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A Note on Differential Geometry of the Curves in $\mathbf{E}^4$

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Abstract: In this note, we prove that every regular curve in four dimensional Euclidean space satisfies a vector differential equation of fifth order. Thereafter, in the same space, a relation among curvatures functions of inclined curves is obtained in terms of harmonic curvatures, which is related with Smarandache geometries ([5]).

Key Words: Euclidean space, Frenet formulas, inclined curves, harmonic curvatures.

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§1. Introduction

At the beginning of the twentieth century, A.Einstein's theory opened a door of use of new geometries. These geometries mostly have higher dimensions. In higher dimensional Euclidean space, researchers treated some topics of classical differential geometry [1], [2] and [3].

It is well-known that, if a curve differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called *Frenet frame* or *moving frame vectors*. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve, is called *Frenet apparatus of the curves*.

In [1], author wrote a relation of inclined curves. In this work, first, we prove that every regular curve in four dimensional Euclidean space satisfies a vector differential equation of fifth order. This result is obtained by means of Frenet formulas. Then using relation of inclined curves written in [1], we express a new relation for inclined curves in Euclidean space E^4, which is related with Smarandache geometries, see [5] for details.

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^4 are briefly presented (a more complete elementary treatment can be found in [4]).

Let $\alpha : I \subset R \to E^4$ be an arbitrary curve in the Euclidean space E^4. Recall that the curve α is said to be of unit speed (or parameterized by arclength function s) if $\langle \alpha'(s), \alpha'(s)\rangle = 1$, where $\langle ., . \rangle$ is the standard scalar (inner) product of E^4given by

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$$\langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4, \qquad (1)$$

for each $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4) \in E^4$. In particular, the norm of a vector $X \in E^4$ is given by

$$\|X\| = \sqrt{\langle X, X\rangle}.$$

Let $\{T(s), N(s), B(s), E(s)\}$ be the moving frame along the unit speed curve α. Then the Frenet formulas are given by [2]

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix}. \qquad (2)$$

Here T, N, B and E are called the *tangent*, the *normal*, the *binormal* and the *trinormal vector fields of the curves*, respectively, and the functions $\kappa(s), \tau(s)$ and $\sigma(s)$ are called the first, the second and the third curvature of a curve in E^4, respectively. Also, the functions $H_1 = \frac{\kappa}{\tau}$ and $H_2 = \frac{H_1'}{\sigma}$ are called *harmonic curvatures* of the curves in E^4, where $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$. Let $\alpha : I \subset R \to E^4$ be a regular curve. If tangent vector field T of α forms a constant angle with unit vector U, this curve is called an inclined curve in E^4.

In the same space, the author wrote a characterization for inclined curves with the following theorem in [1].

Theorem 2.1 *Let $\alpha : I \subset R \to E^4$ be an unit speed regular curve with curvatures $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$. α is an inclined curve if and only if there is a relation*

$$\frac{\kappa}{\tau} = A. \cos \int_0^s \sigma ds. + B. \sin \int_0^s \sigma ds, \qquad (3)$$

where $A, B \in R$.

§3. Vector Differential Equation of Fifth Order Satisfied by Regular Curves in $\mathbf{E}^4$

Theorem 3.1 *Let $X : I \subset R \to E^4$ be an unit speed regular curve with curvatures $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$ in E^4. Position vector and curvatures of α satisfies a vector differential equation of fifth order.*

Proof Let $X : I \subset R \to E^4$ be an unit speed regular curve with curvatures $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$ in E^4. Considering Frenet equations, we write that

$$N = \frac{T'}{\kappa} \qquad (4)$$

and

$$B = \frac{1}{\tau}(\kappa T + N'). \tag{5}$$

Substituting (3) in $(1)_3$, we get

$$B' = -\frac{\tau}{\kappa}T' + \sigma E. \tag{6}$$

Then, differentiating (3) and substituting it to (4), we find

$$B = \frac{1}{\tau}\left[\kappa T + (\frac{T'}{\kappa})'\right]. \tag{7}$$

Taking the integral on both sides of $(1)_4$, we know

$$E = -\int \sigma B ds \tag{8}$$

and substituting (6) to (7), we get

$$E = -\int \frac{\sigma}{\tau}\left[\kappa T + (\frac{T'}{\kappa})'\right] ds. \tag{9}$$

Applying (8) in (5), we have

$$B' = -\frac{\tau}{\kappa}T' - \sigma \int \frac{\sigma}{\tau}\left[\kappa T + (\frac{T'}{\kappa})'\right] ds. \tag{10}$$

Similarly, differentiating (6) and considering (9), then

$$\left\{\begin{array}{c} (\frac{1}{\tau})'\left[\frac{T''\kappa - T'\kappa'}{\kappa^2} + \kappa T\right] + \\ \frac{1}{\tau}\left[\frac{(T'''\kappa + T'\kappa'')\kappa^2 - 2\kappa\kappa'(T''\kappa - T'\kappa')}{\kappa^+} + \kappa' T + \kappa T'\right] \\ +\frac{\tau}{\kappa}T' + \sigma \int \frac{\sigma}{\tau}\left[\kappa T + (\frac{T'}{\kappa})'\right] ds \end{array}\right\} = 0 \tag{11}$$

is obtained. One more differentiating of (10) and simplifying this with $\dot{X} = T, \ddot{X} = T', \dddot{X} = T'', X^{(IV)} = T'''$ and $X^{(V)} = T^{(IV)}$, we know

