining this fact with the equality $(*)$, $\widetilde{D}^{\top}_{Z^{\perp}}(X)$ is a tensor field of type $(1,1)$. Whence, $\widetilde{D}^{\top}_{Z^{\perp}}$ determines a linear transformation $\widetilde{D}^{\top}_{Z^{\perp}} : T_p\widetilde{M} \to T_p\widetilde{M}$ at any point $p \in \widetilde{M}$. Besides, we can also show that $\widetilde{D}^{\top}_{Z^{\perp}}(X)$ posses the tensor properties for $\forall Z^{\perp} \in T^{\perp}\widetilde{M}$. For example, for any $\lambda \in C^{\infty}(\widetilde{M})$ we know that

$$\begin{aligned}\left\langle \widetilde{D}^{\top}_{\lambda Z^{\perp}}(X), Y\right\rangle &= \left\langle \widetilde{D}^{\perp}_{X}Y, \lambda Z^{\perp}\right\rangle = \lambda\left\langle \widetilde{D}^{\perp}_{X}Y, Z^{\perp}\right\rangle \\ &= \left\langle \lambda\widetilde{D}^{\top}_{Z^{\perp}}(X), Y\right\rangle, \quad \forall X, Y \in \mathscr{X}(\widetilde{M})\end{aligned}$$

by applying the equality $(*)$ again. Therefore, we finally get that $\widetilde{D}_{\lambda Z^{\perp}}(X) = \lambda\widetilde{D}_{Z^{\perp}}(X)$.

Combining the symmetry of $\widetilde{D}^{\perp}_{X}Y$ with the equality $(*)$ enables us to know that the linear transformation $\widetilde{D}^{\top}_{Z^{\perp}} : T_p\widetilde{M} \to T_p\widetilde{M}$ at a point $p \in \widetilde{M}$ is self-conjugate. In fact, for $\forall X, Y \in T_p\widetilde{M}$, we get that

$$\begin{aligned}\left\langle \widetilde{D}^{\top}_{Z^{\perp}}(X), Y\right\rangle &= \left\langle \widetilde{D}^{\perp}_{X}Y, Z^{\perp}\right\rangle = \left\langle \widetilde{D}^{\perp}_{Y}X, Z^{\perp}\right\rangle \\ &= \left\langle \widetilde{D}^{\top}_{Z^{\perp}}(Y), X\right\rangle = \left\langle X, \widetilde{D}^{\top}_{Z^{\perp}}(Y)\right\rangle.\end{aligned}$$

Whence, $\widetilde{D}^{\top}_{Z^{\perp}}$ is self-conjugate. This completes the proof. □

Now we look for local forms for $\widetilde{D}^{\top}$ and $\widetilde{D}^{\perp}$. Let $(\widetilde{M}, g, \widetilde{D}^{\top})$ be a combinatorially Riemannian submanifold of $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$. For $\forall p \in \widetilde{M}$, let

$$\begin{aligned}\{e_{AB} | 1 \le A \le d_{\widetilde{N}}(p), 1 \le B \le n_A \quad &\text{and} \quad e_{A_1B} = e_{A_2B}, \\ &\text{for} \quad 1 \le A_1, A_2 \le d_{\widetilde{N}}(p) \text{ if } 1 \le B \le \widehat{d}_{\widetilde{N}}(p)\}\end{aligned}$$

be an orthogonal frame with a dual

$$\begin{aligned}\{\omega^{AB} | 1 \le A \le d_{\widetilde{N}}(p), 1 \le B \le n_A \quad &\text{and} \quad \omega^{A_1B} = \omega^{A_2B}, \\ &\text{for} \quad 1 \le A_1, A_2 \le d_{\widetilde{N}}(p) \text{ if } 1 \le B \le \widehat{d}_{\widetilde{N}}(p)\}\end{aligned}$$

at the point p in $T\widetilde{N}$ abbreviated to $\{e_{AB}\}$ and $\omega^{AB}$. Choose indexes $(AB), (CD), \cdots$, $(ab), (cd), \cdots$ and $(\alpha\beta), (\gamma\delta), \cdots$ satisfying $1 \le A, C \le d_{\widetilde{N}}(p)$, $1 \le B \le n_A$, $1 \le D \le n_C, \cdots$, $1 \le a, c \le d_{\widetilde{M}}(p)$, $1 \le b \le n_a, 1 \le d \le n_c, \cdots$ and $\alpha, \gamma \ge d_{\widetilde{M}}(p) + 1$ or $\beta, \delta \ge n_i + 1$ for $1 \le i \le d_{\widetilde{M}}(p)$. For getting local forms of $\widetilde{D}^{\top}$ and $\widetilde{D}^{\perp}$, we can even assume that $\{e_{AB}\}, \{e_{ab}\}$ and $\{e_{\alpha\beta}\}$ are the orthogonal frame of the point in the tangent vector space $T\widetilde{N}, T\widetilde{M}$ and the normal vector space $T^{\perp}\widetilde{M}$ by Theorems 3.1−3.3. Then the Gauss's and Weinggarten's formula can be expressed by

$$\widetilde{D}_{e_{ab}}e_{cd} = \widetilde{D}^{\top}_{e_{ab}}e_{cd} + \widetilde{D}^{\perp}_{e_{ab}}e_{cd},$$

$$\widetilde{D}_{e_{ab}}e_{\alpha\beta} = \widetilde{D}^{\top}_{e_{ab}}e_{\alpha\beta} + \widetilde{D}^{\perp}_{e_{ab}}e_{\alpha\beta}.$$

When $p$ is varied in $\widetilde{M}$, we know that $\omega^{ab} = \widetilde{i}^{*}(\omega^{ab})$ and $\omega^{\alpha b} = 0, \omega^{a\beta} = 0$. Whence, $\{\omega^{ab}\}$ is the dual of $\{e_{ab}\}$ at the point $p \in T\widetilde{M}$. Notice that $\widetilde{d}\omega^{ab} = \omega^{cd} \wedge \omega^{ab}_{cd}$, $\omega^{ab}_{cd} + \omega^{cd}_{ab} = 0$ in $(\widetilde{M}, g, \widetilde{D}^{\top})$, $\widetilde{d}\omega^{AB} = \omega^{CD} \wedge \omega^{AB}_{CD}$, $\omega^{CD}_{AB} + \omega^{AB}_{CD} = 0$,$\omega^{\alpha\beta}_{ab} + \omega^{ab}_{\alpha\beta} = 0$, $\omega^{\gamma\delta}_{\alpha\beta} + \omega^{\alpha\beta}_{\gamma\delta} = 0$ in $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ by the structural equations and

$$\widetilde{D}e_{AB} = \omega_{AB}^{CD} e_{CD}.$$

We finally get that

$$\widetilde{D}e_{ab} = \omega_{ab}^{cd} e_{cd} + \omega_{ab}^{\alpha\beta} e_{\alpha\beta}, \quad \widetilde{D}e_{\alpha\beta} = \omega_{\alpha\beta}^{cd} e_{cd} + \omega_{\alpha\beta}^{\gamma\delta} e_{\gamma\delta}.$$

