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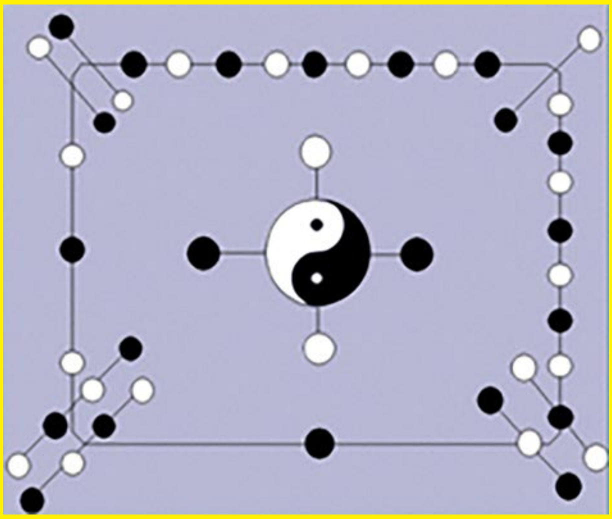

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Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.Linfan MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

Smarandache multi-spaces and Smarandache geometries with applications to other sciences;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

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Famous Words:

Anything one man can imagine, other men can make real.

By Jules Verne, a French novelist, dramatist, also a poet

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Disentangling Smarandache Multispace and Multisystem with Information Decoding

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Abstract: Certainly, a Smarandache multispace or multisystem $\widetilde{S}$ is a union of m distinct spaces or systems $S_1, S_2, \cdots, S_m$ which is an appropriate model on things T in the universe because of the limitation of humans ourselves and a thing T is complex, even overlap with other things. However, nearly all observation data on T is a multiple one $\widetilde{S}$ which implies S_i and S_j are entangled if $S_i \bigcap S_j \neq \emptyset$, we have to disentangle S_i from S_j, $1 \leq i \neq j \leq m$ for hold on the reality of thing T. Thus, disentangling a multi-space $\widetilde{S}$ to self-enclosed spaces or systems $S_1, S_2, \cdots, S_m$ is interesting, also valuable in hold on the reality of things in the universe. The main purpose of this paper is to discuss the disentangling ways on a Smarandache multispace or multisystem if we assume that each self-enclosed space or system of $S_i, 1 \leq i \leq m$ is endowed with mathematical elements such as those of topological, geometrical, algebraic structures or generally, each space or system of S_i has a character χ_i different from others for integers $1 \leq i \leq m$. As it happens, this problem is equivalent to Schrödinger's cat of quantum mechanics in the case of $m = 2$, which are extensively applied in quantum teleportation for preparation, distribution and measurement of the entangled pairs of particles and prospecting us to design a general key carrier on the Smarandachely entangling pairs in commutation.

Key Words: Entanglement, disentangling, Smarandache multispace, Smarandache multisystem, Smarandachely entangling pair, mathematical combinatorics, collapse mapping, Schrödinger's cat, quantum communication, key carrier, information decoding.

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§1. Introduction

Usually, a thing T is complex, even overlap with other things in the universe and it has many characters showing in front of humans such as those of the color, smell, density, states, solubility, still or moving of physical characters; the acidity, alkalinity, oxidizability, reducibility, thermal stability of chemical characters, also a dead or living body with growth, reproduction and habitat of biological characters. Then, *how do we understand the thing T*? Notice that a character of thing T maybe integral or partial, also conditional, the answer is nothing else but

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the union or combination of all known characters of T, i.e., the *Smarandache multispace* or *Smarandache multisystem*! For example, let $\mu_1, \mu_2, \cdots, \mu_n$ be its known and $\nu_i, i \geq 1$ unknown characters at time t. Then, the reality of thing T should be the union

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right)\bigcup\left(\bigcup_{k\geq 1}\{\nu_k\}\right) \tag{1.1}$$

of characters in logic, where a character in (1.1) should be existed, existing or will existing whether or not they are observable or understand by humans. However, T is understood by

$$T[t] = \bigcup_{i=1}^{n}\{\mu_i\} \tag{1.2}$$

for humans at time t, only an approximation on the reality of T, which implies that to hold on the reality of T is a gradual process, little by little. Even so, the Smarandache multispace or multisystem appeared in (1.1) or (1.2) is the basis for systematically understanding a thing T.

Then, *what is a Smarandache multispace or multisystem?* Formally by mathematics, a *Smarandache multispace* or *multisystem* is defined in the following on spaces or systems known by humans.

Definition 1.1([18, 32, 34]) *Let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be m mathematical spaces or systems, different two by two, i.e., for any two spaces or systems $(\Sigma_i;\mathcal{R}_i)$ and $(\Sigma_j;\mathcal{R}_j)$, $\Sigma_i \neq \Sigma_j$ or $\Sigma_i = \Sigma_j$ but $\mathcal{R}_i \neq \mathcal{R}_j$. Then, a Smarandache multispace or multisystem $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m}\Sigma_i$ with rules $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m}\mathcal{R}_i$ on $\widetilde{\Sigma}$, i.e., the union of rules $\mathcal{R}_i$ on Σ_i for integers $1 \leq i \leq m$, denoted by $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.*

Certainly, two spaces or systems S_i and S_j are entangled if $S_i \bigcap S_j \neq \emptyset, 1 \leq i \neq j \leq m$. Notice that any matter inherits a topological structure G^L of 1-dimension by the theory of matter composition. This conclusion also holds on a Smarandache multispace or multisystem $\widetilde{S}$ determined by the definition following, which consists of the element in mathematical combinatorics on the reality of thing in the universe([13], [21]-[29]).

Definition 1.2([18 - 21],) *For an integer $m \geq 1$, let $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ be a Smarandache multispace or system consisting of m mathematical spaces or systems $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$. An inherited topological structure $G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]$ of $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ is a labeled topological graph defined following:*

$$V\left(G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]\right) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E\left(G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]\right) = \{(\Sigma_i, \Sigma_j)\,|\Sigma_i \bigcap \Sigma_j \neq \emptyset,\ 1 \leq i \neq j \leq m\} \text{ with labeling}$$

$$L:\ \Sigma_i \to L(\Sigma_i) = \Sigma_i \quad \text{and} \quad L:\ (\Sigma_i, \Sigma_j) \to L(\Sigma_i, \Sigma_j) = \Sigma_i \bigcap \Sigma_j$$

for integers $1 \leq i \neq j \leq m$.

Notice that a mathematical space or system $(\Sigma_i;\mathcal{R}_i), 1 \leq i \leq m$ is self-closed by definition and generally, the appearance $\widetilde{S}$ of a thing T is multilateral in front of humans, usually out of order, we need to find which self-space or self-system it belong to. This is the disentangling problem on a Smarandache multispace or multisystem $\widetilde{S}$. In fact, it is more important for understanding a thing T in the universe, which advances us to establish the *collapse mappings* $\phi : \widetilde{S} \to S_i$, i.e., disentangle $\widetilde{S}$ to character or self-closed spaces or systems S_i, $1 \leq i \leq m$ for systematically understanding T, and the case of $m = 2$ happens to be a famous problem in quantum mechanics, i.e., the Schrödinger's cat or quantum entanglement which is the foundation of quantum communication ([3], [5]) because any kind of encryption codes in communication is essentially an application of Smarandache multisystems. This fact leads to the possible applications of disentangling a Smarandache multispace or multisystem to communication, particularly, a general model to quantum communication.

The main purpose of this paper is to discuss the disentangling problem of Smarandache multispace or multisystem by the assumption that each self-enclosed space $S_i, 1 \leq i \leq m$ is endowed with a mathematical structure such as those of topological, geometrical, algebraic structures or generally, each space or system S_i has a character χ_i different from others for integers $1 \leq i \leq m$ and its possible application to information encoding and decoding in communication. Certainly, we have known the application of quantum entanglement in quantum teleportation for preparation, distribution and measurement of the entangled pairs of particles. However, a general prospects on communication is the application of entangling Smarandache multispace or multisystem. For this prospection, applying model is suggested in this paper.

For terminologies and notations not mentioned here, we follow reference [1] and [30] for topology, [2] for algebra, [3] and [5] for quantum teleportation, [6], [19] and [32] for Smarandache geometry, [20] for combinatorial manifolds, [18], [33] for Smarandache multispaces and multisystems and [31] for elementary particles.

§2. Schrödinger's Cat with Entangling Pair

2.1.Schrödinger's cat. The first motivation of Schrödinger's cat was as a paradox on the explaining of instantaneous collapse for the strange nature of quantum superpositions in the macro world and then, a reasonable interpretation on this paradox is the Everetts multi-world interpretation (MWI), which maybe the first time for understanding a thing by multispaces.

[**Schrödinger's Cat**] In this paradox, Schrodinger placed a cat in a box along with a radioactive substance, a hammer and Geiger counter and a vial of poison. When the radioactive substance kept in the box decays, the Geiger counter will detect it and will trigger the hammer to release the poison such as those shown in Figure 1. This will subsequently kill the cat. It is not possible to predict when radioactive decay will happen since it is a random process. An observer will not know if the cat is dead or alive until the box is opened. The cats fate is tied to whether the radioactive substance has decayed or not and the cat would be, as claimed

Figure 1. Schrödinger's Cat

by Schrodinger that the cat's "*living and dead* $\cdots$ *in equal parts*" until the box is opened to observe the cat.

The really weird matter on the Schrödinger's cat is that whether the answer is "*living*" or "*dead*" is incomplete, both of them face possible an incorrect ending. However, whether the cat is living or dead is only certainty if the box is opened once, which presents a false impression that the cat's life dependent on the observation of humans. Objectively, *how could a cat's living or dead depend on human observation?* The living or dead of the cat is certain in the nature but it is just because of that one lost a piece of cat information from the close to opening of the box, which results in establishing not the causal relationship on the cat's life. *Why do such ambiguous answers exist?* Because the cat information is incomplete, the fragment from closing the box to opening its lid is lost and there are no logical agreement causal relationship can be established on the cat's life. In this case, the best way is to set a camera inside the box, observe the cat's activity at any time, establish a causal relationship and then to answer the question on the cat being living or dead.

Then, *How to describe the life state of Schrodinger's cat in the box ?* Let $\mathbf{L}$ and $\mathbf{D}$ be respectively the living and dead state of the cat in the box. Then, the Schrodinger cat's state can be expressed as $\mathbf{L}+\mathbf{D}$, which may be a living state, i.e., there is a mapping $\alpha : \mathbf{L}+\mathbf{D} \to \mathbf{L}$ and may also be a dead state mapping $\delta : \mathbf{L}+\mathbf{D} \to \mathbf{D}$. The question lies in how one knows the Schrodinger's cat is living or dead in here. For humans, to determine the life of schrodinger's cat requires lifting the lid of the box to really see if the cat is living or dead. Thus, the state of the cat's $\mathbf{L}+\mathbf{D}$ can not be seen, one can only find the cat is living or dead when open the lid of the box. *why the* $\mathbf{L}+\mathbf{D}$ *into* $\mathbf{L}$ *or* $\mathbf{D}$ *for an instant?* In order to give a logically consistent explanation, Bohr et al. proposed the *state collapse hypothesis* on the cat's life, i.e. $\mathbf{L}+\mathbf{D} \to \mathbf{L}$ or $\mathbf{L}+\mathbf{D} \to \mathbf{D}$ depending on human observation, it is $\mathbf{L}+\mathbf{D}$ when not observed but collapse to $\mathbf{L}$ or $\mathbf{D}$ when observed because it is knowing the cat's life by humans. Thus, it is necessary to examine what the cat state $\mathbf{L}+\mathbf{D}$ is. Generally, it can be interpreted as the sum of two vectors, the superposition of the cat's living and dead states, and furthermore, it can be viewed as the state of a living being, not only the Schrodinger's cat in the box.

Notice that if we define an axioms A: "*the cat is living*" and B: "*the cat is dead*", then the axiom A or B both generate a space $\mathbf{L}$ and $\mathbf{D}$, namely $\mathbf{L}+\mathbf{D}$ is nothing else but a Smarandache multispace $\widetilde{S}$ of $m=2$ with self-closed spaces $\mathbf{L}$ and $\mathbf{D}$, i.e., $\widetilde{S} = \mathbf{L}\bigcup\mathbf{D}$ in the multi-worlds interpretation $\mathbf{L}+\mathbf{D}$ of Schrodinger cat. The cat state $\mathbf{L}+\mathbf{D}$ can be decomposed according to axiom A and B. Notice also that the living state with the dead state are mutually exclusive in the eyes of humans. Thus, there must be $\mathbf{L}\bigcap\mathbf{D} = \emptyset$, which implies the state $\mathbf{L}+\mathbf{D}$ is a special kind of vector addition, i.e., direct sum and the state $\mathbf{L}+\mathbf{D}$ can be expressed by $\mathbf{L}\bigoplus\mathbf{D}$. In this case, there are only 2 self-closed spaces, inherited a topological structure $K_2^L[\mathbf{L}\bigoplus\mathbf{D}]$ of order 2 and the collapse mappings α and δ are also exclusive, i.e., if α appears then δ can not be seen, or in other words, if one is positive then another must be negative in observing.

2.2.Entangling pair. The living state $\mathbf{L}$ and dead state $\mathbf{D}$ of the Schrödinger's cat is in entangling in the multispace $\mathbf{L}\bigoplus\mathbf{D}$, i.e., if one appear then another would be not occur in observing or in other words, we know their states if one state of the pair is determined. Such a pair has the entanglement property, observed in microscopic particles and first discussed by

Einstein A., B. Podolsky and N. Rosen in 1935 ([4], usually called EPR paper).

It should be noted the entangled situation is not only appearing in the microscopic but also in the macroscopic such as the Schrödinger's cat. Certainly, there are many such pairs in classical mechanics. For example, let A and B be respectively two elastic balls with mass of M and m in a vacuum, moving backward along a straight line with velocity of V and v after the positive collision at the origin of O (see Figure 2 for details). In this case, if the velocity of A and B after collision are v_1' and v_2' respectively, then according to the conservation law of momentum

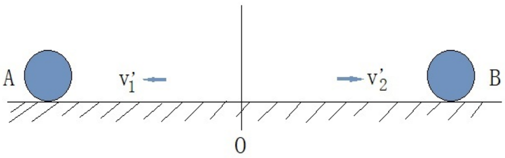

Figure 2. An elastic collision

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$$

in classical mechanics, we immediately get the velocity

$$v_2' = \frac{m_1\left(v_1 - v_1'\right) + m_2v_2}{m_2}$$

of ball B, namely, only one speed of balls A, B needs to measure after the collision, then we know the speed of the other. However, we can not measure exactly both its position and its momentum for a microscopic particle at the same time, asserted in the uncertainty principle of quantum mechanics. Even so, there are also the entanglement property. For example, whenever one of the two separated entangled particles A and B is measured, as long as the spin direction of A is upward then the spin direction of B must be downward and conversely, if the spin direction of A is downward then the spin direction of B must be upward, such as those shown in Figure 3, i.e., the microscopic particles A and B consist of an entangling pair.

