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On Pathos Block Vertex Graph of a Tree

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Abstract: A pathos block vertex graph of a tree T, written $PBV(T)$, is a graph whose vertices are the vertices, blocks (edges), and paths of a pathos of T, with two vertices of $PBV(T)$ adjacent whenever one corresponds to a block B_i of T and the other to a vertex v_j of T such that B_i is incident with v_j or the block lies on the corresponding path of the pathos; two distinct pathos vertices P_m and P_n of $PBV(T)$ are adjacent whenever the corresponding paths of the pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex in T. We study the properties of $PBV(T)$; and present the characterization of graphs whose $PBV(T)$ are planar; outerplanar; and crossing number one. We further show that for any tree T, $PBV(T)$ is not maximal outerplanar and not minimally nonouterplanar.

Key Words: Crossing number, inner vertex number, path, cycle.

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§1. Introduction

Notations and definitions not introduced here can be found in [1]. There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graph, the total graph, and their generalizations.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.

A graph G is connected if between any two distinct vertices there is a path. A maximal connected subgraph of G is called a *component* of G. A *cut-vertex* of a graph is one whose removal increases the number of components. A *non-separable graph* is connected, non-trivial, and has no cut-vertices. A *block* of a graph is a maximal non-separable subgraph. If two distinct blocks B_1 and B_2 are incident with a common cut-vertex, then they are called *adjacent blocks*.

The *block graph* of a graph G, written $B(G)$, is the graph whose vertices are the blocks of

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G and in which two vertices are adjacent whenever the corresponding blocks have a cut-vertex in common.

The *cut-vertex graph* $C(G)$ of a graph G is the graph whose vertices are the cut-vertices of G and in which two vertices are adjacent whenever the corresponding cut-vertices lie on a common block of G.

Harary et al. [3] introduced the concept of block cut-vertex graph of a graph as follows. For a connected graph G with blocks $\{B_i\}$ and cut-vertices $\{c_j\}$, the *block cut-vertex graph* of G, denoted by $bc(G)$, is defined as the graph having vertex set $\{B_i\} \cup \{c_j\}$, with two vertices adjacent if one corresponds to a block B_i and other corresponds to a cut-vertex c_j and c_j is in B_i.

Kulli [5] introduced the concept of block-vertex tree of a graph as follows. The *block-vertex tree BV(G)* of a graph G is the graph whose vertices can be put in one-to-one correspondence with the set of vertices and blocks of G in such a way that two vertices of $BV(G)$ are adjacent if and only if one corresponds to a block B of G and the other to a vertex v of G and v is in B. Clearly, if G_1 is the graph obtained from $BV(G)$ by deleting its end vertices, then $G_1 = bc(G)$.

The following characterization of the block cut-vertex graphs is well known.

Theorem 1.1 (F. Harary and G. Prins, [3]) *A graph G is the block cut-vertex graph of some graph H if and only if it is a tree in which the distance between any two end vertices is even.*

In view of Theorem 1.1, the author in [5] will speak of the block vertex tree of a graph.

If a path P_n of order n $(n \geq 2)$ starts at one vertex and ends at a different vertex, then P_n is called an *open path*. The concept of *pathos* of a graph G was introduced by Harary [2] as a collection of minimum number of edge disjoint open paths whose union is G. The path number of a graph G is the number of paths in any pathos. The path number of a tree T equals k, where $2k$ is the number of odd degree vertices of T. A *pathos vertex* is a vertex corresponding to a path of the pathos of T.

Motivated by the studies above, we now define a new graph operator called a pathos block vertex graph of a tree.

§2. **Preliminaries**

A *graph* $G = (V, E)$ is a pair, consisting of some set V, the so-called *vertex set*, and some subset E of the set of all 2-element subsets of V, the *edge set*. We write $x = (p, q)$ and say that p and q are *adjacent vertices* (sometimes denoted p adj q). A graph G is *connected* if between any two distinct vertices there is a path. A *maximal connected subgraph* of G is called a *component* of G. A *cut-vertex* of a graph is one whose removal increases the number of components. A *nonseperable* graph is connected, nontrivial, and has no cut-vertices. A *block* of a graph is a maximal nonseparable subgraph.

A graph G is *planar* if it has a drawing without crossings. For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane.

