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On Skew Randić Sum Eccentricity Energy of Digraphs

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Abstract: In this paper we introduce the concept of skew Randić sum eccentricity energy of digraphs. We then obtain upper and lower bounds for skew Randić sum eccentricity energy of digraphs. Then we compute the skew Randić sum eccentricity of some digraphs such as star digraph, complete bipartite digraph, $(S_m \wedge P_2)$ digraph and crown digraph.

Key Words: Digraph, skew-adjacency matrix of graph, skew Randić sum eccentricity energy, Smarandachely sum eccentricity energy.

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§1. Introduction

In [1], we have introduced the Randić sum eccentricity energy of a simple graph G as follows. The Randić sum eccentricity energy adjacency matrix of G is a $n \times n$ matrix $A_{rse} = (a_{ij})$, where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{e(v_i)+e(v_j)}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent}, \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent}, \end{cases}$$

where $e(v_i)$ is the eccentricity of the vertex v_i. The Randić sum eccentricity energy of G is defined as the sum of absolute values of the eigenvalues of the Randić sum eccentricity energy adjacency matrix of G. Generally, a Smarandachely sum eccentricity energy adjacency matrix of G is a $n \times n$ matrix $A^s_{rse} = (a^s_{ij})$ with

$$a^s_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt[d+1]{e^d(v_i)+e^d(v_j)}}, & \text{if the distance of vertices } v_i \text{ and } v_j \text{ is } d, \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not connected} \end{cases}$$

which characterizes the non-homogeneity of vertices on a graph by eccentricity. Certainly, the matrix A_{rse} characterizes vertices of G in case of homogeneity which is a submatrix of A^s_{rse}.

In 2010, Adiga, Balakrishnan and Wasin So [5] introduced the skew energy of a digraph

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as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. The skew-adjacency matrix of D is the $n \times n$ matrix $S(D) = (s_{ij})$ where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(D)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(D)$ and $s_{ij} = 0$ otherwise. Hence $S(D)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and λ is a real number. The skew energy of G is the sum of the absolute values of eigenvalues of $S(D)$.

Motivated by these works, we introduce the concept of skew Randić sum eccentricity energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. Then the skew Randić sum eccentricity adjacency matrix of D is the $n \times n$ matrix $A_{srse} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \frac{1}{\sqrt{e(v_i)+e(v_j)}}, & \text{if } (v_i, v_j) \in \Gamma(D), \\ -\frac{1}{\sqrt{e(v_i)+e(v_j)}}, & \text{if } (v_j, v_i) \in \Gamma(D), \\ 0, & \text{otherwise.} \end{cases}$$

Then, the skew Randić sum eccentricity energy $E_{srse}(D)$ of D is defined as the sum of the absolute values of eigenvalues of A_{srse}.

For example Let D be the directed circle on 4 vertices with the arc set $\{(1, 2), (2, 3), (3, 4), (4, 1)\}$. Then

$$A_{srse} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Then, the characteristic equation is given by $\lambda^4 + \lambda^2$. The eigenvalues are i, 0, 0, $-i$ and skew Randić sum eccentricity energy of D is 2.

In Section 2 of this paper we obtain the upper and lower bounds for skew Randić sum eccentricity energy of digraphs. In Section 3 we compute the skew Randić sum eccentricity energy of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph and a crown digraph.

§2. Upper and Lower Bounds for Skew Randić Sum Eccentricity Energy

Theorem 2.1 *Let D be a simple digraph of order n. Then*

$$E_{srse}(D) \leq \sqrt{2n \sum_{j\sim k} \left(\frac{1}{e(v_i) + e(v_j)} \right)}.$$

Proof Let $i\lambda_1, i\lambda_2, i\lambda_3, \cdots, i\lambda_n$, be the eigenvalues of A_{srse}, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq$

$\cdots \geq \lambda_n$. Since

$$\sum_{j=1}^{n}(i\lambda_j)^2 = tr(A_{srse}^2) = -\sum_{j=1}^{n}\sum_{k=1}^{n}a_{jk}^2 = -2\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right),$$

we have

$$\sum_{j=1}^{n}|\lambda_j|^2 = 2\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right). \tag{1}$$