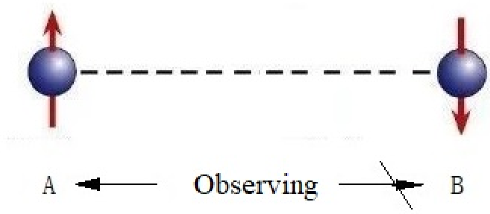

Figure 3. Entanglement particles

Generally, let S and S' be two self-closed spaces or systems. If there are known mappings $f : S \to S'$ with $f(S) \supseteq S'$ and $f' : S' \to S$ with $f'(S') \supseteq S$, then S and S' are called an *entangling pair*, which implies that one of S and S' is known then another is determined. The application of mathematical results immediately enables us getting conclusions on entangling pairs following by definition.

Theorem 2.1 *Let S and S' be two sets with onto mapping $f : S \to S'$ and $f' : S' \to S$. Then, S and S' are entangling. Particularly, if f is $1-1$, S and S' are entangling.*

Proof Notice that an onto mapping implies that $f(S) \supseteq S'$ and $f'(S') \supseteq S$. Therefore, S and S' are entangling by definition. □

By applying mathematical results, we can deduce many entangling pairs by Theorem 2.1.

Corollary 2.2 *Two homeomorphic spaces, isomorphic spaces, isomorphic groups, rings, fields* *or isomorphic algebraic systems, isomorphic vector spaces, isomorphic function or functional spaces, isomorphic operator spaces S and S' are entangling.*

Notice that a microscopic particle has only two directions on the spin, i.e., upward or downward. If we denote the upward direction by $+1$ and the downward direction by 0, then the entangling particles A and B implies the mapping $f : \{A, B\} \to \{0, +1\}$ posses the property that $f(A) \neq f(B)$, i.e, the exclusive property. In fact, there are many cases of binary logic in daily life. For example, a human has a pair of socks with one red and one white. On a dark night, after putting on socks he went to the street lamp and looked at the red sock on his left foot. Then, he does not need to look at his right foot again because he can deduce that it is a white sock in the binary logic case. Thus, we can generally get entangling pairs by binary logic on sets following.

Theorem 2.3 *Let S be a set with a one-valued mapping $f : S \to \{0, +1\}$ and $A = \{x \in S|f(x) = 1\}$, $B = \{x \in S|f(x) = 0\}$. Then, A and B are entangling.*

Proof Notice that f is a one-valued mapping on S, i.e., for any element $x \in S$, $f(x) = 1$ or $f(x) = 0$ and there are no elements $y \in S$ such that $f(y) = 0$ also with $f(y) = 1$. Thus, $A \bigcup B = S$ and $A \bigcap B = \emptyset$. Whence, $A = S \setminus B$ and $B = S \setminus A$. We therefore know that A and B are entangling. □

§3. Disentangling Smarandache Multispace or Multisystem

Let $\widetilde{S}$ be Smarandache multispace or multisystem on a thing T in the universe. The understanding process of humans on T is gradually by holding on characters of thing T. Thus, if we view T as a set of elements, then a character can be viewed as a self-closed space or system consisting of a few elements in T. Consequently, this process is essentially a disentangling process on $\widetilde{S}$ in logic. In fact, if each self-closed space or system is endowed with a mathematical structure, the disentangling process can be carried out immediately.

3.1.Algebraic Structure. An algebraic system $(\mathcal{A}; \circ)$ is a self-closed system under the operation $\circ$, i.e., for any $a, b \in \mathcal{A}$, $a \circ b \in \mathcal{A}$. Now, if a *Smarandache multisystem* $\widetilde{A}$ is the union of algebraic systems $(A_i; O_i)$ with $1 \leq i \leq m$ and operation sets $O_i = \{\circ_{ik}, 1 \leq k \leq s_i\}$, we can disentangling $\widetilde{S}$ to algebraic systems by the ruler that *for* $\forall a, b \in \widetilde{A}$*,* $a, b \in A_i$ *if and only if* $a \circ_{ik} b \in A_i$ *for* $\circ_{ik} \in O_i$ with a programme following:

STEP 1.1. For an integer $i, 1 \leq i \leq m$, let a be an element of $\widetilde{A}$ with definition on $a \circ_{ik} b$ for operation $\circ_{ik} \in O_i$, integer $k, 1 \leq k \leq s_k$ and some elements $b \in \widetilde{A}$;

STEP 1.2. Choose any element $x \in \widetilde{A}$, calculate $a \circ_{ik} x, \circ_{ik} \in O_i$ for integers $1 \leq k \leq s_i$;

STEP 1.3. If $a \circ_{ik} x$ is defined on $\widetilde{A}$ then let $a, x, a \circ_{ik} x \in A_i$. Otherwise, $x \notin A_i$ and if $\widetilde{A} \setminus \{x\} = \emptyset$, then turn to STEP 1.4; if $\widetilde{A} \setminus \{x\} \neq \emptyset$, come back to STEP 1.1 by replacing a with an element of A_i and $\widetilde{A}$ with $\widetilde{A} \setminus \{x\}$;

STEP 1.4. The programming terminated if $\forall x \in \widetilde{A}$ chosen in STEP 1.2.

Clearly, if $a \circ_{ik} x \in A_i$ then there must be $x \circ_{ik} a \in A_i$ if $x \circ_{ik} a$ is defined in $\widetilde{A}$ by this programme. Furthermore, The next result convinces us the disentangling of an algebraic Smarandache multisystem $\widetilde{A}$.

Theorem 3.1 *For any integer $1 \leq i \leq m$, A_i is maximally a self-closed algebraic system of $\widetilde{A}$ by STEP 1.1- STEP 1.4, which establishes the collapse mapping $\phi : \widetilde{A} \to A_i$ for integers $1 \leq i \leq m$.*

Proof By STEP 1.1- STEP 1.4, $A_i \subset \widetilde{A}$ is self-closed for integers $1 \leq i \leq m$. Otherwise, if there exist elements $x, y \in A_i$ with definition $x \circ_{ik} y$ for an integer $1 \leq k \leq s_i$ on $\widetilde{A}$ but $x \circ_i y \notin A_i$, it contradicts to STEP 1.3 with $x \in A_i$. And then, A_i is maximal because if there is an element $x \in \widetilde{A}$ with definition of $a \circ_{ik} x$ for an integer $1 \leq k \leq s_i$ and some elements $a \in \widetilde{A}$ there must be $x \in A_i$ by STEP 1.2. Whence, A_i is maximally a self-closed system. □

Notice that $\widetilde{A}$ is an algebraic Smarandache multisystem in Theorem 3.1, which enables us to get immediately the collapse mapping on the *Smarandache mutigroup, multiring, multifield* and *vector multispace* ([9], [11-12]) following.

Corollary 3.2 *Let $\left(\widetilde{G}; \circ_i, 1 \leq i \leq m\right)$ be a Smarandache multigroup. Then, the collapse mapping $\phi : \widetilde{G} \to G_i$ can be established by STEP 1.1- STEP 1.4 with operation $\circ_i$ of the group G_i, $1 \leq i \leq m$.*

Particularly, if $\widetilde{G}$ is finitely Abelian, i.e., $\left|\widetilde{G}\right| < \infty$ and $a \circ_i b = b \circ_i a$ for $\forall a, b \in \widetilde{G}$ and integers $1 \leq i \leq m$, then the collapse mapping can be not only on groups G_i but also on its cyclic groups with

$$\phi : \widetilde{G} \to G_i, 1 \leq i \leq m \quad \text{and} \quad \phi_{ij} : \widetilde{G} \to \langle a_{ij} \rangle$$

where, $a_{ij} \in G_i, 1 \leq j \leq s$ with a direct product decomposition of group G_i by $G_i = \langle a_{i1} \rangle \otimes \langle a_{i2} \rangle \otimes \cdots \otimes \langle a_{is} \rangle$.

Corollary 3.3 *Let $\left(\widetilde{R}; +_i, \cdot_i, 1 \leq i \leq m\right)$ be a Smarandace multiring. Then, the collapse mapping $\phi : \left(\widetilde{R}; +_i, \cdot_i, 1 \leq i \leq m\right) \to (R_i; +_i, \cdot_i$ can be established by STEP 1.1- STEP 1.4 with operations $+_i, \cdot_i$ of the ring $(R_i; +_i, \cdot_i)$ for integers $1 \leq i \leq m$. Particularly, the collapse mapping $\phi : \widetilde{R} \to R_i$ can be established by STEP 1.1- STEP 1.4 for Smarandache multifields.*

Corollary 3.4 *Let $(\widetilde{V}; \widetilde{F})$ be a vector Smarandache multispace with a vector set $\widetilde{V} = V_1 \bigcup V_i \bigcup \cdots \bigcup V_m$, an operation set $O(\widetilde{V}) = \{(\cdot +_i, \cdot_i) \mid 1 \leq i \leq m\}$ and a Smarandache multifiled $\widetilde{F} = F_1 \bigcup F_2 \bigcup \cdots \bigcup F_m$ with a double operation $O(\widetilde{F}) = \{(+_i, \times_i) \mid 1 \leq i \leq k\}$. Then, the collapse mapping $\phi : (\widetilde{V}; \widetilde{F}) \to (V_i; F_i)$ can be established by STEP 1.1- STEP 1.4 with operations $+_i, \cdot_i$ of the vector space $(V_i; F_i)$ for integers $1 \leq i \leq m$.*

3.2.Geomertical Structure. For an integer $n \geq 1$, a manifold M is a locally Euclidean space of dimension n, i.e., for $\forall x \in M$ there is a neighborhood $U(x)$ homeomorphic to $\mathbb{R}^n$. Now, let $\widetilde{M}$ be a connected *Smarandache multimanifold*, i.e., the union of manifolds $M_i, 1 \leq i \leq m < \infty$ with dimensions $\dim M_i = n_i, 1 \leq i \leq m$ which is connected. Then, we can disentangling $\widetilde{M}$ by the ruler that *if $x \in M_i$ with a neighborhood $U(x)$ homeomorphic to $\mathbb{R}^{n_i}$ for an integer $1 \leq i \leq m$ and $y \in U(x)$, then $y \in M_i$* with a programme following:

STEP 2.1. Let x be a point of $\widetilde{M}$ with a neighborhood $U(x)$ homeomorphic to $\mathbb{R}^{n_i}$ with $1 \leq i \leq m$ and $y \in \widetilde{M}$;

STEP 2.2. If $y \in U(x)$ then let $y \in M_i$. Otherwise, $y \notin M_i$ and if $\widetilde{\mathcal{T}} \backslash \{x, y\} = \emptyset$, then turn to STEP 2.3; if $\widetilde{\mathcal{T}} \backslash \{x, y\} \neq \emptyset$, come back to STEP 2.1 by replacing x with an element of M_i and $\widetilde{M}$ with $\widetilde{M} \backslash \{x, y\}$;

STEP 2.3. The programming terminated if $\forall x \in \widetilde{A}$ chosen in STEP 2.1.

Clearly, if $y \in \mathcal{T}_i$ then there must be $x \in \mathcal{T}_i$ also in this programme. We have the following result on the disentangling topological Smarandache multispaces.

Theorem 3.5 *For any integer $1 \leq i \leq m$, M_i is maximally a manifold of dimension n_i of $\widetilde{M}$ by STEP 2.1- STEP 2.3, which establishes the collapse mapping $\phi : \widetilde{M} \to M_i$ for integers $1 \leq i \leq m$.*

Proof By STEP 2.1-2.3, M_i is clearly a manifold of dimension n_i by definition. For its maximality, if there is a point $y \in \widetilde{M}$ but $y \notin M_i$ with $y \in U(x)$ of a neighborhood of $x \in M_i$ homeomorphic to $\mathbb{R}^{n_i}$, then there must be $y \in M_i$ by STEP 2.2, this programme will not be terminated, a contradiction. Thus, M_i is maximally a dimensional n_i manifold of $\widetilde{M}$. □

Notice that the Smarandache multimanifold $\widetilde{M}$ is called a *finitely combinatorial manifold* in [14] and [20-21], which can be characterized by vertex-edge labeled graphs inherited in $\widetilde{M}$. Furthermore, if the Smarandache multimanifold is differentiable, i.e., a *differentiable combinatorial manifold* $\widetilde{M}$ ([14]), a similar programme can be also established and get a conclusion following.

Theorem 3.6 *Let $\widetilde{M}$ be a differentiable combinatorial manifold consisting of differentiable manifolds $M_i, 1 \leq i \leq m$ of dimension $n_i, 1 \leq i \leq m$, respectively. Then, the collapse mapping $\phi : \widetilde{M} \to M_i$ for integers $1 \leq i \leq m$ can be established.*

Particularly, if all manifold $M_i, 1 \leq i \leq m$ are respectively Euclidean spaces $\mathbb{R}^{n_i}$ for integers $1 \leq i \leq m$, such a Smarandache multispace $\widetilde{M}$ is the *combinatorially Euclidean space* in this case ([14]). We get a conclusion by Theorem 3.5 following.