Since $\widetilde{d}\omega^{\alpha i} = \omega^{ab} \wedge \omega_{ab}^{\alpha i} = 0, \widetilde{d}\omega^{i\beta} = \omega^{ab} \wedge \omega_{ab}^{i\beta} = 0$, by the *Cartan*'s Lemma, i.e., *for vectors* $v_1, v_2, \cdots, v_r,\ w_1, w_2, \cdots, w_r$ *with*

$$\sum_{s=1}^{r} v_s \wedge w_s = 0.$$

*if* $v_1, v_2, \cdots, v_r$ *are linearly independent, then*

$$w_s = \sum_{t=1}^{r} a_{st} v_t, \quad 1 \leq s \leq s,$$

*where* $a_{st} = a_{ts}$, we know that

$$\omega_{ab}^{\alpha i} = h_{(ab)(cd)}^{\alpha i} \omega^{cd}, \quad \omega_{ab}^{i\beta} = h_{(ab)(cd)}^{i\beta} \omega^{cd}$$

with $h_{(ab)(cd)}^{\alpha i} = h_{(ab)(cd)}^{\alpha i}$ and $h_{(ab)(cd)}^{i\beta} = h_{(ab)(cd)}^{i\beta}$. Thereafter, we get that

$$\widetilde{D}_{e_{ab}}^{\perp} e_{cd} = \omega_{ab}^{\alpha\beta} e_{\alpha\beta} = h_{(ab)(cd)}^{\alpha\beta} e_{\alpha\beta},$$

$$\widetilde{D}_{e_{ab}}^{\top} e_{\alpha\beta} = \omega_{\alpha\beta}^{cd} e_{cd} = h_{(ab)(cd)}^{\alpha\beta} e_{\alpha\beta}$$

.

Whence, we get local forms of $\widetilde{D}^{\top}$ and $\widetilde{D}^{\perp}$ in the following.

**Theorem** 3.4 *Let* $(\widetilde{M}, g, \widetilde{D}^{\top})$ *be a combinatorially Riemannian submanifold of* $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$. *For* $\forall p \in \widetilde{M}$ *with locally orthogonal frames* $\{e_{AB}\}$, $\{e_{ab}\}$ *and their dual* $\{\omega^{AB}\}, \{\omega^{ab}\}$ *in* $T\widetilde{N}$, $T\widetilde{M}$,

$$\widetilde{D}_{e_{ab}}^{\top} e_{cd} = \omega_{ab}^{cd} e_{cd}, \quad \widetilde{D}_{e_{ab}}^{\perp} e_{cd} = h_{(ab)(cd)}^{\alpha\beta} e_{\alpha\beta}$$

*and*

$$\widetilde{D}_{e_{ab}}^{\top} e_{\alpha\beta} = h_{(ab)(cd)}^{\alpha\beta} e_{\alpha\beta}, \quad \widetilde{D}_{e_{ab}}^{\perp} e_{\alpha\beta} = \omega_{\alpha\beta}^{\gamma\delta} e_{\gamma\delta}.$$

## §4. **Fundamental Equations**

Applications of these *Gauss*'s and *Weingarten*'s formulae enable one to get fundamental equations such as the *Gauss*'s, *Codazzi*'s and *Ricci*'s equations on curvature tensors for characterizing combinatorially Riemannian submanifolds.

**Theorem** 4.1(Gauss equation) *Let $(\widetilde{M}, g, \widetilde{D}^{\top})$ be a combinatorially Riemannian submanifold of $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ with the induced metric $g = \widetilde{i}^* g_{\widetilde{N}}$ and $\widetilde{R}, \widetilde{R}_{\widetilde{N}}$ curvature tensors on $\widetilde{M}$ and $\widetilde{N}$, respectively. Then for $\forall X, Y, Z, W \in \mathscr{X}(\widetilde{M})$,*

$$\widetilde{R}(X,Y,Z,W) = \widetilde{R}_{\widetilde{N}}(X,Y,Z,W) + \left\langle \widetilde{D}^{\perp}_X Z, \widetilde{D}^{\perp}_Y W \right\rangle - \left\langle \widetilde{D}^{\perp}_Y Z, \widetilde{D}^{\perp}_X W \right\rangle.$$

*Proof* By definition, we know that

$$\widetilde{\mathcal{R}}_{\widetilde{N}}(X,Y)Z = \widetilde{D}_X \widetilde{D}_Y Z - \widetilde{D}_Y \widetilde{D}_X Z - \widetilde{D}_{[X,Y]} Z.$$

Applying the *Gauss* formula, we find that

$$\begin{aligned} \widetilde{\mathcal{R}}_{\widetilde{N}}(X,Y)Z &= \widetilde{D}_X(\widetilde{D}^{\top}_Y Z + \widetilde{D}^{\perp}_Y Z) - \widetilde{D}_Y(\widetilde{D}^{\top}_X Z + \widetilde{D}^{\perp}_X Z) \\ &\quad -(\widetilde{D}^{\top}_{[X,Y]} Z + \widetilde{D}^{\perp}_{[X,Y]} Z) \\ &= \widetilde{D}^{\top}_X \widetilde{D}^{\top}_Y Z + \widetilde{D}^{\perp}_X \widetilde{D}^{\top}_Y Z + \widetilde{D}_X \widetilde{D}^{\perp}_Y Z - \widetilde{D}^{\top}_Y \widetilde{D}^{\top}_X Z \\ &\quad -\widetilde{D}^{\perp}_Y \widetilde{D}^{\top}_X Z - \widetilde{D}_Y \widetilde{D}^{\perp}_X Z - \widetilde{D}^{\top}_{[X,Y]} Z - \widetilde{D}^{\perp}_{[X,Y]} Z \\ &= \widetilde{R}(X,Y)Z + (\widetilde{D}^{\perp}_X \widetilde{D}^{\top}_Y Z - \widetilde{D}^{\perp}_Y \widetilde{D}^{\top}_X Z) \\ &\quad -(\widetilde{D}^{\perp}_{[X,Y]} Z - \widetilde{D}_X \widetilde{D}^{\perp}_Y Z + \widetilde{D}_Y \widetilde{D}^{\perp}_X Z). \quad (4.1) \end{aligned}$$