If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be *outerplanar*, i.e., $i(G) = 0$.

An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing outerplanarity. A graph G is said to be *minimally nonouterplanar* if $i(G)$=1 [4]. A minimally nonouterplanar graph G is said to be *maximal minimally nonouterplanar* if no edge can be added without losing minimally nonouterplanarity. The least number of edge crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by $\mathrm{cr}(G)$.

The *Dutch Windmill graph* $D_3^{(m)}$, also called a *friendship graph*, is the graph obtained by taking m copies of the cycle graph C^3 with a vertex in common and therefore corresponds to the usual *Windmill graph* $W_n^{(m)}$. It is therefore natural to extend the definition to $D_n^{(m)}$, consisting of m copies of C^n. The *Windmill graph* $W_n^{(m)}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common.

§3. **Definition of** $PBV(T)$

A *pathos block vertex graph* of a tree T, written $PBV(T)$, is a graph whose vertices are the vertices, blocks (edges), and paths of a pathos of T, with two vertices of $PBV(T)$ adjacent whenever one corresponds to a block B_i of T and the other to a vertex v_j of T such that B_i is incident with v_j or the block lies on the corresponding path of the pathos; two distinct pathos vertices P_m and P_n of $PBV(T)$ are adjacent whenever the corresponding paths of the pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex in T.

In Figure 1, a tree T and its different pathos block vertex graph $PBV(T)$ are shown.

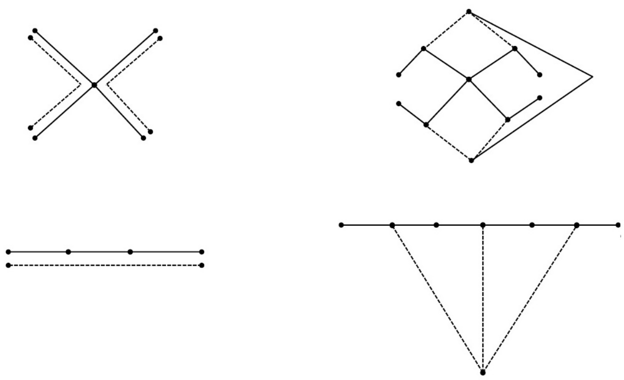


Figure 1

Note that there is freedom in marking the pathos of a tree T in different ways, provided that the path number k of T is fixed. For example, consider the marking of the pathos by dotted lines of a tree (on left) in Figure 1, where $k = 2$. Since the order of marking of the pathos of a tree is not unique, the corresponding pathos block vertex graph is also not unique. This obviously raises the question of the existence of "unique" pathos block vertex graph. One can easily check that if the path number of a tree is exactly one, i.e., k=1, then the corresponding pathos block vertex graph is unique. Since the path number of a path P_n on $n \geq 2$ vertices is one, it follows that pathos block vertex graph of a path is unique. Furthermore, for different ways of marking of pathos of a star graph $K_{1,n}$ on $n \geq 3$ vertices, the corresponding pathos block vertex graphs are isomorphic.

§4. Basic Properties of $PBV(T)$

In this section we present some of the properties of $PBV(T)$.

Property 4.1 *If v is a vertex of degree n in T, then the degree of v in $PBV(T)$ is also n. Consequently, if v is an end-vertex in T, then the corresponding vertex v in $PBV(T)$ is also an end-vertex. Therefore, $PBV(T)$ is non-eulerian and non-hamiltonian.*

Property 4.2 *The degree of every block vertex in $PBV(T)$ is three.*

Property 4.3 *Let T be a tree of order n ($n \geq 3$). Then the number of edges whose end-vertices are the pathos vertices in $PBV(T)$ is at most $\frac{k(k-1)}{2} = \beta$ (say), where k is the path number of T. In particular, if T is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the number of edges whose end-vertices are the pathos vertices in $PBV(T)$ is exactly β, i.e., in a pathos block vertex graph of a star graph, the pathos vertices are pairwise adjacent.*

While defining any class of graphs, it is desirable to know the order and size of each; it is easy to determine for $PBV(T)$.