Corollary 3.7 *Let $\widetilde{E}$ be a combinatorial Euclidean space of $\mathbb{R}^{n_i}, 1 \leq i \leq m$. Then, the collapse mapping $\phi : \widetilde{E} \to \mathbb{R}^{n_i}$ can be established by STEP 2.1- STEP 2.3 for integers $1 \leq i \leq m$.*

Notice that a *metric Smarandache multispace* is the union $\widetilde{\mathcal{S}}$ of spaces $\mathcal{S}_i$ with a metric ρ_i for integers $1 \leq i \leq m$ which is connected. Then, we can disentangling $\widetilde{\mathcal{S}}$ by the ruler that *for $\forall x, y \in \widetilde{\mathcal{S}}$ if $\rho_i(x, y)$ is defined in $\widetilde{\mathcal{S}}$ then $x, y \in \mathcal{S}_i$ for an integer* $1 \leq i \leq m$ with a programme following:

STEP 3.1. Let x, y be points of $\widetilde{\mathcal{S}}$ and i an integer with $1 \leq i \leq m$;

STEP 3.2. If $\rho_i(x, y)$ is defined in $\widetilde{\mathcal{S}}$ then let $y \in \mathcal{S}_i$. Otherwise, $y \notin \mathcal{T}_i$ and if $\widetilde{\mathcal{T}} \backslash \{x, y\} = \emptyset$, then turn to STEP 3.3; if $\widetilde{\mathcal{T}} \backslash \{x, y\} \neq \emptyset$, come back to STEP 3.1 by replacing x with an element of $\mathcal{S}_i$ and $\widetilde{\mathcal{S}}$ with $\widetilde{\mathcal{S}} \backslash \{x, y\}$;

STEP 3.3. The programming terminated if $\forall x, y \in \widetilde{\mathcal{S}}$ chosen in STEP 3.1.

Clearly, by definition if $y \in \mathcal{S}_i$ in STEP 3.2, then there must be $x \in \mathcal{T}_i$ by STEP 3.1-STEP 3.3 and a conclusion on disentangling the metric Smarandache multispaces $\widetilde{\mathcal{S}}$ following.

Theorem 3.8 *Let $\widetilde{\mathcal{S}}$ be a metric Smarandache multispace of metrics $\mathcal{S}_i, 1 \leq i \leq m$. Then, for any integer $1 \leq i \leq m$, $\mathcal{S}_i$ is maximally a metric space of $\widetilde{M}$ by STEP 3.1- STEP 3.3, which establishes the collapse mapping $\phi : \widetilde{\mathcal{S}} \to \mathcal{S}_i$ for integers $1 \leq i \leq m$.*

Proof The proof is similar to that of Theorem 3.1. □

As we known, a topological group $(G; \circ)$ is a Smarandache multispace $G \cup G$ in the case of $m = 2$, endowed both with the topological and group properties. By definition, a topological group is a Hausdorff topological space G together with an algebraic group structure on $(G; \circ)$, namely, ① the group multiplication $\circ :\ (a, b) \to a \circ b$ of $G \times G \to G$ is continuous; ② the group inversion $g \to g^{-1}$ of $G \to G$ is continuous. That is, the identity mapping $1_G : G \to G$ is both a collapse mapping of the topological group $(G; \circ)$ to its topological space G and algebraic group $(G; \circ)$. Similarly, a *topological Smarandache multigroup* $(\widetilde{A}; \mathscr{O})$ is an algebraic Smarandache multisystem $(\widetilde{A}; \mathscr{O})$ with $\widetilde{A} = H_1 \cup H_2 \cup \cdots \cup H_m$ and $\mathscr{O} = \{\circ_i; 1 \leq i \leq m\}$ hold with conditions: ① $(H_i; \circ_i)$ is a group for each integer i, $1 \leq i \leq m$, namely, $(H, \mathscr{O})$ is a Smarandache multigroup; ② $\widetilde{A}$ is itself a connected topological Smarandache multispace; ③ the mapping $(a, b) \to a \circ b^{-1}$ is continuous for $\forall a, b \in H_i$ and $\forall \circ \in \mathcal{O}_i$, $1 \leq i \leq m$.

For example, let $\mathbb{R}^{n_i}$ be Euclidean spaces of dimension n_i with an additive operation $+_i$ for integers $1 \leq i \leq m$ and scalar multiplication $\cdot$ determined by

$$
\begin{aligned}
&(\lambda_1 \cdot x_1, \lambda_2 \cdot x_2, \cdots, \lambda_{n_i} \cdot x_{n_i}) +_i (\zeta_1 \cdot y_1, \zeta_2 \cdot y_2, \cdots, \zeta_{n_i} \cdot y_{n_i}) \\
&= (\lambda_1 \cdot x_1 + \zeta_1 \cdot y_1, \lambda_2 \cdot x_2 + \zeta_2 \cdot y_2, \cdots, \lambda_{n_i} \cdot x_{n_i} + \zeta_{n_i} \cdot y_{n_i})
\end{aligned}
$$

for $\forall \lambda_l, \zeta_l \in \mathbb{R}$, where $1 \leq \lambda_l, \zeta_l \leq n_i$. Then, each $\mathbb{R}^{n_i}$ is a continuous group under $+_i$. Whence, the algebraic Smarandache multisystem $(\widetilde{A}; \mathscr{O})$ is a topological multigroup by definition, where $\mathscr{O} = \{+_i; 1 \leq i \leq m\}$. Particularly, if $m = 1$, i.e., an n-dimensional Euclidean space $\mathbb{R}^n$ with the vector additive $+$ and multiplication $\cdot$ is nothing else but a topological group.

The next conclusion on the collapse mapping $\phi : (\widetilde{A}; \mathscr{O}) \to (H_i; \circ_i)$ can be obtained by STEP 1.1-STEP 1.4 similar to that of Theorems 3.1.

Theorem 3.9 *Let $(\widetilde{A}; \mathscr{O})$ be a topological Smarandache multispace of topological groups $(H_i; \circ_i)$, $1 \leq i \leq m$. Then, for any integer $1 \leq i \leq m$, $(H_i; \circ_i)$ is maximally a topological group of $\widetilde{A}$ by STEP 1.1- STEP 1.4, which establishes the collapse mapping $(\widetilde{A}; \mathscr{O}) \to (H_i; \circ_i)$ for integers $1 \leq i \leq m$.*

Certainly, Theorems 3.1-3.9 show that the collapse mapping on an algebraic or geometrical Smarandache multispace or multisystem can be established by the structure inherited in the spaces or systems, which proposes a question on the collapse mapping of a Smarandache multispace or multisystem naturally, i.e., *could we find a unified form on collapse mappings of a Smarandache multispace or multisystem in mathematics?* The answer is positive! Generally, let $\widetilde{S}$ be the union of spaces or systems $S_1, S_2, \cdots, S_m$, i.e., a Smarandache multispace of multisystem. If we view that each S_i of space or system is independent. Then, the Smarandache multispace or multisystem $\widetilde{S}$ can be represented by a tensor product $\widetilde{S} = S_1 \otimes S_2 \otimes \cdots \otimes S_m$ formally and view S_i to be

$$S_i \simeq 1_{S_1} \otimes 1_{S_2} \otimes \cdots \otimes S_i \otimes \cdots \otimes S_m, \tag{3.1}$$

where 1_{S_k} denotes the unit or origin of S_k for integers $1 \leq k \leq m$. In this case, we can present a unified form for collapse mapping.

By definition, a projection π_i is determined by $\pi_i : \widetilde{S} \to S_i$, i.e., $\pi_i : s_1 \otimes s_2 \otimes \cdots \otimes s_m \to s_i$, where $s_i \in S_i$ for integers $1 \leq i \leq m$. Thus, the collapse mapping $\phi : \widetilde{S} \to S_i$ can be represented by $\pi_i, 1 \leq i \leq m$. Furthermore, for an integer $1 \leq i \leq m$ define an identity projection

$$1_{\pi_i}(x) = \begin{cases} x, & \text{if } x \in S_i \\ 1_{S_k}, & \text{if } x \notin S_i \text{ but } x \in S_k, k \neq i. \end{cases}$$

Then, we can get the unified form of collapse mapping of a Smarandache multispace or multisystem following.

Theorem 3.10 *Let $\widetilde{S} = S_1 \otimes S_2 \otimes \cdots \otimes S_m$ be a Smaradache multispace or multisystem with convention* (3.1)*. Then, all collapse mappings can be represented by projections*

$$\pi_i : S_1 \otimes S_2 \otimes \cdots \otimes S_m \to S_i, \quad 1 \leq i \leq m \tag{3.2}$$

and particularly, the identity projections 1_{π_i} for integers $1 \leq i \leq m$.

3.3.Character Observing. A more general question on collapse mapping of Smarandache multispace or multisystem is on the understanding model, i.e., *how to hold on the collapse mapping of Smarandache multispace or multisystem* (1.1) *or* (1.2)*?* For answering this question we consider the case of Schrödinger's cat again. According to the interpretation of Bohr et al., the collapse of the Schrödinger's cat happened in the observing of a human opening the lid of the box to hold on the living or dead of the cat. It is at this time that the superposition state of the cat's state, maybe living or dead instantly collapsed to a determined situation of "*living*" or "*dead*" that a human could understanding or in the words of Smarandache multispace or multisystem, the collapse mapping $\phi : \mathbf{L} + \mathbf{D} \to \mathbf{L}$ or $\mathbf{D}$ appears instantly at the time of a human lifting the lid of the box and observing the cat's living or dead inside the box. And then, *what time happens that a Smarandache multispace or multisytsem disentangles in the understanding things T in the universe?* It happens at the time that a thing T is understood by a character.

Certainly, we have known a Smarandache mutispace or multisystem $\widetilde{S}$ can be disentangled by its inside mathematical structure in Subsections 3.1-3.2. However, all the mathematical structures inside $\widetilde{S}$ are only a hypothesis by humans for simulating its behavior observed. We do not know if there really is one even though there is a mathematical universe hypothesis claims that our external physical reality is a mathematical structure proposed by Max Tegmark [35] in 2003. It can not be verified ([25]) because it is essentially a special case of the *Theory of*

Everything. That is, although the reality of thing T is determined by a Smarandache multispace or multisystem (1.1) or an approximation (1.2) we can not conclude T inherits itself unless endowed a mathematics on it by humans. Thus, we can not assume the spaces or systems determined in (1.1) or (1.2) by characters $\mu_i, 1 \leq i \leq n$ or $\nu_k, k \geq 1$ are self-closed mathematical spaces or systems. In this case, *how to we get the entangling mapping of the Smarandache multispace of multisystem on T?* The answer lies in how to understand the characters of thing T even though it maybe not posses a mathematical structure.

If a thing T is characterized by (1.1) or (1.2) whether or not it has a mathematical structure, *what does the characters $\{\mu_i; 1 \leq i \leq n\}$ and $\{\nu_k, k \geq 1\}$ means?* Certainly, we can view each of them as a parameter or feature. However, if we equate thing T with a Smarandache multispace or multisystem $\widetilde{T}$ consisting of elements, i.e., $\widetilde{T} = \{a_\lambda | \lambda \in \Lambda\}$, where Λ denotes an index set associated with elements in T and the character of an element a_λ is $\chi(a_\lambda)$, then each character μ_i or ν_k is essentially in classifying elements of T into subsets $\{\mu_i\}$ or $\{\nu_k\}$, i.e.,

$$\{\mu_i\} = \{a_\lambda \in T | \chi(a_\lambda) = \mu_i\}, \quad \{\nu_k\} = \{a_\lambda \in T | \chi(a_\lambda) = \nu_k\}, \tag{3.3}$$

namely, $\{\mu_i\}$ and $\{\nu_k\}$ are respectively the sets consisting of elements in T with the same character μ_i or ν_k for integers $1 \leq i \leq n$ and $k \geq 1$. In this case, the collapse mappings on T are nothing else but determined by characters

$$\mu_i : T \to \{\mu_i\} \quad \text{and} \quad \nu_k : T \to \{\nu_k\} \tag{3.4}$$

and similar to the case of Schrödinger's cat, each of them happens in observing character μ_i or ν_k of humans for an integer $1 \leq i \leq n$ or $k \geq 1$. Certainly, $\{\mu_i\}, 1 \leq i \leq n$, $\{\nu_k\}, k \geq 1$ do not have the exclusive property if $n \geq 2$, different from the case of the Schrödinger's cat in general.

For example, let a ripple curve L of water is the composition that of L_1 and L_2 such as those shown in Figure 4.

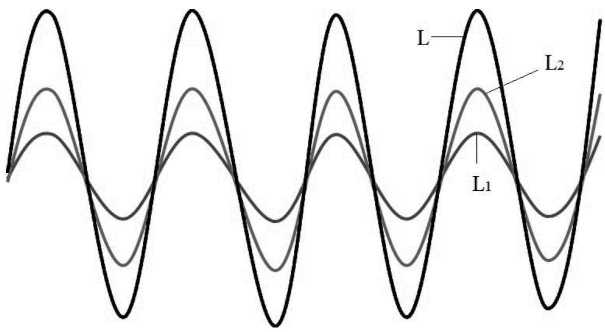

Figure 4

Then, *could one decompose L into L_1 and L_2 for hold on the collapse mapping?* The answer is positive if one knows the characters of the ripple curves of L_1 and L_2 such as those of starting point, the highest and lowest points, spacing, velocity, etc., then it is easily to get the collapse mapping $\phi : L \to L_1$ or L_2 by the characters of L_1 and L_2.

§4. Application to Information Encoding and Decoding

A transmission of information from a sender to a receiver includes information encoding, channel transmission and information decoding by a string consisting of digital numbers. Generally, let S and S' be an entangling pair. Then, one know S' if the onto mapping f is known and vice versa, know S if the onto mapping f' is known by Theorem 2.1. Thus, f, f' are keys in the information encoding and decoding if $ff' = f'f = 1_{id}$, denoted by $f' = f^{-1}$ or $f = f'^{-1}$. Usually, an information is first transformed to a digital form I and then, encode by the action of f on I to get a mixed state $f(I)$ for transmission on the channel. After received $f(I)$, the receiver decodes $f(I)$ by the action f^{-1} on $f(I)$ to know the information I.

As is known to all, a central job in the transmission of information is the encoding and decoding with the information not declassified unless the sender and the receiver. In fact, what are lots of humans value quantum entanglement in disentangling because it can provides one with a key that believed randomly for decoding in quantum teleportation. Then, *can we generalize the encoding and decoding of information by the Smarandace multisystems with disentangling in communication?* Certainly, we can generalize the usual transforming model by Smarandache multisystems.