By the *Weingarten* formula,

$$\widetilde{D}_X \widetilde{D}^{\perp}_Y Z = \widetilde{D}^{\top}_X \widetilde{D}^{\perp}_Y Z + \widetilde{D}^{\perp}_X \widetilde{D}^{\perp}_Y Z, \quad \widetilde{D}_Y \widetilde{D}^{\perp}_X Z = \widetilde{D}^{\top}_Y \widetilde{D}^{\perp}_X Z + \widetilde{D}^{\perp}_Y \widetilde{D}^{\perp}_X Z.$$

Therefore, we get that

$$\left\langle \widetilde{R}(X,Y)Z, W \right\rangle = \left\langle \widetilde{R}_{\widetilde{N}}(X,Y)Z, W \right\rangle + \left\langle \widetilde{D}^{\perp}_X Z, \widetilde{D}^{\perp}_Y W \right\rangle - \left\langle \widetilde{D}^{\perp}_Y Z, \widetilde{D}^{\perp}_X W \right\rangle$$

by applying the equality $(*)$ in Theorem 2.4, i.e.,

$$\widetilde{R}(X,Y,Z,W) = \widetilde{R}_{\widetilde{N}}(X,Y,Z,W) + \left\langle \widetilde{D}^{\perp}_X Z, \widetilde{D}^{\perp}_Y W \right\rangle - \left\langle \widetilde{D}^{\perp}_Y Z, \widetilde{D}^{\perp}_X W \right\rangle. \quad \square$$

For $\forall X, Y, Z \in \mathscr{X}(\widetilde{M})$, define the covarint differential $\widetilde{D}_X$ on $\widetilde{D}^{\perp}_Y Z$ by

$$(\widetilde{D}_X \widetilde{D}^{\perp})_Y Z = \widetilde{D}^{\perp}_X(\widetilde{D}^{\perp}_Y Z) - \widetilde{D}^{\perp}_{\widetilde{D}^{\top}_X Y} Z - \widetilde{D}^{\perp}_Y(\widetilde{D}^{\top}_X Z).$$

Then we get the *Codazzi equation* in the following.

**Theorem** 4.2 (Codazzi equation) *Let $(\widetilde{M}, g, \widetilde{D}^{\top})$ be a combinatorially Riemannian submanifold of $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ with the induced metric $g = \widetilde{i}^* g_{\widetilde{N}}$ and $\widetilde{R}, \widetilde{R}_{\widetilde{N}}$ curvature tensors on $\widetilde{M}$ and $\widetilde{N}$, respectively. Then for $\forall X, Y, Z \in \mathscr{X}(\widetilde{M})$,*

$$(\widetilde{D}_X \widetilde{D}^{\perp})_Y Z - (\widetilde{D}_Y \widetilde{D}^{\perp})_X Z = \widetilde{R}^{\perp}(X,Y)Z$$

*Proof* Decompose the curvature tensor $\widetilde{R}_{\widetilde{N}}(X,Y)Z$ into

$$\widetilde{R}_{\widetilde{N}}(X,Y)Z = \widetilde{R}^{\top}_{\widetilde{N}}(X,Y)Z + \widetilde{R}^{\perp}_{\widetilde{N}}(X,Y)Z.$$

Notice that

$$\widetilde{D}^{\top}_X Y - \widetilde{D}^{\top}_Y Z = [X,Y].$$

By the formula (4.1), we know that

$$\begin{aligned}
\widetilde{R}^{\perp}_{\widetilde{N}}(X,Y)Z &= \widetilde{D}^{\perp}_X\widetilde{D}^{\top}_Y Z - \widetilde{D}^{\perp}_Y\widetilde{D}^{\top}_X Z - \widetilde{D}^{\perp}_{[X,Y]}Z + \widetilde{D}^{\perp}_X\widetilde{D}^{\perp}_Y Z - \widetilde{D}^{\perp}_Y\widetilde{D}^{\perp}_X Z \\
&= \widetilde{D}^{\perp}_X\widetilde{D}^{\perp}_Y Z - \widetilde{D}^{\perp}_Y\widetilde{D}^{\top}_X Z - \widetilde{D}_{\widetilde{D}^{\top}_X Y}Z + \widetilde{D}^{\perp}_Y\widetilde{D}^{\perp}_X Z - \widetilde{D}^{\perp}_X\widetilde{D}^{\top}_Y Z - \widetilde{D}_{\widetilde{D}^{\top}_Y X}Z \\
&= (\widetilde{D}_X\widetilde{D}^{\perp})_Y Z - (\widetilde{D}_Y\widetilde{D}^{\perp})_X Z. \quad \square
\end{aligned}$$

For $\forall X,Y \in \mathscr{X}(\widetilde{M})$, $Z^{\perp} \in T^{\perp}(\widetilde{M})$, the curvature tensor $\widetilde{R}^{\perp}$ determined by $\widetilde{D}^{\perp}$ in $T^{\perp}\widetilde{M}$ is defined by

$$\widetilde{R}^{\perp}(X,Y)Z^{\perp} = \widetilde{D}^{\perp}_X\widetilde{D}^{\perp}_Y Z^{\perp} - \widetilde{D}^{\perp}_Y\widetilde{D}^{\perp}_X Z^{\perp} - \widetilde{D}^{\perp}_{[X,Y]}Z^{\perp}.$$

Similarly, we get the next result.