Applying the Cauchy-Schwartz inequality

$$\left(\sum_{j=1}^{n}a_jb_j\right)^2 \leq \left(\sum_{j=1}^{n}a_j^2\right)\cdot\left(\sum_{j=1}^{n}b_j^2\right)$$

with $a_j = 1$, $b_j =| \lambda_j |$, we obtain

$$E_{srse}(D) = \sum_{j=1}^{n}|\lambda_j| = \sqrt{(\sum_{j=1}^{n}|\lambda_j|)^2} \leq \sqrt{n\sum_{j=1}^{n}|\lambda_j|^2} = \sqrt{2n\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right)}.$$

This completes the proof. $\square$

Theorem 2.2 *Let D be a simple digraph of order n. Then*

$$E_{srse}(D) \geq \sqrt{2\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right) + n(n-1)p^{\frac{2}{n}}},\ \ where\ \ p = |detA_{srse}| = \prod_{j=1}^{n}|\lambda_j|. \tag{2}$$

Proof Notice that

$$(E_{srse}(D))^2 = \left(\sum_{j=1}^{n}|\lambda_j|\right)^2 = \sum_{j=1}^{n}|\lambda_j|^2 + \sum_{1\leq j\neq k\leq n}|\lambda_j||\lambda_k|.$$

By arithmetic-geometric mean inequality, we get

$$\begin{aligned}
\sum_{1\leq j\neq k\leq n}|\lambda_j||\lambda_k| &= |\lambda_1|(|\lambda_2|+|\lambda_3|+\cdots+|\lambda_n|)\\
&\quad +|\lambda_2|(|\lambda_1|+|\lambda_3|+\cdots+|\lambda_n|)+\cdots\\
&\quad +|\lambda_n|(|\lambda_1|+|\lambda_2|+\cdots+|\lambda_{n-1}|)\\
&\geq n(n-1)(|\lambda_1||\lambda_2|...|\lambda_n|)^{\frac{1}{n}}(|\lambda_1|^{n-1}|\lambda_2|^{n-1}...|\lambda_n|^{n-1})^{\frac{1}{n(n-1)}}\\
&= n(n-1)(\prod_{j=1}^{n}|\lambda_j|)^{\frac{1}{n}}(\prod_{j=1}^{n}|\lambda_j|)^{\frac{1}{n}} = n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}}.
\end{aligned}$$

Thus,

$$(E_{srse}(D))^2 \geq \sum_{j=1}^{n} |\lambda_j|^2 + n(n-1)\left(\prod_{j=1}^{n} |\lambda_j|\right)^{\frac{2}{n}}.$$

From the equation (1), we get

$$(E_{srse}(D))^2 \geq 2\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right) + n(n-1)p^{\frac{2}{n}},$$

which gives (2). □

§3. Skew Randić Sum Eccentricity Energies of Some Families of Graphs

We begin with some basic definitions and notations.

Definition 3.1([3]) *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 3.2([3]) *A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 3.3([2]) *The crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_i v_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore S_n^0 coincides with complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Definition 3.4([4]) *The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m)$ and $v_j v_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Now we compute skew Randić sum eccentricity energies of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ and a crown digraph.

Theorem 3.5 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $K_{m,n}$ complete bipartite digraph be respectively given by*

$$\begin{aligned} V(D) &= \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}, \\ \Gamma(D) &= \{(u_i, v_j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}. \end{aligned}$$

Then, the skew Randić sum eccentricity energy of the complete bipartite digraph is $\sqrt{mn}$.

Proof The skew Randić sum eccentricity matrix of complete bipartite digraph is given by

$$A_{srse} = \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 & \cdots & 0 \end{pmatrix}$$

with a characteristic polynomial

$$|\lambda I - A_{srse}| = \begin{vmatrix} \lambda I_m & -\frac{1}{2}J^T \\ \frac{1}{2}J & \lambda I_n \end{vmatrix},$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\frac{1}{2}J^T \\ \frac{1}{2}J & \lambda I_n \end{vmatrix} = 0,$$

which can be written as

$$|\lambda I_m| \left| \lambda I_n - \left(\frac{1}{2}J\right) \frac{I_m}{\lambda} \left(-\frac{1}{2}J^T\right) \right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{(4)^n} \left| (4)\lambda^2 I_n + JJ^T \right| = 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{(4)^n} P_{JJ^T}(4\lambda^2) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix JJ^T. Thus, we have