4.1.Information Encoding and Decoding. Let $\widetilde{S}$ be a Smarandache multisystem of systems $S_1, S_2, \cdots, S_m$ with respective characters $\chi_1, \chi_2, \cdots, \chi_m$. If one or more systems of $S_1, S_2, \cdots, S_m$ are information, we can naturally view the Smarandache multisystem $\widetilde{S}$ to be a disorganized string of numbers or an encoding of the information which can be transmitted in a channel by the sender. After disentangling $\widetilde{S}$ to systems $S_1, S_2, \cdots, S_m$ by different character $\chi_1, \chi_2, \cdots, \chi_m$, the receiver knows the information I such as those shown in Figure 5, where

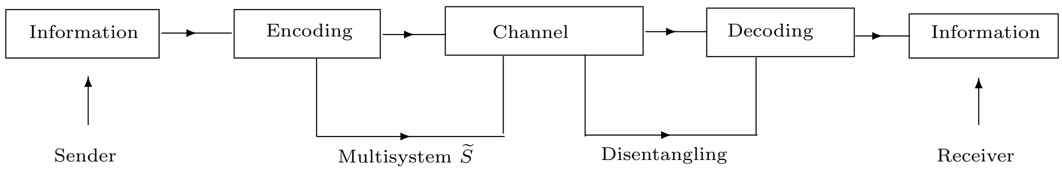

Figure 5

the type of systems $S_i, 1 \leq i \leq m$ maybe the same or district, mathematical or not, finite or infinite, determined or randomly, also be the variables as the sender and receiver wish. For example, let $S_i, 1 \leq i \leq m$ be one of finite fields

$$(\mathbb{Z}_{p_i}; +, \cdot), \qquad \left\{1, t, 2t^2, \cdots, p_i t^{p_i}\right\} \qquad \text{or} \qquad \left\{\begin{pmatrix} m \\ i \end{pmatrix} p^i q^{m-i}\right\} \tag{4.1}$$

for integers $1 \leq i \leq m$ and define the Smarandache multisystem $\widetilde{S} = S_1 \otimes S_1 \otimes \cdots \otimes S_m$ for m primes $p_1, p_2, \cdots, p_m$ and a real number $0 < p < 1$ with $p + q = 1$. Then, *how to encode and decode an information by Smarandache multisystem?* Certainly, the encoding and decoding of an information by a Smarandache multisystem are easily carried out. For example, the typical case that some systems are the transmitted information but others are all bewitching is shown in the programme following.

STEP 4.1. For a transforming Information I, choose a Smarandachely multisystem $\widetilde{S} = S_1 \cup S_2 \cup \cdots \cup S_m$ with respective characters $\chi_1, \chi_2, \cdots, \chi_m$, $m \geq 1$;

STEP 4.2. Encode information I by some systems of $S_i, 1 \leq i \leq m$;

STEP 4.3. Encode $\widetilde{S}$ by a public coding system to a digital form $I(\widetilde{S})$;

STEP 4.4. Transmit $I(\widetilde{S})$ on an opened channel;

STEP 4.5. Decode $I(\widetilde{S})$ by the public coding system to get Smarandache multisystem $\widetilde{S}$;

STEP 4.6. Disentangle $\widetilde{S}$ by characters $\chi_i, 1 \leq i \leq m$ to get systems $S_1, S_2, \cdots, S_m$.

Notice that if $m = 1$, i.e., encode information by one system S, it is the usual case in pubic. Thus, this programme includes the public case in communication. However, it applies to the secret transmitting in case of $m \geq 2$ with a property that the bigger of m or the more complex of systems $S_i, 2 \leq i \leq m$, the higher the security for transmitting of the information. In this model, all characters $\chi_i, 1 \leq i \leq m$ are keys for decoding. Certainly, we can encrypt purposely the information by applying the Smarandachely entangling pairs.

4.2.Smarandachely Entangling Pair. Let $\widetilde{A}$ and $\widetilde{A}'$ be two Smarandache multisystems. They are called *Smarandachely entangling pair* if $\widetilde{A}$ and $\widetilde{A}'$ are entangling. In this case, there must be the known onto mappings $f : \widetilde{A} \to \widetilde{A}'$ and $f' : \widetilde{A}' \to \widetilde{A}$ holding by the sender and the receiver, respectively. Particularly, $f' = f^{-1}$ and both variable on the same Geiger counter t, i.e., $f(t)$ and $f^{-1}(t)$ beginning from an initial number $t = 0$. For example, the quantum teleportation by the pair of entangled particles A,B shown in Figure 6 is in the case.

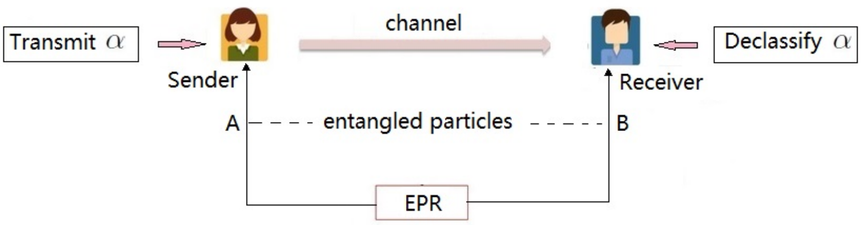

Figure 6

Then, *how to apply a Smarandachely entangling pair in an encrypting transmission of information?* A general model for the encrypting transmission by applying Smarandachely entangling pairs $\widetilde{A}, \widetilde{A}'$ is associating $\widetilde{A}$ with a transmitted information I, encoding I by $f(\widetilde{A})$ and then, the receiver decodes $f(\widetilde{A})$ by f^{-1}, such as the case of entangling particles A, B in Figure 6. By the different applying cases of entangling pairs, there are two models following.

Case 1. Apply one Smarandachely entangling pairs $\widetilde{A}, \widetilde{A}'$.

In this case, a generalized model for transmission of information I by Smarandache multi-system is shown in the following.

STEP 5.1. For a transmitted information I, choose a Smarandachely multisystem $\widetilde{A} = A_1 \cup A_2 \cup \cdots \cup A_m$ with an entangling Smarandache multisystem $\widetilde{A}'$;

STEP 5.2. Encode information I by some of systems $A_1, A_2, \cdots, A_m$ with respective characters $\chi_1, \chi_2, \cdots, \chi_m$ to get a Smarandache multisystem $\widetilde{A}$ and then, applying the entangling pair to get the Smarandache multisystem $\widetilde{A}'$;

STEP 5.3. Encode $\widetilde{A}'$ by a public coding system to a digital form $I(\widetilde{A}')$;

STEP 5.4. Transmit $I(\widetilde{A}')$ on an opened channel;

STEP 5.5. Decode $I(\widetilde{A}')$ by the public coding system to get Smarandache multisystem $\widetilde{A}'$;

STEP 5.6. Disentangle $\widetilde{A}'$ its entangling pair to get $\widetilde{A}$ and then by characters $\chi_i, 1 \leq i \leq m$ to get systems $A_1, A_2, \cdots, A_m$.

Case 2. Apply m entangling pairs $A_i, A_i', 1 \leq i \leq m$ of algebraic systems.

In this case, a generalized model for transmission of information I by Smarandache multisystem is shown in the following.

STEP 6.1. For a transmitted information I, choose an entangling pair A_i, A_i' with respective characters χ_i, χ_i' for integers $1 \leq i \leq m$;

STEP 6.2. Encode information I by some of systems $A_1, A_2, \cdots, A_m$ and then, applying the entangling pair to get the Smarandache multisystem $\widetilde{A}' = A_1' \cup A_2' \cup \cdots \cup A_m'$;

STEP 6.3. Encode $\widetilde{A}'$ by a public coding system to a digital form $I(\widetilde{A}')$;

STEP 6.4. Transmit $I(\widetilde{A}')$ on an opened channel;

STEP 6.5. Decode $I(\widetilde{A}')$ by the public coding system to get Smarandache multisystem $\widetilde{A}'$;

STEP 6.6. Disentangle $\widetilde{A}'$ by characters $\chi_i', 1 \leq i \leq m$ to get systems $A_1', A_2', \cdots, A_m'$ and then, apply the entangling pairs to get system $A_1, A_2, \cdots, A_m$.

Notice that each of systems $A_1, A_2, \cdots, A_m$ and $A_1', A_2', \cdots, A_m'$ could be constantly or variable systems in case. Particularly, if the Smarandache multisystem $\widetilde{A}$ or systems $A_1, A_2, \cdots, A_m$ are variable on $\mathbf{x}$ with known f, f^{-1}, then both of Cases 1 and 2 include the applying case of quantum entangling particles in communication by the hidden variable theory of Bohm D. and Y. Aharonov in [3]. For example, let

$$A = \left\{x_1^2(t), x_2^2(t), \cdots, x_n^2(t), \cdots\right\} \quad \text{and} \quad A' = \left\{\sqrt{x_1(t)}, \sqrt{x_2(t)}, \cdots, \sqrt{x_n(t)}, \cdots\right\}, \tag{4.2}$$

where t is determined by a Geiger counter. Then, A and A' consist of an entangling pair variable on variable t. Then, *what is the implication included in this example?* It implies that the particles in a quantum entangling pair is only an information or a key carrier if we cast off the mystery of microscopic particles and the key is in fact on hidden variables determined by observing. Thus, a general carrier for encoding and decoding of information should be designed on the Smarandachely entangling pairs and then, we can apply it to communication.

§5. Conclusion

A central topic of this paper is to disentangle Smarandache multispaces or multisystems by its mathematical structures or characters and then, generalizes the quantum entangling pairs by Smarandachely entangling pairs with possible applications to communication. In fact, the application of quantum entanglement is a hot topic in communication until today but hardly one noted its mathematical nature, bewitched by its appearance of the microscopic particles. For unraveling the mysteries of the entangling state, we discuss its general case, i.e., Smarandache multispace or multisystem and show how to disentangle a Smarandache multispace or multisystem to self-closed spaces or systems by their mathematical structures or characters, and generalize the entangling pair of particles to Smarandachely entangling pair for application of Smarandache multispace or multisystem in communication. Certainly, the application of encoding and decoding by Smarandache multispaces or multisystems needs one to design the key carrier, likewise the entangled quanta. However, we believe such a key carrier will come true in the near future by the notion.

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Fixed Point Results for $\mathcal{F}_{(S,\mathcal{T})}$-Contraction in S-Metric Spaces Using Implicit Relation with Applications

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Abstract: The main purpose of this paper is to study and establish some fixed point theorems for $\mathcal{F}_{(S,\mathcal{T})}$-contraction in the setting of S-metric space via an implicit relation. The results presented in this paper extend, unify and generalize several known results from the existing literature. Also, we give one of the possible applications of our result to well-posed and limit shadowing property of fixed point problems.

Key Words: Fixed point, implicit relation, $\mathcal{F}_{(S,\mathcal{T})}$-contraction, S-metric space.

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§1. Introduction

Fixed point theory is one of the most important topic in the development of nonlinear analysis. As it is well known, one of the most useful theorems in nonlinear analysis is the Banach contraction principle [9]. A mapping $\mathcal{T}\colon X \to X$ where (X,d) is a metric space, is said to be a contraction if there exists $c \in [0,1)$ such that for all $x, y \in X$,

$$d(\mathcal{T}(x), \mathcal{T}(y)) \leq c\, d(x,y). \tag{1.1}$$

If the metric space (X,d) is complete then the mapping satisfying (1.1) has a unique fixed point. Inequality (1.1) implies continuity of $\mathcal{T}$. Many authors generalized this famous result in different ways. In recent time the study of fixed point theory in metric space is very interesting field and attract many researchers to investigated different results on it.

In 2006, Mustafa and Sims [2] introduced a new notion of generalized metric space, called G-metric space and gave a modification to the contraction principle of Banach. After then, several authors studied various fixed and common fixed point problems for adequate classes of contractive mappings in generalized metric spaces (see, [1, 2, 3, 4, 8, 10, 12, 16, 22, 23, 24, 26, 33, 40, 41, 42]).

In 2012, Sedghi et al. [38] introduced the notion of S-metric space which is a generalization of a G-metric space and D^*-metric space. In [38] the authors proved some basic properties of S-metric spaces. Also, they obtained some fixed point theorems in S-metric space for a self-

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map. Afterwards, a multitude of results was obtained in these spaces (see, e.g., [13, 39, 32]) and many others.

Sedghi et al. [38] introduced the notion of S-metric spaces as follows:

Definition 1.1([38]) *Let X be a nonempty set and $S\colon X^3 \to \mathbb{R}^+$ be a function satisfying the following conditions*

$(S1)$ $S(x,y,z) = 0$ *if and only if* $x = y = z$*;*

$(S2)$ $S(x,y,z) \le S(x,x,t) + S(y,y,t) + S(z,z,t)$ *for all* $x,\ y,\ z,\ t \in X$*, where* $\mathbb{R}^+ = [0,\infty)$*,* *Then, the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.*

Example 1.2([38]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an S-metric on X.

Example 1.3([38]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|x-z\| + \|y-z\|$ is an S-metric on X.

Example 1.4([39]) Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.

Lemma 1.5 ([38], Lemma 2.5) *If (X,S) be an S-metric space, then we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Lemma 1.6 ([38], Lemma 2.12) *Let (X,S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.*

Definition 1.7([38]) *Let (X,S) be an S-metric space.*

$(a1)$ *A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $S(x_n,x_n,x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;*

$(a2)$ *A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \ge n_0$ we have $S(x_n,x_n,x_m) < \varepsilon$;*

$(a3)$ *The S-metric space (X,S) is called complete if every Cauchy sequence in (X,S) is convergent in (X,S).*

Definition 1.8 *Let T be a self mapping on an S-metric space (X,S). Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.*

Definition 1.9([38]) *Let (X,S) be an S-metric space. A mapping $T\colon X \to X$ is said to be a contraction if there exists a constant $0 \le L < 1$ such that*

$$S(\mathcal{T}x,\mathcal{T}y,\mathcal{T}z) \le L\,S(x,y,z) \tag{1.2}$$

for all $x,y,z \in X$.

Notice that if the S-metric space (X,S) is complete, then the mapping defined in the Definition 1.9 has a unique fixed point ([38]).

Moradi and Beiranvand [19] introduced the following notion.

Definition 1.10([19]) *Let (X,d) be a metric space and $f,\mathcal{T}\colon X\to X$ be two mappings. The mapping f is said to be a $\mathcal{T}_{\mathcal{F}}$-contraction if there exists $a\in[0,1)$ such that for all $x,y\in X$*

$$\mathcal{F}\Big(d(\mathcal{T}fx,\mathcal{T}fy)\Big)\le a\,\mathcal{F}\Big(d(\mathcal{T}x,\mathcal{T}y)\Big), \tag{1.3}$$

where,

(1) *$\mathcal{F}\colon[0,\infty)\to[0,\infty)$, $\mathcal{F}$ is nondecreasing continuous from the right and $\mathcal{F}^{-1}(0)=\{0\}$;*
(2) *$\mathcal{T}$ is one to one and graph closed.*

We introductions the definition of $\mathcal{F}_{(S,\mathcal{T})}$-contraction following.