**Theorem** 4.3(Ricci equation) *Let $(\widetilde{M}, g, \widetilde{D}^{\top})$ be a combinatorially Riemannian submanifold of $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ with the induced metric $g = \widetilde{i}^* g_{\widetilde{N}}$ and $\widetilde{R}, \widetilde{R}_{\widetilde{N}}$ curvature tensors on $\widetilde{M}$ and $\widetilde{N}$, respectively. Then for $\forall X,Y \in \mathscr{X}(\widetilde{M})$, $Z^{\perp} \in T\widetilde{M}$,*

$$\widetilde{R}^{\perp}(X,Y)Z^{\perp} = \widetilde{R}^{\perp}_{\widetilde{N}}(X,Y)Z^{\perp} + (\widetilde{D}_X\widetilde{D^{\perp}})_Y Z^{\perp} - (\widetilde{D}_Y\widetilde{D^{\perp}})_X Z^{\perp}).$$

*Proof* Similar to the proof of Theorem 4.1, we know that

$$\begin{aligned}
\widetilde{R}_{\widetilde{N}}(X,Y)Z^{\perp} &= \widetilde{D}_X\widetilde{D}_Y Z^{\perp} - \widetilde{D}_Y\widetilde{D}_X Z^{\perp} - \widetilde{D}_{[X,Y]}Z^{\perp} \\
&= \widetilde{R}^{\perp}(X,Y)Z^{\perp} + \widetilde{D}^{\perp}_X\widetilde{D}^{\top}_Y Z^{\perp} - \widetilde{D}^{\perp}_Y\widetilde{D}^{\top}_X Z^{\perp} \\
&\quad + \widetilde{D}_X\widetilde{D}^{\perp}_Y Z^{\perp} - \widetilde{D}_Y\widetilde{D}^{\perp}_X Z^{\perp} \\
&= (\widetilde{R}^{\perp}(X,Y)Z^{\perp} + (\widetilde{D}_X\widetilde{D^{\perp}})_Y Z^{\perp} - (\widetilde{D}_Y\widetilde{D^{\perp}})_X Z^{\perp}) \\
&\quad + \widetilde{D}^{\top}_X\widetilde{D}^{\perp}_Y Z^{\perp} - \widetilde{D}^{\top}_Y\widetilde{D}^{\perp}_X Z^{\perp}.
\end{aligned}$$

Whence, we get that

$$\widetilde{R}^{\perp}(X,Y)Z^{\perp} = \widetilde{R}^{\perp}_{\widetilde{N}}(X,Y)Z^{\perp} + (\widetilde{D}_X\widetilde{D^{\perp}})_Y Z^{\perp} - (\widetilde{D}_Y\widetilde{D^{\perp}})_X Z^{\perp}). \quad \square$$

Certainly, we can also find local forms for these *Gauss*'s, *Codazzi*'s and Ricci's equations in a locally orthogonal frames $\{e_{AB}\}$, $\{e_{ab}\}$ of $T\widetilde{N}$ and $T\widetilde{M}$ at a point $p \in \widetilde{M}$.

**Theorem** 4.4 *Let $(\widetilde{M}, g, \widetilde{D}_{\widetilde{M}})$ be a combinatorially Riemannian submanifold of $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ with $g = \widetilde{i}^* g_{\widetilde{N}}$ and for $p \in \widetilde{M}$, let $\{e_{AB}\}$, $\{e_{ab}\}$ be locally orthogonal frames of $T\widetilde{N}$ and $T\widetilde{M}$ at $p$ with dual $\{\omega^{AB}\}$, $\{\omega^{ab}\}$. Then*

$$\widetilde{R}_{(ab)(cd)(ef)(gh)} = (\widetilde{R}_{\widetilde{N}})_{(ab)(cd)(ef)(gh)} - \sum_{\alpha,\beta}(h^{\alpha\beta}_{(ab)(ef)}h^{\alpha\beta}_{(cd)(gh)} - h^{\alpha\beta}_{(ab)(gh)}h^{\alpha\beta}_{(cd)(ef)}) \quad (\ Gauss\ ),$$

$$h^{\alpha\beta}_{(ab)(cd)(ef)} - h^{\alpha\beta}_{(ab)(ef)(cd)} = (\widetilde{R}_{\widetilde{N}})_{(\alpha\beta)(ab)(cd)(ef)} \quad (\ Codazzi\ )$$

*and*

$$\widetilde{R}^{\perp}_{(\alpha\beta)(\gamma\delta)(ab)(cd)} = (\widetilde{R}_{\widetilde{N}})_{(\alpha\beta)(\gamma\delta)(ab)(cd)} - \sum_{e,f}(h^{\alpha\beta}_{(ab)(ef)}h^{\gamma\delta}_{(cd)(gh)} - h^{\alpha\beta\beta}_{(cd)(ef)}h^{\gamma\delta}_{(ab)(gh)}) \quad (\ Ricci\ )$$

*with* $\widetilde{R}^{\perp}_{(\alpha\beta)(\gamma\delta)(ab)(cd)} = \left\langle \widetilde{R}(e_{ab}, e_{cd})e_{\alpha\beta}, e_{\gamma\delta}\right\rangle$ *and*

$$h^{\alpha\beta}_{(ab)(cd)(ef)}\omega^{ef} = \widetilde{d}h^{\alpha\beta}_{(ab)(cd)} - \omega^{ef}_{ab}h^{\alpha\beta}_{(ef)(cd)} - \omega^{ef}_{cd}h^{\alpha\beta}_{(ab)(ef)} + \omega^{\alpha\beta}_{\gamma\delta}h^{\gamma\delta}_{(ab)(cd)}.$$

*Proof* Let $\widetilde{\Omega}$ and $\widetilde{\Omega}_{\widetilde{N}}$ be curvature forms in $\widetilde{M}$ and $\widetilde{N}$. Then by the structural equations in $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ ([10]), we know that

$$(\widetilde{\Omega}_{\widetilde{N}})^{CD}_{AB} = \widetilde{d}\omega^{CD}_{AB} - \omega^{EF}_{AB} \wedge \omega^{CD}_{EF} = \frac{1}{2}(\widetilde{R}\widetilde{N})_{(AB)(CD)(EF)(GH)}\omega^{EF} \wedge \omega^{GH}$$

and $\widetilde{R}(e_{AB}, e_{CD})e_{EF} = \widetilde{\Omega}^{GH}_{EF}(e_{AB}, e_{CD})e_{GH}$. Let $\widetilde{i} : \widetilde{M} \rightarrow \widetilde{N}$ be an embedding mapping. Applying $\widetilde{i}^*$ action on the above equations, we find that

$$\begin{aligned}(\widetilde{\Omega}_{\widetilde{N}})^{cd}_{ab} &= \widetilde{d}\omega^{cd}_{ab} - \omega^{ef}_{ab} \wedge \omega^{cd}_{ef} - \omega^{\alpha\beta}_{ab} \wedge \omega^{cd}_{\alpha\beta} \\ &= \widetilde{\Omega}^{cd}_{ab} + \sum_{\alpha,\beta} h^{\alpha\beta}_{(ab)(ef)}h^{\alpha\beta}_{(cd)(gh)}\omega^{ef} \wedge \omega^{gh}.\end{aligned}$$