Proposition 4.4 *Let T be a tree with vertex set $V(T) = \{v_1, v_2, \cdots, v_n\}$ and edge (block) set $E(T) = \{e_1, e_2, \cdots, e_{n-1}\}$. Then the order and size of $PBV(T)$ are*

$$2n + k - 1 \quad \textit{and} \quad 3(n-1) + \frac{k(k-1)}{2},$$

respectively, where k is the path number of T.

Proof Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$. Then the order of $PBV(T)$ equals the sum of order, size, and the path number of T. Thus $V(PBV(T)) = 2n + k - 1$. The size of $PBV(T)$ is equal to thrice the size of T and the number of edges whose end-vertices are the pathos vertices. By Property 4.3,

$$E(PBV(T)) = 3(n-1) + \frac{k(k-1)}{2}. \qquad \square$$

§5. Characterization of $PBV(T)$

5.1 Planar Pathos Block Vertex Graphs

We now characterize the graphs whose $PBV(T)$ is planar.

Theorem 5.1 *A pathos block vertex graph $PBV(T)$ of a tree T is planar if and only if $\Delta(T) \leq 6$,* *for every vertex $v \in T$.*

Proof Suppose $PBV(T)$ is planar. Assume that $\Delta(T) > 6$, for every vertex $v \in T$. If there exists a vertex v of degree seven in T, i.e., $T = K_{1,7}$, where v is the central vertex. By definition, $BV(T)$ is a graph obtained by adjoining a pendant edge at each pendant vertex of the star graph $K_{1,7}$. Let $P(T) = \{P_1, P_2, P_3, P_4\}$ be a pathos set of T. Then $D_4^{(4)} - v_1$ is an induced subgraph of $PBV(T)$, where v_1 is a vertex at distance one from v. Clearly $\mathrm{cr}(PBV(T)) = 0$. Furthermore, the pathos vertices P_1, P_2, P_3, and P_4 of $PBV(T)$ are pairwise adjacent. This shows that $\mathrm{cr}(PBV(T)) = 1$, a contradiction.

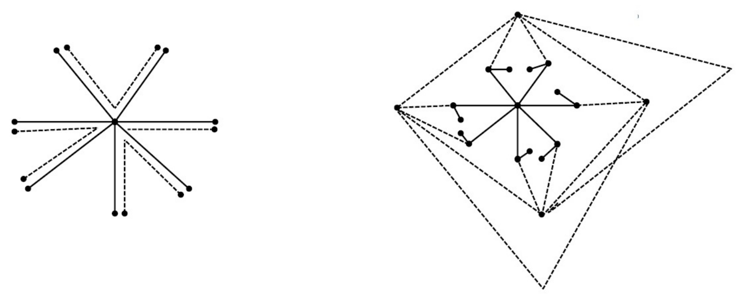
For sufficiency, we consider the following cases.

Case 1. Suppose that T is a path of order n ($n \geq 2$). Let $V(T) = \{v_1, v_2, \cdots, v_n\}$ and $E(T) = \{e_1, e_2, \cdots, e_{n-1}\}$ be the vertex set and edge set of T, respectively. Then $BV(T)$ is a path with edges (v_i, e_i) and (e_i, v_{i+1}) for $1 \leq i \leq n-1$. The path number of T is one, say P_1, and the corresponding pathos vertex P_1 is adjacent to every vertex e_i ($1 \leq i \leq n-1$) of $BV(T)$. This shows that $\mathrm{cr}(PBV(T)) = 0$.

Case 2. Suppose that T is $K_{1,2}$ (or the path P_3). Then $BV(T)$ is the path P_5. The path number of T is one. Then $PBV(T)$ is a graph obtained by adjoining a pendant edge at any two consecutive vertices of the cycle C^4. Clearly $\mathrm{cr}(PBV(T)) = 0$.

Case 3. Suppose that T is a star graph $K_{1,n}$ ($3 \leq n \leq 6$). Then $BV(T)$ is a graph obtained by adjoining a pendant edge at each pendant vertex of $K_{1,n}$. The path number of T is at most three. For $n = 3$ and 5, $D_4^{(2)} - v_1$ and $D_4^{(3)} - v_1$, respectively, are the induced subgraphs of $PBV(T)$, where v_1 is a vertex at distance one from the central vertex of $K_{1,n}$. Next, for $n = 4$ and 6, $D_4^{(2)}$ and $D_4^{(3)}$, respectively, are the induced subgraphs of $PBV(T)$. Clearly $\mathrm{cr}(PBV(T)) = 0$. Furthermore, the pathos vertices of these induced subgraphs are pairwise adjacent and does not increase the crossing number of $PBV(T)$. Thus $\mathrm{cr}(PBV(T)) = 0$.