Definition 1.11 *Let (X,S) be an S-metric space and $\mathcal{T}\colon X\to X$ be a mapping. The mapping $\mathcal{T}$ is said to be a $\mathcal{F}_{(S,\mathcal{T})}$-contraction if there exists $a\in[0,1)$ such that for all $x,y,z\in X$ and*

$$\mathcal{F}\Big(S(\mathcal{T}x,\mathcal{T}y,\mathcal{T}z)\Big)\le a\,\mathcal{F}\Big(S(x,y,z)\Big), \tag{1.4}$$

where $\mathcal{F}\colon[0,\infty)\to[0,\infty)$ is a function satisfying the following conditions

($\mathcal{F}_1$) *$\mathcal{F}$ is nondecreasing;*
($\mathcal{F}_2$) *$\mathcal{F}$ is continuous from the right and;*
($\mathcal{F}_3$) *$\mathcal{F}^{-1}(0)=\{0\}$.*

Remark 1.12 *If we take $\mathcal{F}(t)=t$ in equation (1.4), then we obtain Banach contraction type condition (1.2) in S-metric space (X,S) with $a=L$ and if X is complete then $\mathcal{T}$ has a unique fixed point.*

Now, we introduce an implicit relation to investigate some fixed point theorems in S-metric spaces.

Definition 1.13 (Implicit Relation) *Let Φ be the family of all real valued continuous functions $\phi\colon\mathbb{R}_+^3\to\mathbb{R}_+$, for three variables. For some $h\in[0,1)$, we consider the following conditions*

($R1$) *For $x,y\in\mathbb{R}_+$, if $x\le\phi(y,y,x)$, then $x\le hy$;*
($R2$) *For $x\in\mathbb{R}_+$, if $x\le\phi(0,0,x)$, then $x=0$;*
($R3$) *For $x\in\mathbb{R}_+$, if $x\le\phi(x,0,0)$, then $x=0$ since $h\in[0,1)$.*

The main purpose of this paper is to study $\mathcal{F}_{(S,\mathcal{T})}$-contraction in S-metric space and establish some fixed point theorems under an implicit relation. The results presented in this paper extend, generalize and unify several known results from the existing literature. Also, we give one of the possible applications of our result to well-posed and limit shadowing property of fixed point problems.

§2. Main Results

In this section, we shall prove some fixed point theorems for $\mathcal{F}_{(S,\mathcal{T})}$-contraction under an implicit relation in the setting of S-metric spaces.

Theorem 2.1 *Let (X, S) be a complete S-metric space and $\mathcal{T}\colon X \to X$ be a mapping. If for all $x, y \in X$ and*

$$\mathcal{F}\Big(S(\mathcal{T}x,\mathcal{T}x,\mathcal{T}y)\Big) \leq \phi\Big\{\mathcal{F}\Big(S(x,x,y)\Big),\mathcal{F}\Big(S(x,x,\mathcal{T}x)\Big),\mathcal{F}\Big(S(y,y,\mathcal{T}y)\Big)\Big\} \tag{2.1}$$

where $\mathcal{F}\colon [0,\infty) \to [0,\infty)$ is nondecreasing continuous function and $\mathcal{F}(t) = 0$ if and only if $t = 0$ and some $\phi \in \Phi$. Then, we have

(1) *If ϕ satisfies the conditions $(R1)$ and $(R2)$, then $\mathcal{T}$ has a fixed point;*

(2) *If ϕ satisfies the condition $(R3)$ and $\mathcal{T}$ has a fixed point, then the fixed point is unique.*

Proof (1) Let $x_0 \in X$ be an arbitrary point and $x_n = \mathcal{T}x_{n-1} = \mathcal{T}^n x_0$, $n = 0, 1, 2, \cdots$. Now, from (2.1) we have

$$\begin{aligned}
&\mathcal{F}\Big(S(x_{n+1},x_{n+1},x_{n+2})\Big) = \mathcal{F}\Big(S(\mathcal{T}x_n,\mathcal{T}x_n,\mathcal{T}x_{n+1})\Big)\\
&\leq \phi\Big\{\mathcal{F}\Big(S(x_n,x_n,x_{n+1})\Big),\mathcal{F}\Big(S(x_n,x_n,\mathcal{T}x_n)\Big),\mathcal{F}\Big(S(x_{n+1},x_{n+1},\mathcal{T}x_{n+1})\Big)\Big\}\\
&= \phi\Big\{\mathcal{F}\Big(S(x_n,x_n,x_{n+1})\Big),\mathcal{F}\Big(S(x_n,x_n,x_{n+1})\Big),\mathcal{F}\Big(S(x_{n+1},x_{n+1},x_{n+2})\Big)\Big\}.
\end{aligned} \tag{2.2}$$

Since ϕ satisfies the condition $(R1)$, there exists $h \in [0, 1)$ such that

$$\begin{aligned}
\mathcal{F}\Big(S(x_{n+1},x_{n+1},x_{n+2})\Big) &\leq h\,\mathcal{F}\Big(S(x_n,x_n,x_{n+1})\Big) \leq \cdots\\
&\leq h^{n+1}\mathcal{F}\Big(S(x_0,x_0,x_1)\Big).
\end{aligned} \tag{2.3}$$

Thus, for all $n < m$, by using $(S2)$ Lemma 1.5 and equation (2.3) we have

$$\begin{aligned}
\mathcal{F}\Big(S(x_n,x_n,x_m)\Big) &\leq \mathcal{F}\Big(2S(x_n,x_n,x_{n+1}) + S(x_m,x_m,x_{n+1})\Big)\\
&= \mathcal{F}\Big(2S(x_n,x_n,x_{n+1}) + S(x_{n+1},x_{n+1},x_m)\Big)\\
&\cdots\\
&\leq \mathcal{F}\Big(2[h^n + \cdots + h^{m-1}]S(x_0,x_0,x_1)\Big)\\
&\leq \mathcal{F}\Big(\Big(\frac{2h^n}{1-h}\Big)S(x_0,x_0,x_1)\Big).
\end{aligned}$$

Taking the limit as $n, m \to \infty$ and using the property of $\mathcal{F}$, we get $\mathcal{F}\Big(S(x_n,x_n,x_m)\Big) \to 0^+$, since $0 < h < 1$. As $\mathcal{F}$ is continuous, we obtain $S(x_n,x_n,x_m) = 0$. This proves that the sequence $\{x_n\}$ is a Cauchy sequence in the complete S-metric space (X, S). By the completeness of the space, there exists $v \in X$ such that $\{x_n\}$ converges to $v \in X$. Now, we prove that v is a fixed point of $\mathcal{T}$. Again by using inequality (2.1), we obtain

$$\begin{aligned}
&\mathcal{F}\Big(S(x_{n+1}, x_{n+1}, \mathcal{T}v)\Big) = \mathcal{F}\Big(S(\mathcal{T}x_n, \mathcal{T}x_n, \mathcal{T}v)\Big)\\
&\leq \phi\Big\{\mathcal{F}\Big(S(x_n, x_n, v)\Big), \mathcal{F}\Big(S(x_n, x_n, \mathcal{T}x_n)\Big), \mathcal{F}\Big(S(v, v, \mathcal{T}v)\Big)\Big\}\\
&= \phi\Big\{\mathcal{F}\Big(S(x_n, x_n, v)\Big), \mathcal{F}\Big(S(x_n, x_n, x_{n+1})\Big), \mathcal{F}\Big(S(v, v, \mathcal{T}v)\Big)\Big\}. \qquad (2.4)
\end{aligned}$$

Indeed, as $\mathcal{F}$ is continuous and note that $\phi \in \Phi$, then using the property of $\mathcal{F}$ and taking the limit as $n \to \infty$, we get

$$\begin{aligned}
\mathcal{F}\Big(S(v, v, \mathcal{T}v)\Big) &\leq \phi\Big\{\mathcal{F}\Big(S(v, v, v)\Big), \mathcal{F}\Big(S(v, v, v)\Big), \mathcal{F}\Big(S(v, v, \mathcal{T}v)\Big)\Big\}\\
&= \phi\Big\{0, 0, \mathcal{F}\Big(S(v, v, \mathcal{T}v)\Big)\Big\}.
\end{aligned}$$

Since ϕ satisfies the condition $(R2)$, then $\mathcal{F}\Big(S(v, v, \mathcal{T}v)\Big) \leq h.0 = 0$. This implies that $S(v, v, \mathcal{T}v) = 0$. Thus, $v = \mathcal{T}v$. Hence v is a fixed point of $\mathcal{T}$.

(2) Let u_1, u_2 be fixed points of f with $u_1 \neq u_2$. We shall prove that $u_1 = u_2$. It follows from equation (2.1) and property of $\mathcal{F}$ that

$$\begin{aligned}
&\mathcal{F}\Big(S(u_1, u_1, u_2)\Big) = \mathcal{F}\Big(S(\mathcal{T}u_1, \mathcal{T}u_1, \mathcal{T}u_2)\Big)\\
&\leq \phi\Big\{\mathcal{F}\Big(S(u_1, u_1, u_2)\Big), \mathcal{F}\Big(S(u_1, u_1, \mathcal{T}u_1)\Big), \mathcal{F}\Big(S(u_2, u_2, \mathcal{T}u_2)\Big)\Big\}\\
&= \phi\Big\{\mathcal{F}\Big(S(u_1, u_1, u_2)\Big), \mathcal{F}\Big(S(u_1, u_1, u_1)\Big), \mathcal{F}\Big(S(u_2, u_2, u_2)\Big)\Big\}\\
&= \phi\Big\{\mathcal{F}\Big(S(u_1, u_1, u_2)\Big), 0, 0\Big\}.
\end{aligned}$$

Since ϕ satisfies the condition $(R3)$, we get

$$\begin{aligned}
\mathcal{F}\Big(S(u_1, u_1, u_2)\Big) &\leq h\,\mathcal{F}\Big(S(u_1, u_1, u_2)\Big)\\
&\Rightarrow \mathcal{F}\Big(S(u_1, u_1, u_2)\Big) = 0, \text{ because of } 0 < h < 1.
\end{aligned}$$

This implies that $S(u_1, u_1, u_2) = 0$. Thus, $u_1 = u_2$. This shows that the fixed point of $\mathcal{T}$ is unique. This completes the proof. □

Theorem 2.2 *Let (X, S) be a complete S-metric space such that for positive integer n, $\mathcal{T}^n$ satisfies the contraction condition (2.1) for all $x, y \in X$, where $\mathcal{F}$ and ϕ are as in Theorem 2.1. Then $\mathcal{T}$ has a unique fixed point in X.*

Proof From Theorem 2.1, let u_0 be the unique fixed point of $\mathcal{T}^n$. Then

$$\mathcal{T}(\mathcal{T}^n u_0) = \mathcal{T}u_0 \quad \text{or} \quad \mathcal{T}^n(\mathcal{T}u_0) = \mathcal{T}u_0,$$

which gives $\mathcal{T}u_0 = u_0$. This shows that u_0 is a unique fixed point of $\mathcal{T}$. This completes the proof. □

In Theorem 2.1, if we consider $\mathcal{F}$ is an identity map, then we obtain the following result as corollary.

Corollary 2.3 *Let (X,S) be a complete S-metric space and $\mathcal{T}\colon X\to X$ be a mapping satisfying the inequality*

$$S(\mathcal{T}x,\mathcal{T}x,\mathcal{T}y)\le \phi\Big\{S(x,x,y),S(x,x,\mathcal{T}x),S(y,y,\mathcal{T}y)\Big\}$$

for all $x,y\in X$ and some $\phi\in\Phi$. If ϕ satisfies the conditions $(R1)$, $(R2)$ and $(R3)$, then $\mathcal{T}$ has a unique fixed point in X.

Next, we give an analogues of fixed point theorems in metric spaces for S-metric spaces by combining Theorem 2.1 with $\phi\in\Phi$ and ϕ satisfies conditions $(R1)$, $(R2)$ and $(R3)$. The following corollary is an analogue of Banach's type contraction principle.

Corollary 2.4 *Let (X,S) be a complete S-metric space and $\mathcal{T}\colon X\to X$ be a mapping. If for all $K_1\in[0,1)$ and $x,y\in X$ and satisfying the inequality*

$$\mathcal{F}\Big(S(\mathcal{T}x,\mathcal{T}x,\mathcal{T}y)\Big)\le K_1\,\mathcal{F}\Big(S(x,y,z)\Big)$$

where $\mathcal{F}\colon[0,\infty)\to[0,\infty)$ is nondecreasing continuous function and $\mathcal{F}(t)=0$ if and only if $t=0$. Then $\mathcal{T}$ has a unique fixed point in X.

Proof The assertion follows using Theorem 2.1 with $\phi(p,q,r)=K_1p$ for some $K_1\in[0,1)$ and all $p,q,r\in\mathbb{R}_+$. □

The following corollary is an analogue of R. Kannan's type result [15].

Corollary 2.5 *Let (X,S) be a complete S-metric space and $\mathcal{T}\colon X\to X$ be a mapping. If for all $K_2\in[0,\frac{1}{2})$ and $x,y\in X$ and satisfying the inequality*

$$\mathcal{F}\Big(S(\mathcal{T}x,\mathcal{T}x,\mathcal{T}y)\Big)\le K_2\left[\mathcal{F}\Big(S(x,x,\mathcal{T}x)\Big)+\mathcal{F}\Big(S(y,y,\mathcal{T}y)\Big)\right]$$

where $\mathcal{F}\colon[0,\infty)\to[0,\infty)$ is nondecreasing continuous function and $\mathcal{F}(t)=0$ if and only if $t=0$. Then $\mathcal{T}$ has a unique fixed point in X.

Proof The assertion follows using Theorem 2.1 with $\phi(p,q,r)=K_2(q+r)$ for some $K_2\in[0,\frac{1}{2})$ and all $p,q,r\in\mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(y,y,x)=K_2(y+x)$. So, if $x\le\phi(y,y,x)$, then $x\le\Big(\frac{K_2}{1-K_2}\Big)y$ with $\Big(\frac{K_2}{1-K_2}\Big)<1$. Thus, $\mathcal{T}$ satisfies the condition $(R1)$.