Whence, we get that

$$\widetilde{\Omega}^{cd}_{ab} = (\widetilde{\Omega}_{\widetilde{N}})^{cd}_{ab} - \frac{1}{2}\sum_{\alpha,\beta}(h^{\alpha\beta}_{(ab)(ef)}h^{\alpha\beta}_{(cd)(gh)} - h^{\alpha\beta}_{(ab)(gh)}h^{\alpha\beta}_{(cd)(ef)})\omega^{ef} \wedge \omega^{gh}.$$

This is the *Gauss*'s equation

$$\widetilde{R}_{(ab)(cd)(ef)(gh)} = (\widetilde{R}_{\widetilde{N}})_{(ab)(cd)(ef)(gh)} - \sum_{\alpha,\beta}(h^{\alpha\beta}_{(ab)(ef)}h^{\alpha\beta}_{(cd)(gh)} - h^{\alpha\beta}_{(ab)(gh)}h^{\alpha\beta}_{(cd)(ef)}).$$

Similarly, we also know that

$$\begin{aligned}(\widetilde{\Omega}_{\widetilde{N}})^{\alpha\beta}_{ab} &= \widetilde{d}\omega^{\alpha\beta}_{ab} - \omega^{cd}_{ab} \wedge \omega^{\alpha\beta}_{cd} - \omega^{\gamma\delta}_{ab} \wedge \omega^{\alpha\beta}_{\gamma\delta} \\ &= \widetilde{d}(h^{\alpha\beta}_{(ab)(cd)}\omega^{cd}) - h^{\alpha\beta}_{(cd)(ef)}\omega^{cd}_{ab} \wedge \omega^{ef} - h^{\gamma\delta}_{(ab)(ef)}\omega^{ef} \wedge \omega^{\alpha\beta}_{\gamma\delta} \\ &= (\widetilde{d}h^{\alpha\beta}_{(ab)(cd)} - h^{\alpha\beta}_{(ab)(ef)}\omega^{ef}_{cd}) - h^{\alpha\beta}_{(ef)(cd)}\omega^{ef}_{ab} + h^{\gamma\delta}_{(ab)(cd)}\omega_{\alpha\beta}) \wedge \omega^{cd} \\ &= h^{\alpha\beta}_{(ab)(cd)(ef)}\omega^{ef} \wedge \omega^{cd} \\ &= \frac{1}{2}(h^{\alpha\beta}_{(ab)(cd)(ef)} - h^{\alpha\beta}_{(ab)(ef)(cd)})\omega^{ef} \wedge \omega^{cd}\end{aligned}$$

and

$$
\begin{aligned}
(\widetilde{\Omega}_{\widetilde{N}})^{\gamma\delta}_{\alpha\beta} &= \widetilde{d}\omega^{\gamma\delta}_{\alpha\beta} - \omega^{ef}_{\alpha\beta}\wedge\omega^{\gamma\delta}_{ef} - \omega^{\zeta\eta}_{\alpha\beta}\wedge\omega^{\gamma\delta}_{\zeta\eta} \\
&= \widetilde{\Omega}^{\perp\gamma\delta}_{\alpha\beta} + \frac{1}{2}\sum_{e,f}(h^{\alpha\beta}_{(ef)(ab)}h^{\gamma\delta}_{(ef)(cd)} - h^{\alpha\beta}_{(ef)(cd)}h^{\gamma\delta}_{(ef)(ab)})\omega^{ab}\wedge\omega^{cd}.
\end{aligned}
$$

These equalities enables us to get

$$h^{\alpha\beta}_{(ab)(cd)(ef)} - h^{\alpha\beta}_{(ab)(ef)(cd)} = (\widetilde{R}_{\widetilde{N}})_{(\alpha\beta)(ab)(cd)(ef)},$$

and

$$\widetilde{R}^{\perp}_{(\alpha\beta)(\gamma\delta)(ab)(cd)} = (\widetilde{R}_{\widetilde{N}})_{(\alpha\beta)(\gamma\delta)(ab)(cd)} - \sum_{e,f}(h^{\alpha\beta}_{(ab)(ef)}h^{\gamma\delta}_{(cd)(gh)} - h^{\alpha\beta\beta}_{(cd)(ef)}h^{\gamma\delta}_{(ab)(gh)}).$$

These are just the *Codazzi*'s or *Ricci*'s equations. □

## §5. Embedding in Combinatorially Euclidean Spaces

For a given integer sequence $k_1, n_2, \cdots, k_l, l \geq 1$ with $0 < k_1 < k_2 < \cdots < k_l$, a *combinatorially Euclidean space* $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$ is a union of finitely Euclidean spaces $\bigcup\limits_{i=1}^{l}\mathbf{R}^{k_i}$ such that for $\forall p \in \widetilde{\mathbf{R}}(k_1, \cdots, k_l)$, $p \in \bigcap\limits_{i=1}^{l}\mathbf{R}^{k_i}$ with $\widehat{l} = \dim(\bigcap\limits_{i=1}^{l}\mathbf{R}^{k_i})$ a constant. For a given combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, *wether it can be realized in a combinatorially Euclidean space* $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$*?* We consider this problem with twofold in this section, i.e., topological or isometry embedding of a combinatorial manifold in combinatorially Euclidean spaces.

### 5.1. Topological embedding

Given two topological spaces $\mathscr{C}_1$ and $\mathscr{C}_2$, a *topological embedding* of $\mathscr{C}_1$ in $\mathscr{C}_2$ is a one-to-one continuous map

$$f:\ \mathscr{C}_1 \rightarrow \mathscr{C}_2.$$

When $f : \widetilde{M}(n_1, n_2, \cdots, n_m) \rightarrow \widetilde{\mathbf{R}}(k_1, \cdots, k_l)$ maps each manifold of $\widetilde{M}$ to an Euclidean space of $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$, we say that $\widetilde{M}$ is in-embedded into $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$.