Case 4. Suppose that the degree of each vertex of T is at most six. Then $BV(T)$ is a graph obtained by adjoining a pendant edge at each pendant vertex of T such that $\mathrm{cr}(BV(T)) = 0$. The path number of T is at least one. Then $PBV(T)$ contains either C^4 or P_2 or the product of P_2 and P_3 as subgraphs, which shows that $\mathrm{cr}(PBV(T)) = 0$. Finally, the edges joining pathos vertices of $PBV(T)$ does not increase crossing number of $PBV(T)$. Hence by all the cases above, $PBV(T)$ is planar. This completes the proof. □



(a) (b)

Figure 2 Star graph $K_{1,7}$ and $PBV(K_{1,7})$

Note that the path number of a star graph $T = K_{1,8}$ is four and the corresponding pathos vertices are pairwise adjacent in $PBV(T)$. This shows that the crossing number of $PBV(T)$ is one. Therefore, the necessity of Theorem 5.1 can also be proved by assuming $T = K_{1,8}$.

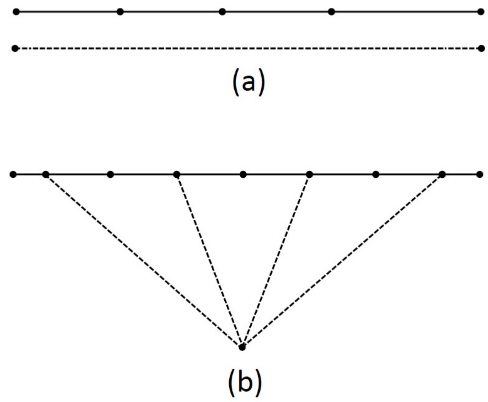


Figure 3 The path P_5 and $PBV(P_5)$

We now establish a characterization of graphs whose $PBV(T)$ are outerplanar, maximal outerplanar and minimally nonouterplanar.

Theorem 5.2 *A pathos block vertex graph $PBV(T)$ of a tree T is outerplanar if and only if T is a path of order n $(n \geq 2)$.*

Proof Suppose $PBV(T)$ is outerplanar. Assume that there exists a vertex of degree three in T, i.e., $T = K_{1,3}$. Let $P(T) = \{P_1, P_2\}$ be a pathos set of T. Then $PBV(T)$ contains $D_4^{(2)} - v_1$ as an induced subgraph. Furthermore, the pathos vertices P_1 and P_2 are adjacent. Clearly

$$i(PBV(T)) > 1,$$

a contradiction.

Conversely, suppose that T is a path of order n $(n \geq 2)$. We consider the following cases.

Case 1. Suppose that T is the path P_2. Then $PBV(T) = K_{1,3}$, which is outerplanar.

Case 2. Suppose that T is the path P_3. By Case 2 of sufficiency of Theorem 5.1, $PBV(T)$ is a graph obtained by adjoining a pendant edge at any two consecutive vertices of the cycle C^4. This shows that

$$i(PBV(T)) = 0.$$

Thus, $PBV(T)$ is outerplanar.

Case 3. Suppose that T is a path of order n $(n \geq 4)$. By definition, $BV(T)$ is a path of order $2\alpha + 5$, where $\alpha = (n - 3)$, $n \geq 4$. The path number of T is one, say P_1. Then $PBV(T)$ is a graph obtained by taking the join of alternative vertices of the path (of order $2\alpha + 5$) and P_1. This shows that