$$\left\{\begin{array}{c} \left[\frac{1}{\kappa\tau}\right] X^{(V)} + \left[\frac{\kappa'}{\kappa^2\tau} + (\frac{1}{\tau})'\frac{1}{\kappa} + (\frac{1}{\kappa^4\tau})'\kappa + \frac{2}{\kappa^3\tau}\right] .X^{(IV)} + \\ \left[(\frac{1}{\tau})''\frac{1}{\kappa} - 2(\frac{1}{\tau})'\frac{\kappa'}{\kappa^2} - \kappa^2\kappa'\tau(\frac{1}{\kappa^4\tau})' - \frac{\kappa''}{\kappa^2\tau} - \frac{2}{\kappa^3\tau}{\kappa'}^2 + \kappa^5 + \frac{\tau}{\kappa} + \frac{\sigma^2}{\kappa\tau}\right] .\dddot{X} \\ \left[\begin{array}{c} -\frac{\kappa'}{\kappa^2}(\frac{1}{\tau})'' - \frac{\kappa''}{\kappa^2}(\frac{1}{\tau})' + \frac{2{\kappa'}^2}{\kappa^3}(\frac{1}{\tau})' + \kappa(\frac{1}{\tau})' + 2{\kappa'}^2\kappa(\frac{1}{\kappa^4\tau}) + \kappa^5(\frac{1}{\kappa^4\tau})' \\ -\frac{2\kappa'\kappa''}{\kappa^3\tau} + \frac{\kappa'\sigma}{\tau} + (\frac{\tau}{\kappa})' + \frac{\sigma^2}{\kappa\tau} - \kappa^2\kappa''(\frac{1}{\kappa^4\tau})' - \frac{\sigma^2\kappa'}{\kappa\tau} \end{array}\right] .\ddot{X} \\ \left[\kappa(\frac{1}{\tau})'' + \kappa'(\frac{1}{\tau})'' + \kappa'(\frac{1}{\tau})' + \kappa^4\kappa'(\frac{1}{\kappa^4\tau})' + \frac{4{\kappa'}^2}{\kappa\tau} + \frac{\kappa''}{\tau} + \frac{\sigma^2\kappa'}{\tau}\right] .\dot{X} \end{array}\right\} = 0. \tag{12}$$

The formula (12) proves the theorem as desired. □

§4. A Characterization of Inclined Curves in $\mathbf{E}^4$

Theorem 4.1 *Let $\alpha : I \subset R \to E^4$ be an unit speed regular curve with curvatures $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$ in E^4. α is an inclined curve if and only if*

$$H_1^2 + H_2^2 = constant, \tag{13}$$

where H_1 and H_2 are harmonic curvatures.

Proof Let α be an regular inclined curve in E^4. In this case, we can write

$$\frac{\kappa}{\tau} = A.\cos\int_0^s \sigma ds. + B.\sin\int_0^s \sigma ds, \tag{14}$$

where $A, B \in R$. If we differentiate (14) respect to s, we get

$$\frac{1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau}) = -A.\sin\int_0^s \sigma ds. + B.\cos\int_0^s \sigma ds. \tag{15}$$

Similarly, one more differentiating (15) respect to s, we have

$$\frac{d}{ds}\left[\frac{1}{\sigma}\frac{d}{ds}(\frac{\kappa}{\tau})\right] = -A\sigma\sin\int_0^s \sigma ds. - B\sigma\sin\int_0^s \sigma ds. \tag{16}$$

Using notations $\sigma H_1 = \sigma\frac{\kappa}{\tau}$ and $\frac{dH_2}{ds}$ in (16), we find

$$\sigma H_1 + \frac{dH_2}{ds} = 0. \tag{17}$$

Multiplying both sides of (17) with $\frac{1}{\sigma}H_1' = H_2$, we obtain

$$H_1H_1' + H_2H_2' = 0. \tag{18}$$

The formula (18) yields that

$$H_1^2 + H_2^2 = \text{constant}. \tag{19}$$

Conversely, let relation (19) hold. Differentiating (19) respect to s, we know

$$H_1H_1' + H_2H_2' = 0. \tag{20}$$

Similarly differentiating of expressions of harmonic curvatures and using these in (20), we have the following differential equation

$$\frac{1}{\sigma^2}H_1'' + \frac{1}{\sigma}(\frac{1}{\sigma})'H_1' + H_1 = 0. \tag{21}$$

Using an exchange variable $t = \int_0^s \sigma ds$ in (20),

$$\ddot{H}_1 + H_1 = 0. \tag{22}$$

Here, the notation $\ddot{H}_1$ indicates derivative of H_1 according to t. Solution of (22) follows that

$$H_1 = A\cos t + B\sin t, \tag{23}$$

where $A, B \in R$. Therefore, we write that

$$\frac{\kappa}{\tau} = A.\cos\int_0^s \sigma ds. + B.\sin\int_0^s \sigma ds. \tag{24}$$

By Theorem 2.1, (24) implies that α is an inclined curve in E^4.

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References

[1] A. Magden, Characterizations of some Special Curves in E^4, Dissertation, Dept.Math Ataturk University, Erzurum,Turkey, 1990.

[2] A. Sabuncuoglu and H.H.Hacisalihoglu, On Higher Curvature of a Curve, *Communications de la Fac. Sci. Uni. Ankara*, **24** (5):33-46, (1975).

[3] H. Gluck, Higher Curvatures of Curves in Euclidean Space, *Amer. Math. Monthly*, 73:699-704, (1966).

[4] H.H.Hacisalihoglu, Differential Geometry I, Ankara University Faculty of Science Press, 2000.

[5] L.F.Mao, Pseudo-manifold geometries with applications, *International J.Math. Combin.*, Vol.1(2007), No.1, 45-58.