Whitney had proved once that *any n-manifold can be topological embedded as a closed submanifold of* $\mathbf{R}^{2n+1}$ *with a sharply minimum dimension* $2n + 1$ in 1936[1]. Applying Whitney's result enables us to find conditions of a finitely combinatorial manifold embedded into a combinatorially Euclidean space $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$.

Firstly, We thereafter get a result for the case $l = 1$, which completely answers the problem 4.1 raised in [7].

**Theorem** 5.1 *Any finitely combinatorial manifold* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *can be embedded into* $\mathbf{R}^{2n_m+1}$.

*Proof* According to Whitney's result, each manifold $M^{n_i}, 1 \leq i \leq m$, in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ can be topological embedded into an Euclidean space $\mathbf{R}^\eta$ for any $\eta \geq 2n_i + 1$. By assumption, $n_1 < n_2 < \cdots < n_m$. Whence, any manifold in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ can be embedded into $\mathbf{R}^{2n_m+1}$. Applying Theorem 2.2, we know that $\widetilde{M}(n_1, n_2, \cdots, n_m)$ can be embedded into $\mathbf{R}^{2n_m+1}$. □

For in-embedding a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ into combinatorially Euclidean spaces $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$, we get the next result.

**Theorem** 5.2 *Any finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ can be in-embedded into a combinatorially Euclidean space $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$ if there is an injection*

$$\varpi : \{n_1, n_2, \cdots, n_m\} \rightarrow \{k_1, k_2, \cdots, k_l\}$$

*such that*

$$\varpi(n_i) \geq \max\{2\epsilon + 1 |\ \forall \epsilon \in \varpi^{-1}(\varpi(n_i))\}$$

*and*

$$dim(\mathbf{R}^{\varpi(n_i)} \bigcap \mathbf{R}^{\varpi(n_j)}) \geq 2dim(M^{n_i} \bigcap M^{n_j}) + 1$$

*for any integer $i, j, 1 \leq i, j \leq m$ with $M^{n_i} \cap M^{n_j} \neq \emptyset$.*

*Proof* Notice that if

$$\varpi(n_i) \geq \max\{2\epsilon + 1 |\ \forall \epsilon \in \varpi^{-1}(\varpi(n_i))\}$$

for any integer $i, 1 \leq i \leq m$, then each manifold $M^\epsilon, \forall \epsilon \in \varpi^{-1}(\varpi(n_i))$ can be embedded into $\mathbf{R}^{\varpi(n_i)}$ and for $\forall \epsilon_1 \in \varpi^{-1}(n_i)$, $\forall \epsilon_2 \in \varpi^{-1}(n_j)$, $M^{\epsilon_1} \cap M^{\epsilon_2}$ can be in-embedded into $\mathbf{R}^{\varpi(n_i)} \cap \mathbf{R}^{\varpi(n_j)}$ if $M^{\epsilon_1} \cap M^{\epsilon_2} \neq \emptyset$ by Whitney's result. In this case, a few manifolds in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ may be in-embedded into one Euclidean space $\mathbf{R}^{\varpi(n_i)}$ for any integer $i, 1 \leq i \leq m$. Therefore, by applying Theorem 2.3 we know that $\widetilde{M}(n_1, n_2, \cdots, n_m)$ can be in-embedded into a combinatorially Euclidean space $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$. □

If $l = 1$ in Theorem 5.2, then we obtain Theorem 5.1 once more since $\varpi(n_i)$ is a constant in this case. But on a classical viewpoint, Theorem 5.1 is more accepted for it presents the appearances of a combinatorial manifold in a classical space. Certainly, we can also get concrete conclusions for practical usefulness by Theorem 5.2, such as the next result.

**Corollary** 5.1 *Any finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ can be in-embedded into a combinatorially Euclidean space $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$ if*

*(i)* $l \geq m$;

*(ii) there exists $m$ different integers $k_{i_1}, k_{i_2}, \cdots, k_{i_m} \in \{k_1, k_2, \cdots, k_l\}$ such that*

$$k_{i_j} \geq 2n_j + 1$$

*and*

$$dim(\mathbf{R}^{k_{i_j}}\bigcap\mathbf{R}^{k_{i_r}})\geq 2dim(M^{n_j}\bigcap M^{n_r})+1$$

*for any integer* $i,j,1\leq i,j\leq m$ *with* $M^{n_j}\cap M^{n_r}\neq\emptyset$.

*Proof* Choose an injection

$$\pi:\{n_1,n_2,\cdots,n_m\}\rightarrow\{k_1,k_2,\cdots,k_l\}$$

by $\pi(n_j)=k_{i_j}$ for $1\leq j\leq m$. Then conditions $(i)$ and $(ii)$ implies that $\pi$ is an injection satisfying conditions in Theorem 5.2. Whence, $\widetilde{M}(n_1,n_2,\cdots,n_m)$ can be in-embedded into $\widetilde{\mathbf{R}}(k_1,\cdots,k_l)$. □

### 5.2. Isometry embedding

For two given combinatorially Riemannian $C^r$-manifolds $(\widetilde{M},g,\widetilde{D}_{\widetilde{M}})$ and $(\widetilde{N},g_{\widetilde{N}},\widetilde{D})$, an *isometry embedding*

$$\widetilde{i}:\widetilde{M}\rightarrow\widetilde{N}$$

is an embedding with $g=\widetilde{i}^{*}g_{\widetilde{N}}$. By those discussions in Sections 3 and 4, let the local charts of $\widetilde{M}$, $\widetilde{N}$ be $(U,[x])$, $(V,[y])$ and the metrics in $\widetilde{M}$, $\widetilde{N}$ to be respective

$$g_{\widetilde{N}}=\sum_{(\varsigma\tau),(\vartheta\iota)}g_{\widetilde{N}_{(\varsigma\tau)(\vartheta\iota)}}dy^{\varsigma\tau}\otimes dy^{\vartheta\iota},\ g=\sum_{(\mu\nu),(\kappa\lambda)}g_{(\mu\nu)(\kappa\lambda)}dx^{\mu\nu}\otimes dx^{\kappa\lambda},$$

then an isometry embedding $\widetilde{i}$ form $\widetilde{M}$ to $\widetilde{N}$ need us to determine wether there are functions

$$y^{\kappa\lambda}=i^{\kappa\lambda}[x^{\mu\nu}],1\leq\mu\leq s(p),1\leq\nu\leq n_{s(p)}$$