Next, if $x\le\phi(0,0,x)=K_2(0+x)=K_2x$, then $x=0$, since $K_2<\frac{1}{2}<1$. Thus, $\mathcal{T}$ satisfies the condition $(R2)$.

Finally, if $x\le\phi(x,0,0)=K_2.0=0$, then $x=0$. Thus, $\mathcal{T}$ satisfies the condition $(R3)$. □

The following corollary is an analogue of S. Reich's type result [34].

Corollary 2.6 *Let (X,S) be a complete S-metric space and $\mathcal{T}\colon X\to X$ be a mapping. If for* *all* $A_1, A_2, A_3 \geq 0$ *with* $A_1 + A_2 + A_3 < 1$ *and* $x, y \in X$ *and satisfying the inequality*

$$\begin{aligned}\mathcal{F}\Big(S(\mathcal{T}x,\mathcal{T}x,\mathcal{T}y)\Big) &\leq A_1\,\mathcal{F}\Big(S(x,x,y)\Big) + A_2\,\mathcal{F}\Big(S(x,x,\mathcal{T}x)\Big)\\ &\quad + A_3\,\mathcal{F}\Big(S(y,y,\mathcal{T}y)\Big)\end{aligned}$$

where $\mathcal{F}\colon [0,\infty) \to [0,\infty)$ *is nondecreasing continuous function and* $\mathcal{F}(t) = 0$ *if and only if* $t = 0$. *Then* $\mathcal{T}$ *has a unique fixed point in* X.

Proof The assertion follows using Theorem 2.1 with $\phi(p,q,r) = A_1p + A_2q + A_3r$ for some $A_1, A_2, A_3 \geq 0$ are constants with $A_1+A_2+A_3 < 1$ and all $p, q, r \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(y,y,x) = A_1y + A_2y + A_3x$. So, if $x \leq \phi(y,y,x)$, then $x \leq \left(\frac{A_1+A_2}{1-A_3}\right)y$ with $\left(\frac{A_1+A_2}{1-A_3}\right) < 1$. Thus, $\mathcal{T}$ satisfies the condition $(R1)$.

Next, if $x \leq \phi(0,0,x) = A_1.0 + A_2.0 + A_3.x = A_3x$, then $x = 0$ since $A_3 < 1$. Thus, $\mathcal{T}$ satisfies the condition $(R2)$.

Finally, if $x \leq \phi(x,0,0) = A_1.x + A_2.0 + A_3.0 = A_1x$, then $x = 0$ since $A_1 < 1$. Thus, $\mathcal{T}$ satisfies the condition $(R3)$. □

Again, we give an analogues of fixed point theorems in metric spaces for S-metric spaces by combining Corollary 2.3 with $\phi \in \Phi$ and ϕ satisfies conditions $(R1)$, $(R2)$ and $(R3)$. The following corollary is an analogue of Banach's type contraction principle.

Corollary 2.7 *Let* (X, S) *be a complete* S*-metric space. Suppose that the mapping* $\mathcal{T}\colon X \to X$ *satisfies the following condition:*

$$S(\mathcal{T}x,\mathcal{T}x,\mathcal{T}y) \leq L\,S(x,x,y)$$

for all $x, y \in X$*, where* $L \in [0,1)$ *is a constant. Then* $\mathcal{T}$ *has a unique fixed point in* X*. Moreover,* $\mathcal{T}$ *is continuous at the fixed point.*

Proof The assertion follows using Corollary 2.3 with $\phi(p,q,r) = Lp$ for some $L \in [0,1)$ and all $p, q, r \in \mathbb{R}_+$. □

The following corollary is an analogue of R. Kannan's type result [15].

Corollary 2.8 *Let* (X, S) *be a complete* S*-metric space. Suppose that the mapping* $\mathcal{T}\colon X \to X$ *satisfies the following condition:*

$$S(\mathcal{T}x,\mathcal{T}x,\mathcal{T}y) \leq M\,[S(x,x,\mathcal{T}x) + S(y,y,\mathcal{T}y)]$$

for all $x, y \in X$*, where* $M \in [0, \frac{1}{2})$ *is a constant. Then* $\mathcal{T}$ *has a unique fixed point in* X*. Moreover,* $\mathcal{T}$ *is continuous at the fixed point.*

Proof The assertion follows using Corollary 2.3 with $\phi(p,q,r) = M(q+r)$ for some $M \in [0, \frac{1}{2})$ and all $p, q, r \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(y,y,x) = M(y+x)$. So, if $x \leq \phi(y,y,x)$, then $x \leq \left(\frac{M}{1-M}\right)y$ with $\left(\frac{M}{1-M}\right) < 1$. Thus, f satisfies the condition $(R1)$.

Next, if $x \leq \phi(0,0,x) = M(0+x) = Mx$, then $x = 0$, since $M < \frac{1}{2} < 1$. Thus, $\mathcal{T}$ satisfies the condition $(R2)$.

Finally, if $x \leq \phi(x,0,0) = M.0 = 0$, then $x = 0$. Thus, $\mathcal{T}$ satisfies the condition $(R3)$. □

The following corollary is an analogue of S. Reich's type result [34].

Corollary 2.9 *Let (X, S) be a complete S-metric space. Suppose that the mapping $\mathcal{T}\colon X \to X$ satisfies the following condition:*

$$S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) \leq k_1\, S(x,x,y) + k_2\, S(x,x,\mathcal{T}x) + k_3\, S(y,y,\mathcal{T}y)$$

for all $x, y \in X$, where $k_1, k_2, k_3 \geq 0$ are constants with $k_1 + k_2 + k_3 < 1$. Then $\mathcal{T}$ has a unique fixed point in X. Moreover, if $k_3 < \frac{1}{2}$, then $\mathcal{T}$ is continuous at the fixed point.

Proof: The assertion follows using Corollary 2.3 with $\phi(p,q,r) = k_1 p + k_2 q + k_3 r$ for some $k_1, k_2, k_3 \geq 0$ are constants with $k_1 + k_2 + k_3 < 1$ and all $p, q, r \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(y,y,x) = k_1 y + k_2 y + k_3 x$. So, if $x \leq \phi(y,y,x)$, then $x \leq \Big(\frac{k_1+k_2}{1-k_3}\Big) y$ with $\Big(\frac{k_1+k_2}{1-k_3}\Big) < 1$. Thus, $\mathcal{T}$ satisfies the condition $(R1)$.

Next, if $x \leq \phi(0,0,x) = k_1.0 + k_2.0 + k_3.x = k_3 x$, then $x = 0$ since $k_3 < 1$. Thus, $\mathcal{T}$ satisfies the condition $(R2)$.

Finally, if $x \leq \phi(x,0,0) = k_1.x + k_2.0 + k_3.0 = k_1 x$, then $x = 0$ since $k_1 < 1$. Thus, $\mathcal{T}$ satisfies the condition $(R3)$. □

Example 2.10 Let $X = \mathbb{R}$ be the usual S-metric space as in Example 1.4. Now, we consider the mapping $\mathcal{T}\colon X \to X$ by $\mathcal{T}(x) = \frac{x}{10}$ for all $x \in [0,1]$. Then

$$\begin{aligned} S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) &= |\mathcal{T}x - \mathcal{T}y| + |\mathcal{T}x - \mathcal{T}y| \\ &= 2|\mathcal{T}x - \mathcal{T}y| = 2\Big|\Big(\frac{x}{10}\Big) - \Big(\frac{y}{10}\Big)\Big| \\ &= \frac{1}{5}|x-y| \leq \frac{2}{5}|x-y| \\ &= \frac{1}{5}\big(2|x-y|\big) = \alpha\, S(x,x,y) \end{aligned}$$

where $\alpha = \frac{1}{5} < 1$. Thus $\mathcal{T}$ satisfies all the conditions of Corollary 2.7 and clearly $0 \in X$ is the unique fixed point of $\mathcal{T}$.

Example 2.11 Let $X = \mathbb{R}$ be the usual S-metric space as in Example 1.4. Now, we consider the mapping $T\colon X \to X$ by $\mathcal{T}(x) = \frac{x}{5}$ for all $x \in [0,1]$. Then

$$\begin{aligned} S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) &= |\mathcal{T}x - \mathcal{T}y| + |\mathcal{T}x - \mathcal{T}y| \\ &= 2|\mathcal{T}x - \mathcal{T}y| = 2\Big|\Big(\frac{x}{5}\Big) - \Big(\frac{y}{5}\Big)\Big| \\ &= \frac{2}{5}|x-y| \leq \frac{8}{15}|x-y| \\ &\leq \frac{1}{3}\Big[\frac{8}{5}|x| + \frac{8}{5}|y|\Big]. \end{aligned}$$

$$\begin{aligned} S(x,x,Tx) &= 2|x-\mathcal{T}x| = \frac{8}{5}|x|, \\ S(y,y,Ty) &= 2|y-\mathcal{T}y| = \frac{8}{5}|y|. \end{aligned}$$

Now, we have

$$\begin{aligned} S(\mathcal{T}x,\mathcal{T}x,\mathcal{T}y) &\leq \frac{1}{3}[S(x,x,\mathcal{T}x)+S(y,y,\mathcal{T}y)] \\ &= \beta[S(x,x,\mathcal{T}x)+S(y,y,\mathcal{T}y)] \end{aligned}$$

where $\beta = \frac{1}{3} < \frac{1}{2}$. Thus $\mathcal{T}$ satisfies all the conditions of Corollary 2.8 and clearly $0 \in X$ is the unique fixed point of $\mathcal{T}$.

Example 2.12 Let $X = [0,1]$. We define $S\colon X^3 \to \mathbb{R}_+$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x=y=z, \\ \max\{x,y,z\} & \text{if otherwise.} \end{cases}$$

for all $x,y,z \in X$. Then (X,S) is a complete S-metric space. Let $\mathcal{T}\colon X \to X$ be a mapping defined as $\mathcal{T}(x) = \frac{x}{2}$ for all $x \in X$.

Without loss of generality we may assume that $x > y > z$, then we have

$$S(\mathcal{T}x,\mathcal{T}x,\mathcal{T}y) = \max\left\{\frac{x}{2},\frac{x}{2},\frac{y}{2}\right\} = \frac{x}{2},$$

$$S(x,x,y) = \max\{x,x,y\} = x,$$

$$S(x,x,\mathcal{T}x) = \max\left\{x,x,\frac{x}{2}\right\} = x,$$

$$S(y,y,\mathcal{T}y) = \max\left\{y,y,\frac{y}{2}\right\} = y,$$

Now, we consider the inequality of Corollary 2.9, we have

$$S(\mathcal{T}x,\mathcal{T}y,\mathcal{T}z) = \frac{x}{2} \leq k_1.x + k_2.x + k_3.y,$$

taking $x = 1$ and $y = 0$ in the above inequality, we obtain

$$\frac{1}{2} \leq k_1 + k_2,$$

the above inequality is satisfied for $k_1 = \frac{1}{4}$, $k_2 = \frac{2}{5}$ and $k_3 = 0$ with $k_1 + k_2 + k_3 = \frac{13}{20} < 1$. Thus $\mathcal{T}$ satisfies all the conditions of Corollary 2.9 and clearly $0 \in X$ is the unique fixed point of $\mathcal{T}$.

§3. Application to Well Posedness and Limit Shadowing of Fixed Point Problem

The concept of well posedness of a fixed point problem has generated much interest to several mathematicians, for example [6, 7, 11, 18, 30, 31, 35]. Here, we study well posedness and limit shadowing of a fixed point problem of mappings in Theorem 2.1.

Definition 3.1([11]) *Let (X, d) be a metric space and $\mathcal{T}\colon X \to X$ be a mapping. The fixed point problem of $\mathcal{T}$ is said to be well-posed if*

(i) $\mathcal{T}$ has a unique fixed point z in X;

(ii) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(\mathcal{T}x_n, x_n) = 0$, $\lim_{n\to\infty} d(x_n, z) = 0$.

The limit shadowing property of fixed point problems has been discussed in the papers [27, 28, 36] and others.

Definition 3.2([29]) *Let (X, d) be a metric space and $\mathcal{T}\colon X \to X$ be a mapping. The fixed point problem of $\mathcal{T}$ is said to have limit shadowing property in X if assuming that sequence $\{x_n\}$ in X satisfies $d(\mathcal{T}x_n, x_n) = 0$ as $n \to \infty$ it follows that there exists $x \in X$ such that $d(\mathcal{T}^n x, x_n) = 0$ as $n \to \infty$.*

Now, we define the above notion in S-metric space.

Definition 3.3 *Let (X, S) be a S-metric space and $\mathcal{T}\colon X \to X$ be a mapping. The fixed point problem of $\mathcal{T}$ is said to be well-posed if*

(i) $\mathcal{T}$ has a unique fixed point z in X;

(ii) for any sequence $\{x_n\}$ of points in X such that

$$\lim_{n\to\infty} S(\mathcal{T}x_n, \mathcal{T}x_n, x_n) = 0 = \lim_{n\to\infty} S(x_n, x_n, \mathcal{T}x_n),$$

we have $\lim_{n\to\infty} S(x_n, x_n, z) = 0 = \lim_{n\to\infty} S(z, z, x_n)$.

Definition 3.4 *Let (X, S) be an S-metric space and $\mathcal{T}\colon X \to X$ be a mapping. The fixed point problem of $\mathcal{T}$ is said to have limit shadowing property in X if assuming that sequence $\{x_n\}$ in X satisfies $\lim_{n\to\infty} S(\mathcal{T}x_n, \mathcal{T}x_n, x_n) = 0 = \lim_{n\to\infty} S(x_n, x_n, \mathcal{T}x_n)$ it follows that there exists $z \in X$ such that $\lim_{n\to\infty} S(\mathcal{T}^n z, \mathcal{T}^n z, x_n) = 0 = \lim_{n\to\infty} S(x_n, x_n, \mathcal{T}^n z)$.*

Concerning the well-posedness and limit shadowing of the fixed point problem for a mapping in a S-metric space satisfying the conditions of Theorem 2.1, we have the following result.