$$\frac{\lambda^{m-n}}{(4)^n}(4\lambda^2 + mn)(4\lambda^2)^{n-1} = 0,$$

which is same as

$$\lambda^{m+n-2}(\lambda^2 + \frac{mn}{4}) = 0.$$

Therefore, the spectrum of $K_{m,n}$ is given by

$$Spec\,(K_{m,n}) = \begin{pmatrix} 0 & i\sqrt{\frac{mn}{4}} & -i\sqrt{\frac{mn}{4}} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Hence, the skew Randić sum eccentricity energy of complete bipartite digraph is

$$E_{srse}(K_{m,n}) = \sqrt{mn},$$

as desired. $\square$

Theorem 3.6 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of S_n star digraph be respectively given by*

$$V(D) \quad = \quad \{v_1, v_2, \cdots, v_n\}, \quad \Gamma(D) = \{(v_1, v_j) \,|\, 2 \le j \le n\}$$

Then, the skew Randić sum eccentricity energy of D is

$$E_{srse}(S_n) = 2\sqrt{\frac{n-1}{3}}.$$

Proof The skew Randić sum eccentricity matrix of the star digraph D is given by

$$A_{srse} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \cdots & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with a characteristic polynomial

$$|\lambda I - A_{srse}| \quad = \quad \begin{vmatrix} \lambda & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \cdots & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \lambda & 0 & \cdots & 0 \\ \frac{1}{\sqrt{3}} & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \quad \left(\frac{1}{\sqrt{3}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{3}$. Then

$$|\lambda I - A_{srse}| = \phi_n(\mu)\left(\frac{1}{\sqrt{3}}\right)^n,$$

where

$$\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.$$

Using the properties of the determinants, we obtain

$$\phi_n(\mu) = (\mu\phi_{n-1}(\mu) + \mu^{n-2})$$

after some simplifications. Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 + (n-1)).$$

Therefore

$$|\lambda I - A_{srse}| = \left(\frac{1}{\sqrt{3}}\right)^n \left[\left((3)\lambda^2 + (n-1)\right)(\lambda\sqrt{3})^{n-2}\right].$$

Thus, the characteristic equation is given by

$$\lambda^{n-2}\left(\lambda^2 + \frac{n-1}{3}\right) = 0.$$

Hence,

$$Spec\,(S_n) = \begin{pmatrix} 0 & i\sqrt{\frac{n-1}{3}} & -i\sqrt{\frac{n-1}{3}} \\ n-2 & 1 & 1 \end{pmatrix}$$

and the skew Randić sum eccentricity energy of S_n is

$$E_{srse}(S_n) = 2\sqrt{\frac{n-1}{3}}.$$ □

Theorem 3.7 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $(S_m \wedge P_2)(m > 1)$ digraph be respectively given by*

$$\begin{aligned} V(D) &= \{v_1, v_2, \cdots, v_{2m+2}\}, \\ \Gamma(D) &= \{(v_1, v_j), (v_{m+2}, v_k)\,|\,2 \le k \le m+1, m+3 \le j \le 2m+2\}. \end{aligned}$$

Then, the skew Randić sum eccentricity energy of D is

$$E_{srse}(D) = 4\sqrt{\frac{n-1}{3}}.$$

Proof The skew Randić sum eccentricity matrix of $(S_m \wedge P_2)$ digraph is given by

$$A_{srse} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ 0 & \gamma & \cdots & \gamma & 0 & 0 & \cdots & 0 \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where $m+1=n$ and $\gamma = \frac{1}{\sqrt{3}}$. Then, its characteristic polynomial is given by

$$|\lambda I - A_{srse}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -\gamma & \cdots & -\gamma \\ 0 & \lambda & \cdots & 0 & \gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & \gamma & 0 & \cdots & 0 \\ 0 & -\gamma & \cdots & -\gamma & \lambda & 0 & \cdots & 0 \\ \gamma & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence, the characteristic equation is given by

$$\left(\frac{1}{\sqrt{3}}\right)^{2n} \begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

where $\Lambda = \sqrt{3}\lambda$.