for $\forall p\in\widetilde{M}$ such that

$$\widetilde{R}_{(ab)(cd)(ef)(gh)}=(\widetilde{R}_{\widetilde{N}})_{(ab)(cd)(ef)(gh)}-\sum_{\alpha,\beta}(h^{\alpha\beta}_{(ab)(ef)}h^{\alpha\beta}_{(cd)(gh)}-h^{\alpha\beta}_{(ab)(gh)}h^{\alpha\beta}_{(cd)(ef)}),$$

$$h^{\alpha\beta}_{(ab)(cd)(ef)}-h^{\alpha\beta}_{(ab)(ef)(cd)}=(\widetilde{R}_{\widetilde{N}})_{(\alpha\beta)(ab)(cd)(ef)},$$

$$\widetilde{R}^{\perp}_{(\alpha\beta)(\gamma\delta)(ab)(cd)}=(\widetilde{R}_{\widetilde{N}})_{(\alpha\beta)(\gamma\delta)(ab)(cd)}-\sum_{e,f}(h^{\alpha\beta}_{(ab)(ef)}h^{\gamma\delta}_{(cd)(gh)}-h^{\alpha\beta\beta}_{(cd)(ef)}h^{\gamma\delta}_{(ab)(gh)})$$

*with* $\widetilde{R}^{\perp}_{(\alpha\beta)(\gamma\delta)(ab)(cd)}=\left\langle\widetilde{R}(e_{ab},e_{cd})e_{\alpha\beta},e_{\gamma\delta}\right\rangle$,

$$h^{\alpha\beta}_{(ab)(cd)(ef)}\omega^{ef}=\widetilde{d}h^{\alpha\beta}_{(ab)(cd)}-\omega^{ef}_{ab}h^{\alpha\beta}_{(ef)(cd)}-\omega^{ef}_{cd}h^{\alpha\beta}_{(ab)(ef)}+\omega^{\alpha\beta}_{\gamma\delta}h^{\gamma\delta}_{(ab)(cd)}$$

and

$$\sum_{(\varsigma\tau),(\vartheta\iota)} g_{\widetilde{N}_{(\varsigma\tau)(\vartheta\iota)}}(\widetilde{i}[x])\frac{\partial i^{\varsigma\tau}}{\partial x^{\mu\nu}}\frac{\partial i^{\vartheta\iota}}{\partial x^{\kappa\lambda}} = g_{(\mu\nu)(\kappa\lambda)}[x].$$

For embedding a combinatorial manifold into a combinatorially Euclidean space $\widetilde{\mathbf{R}}(k_1,\cdots,k_l)$, the last equation can be replaced by

$$\sum_{(\varsigma\tau)} \frac{\partial i^{\varsigma\tau}}{\partial y^{\mu\nu}}\frac{\partial i^{\varsigma\tau}}{\partial y^{\kappa\lambda}} = g_{(\mu\nu)(\kappa\lambda)}[y]$$

since a combinatorially Euclidean space $\widetilde{\mathbf{R}}(k_1,\cdots,k_l)$ is equivalent to an Euclidean space $\mathbf{R}^{\widetilde{k}}$ with a constant $\widetilde{k} = \widehat{l}(p) + \sum\limits_{i=1}^{l(p)}(k_i - \widehat{l}(p))$ for $\forall p \in \mathbf{R}^{\widetilde{k}}$ but not dependent on $p$ (see [9] for details) and the metric of an Euclidean space $\mathbf{R}^{\widetilde{k}}$ to be

$$g_{\widetilde{\mathbf{R}}} = \sum_{\mu,\nu} dy^{\mu\nu} \otimes dy^{\mu\nu}.$$

These combined with additional conditions enable us to find necessary and sufficient conditions for existing particular combinatorially Riemannian submanifolds.

Similar to Theorems 5.1 and 5.2, we can also get sufficient conditions on isometry embedding by applying Lemma 2.1, i.e., the decomposition lemma on unit. Firstly, we need two important lemmas following.

**Lemma** 5.1([2]) *For any integer $n \geq 1$, a Riemannian $C^r$-manifold of dimensional $n$ with $2 < r \leq \infty$ can be isometry embedded into the Euclidean space $\mathbf{R}^{n^2+10n+3}$.*

**Lemma** 5.2 *Let $(\widetilde{M}, g, \widetilde{D}_{\widetilde{M}})$ and $(\widetilde{N}, g_{\widetilde{N}}, \widetilde{D})$ be combinatorially Riemannian manifolds. If for $\forall M \in V(G[\widetilde{M}])$, there exists isometry embedding $F_M : M \to \widetilde{N}$, then $\widetilde{M}$ can be isometry embedded into $\widetilde{N}$.*

*Proof* Similar to the proof of Theorems 2.2 and 2.3, we only need to prove that the mapping $\widetilde{F} : \widetilde{M} \to \widetilde{N}$ defined by

$$\widetilde{F}(p) = \sum_{i=1}^{\widehat{s}(p)} f_{M_i} F_{M_i}$$

is an isometry embedding. In fact, for $p \in \widetilde{M}$ we have already known that

$$g_{\widetilde{N}}((F_{M_i})_*(v), (F_{M_i})_*(w)) = g(v, w)$$

for $\forall v, w \in T_p\widetilde{M}$ and $i, 1 \leq i \leq \widehat{s}(p)$. By definition we know that

$$\begin{aligned}
g_{\widetilde{N}}(\widetilde{F}_*(v),\widetilde{F}_*(w)) &= g_{\widetilde{N}}(\sum_{i=1}^{\widehat{s}(p)} f_{M_i}(F_{M_i})(v),\sum_{j=1}^{\widehat{s}(p)} f_{M_j}(F_{M_j})(w)) \\
&= \sum_{i=1}^{\widehat{s}(p)}\sum_{j=1}^{\widehat{s}(p)} g_{\widetilde{N}}(f_{M_i}(F_{M_i})(v), f_{M_j}(F_{M_j})(w))) \\
&= \sum_{i=1}^{\widehat{s}(p)}\sum_{j=1}^{\widehat{s}(p)} g(f_{M_i}(F_{M_i})(v), f_{M_j}(F_{M_j})(w))) \\
&= g(\sum_{i=1}^{\widehat{s}(p)} f_{M_i}v,\sum_{j=1}^{\widehat{s}(p)} f_{M_j}w) \\
&= g(v,w).
\end{aligned}$$

Therefore, $\widetilde{F}$ is an isometry embedding. □

Applying Lemmas 5.1 and 5.2, we get results on isometry embedding of a combinatorial manifolds into combinatorially Euclidean spaces following.