Theorem 3.5 *Let $\mathcal{T}\colon X \to X$ be a self mapping as in Theorem 2.1. Then, the fixed point problem for $\mathcal{T}$ is well posed.*

Proof According to Theorem 2.1, we know that $\mathcal{T}$ has a unique fixed point $z = \mathcal{T}z \in X$. Let $\{x_n\} \subset X$ be such that $\lim_{n\to\infty} S(\mathcal{T}x_n, \mathcal{T}x_n, x_n) = 0 = \lim_{n\to\infty} S(x_n, x_n, \mathcal{T}x_n)$. Then, we have

$$\begin{aligned} S(x_n, x_n, z) &\leq 2S(x_n, x_n, \mathcal{T}x_n) + S(z, z, \mathcal{T}x_n) \\ &= 2S(x_n, x_n, \mathcal{T}x_n) + S(\mathcal{T}x_n, \mathcal{T}x_n, \mathcal{T}z). \end{aligned}$$

Now, we have

$$\begin{aligned}\mathcal{F}\Big(S(x_n,x_n,z)\Big) &\leq \mathcal{F}\Big(2S(x_n,x_n,\mathcal{T}x_n)+S(\mathcal{T}x_n,\mathcal{T}x_n,z)\Big)\\ &= \mathcal{F}\Big(2S(x_n,x_n,\mathcal{T}x_n)+S(\mathcal{T}x_n,\mathcal{T}x_n,\mathcal{T}z)\Big).\end{aligned}$$

Indeed, as $\mathcal{F}$ is continuous, then using the property of $\mathcal{F}$ and taking the limit as $n\to\infty$, we get

$$\mathcal{F}\Big(S(x_n,x_n,z)\Big) \leq \mathcal{F}\Big(S(\mathcal{T}x_n,\mathcal{T}x_n,\mathcal{T}z)\Big).$$

Now, using inequality (2.1) we obtain

$$\begin{aligned}&\mathcal{F}\Big(S(x_n,x_n,z)\Big)\leq\phi\Big\{\mathcal{F}\Big(S(x_n,x_n,z)\Big),\mathcal{F}\Big(S(x_n,x_n,\mathcal{T}x_n)\Big),\mathcal{F}\Big(S(z,z,\mathcal{T}z)\Big)\Big\}\\ &=\phi\Big\{\mathcal{F}\Big(S(x_n,x_n,z)\Big),\mathcal{F}\Big(S(x_n,x_n,\mathcal{T}x_n)\Big),\mathcal{F}\Big(S(z,z,z)\Big)\Big\}.\end{aligned}$$

Since $\mathcal{F}$ is continuous, then using the property of $\mathcal{F}$ and taking the limit as $n\to\infty$ in the above inequality, we get

$$\mathcal{F}\Big(S(x_n,x_n,z)\Big) \leq \phi\Big\{\mathcal{F}\Big(S(x_n,x_n,z)\Big),0,0\Big\}.$$

Because ϕ satisfies the condition $(R3)$ by assumption, we obtain

$$\begin{aligned}\mathcal{F}\Big(S(x_n,x_n,z)\Big) &\leq h\,\mathcal{F}\Big(S(x_n,x_n,z)\Big)\\ &\Rightarrow \mathcal{F}\Big(S(x_n,x_n,z)\Big)=0 \text{ because of } 0<h<1.\end{aligned}$$

Using the property of $\mathcal{F}$, this implies that $S(x_n,x_n,z)\to 0$ as $n\to\infty$ which is equivalent to saying that $x_n\to z$ as $n\to\infty$. This completes the proof. □

Theorem 3.6 *Let $\mathcal{T}\colon X\to X$ be a self mapping as in Theorem* **??**. *Then $\mathcal{T}$ has the limit shadowing property.*

Proof According to Theorem 2.1, we know that $\mathcal{T}$ has a unique fixed point $z=\mathcal{T}z\in X$. Let $\{x_n\}\subset X$ be such that $\lim_{n\to\infty} S(\mathcal{T}x_n,\mathcal{T}x_n,x_n)=0=\lim_{n\to\infty} S(x_n,x_n,\mathcal{T}x_n)$. Then, as in the previous proof,

$$\mathcal{F}\Big(S(x_n,x_n,z)\Big) \leq \phi\Big\{\mathcal{F}\Big(S(x_n,x_n,z)\Big),0,0\Big\}.$$

Since ϕ satisfies the condition $(R3)$, then we obtain

$$\begin{aligned}\mathcal{F}\Big(S(x_n,x_n,z)\Big) &\leq h\,\mathcal{F}\Big(S(x_n,x_n,z)\Big)\\ &\Rightarrow \mathcal{F}\Big(S(x_n,x_n,z)\Big)=0 \text{ because of } 0<h<1.\end{aligned}$$

Using the property of $\mathcal{F}$, it follows that $S(x_n,x_n,\mathcal{T}^n z)=S(x_n,x_n,z)\to 0$ as $n\to\infty$.

This completes the proof. □

§4. Conclusion

In this paper, we establish some fixed point theorems for $\mathcal{F}_{(S,\mathcal{T})}$-contraction under an implicit relation in the framework of complete S-metric spaces and obtained some well-known results as corollaries. Also, we give some examples in support of our results and one of the possible applications of our result to well-posed and limit shadowing property of fixed point problems.

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Comparable Graphs of Finite Groups

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Abstract: Let H and K be two subgroups of a finite group G. Then the pair (H, K) is called comparable in G if either H is a subgroup of K or K is a subgroup of H. For any finite group G, there is a comparable graph $CG(G)$ of G whose vertices are all subgroups $Sub(G)$ of G and in which two distinct vertices H and K are adjacent if and only if the pair (H, K) is comparable in G. The purpose of this paper is to give a general and a simple approach to describe comparable pairs in a finite group and structural properties of comparable graphs.

Key Words: Finite group, comparable pair, comparable graphs.

AMS(2010): 14G32, 19B37, 05C25, 05C75, 05C45.

§1. Introduction

It is well known that a logic in studying any algebraic structure is to consider substructures with the same structure. The strategy is that small structures should be easier to study than large ones and that by understanding enough parts of the whole structure, so we can questions and about it more easily. For this reason, the basic inter relation between the structure of the group and the corresponding structure of its subgroups constitutes at most important field of research in both modern algebra and algebraic graph theory. Many researchers generalized the graphical and design problems by defining the concept of the various algebraic graphs. It is a main research object in algebraic theory and the topological graph theory, and further it has important applications to design and network theory, see [1] and [2].

Associating algebraic graphs to subgroup structures and establishing their algebraic concepts and properties implying the algebraic methods in graph theory has been a fascinating field for modern and discrete mathematics in the last decades and consequently arousing researchers wide attention. For many group theoretic graphs, some are play most important role in the theory of codes, securities and designs. For example, directed Cayley graphs of groups [3], power graphs of groups [4], the cyclic graph of a finite group [5], the graph of subgroups of a finite group [6], inclusion graph of subgroups of a group [7], the subgroup graph of a group [8], order divisor graphs of finite groups [9], some metrical properties of lattice graphs of finite

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groups [10].

We have the algebraic system of integers modulo n, Z_n is partitioned into two disjoint non-empty subsets, in which one is $U(Z_n)$, consists only of multiplicative inverse elements called units, that is, $a \in U(Z_n)$ implies that there exists $b \in U(Z_n)$ such that $ab = ba = 1$. Other than $U(Z_n)$, there is another non-empty subset $Z(Z_n)$, that is $a \in Z(Z_n)$ means that there exists $b \in Z(Z_n)$ such that $ab = ba = 0$. These two concepts shows that $Z_n = U(Z_n) \cup Z(Z_n)$.

Multiples and divisors are two focal classes of positive integers which have appreciated incredible regard in the hypothesis of numbers. Now we turn our attention to the elements in the finite group Z_n, where $Z_n = \{0, 1, 2, \cdots, n-1\}$ and generalize the enumeration process of finding comparable pairs in Z_n. Further obtain a formula for enumerating the number of comparable pairs in Z_n where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 1$.

The main objective of this paper is to study and enumerate the combinatorial facts of the comparable pairs of a finite group. Every finite or infinite group has a corresponding comparable graph depending upon the order $|G|$ of a graph G. This algebraic graphical study helps to illuminate the comparable collection structure $Sub(G)$ of the subgroups of G.

In this paper G denotes a nontrivial ($|G| \geq 1$) finite group and $H < K$ denotes H is a subgroup of K. Our aim is to investigate the simple undirected graph $CG(G)$ which is associated with the subgroups of G. The vertices of $CG(G)$ are the subgroups of G, and we join two distinct vertices H, K, whenever either $H < K$ or $K < H$. This algebraic graph will be called the comparable graph. In this paper p and q are distinct primes, and n will always denote positive integer.

Our main aim in this paper is three fold. First, we classify all finite groups whose comparable pairs are finite and enumerated. Our second main aim is to study structural properties and to determine the diameter of $CG(G)$, denoted by $diam(CG(G))$, is bounded. Our bound 2, for $diam(CG(G))$ in the finite simple case. Finally, we describe and illustrate traversability properties of the graph $CG(G)$.

Let us consider some basic notations and definitions in the graph theory. Suppose X is a graph with vertex set $V(X)$ and edge set $E(X)$, and all graphs are simple and undirected, that are contains no loops and no multiple edges. We use the symbol K_n for the complete graph on n vertices with $\frac{n(n-1)}{2}$ edges. The number of vertices incident to the vertex x in X is called degree of x, and is denoted by $deg(x)$. Specifically, if $deg(x) = r$ for every vertex x in X, then X is called r - regular graph. Graph coloring is a simple way of labelling graph vertices with different colors. In a simple graph, no two adjacent vertices are colored with minimum number of colors, and this minimum number is called the chromatic number and the corresponding graph is called a properly colored graph. A graph X is called Eulerian if there exists a Eulerian path in which we can start at a vertex, traverse through every edge only once, and return to the same vertex where we started. A connected simple graph X is Eulerian if each vertex has even degree.

§2. Comparable Pairs in Groups

This section is concerned with the combinatorial facts of the comparable pairs in various finite groups. First we have studied subgroups and seen how to determine such pairs of subgroups when they comparable in Z_n.

Let us recall some basic definitions and notations in subgroups of a group. A nonempty subset H of a group G is called a subgroup of G if H is also a group under the same binary operation defined on G. Particularly, if $|G| = 1$ then $H = \{e\} = G$, where e is the identity element in G, is called trivial subgroup of G. If $|G| = 2$ then G has exactly two subgroups, namely $\{e\}$ and G itself.

Further, a subgroup H of G is called a non-trivial proper subgroup of G if $H \neq \{e\}$ and $H \neq G$. So, generally $H < G$ is denoted as H is a subgroup of G.

We now turn to define comparable pair in a finite group.

Definition 2.1 *Let H and K be two subgroups of a finite group G. Then the pair (H, K) is called comparable in G if either H is a subgroup of K or K is a subgroup of H.*

The set of all subgroups in a group G is denoted by $Sub(G)$ and the set of all comparable pairs in G is denoted by $\varsigma(G)$.

The following example illustrates the above definition.

Example 2.2 For the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k : i^2 = j^2 = k^2 = -1,\ ij = k, ji = -k, etc.\}$, $Sub(Q_8) = \{(1), (-1), (i), (j), (k), Q_8\}$ where $(1) = \{1\}, (-1) = \{1, -1\}, (i) = \{1, -1, i, -i\}, (j) = \{1, -1, j, -j\}, (k) = \{1, -1, k, -k\}$ and Q_8 are the subgroups of Q_8. So, we have $\varsigma(Q_8) = \{((1), (-1)),\ ((1), (i)), ((1), (j)), ((1), (k)),\ ((1), Q_8), ((-1), (i)), ((-1), (j)), ((-1), (k)), ((-1), Q_8),\ ((i), Q_8), ((j), Q_8), ((k), Q_8)\}$ and $|\ \varsigma(Q_8)\ | = 12$. Because $(1) < (-1), (1) < (i), (1) < (j), (1) < (k),\ (1) < Q_8, (-1) < (i), (-1) < (j),\ (-1) < (k), (1) < Q_8, (i) < Q_8, (j) < Q_8$ and $(k) < Q_8$.

Now we wish to find the comparable pairs in the group Z_n. We know that (u) is a subgroup of Z_n for every u in Z_n.

Lemma 2.3 *If u and v are two distinct units of the group Z_n, then $((u), (v))$ is not a comparable pair in Z_n.*

Proof Suppose $((u), (v))$ is a comparable pair in Z_n. Then $(u) < (v)$ or $(v) < (u)$. This is contradiction, because $(u) = (v)$ for any two distinct units of the group Z_n. So, our assumption is not true and hence $((u), (v))$ is not a comparable pair in Z_n. □

By Lemma 2.3 we conclude that every pair of two distinct elements u and v in $U(Z_n)$, $((u), (v))$ does not form a comparable pair. So, our required comparable pair $((u), (v))$ exists for u and v in $Z(Z_n)$ only.

Remark 2.4 *For every element $u \in U(Z_n)$, $(u) = Z_n$ and $v \in Z(Z_n)$, $(v) \subset Z_n$.*

Recently, the authors Sajana and Bharathi explored many results in [15]. The set of all elements in Z_n can be written as the disjoint union of the sets $S_d^{'}$ for all d in D, where $S_d = \{x \in Z_n : (x) = (d)\}$and the set D denotes the set of all divisors of the positive integer n. So for every non unit element in Z_n is an element in some S_d, where $d \neq 1, d \in D$.

For every positive integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 1$, the set of divisors of n is denoted by $D(n)$ and is its cardinality defined as $|D(n)| = d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_m + 1)$. Also the number of subgroups of Z_n is $d(n)$. For instance, if $n = 6 = 2^1 3^1$, then $d(6) = (1+1)(1+1) = 4$, since $D(6) = \{1, 2, 3, 6\}$.

Remark 2.5 *For any divisor d of n, (d) is a subgroup of Z_n. If x is a proper divisor of d, then (d) is a subgroup of (x) under modulo n. Similarly, if y is a proper multiple of d which is a divisor of n, then (y) is a subgroup of (d) under modulo n. This shows that for any divisor d of n, $((d), (x))$ is a comparable pair, where x is a proper divisor of d or proper multiple of d which is a divisor of n.*

Definition 2.6 *Let n be a positive integer and d be a divisor of n. Then the set of all proper multiples of d which is a divisor of n under modulo n is denoted by $M_n(d)$ with cardinality $|M_n(d)|$.*

Now, first we wish to find the number of proper multiples of a divisor d of n, which is divisor of n. This clearly gives the number of proper subgroups of the subgroup (d) in the group Z_n. Now first we can find these proper multiples by the method of induction.