$$i(PBV(T)) = 0.$$

This completes the proof. □

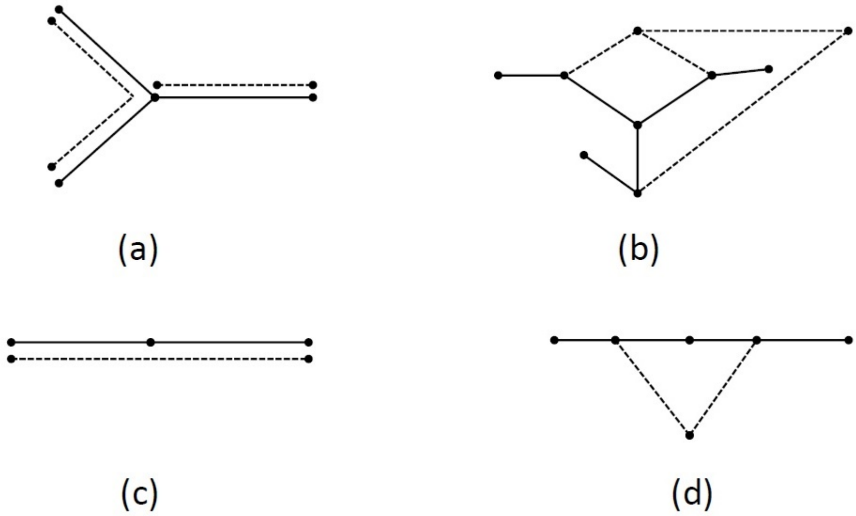


Figure 4

Theorem 5.3 (F. Harary, [1]) *Every maximal outerplanar graph G with n vertices has $2n-3$ edges.*

Theorem 5.4 *For any tree T, $PBV(T)$ is not maximal outerplanar.*

Proof We use contradiction. Suppose $PBV(T)$ is maximal outerplanar. Assume that T is a path of order n ($n \geq 2$). Then the order and size of $PBV(T)$ are $2\alpha+2$ and 3α, respectively, where $\alpha = (n-1)$, $n \geq 2$. But $3\alpha < 4\alpha + 1 = 2(2\alpha+2) - 3$. Since the size of $PBV(T)$ is 3α, Theorem 5.3 implies that $PBV(T)$ is not maximal outerplanar, a contradiction. This completes the proof. □

Theorem 5.5 *For any tree T, $PBV(T)$ is not minimally nonouterplanar.*

Proof We use contradiction. Suppose that $PBV(T)$ is minimally nonouterplanar. We consider the following three cases.

Case 1. Suppose that $\Delta(T) \geq 7$, for every vertex $v \in T$. By Theorem 5.1, $PBV(T)$ is planar, a contradiction.

Case 2. Suppose that $\Delta(T) \leq 2$, for every vertex $v \in T$. By Theorem 5.2, $PBV(T)$ is outerplanar, a contradiction.

Case 3. Suppose that $\Delta(T) \geq 3$. If there exists a vertex of degree three in T. By necessity of Theorem 5.2, $i(PBV(T)) > 1$, a contradiction. Consequently, if there exists a vertex of degree n ($4 \leq n \leq 6$), $i(PBV(T)) > 2$, again a contradiction. Hence by all the cases above, $PBV(T)$ is not minimally nonouterplanar. This completes the proof. □

Remark 5.6 *By Theorem 5.5, for any tree T, $PBV(T)$ is not minimally nonouterplanar.*

Therefore, $PBV(T)$ can never be maximal minimally nonouterplanar.

Theorem 5.7 *A pathos block vertex graph $PBV(T)$ of a tree T has crossing number one if and only if T is either $K_{1,7}$ or $K_{1,8}$.*

Proof Suppose that $PBV(T)$ has crossing number one. Assume that $T = K_{1,9}$, where v is the central vertex. By definition, $BV(T)$ is a graph obtained by adjoining a pendant edge at each pendant vertex of the star graph $K_{1,9}$. Let $P(T) = \{P_1, P_2, P_3, P_4, P_5\}$ be a pathos set of T. Then $D_4^{(5)} - v_1$ is an induced subgraph of $PBV(T)$, where v_1 is a vertex at distance one from v. Furthermore, since the pathos vertices P_1, P_2, P_3, P_4, and P_5 of $PBV(T)$ are pairwise adjacent, $\mathrm{cr}(PBV(T)) > 1$, a contradiction.

Conversely, suppose that T is either $K_{1,7}$ or $K_{1,8}$. By necessity of Theorem 5.1, the crossing number of $PBV(T)$ is one. This completes the proof. □

§6. Open Question

One can naturally extend these concepts to the directed graph version. What can one say about the properties of the directed version?

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