**Theorem** 5.3 *Any combinatorial Riemannian manifold $\widetilde{M}(n_1,n_2,\cdots,n_m)$ can be isometry embedded into $\mathbf{R}^{n_m^2+10n_m+3}$.*

*Proof* According to Lemma 2.1, each manifold $M^{n_i}, 1\le i\le m$, in $\widetilde{M}(n_1,n_2,\cdots,n_m)$ can be isometry embedded into an Euclidean space $\mathbf{R}^{\eta}$ for any $\eta\ge n_i^2+10n_i+3$. By assumption, $n_1<n_2<\cdots<n_m$. Thereafter, each manifold in $\widetilde{M}(n_1,n_2,\cdots,n_m)$ can be embedded into $\mathbf{R}^{n_m^2+10n_m+3}$. Applying Lemma 5.2, we know that $\widetilde{M}(n_1,n_2,\cdots,n_m)$ can be isometry embedded into $\mathbf{R}^{n_m^2+10n_m+3}$. □

**Theorem** 5.4 *A combinatorially Riemannian manifold $\widetilde{M}(n_1,n_2,\cdots,n_m)$ can be isometry embedded into a combinatorially Euclidean space $\widetilde{\mathbf{R}}(k_1,\cdots,k_l)$ if there is an injection*

$$\varpi:\{n_1,n_2,\cdots,n_m\}\to\{k_1,k_2,\cdots,k_l\}$$

*such that*

$$\varpi(n_i)\ge\max\{\epsilon^2+10\epsilon+3|\ \forall\epsilon\in\varpi^{-1}(\varpi(n_i))\}$$

*and*

$$dim(\mathbf{R}^{\varpi(n_i)}\bigcap\mathbf{R}^{\varpi(n_j)})\ge dim^2(M^{n_i}\bigcap M^{n_j})+10dim(M^{n_i}\bigcap M^{n_j})+3$$

*for any integer $i,j,1\le i,j\le m$ with $M^{n_i}\cap M^{n_j}\neq\emptyset$.*

*Proof* If

$$\varpi(n_i)\ge\max\{\epsilon^2+10\epsilon+3|\ \forall\epsilon\in\varpi^{-1}(\varpi(n_i))\}$$

for any integer $i, 1 \leq i \leq m$, then each manifold $M^{\epsilon}, \forall\epsilon \in \varpi^{-1}(\varpi(n_i))$ can be isometry embedded into $\mathbf{R}^{\varpi(n_i)}$ and for $\forall\epsilon_1 \in \varpi^{-1}(n_i)$, $\forall\epsilon_2 \in \varpi^{-1}(n_j)$, $M^{\epsilon_1} \cap M^{\epsilon_2}$ can be isometry embedded into $\mathbf{R}^{\varpi(n_i)} \cap \mathbf{R}^{\varpi(n_j)}$ if $M^{\epsilon_1} \cap M^{\epsilon_2} \neq \emptyset$ by Lemma 5.1. Notice that in this case, serval manifolds in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ may be isometry embedded into one Euclidean space $\mathbf{R}^{\varpi(n_i)}$ for any integer $i, 1 \leq i \leq m$. Now applying Lemma 5.2 we know that $\widetilde{M}(n_1, n_2, \cdots, n_m)$ can be isometry embedded into a combinatorially Euclidean space $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$. □

Similar to the proof of Corollary 5.1, we can get a more clearly condition for isometry embedding of combinatorially Riemannian manifolds into combinatorially Euclidean spaces.

**Corollary** 5.2 *A combinatorially Riemannian manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ can be isometry embedded into a combinatorially Euclidean space $\widetilde{\mathbf{R}}(k_1, \cdots, k_l)$ if*

*(i)* $l \geq m$;

*(ii) there exists $m$ different integers $k_{i_1}, k_{i_2}, \cdots, k_{i_m} \in \{k_1, k_2, \cdots, k_l\}$ such that*

$$k_{i_j} \geq n_j^2 + 10n_j + 3$$

*and*

$$dim(\mathbf{R}^{k_{i_j}} \bigcap \mathbf{R}^{k_{i_r}}) \geq dim^2(M^{n_j} \bigcap M^{n_r}) + 10dim(M^{n_j} \bigcap M^{n_r}) + 3$$

*for any integer $i, j, 1 \leq i, j \leq m$ with $M^{n_j} \cap M^{n_r} \neq \emptyset$.*

## References


[1] R.Abraham and J.E.Marsden, *Foundation of Mechanics*(2nd edition), Addison-Wesley, Reading, Mass, 1978.

[2] W.H.Chern and X.X.Li, *Introduction to Riemannian Geometry*(in Chinese), Peking University Press, 2002.

[3] L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, *JP Journal of Geometry and Topology*, 5(1), 2005,77-81.

[4] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[5] L.F.Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ2006.

[6] L.F.Mao, Combinatorial speculation and the combinatorial conjecture for mathematics, *Selected Papers on Mathematical Combinatorics*(I), 1-22, World Academic Union, 2006.

[7] L.F.Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[8] L.F.Mao, An introduction to Smarandache multi-spaces and mathematical combinatorics, *Scientia Magna*, Vol.3, No.1(2007), 54-80.

[9] L.F.Mao, A combinatorially generalized Stokes' theorem on integrations, *International J.Mathematical Combinatorics*, Vol.1, No.1(2007), 67-86.

[10] L.F.Mao, Curvature equations on combinatorial manifolds with applications to theoretical physics, *International J.Mathematical Combinatorics*, Vol.1(2008), No.1, 16-35.

[11] W.S.Massey, *Algebraic Topology - An Introduction*, Springer-Verlag, New York Inc., 1977.

[12] F.Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math.GM/0010119*, 10/2000.

[13] F.Smarandache, A unifying field in logic: neutrosophic field, *Multi-value Logic*, Vol.8, No.3(2002), 385-438.

[14] J.Stillwell, *Classical Topology and Combinatorial Group Theory*, Springer-Verlag, New York Inc., 1980.

[15] Chuanxi Wu and Guanghan Li, *Geometry on Submanifolds*(in Chinese), Science Press, Beijing, 2002.