Let

$$\begin{aligned}
\phi_{2n}(\Lambda) &= \begin{vmatrix}
\Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\
0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\
0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda
\end{vmatrix}_{2n\times 2n} \\
&= (-1)^{2n+2n}\Lambda \begin{vmatrix}
\Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\
0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\
0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda
\end{vmatrix}_{(2n-1)\times(2n-1)} \\
&\quad +(-1)^{2n+1} \begin{vmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\
\Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\
-1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0
\end{vmatrix}_{(2n-1)\times(2n-1)}.
\end{aligned}$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+1} \begin{vmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\
\Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\
-1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0
\end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain

$$\Psi_{2n-1}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda)$$

after some simplifications, where

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 1 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}.$$

Then,

$$\phi_{2n}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned} \phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+1}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= \Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda). \end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda) = (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda),$$

where,

$$\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}.$$

$$\begin{aligned} \phi_{2n}(\Lambda) &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda) \\ &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda) \\ &= ((n-1)\Lambda^{n-2} + \Lambda^n)\Theta_n(\Lambda). \end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda) = (n-1)\Lambda^{n-2} + \Lambda^n.$$

Therefore

$$\phi_{2n}(\Lambda) = ((n-1)\Lambda^{n-2} + \Lambda^n)^2.$$

Hence characteristic equation becomes

$$\left(\frac{1}{\sqrt{3}}\right)^{2n}\phi_{2n}(\Lambda) = 0,$$

which is same as

$$\left(\frac{1}{\sqrt{3}}\right)^{2n}((n-1)\Lambda^{n-2}+\Lambda^{n})^2=0$$

and can be reduced to

$$\lambda^{2n-4}((n-1)+(3)\lambda^2)^2=0.$$

Therefore

$$Spec\,(D)=\begin{pmatrix} 0 & i\sqrt{\frac{n-1}{3}} & -i\sqrt{\frac{n-1}{3}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence, the skew Randić sum eccentricity energy of $(S_m \wedge P_2)$ digraph is

$$E_{srse}(D)=4\sqrt{\frac{n-1}{3}}.$$ □

Theorem 3.8 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $S_n^0(n>2)$ crown digraph be respectively given by*

$$V(D)=\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\},\quad \Gamma(D)=\{(u_i,v_j)\,|\,1\le i\le n,1\le j\le n,i\ne j\}.$$

Then, the skew Randić sum eccentricity energy of the crown digraph is $2(n-1)$.

Proof The skew Randić sum eccentricity matrix of crown digraph is given by

$$A_{srse}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & X & \cdots & X \\ 0 & 0 & \cdots & 0 & X & 0 & \cdots & X \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & X & X & \cdots & 0 \\ 0 & -X & \cdots & -X & 0 & 0 & \cdots & 0 \\ -X & 0 & \cdots & -X & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -X & -X & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $X=\frac{1}{\sqrt{4}}$. Its characteristic polynomial is

$$|\lambda I-A_{srse}|=\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{4}}K^T \\ \frac{1}{\sqrt{4}}K & \lambda I_n \end{vmatrix},$$

where K is an $n\times n$ matrix. Hence, the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{4}}K^T \\ \frac{1}{\sqrt{4}}K & \lambda I_n. \end{vmatrix}=0,$$

which is the same as

$$|\lambda I_n| \left| \lambda I_n - \left(\frac{K}{\sqrt{4}}\right) \frac{I_n}{\lambda} \left(-\frac{K^T}{\sqrt{4}}\right) \right| = 0$$

and can be written as

$$\frac{1}{(4)^n} P_{KK^T}((4)\lambda^2) = 0,$$

where $P_{KK^T}(\lambda)$ is the characteristic polynomial of the matrix KK^T. Thus, we have

$$\frac{1}{(4)^n}[4\lambda^2 + (n-1)^2][4\lambda^2 + 1]^{n-1} = 0,$$

which is same as

$$\left(\lambda^2 + \frac{(n-1)^2}{4}\right)\left(\lambda^2 + \frac{1}{4}\right)^{n-1} = 0.$$

Therefore

$$Spec\,(S_n^0) = \begin{pmatrix} i\sqrt{\frac{(n-1)^2}{4}} & -i\sqrt{\frac{(n-1)^2}{4}} & i\frac{1}{\sqrt{4}} & -i\frac{1}{\sqrt{4}} \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Hence, the skew Randić sum-eccentricity energy of crown digraph is

$$E_{srse}(S_n^0) = 2(n-1)$$

as desired. □

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