Theorem 2.7 *If $n = p^\alpha, \alpha \geq 1$, then the number of proper multiples of the divisor $d = p^\beta, 0 \leq \beta \leq \alpha$ of n under modulo n is $((\alpha - \beta) + 1) - 1$.*

Proof The set of all proper multiples of the divisor d of n under modulo n is $M_n(d) = \{p^{\beta+1}, p^{\beta+2}, \cdots, p^\alpha\} = 0$. This implies that the number of proper multiples of d under modulo n is $|M_n(d)| = (\alpha - \beta) = ((\alpha - \beta) + 1) - 1$. □

Example 2.8 For $n = 8, 8 = 2^3$ and $d = 2 = 2^1$. Then the number of proper multiples of the divisor 2 of 8 under modulo 8 is $|M_8(2)| = ((3 - 1) + 1) - 1 = 2$, because $M_8(2) = \{2, 4 = 2^2, 8 = 2^3 = 0\}$.

Remark 2.9 *For $\beta = \alpha$, the number of proper multiples of the divisor $d = p^\beta, 0 \leq \beta \leq \alpha$ of $n = p^\alpha, \alpha \geq 1$ under modulo n is 0.*

For example, if $n = 8, 8 = 2^3$ and $d = 8, 8 = 2^3$, then d has no proper multiples under modulo 8.

Theorem 2.10 *If $n = p_1^{\alpha_1} p_2^{\alpha_2}, \alpha_i \geq 1, 1 \leq i \leq 2$, then the number of proper multiples of the divisor $d = p_1^{\beta_1} p_2^{\beta_2}, 0 \leq \beta_i \leq \alpha_i, 1 \leq i \leq 2$ of n under modulo n is $\prod_{i=1}^{2}((\alpha_i - \beta_i) + 1) - 1$.*

Proof The set of all proper multiples of the divisor d of n under modulo n is $M_n(d) = \{p_1^{\beta_1+1} p_2^{\beta_2}, p_1^{\beta_1+2} p_2^{\beta_2}, \cdots, p_1^{\alpha_1} p_2^{\beta_2}$ ($\alpha_1 - \beta_1$ terms), $p_1^{\beta_1} p_2^{\beta_2+1}, p_1^{\beta_1+1} p_2^{\beta_2+1}, \cdots, p_1^{\alpha_1} p_2^{\beta_2+1}$ ($\alpha_1 - \beta_1 + 1$ terms), $p_1^{\beta_1} p_2^{\beta_2+2}, p_1^{\beta_1+1} p_2^{\beta_2+2}, \cdots, p_1^{\alpha_1} p_2^{\beta_2+2}$ ($\alpha_1 - \beta_1 + 1$ terms),$\cdots$, $p_1^{\beta_1} p_2^{\alpha_2}, p_1^{\beta_1+1} p_2^{\alpha 2+2}, \cdots, p_1^{\alpha_1} p_2^{\alpha_2}$ ($\alpha_1 - \beta_1 + 1$ terms)$\}$.

Now the total number of number of proper multiples of the divisor $d = p_1^{\beta_1} p_2^{\beta_2}$ of n under modulo n is, $|M_n(d)| = (\alpha_1 - \beta_1) + (\alpha_1 - \beta_1 + 1)(\alpha_2 - \beta_2) = (\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2)$ $= ((\alpha_1 - \beta_1) + 1)((\alpha_2 - \beta_2) + 1) - 1 = \prod_{i=1}^{2}((\alpha_i - \beta_i) + 1) - 1$. □

Example 2.11 For $n = 36, 36 = 2^2 \cdot 3^2$ and $d = 6, 6 = 2^1 \cdot 3^1$. Then the number of proper multiples of the divisor 6 of 36 under modulo 36 is $|M_{36}(6)| = \prod_{i=1}^{2}((\alpha_i - \beta_i) + 1) - 1 = ((2-1)+1)((2-1)+1) - 1 = 3$ because $M_{36}(6) = \{12 = 2^2 \cdot 3,\ 18 = 2 \cdot 3^2, 2^2 \cdot 3^2 = 0\}$.

Clearly, Theorem 2.12 follows by Theorem 2.10.

Theorem 2.12 *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 1$, then the number of proper multiples of the divisor $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}, m \geq 1, 0 \leq \beta_i \leq \alpha_i, 1 \leq i \leq m$ of n under modulo n is $\prod_{i=1}^{m}((\alpha_i - \beta_i)+1)-1$.*

Theorem 2.13 *In the group $Z_n, n > 0$ and y is a divisor of n, then the number of subgroups which are comparable to the subgroup (y) in Z_n is $|D(y)| + |M_n(y)| - 1$.*

Proof By the Remark 2.5, clearly for any divisor y of n, (y) is a subgroup of Z_n. If x is a proper divisor of y, then (y) is a subgroup of (x) under modulo n. Similarly, if x is a proper multiple of y which is a divisor of n, then (x) is a subgroup of (y) under modulo n. This shows that for any divisor y of n, $((y), (x))$ is a comparable pair, where x is a proper divisor of y or proper multiple of y which is a divisor of n. Therefore the number of subgroups in Z_n which are comparable to the subgroup (y) is $|D(y)| - 1 + |M_n(y)| = |D(y)| + |M_n(y)| - 1$. □

Example 2.14 For $n = 6, 6 = 2.3$ and $d = 2$, then the number of subgroups which are comparable to the subgroup (2) in Z_6 is $|D(2)|+|M_6(2)|-1 = 2+1-1 = 2$, where $D(2) = \{1\}$, $M_6(2) = \{6 \equiv 0(mod 6)\}$. Clearly, $(1) = Z_6$ and $(0) = \{0\}$ are two subgroups which are comparable to the subgroup $(2) = \{0, 2\}$ in Z_6, because $(2) < (1)$ and $(0) < (2)$.

Theorem 2.15 *The number of comparable pairs of subgroups of a finite group $Z_n, n > 0$ is $\frac{1}{2}\Sigma_{y|n}(|D(y)| + |M_n(y)| - 1)$.*

Proof By the Theorem 2.13, the number of subgroups which are comparable to the subgroup (y) in Z_n is $|D(y)|+|M_n(y)|-1$. So twice the number of comparable pairs of subgroups of a finite group Z_n is equal to the sum of the number of subgroups which are comparable to every subgroup (y) in Z_n. Therefore, the proof follows. □

Example 2.16 For $n = 6, 6 = 2.3$, the number of comparable pairs of subgroups of a finite group Z_6 is $\frac{1}{2}\Sigma_{y|6}(|D(y)| + |M_6(y)| - 1) = \frac{1}{2}[(|D(1)| + |M_6(1)| - 1) + (|D(2)| + |M_6(2)| - 1) + (|D(3)|+|M_6(3)|-1)+(|D(6)|+|M_6(6)|-1)] = \frac{1}{2}[(1+3-1)+(2+1-1)+(2+1-1)+(4+0-1)] = \frac{1}{2}[10] = 5$. These comparable pairs are $((0),(2)), ((0),(3)), ((0),(1)), ((2),(1))$ and $((3),(1))$, where $(1) = Z_6$.

§3. Structural Properties of Comparable Graphs of Groups

In this section G denotes a non-trivial finite group and $H < K$ denotes H is a subgroup of K. Our aim is to consider and we study their structural properties of the simple undirected graph $CG(G)$ which is associated with the subgroups of G. In this paper p and q are distinct primes, and n will always denote positive integer.

Definition 3.1 *For a finite group G, the comparable graph $CG(G)$ is a simple undirected graph* *whose vertex set is* $Sub(G)$*, subgroups of* G *and we join two distinct vertices* H_i, H_j *in* $CG(G)$*, whenever* $H_i < H_j$ *or* $H_j < H_i$.

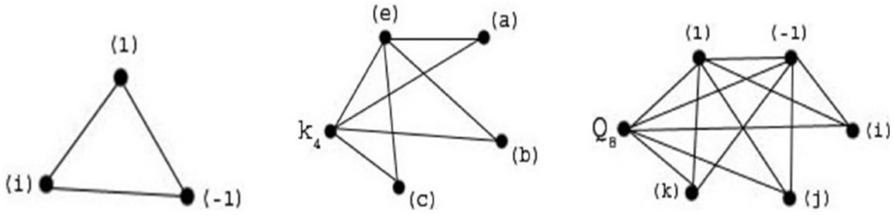

Figure 1. The comparable graph of fourth roots of unity $\{1, i\}$ Kleins four group K_4 and quaternion group Q_8

If G is a finite cyclic group, then it is easy to check that $CG(G)$ has exactly two vertices if and only if G is isomorphic to cyclic group of order p. Another specific trivial result is that the comparable graph of a group G is connected if and only if $|G| > 1$. Hence, for all non elementary finite cyclic p-group, graph $CG(G)$ is complete.

If $|G|$ is not a power of prime and if G is a non-Abelian group then every comparable graph of G is not complete. For instance, $CG(Q_8)$ is not complete. Therefore, we conclude that $CG(G)$ is complete if and only if G is any finite cyclic group of prime power order.

Before proving structured properties of comparable graphs, we introduce the appropriate distance notion of these graphs. Let H_i and H_j be two distinct vertices of the graph $CG(G)$. If either $H_i < H_j$ or $H_j < H_i$ then we say that H_i and H_j are always adjacent in $CG(G)$ and we write $H_i - H_j$ or equivalently, (H_i, H_j). Further, we say that H_i and H_j are connected; and we write $H_i - H_j$, if there exists a finite path in $CG(G)$ between them; otherwise they will be called disconnected Vertices. A comparable graph will be called disconnected if it contains at least two disconnected vertices. Otherwise it will be called connected. If $H_i = H_j$ then we define $d(H_i, H_j) = 0$. Thus $d(H_i, H_j) \leq n - 1$ for some positive integer n will mean that either $H_i = H_j$ or $H_i \neq H_j$ and they can be connected by a path of length up to $n - 1$. Clearly, $d(H_i, H_j) = 1$ if and only if (H_i, H_j) is an edge in $CG(G)$. If H_i and H_j are disconnected then $d(H_i, H_j) = \infty > n - 1$ for all n. If $CG(G)$ is connected then $diam(CG(G))$ denotes its diameter.

We wish to show that if G is a finite group, then $CG(G)$ is connected and the $diam(CG(G))$ is bounded, and its bound is 2. First we prove the following theorem.

Theorem 3.2 *Let* G *be a finite group. Then* $|G| > 1$ *if and only if* $CG(G)$ *is connected.*

Proof If G is a group of prime order, then the theorem certainly holds. Similarly, by the definition of comparable graph, the theorem also holds if $|G| = p^n$. So we may assume that G is a finite group and $|G| \neq p^n$. The distance in $CG(G)$ will be denoted by d. We shall prove each item separately.

(i) It is obvious since (e) is a subgroup of each non-trivial subgroup H of G, so the vertex (e) is adjacent with remaining all the vertices of $CG(G)$.

(ii) Suppose that the result is false, and the subgroups (e) and G itself of G satisfying

$d((e), G) > 2$. Then there exists non-trivial proper subgroups $H_1, H_2, \cdots, H_k$ of G such that either each H_i is a subgroup of H_j or each H_j is a subgroup of H_i or each H_i is not a subgroup of H_j for $1 \leq i \neq j \leq k$.

Let H_i is not a subgroup of H_j (or H_j is not a subgroup of H_i) for each $i \neq j$. Then $d((e), G) \leq d((e), H_i) + d(H_j, G) \leq 1 + 1 = 2$, a contradiction. Therefore, H_i is adjacent to H_j. So $(e) - H_1 - H_2 - \cdots - H_k - G$ in $CG(G)$. This implies that $o(e) \mid o(H_1)$, $o(H_1)|o(H_2), \cdots$, $o(H_{k-1})|o(H_k)$ and $o(H_k)|o(G)$. Therefore, order of G is a power of a single prime, which is a contradiction to the fact that $|G| \neq p^n$. The proof of the theorem is complete. □

For example, Figure 1 shows that the Comparable graphs of Fourth roots of unity $\{\pm 1, \pm i\}$, Kleins four group K_4 and Quaternion group Q_8 are connected.

The following results are immediate consequences of the above theorem.

Corollary 3.3 *Let G be a finite group of $|G| > 1$. Then there are at least two vertices of $CG(G)$ which are adjacent to each and every other vertex.*

Obviously, for any finite group G the vertices (e) and G of $Sub(G)$ are adjacent to remaining all the vertices in the comparable graph $CG(G)$.

Corollary 3.4 *The comparable graph $CG(G) \cong K_2$ if and only if G is a group of order p.*

Remark 3.5 *If $|G| \neq p$ then $|Sub(G)| > 2$, where p is a prime. This implies that*

$$|E(CG(G))| > 2.$$

The following result associates the set of comparable pairs in G and cycles of length three in $CG(G)$.

Theorem 3.6 *For any finite group G with $|G| \neq p$, the comparable graph $CG(G)$ has at least one cycle of length* 3.

Proof Suppose $|G| \neq p$. Then there exists a non trivial subgroup H of G such that the vertex H is adjacent with (e) and G in $CG(G)$. Hence we have the cycle $(e) - H - G - (e)$ of length 3 in $CG(G)$. □

In the light of the above Theorem 3.2 the following result is clear. For finite cyclic groups we have the following necessary and sufficient condition for completeness of comparable graphs.

Theorem 3.7 *For any finite cyclic group G of order n, the comparable graph $CG(G)$ is complete if and only if no two non trivial proper subgroup of G are of relatively prime orders.*

Proof Suppose that $CG(G)$ is a complete graph of a cyclic group G of order n. Then any two vertices H_i and H_j are adjacent in $CG(G)$, $i \neq j$. Consequently, either $H_i < H_j$ or $H_j < H_i$. This implies that, by the Lagranges theorem [11] for finite groups, either $o(H_i)$ divides $o(H_j)$, or $o(H_j)$ divides $o(H_i)$. Hence no two non trivial proper subgroups of G are of relatively prime orders.

Conversely, suppose no two non trivial proper subgroups of a cyclic group G are of relatively prime orders, that is, $gcd(o(H_i), o(H_j)) \neq 1$ for $i \neq j$. Then we claim that $CG(G)$ is complete. Assume that $CG(G)$ is not complete. There exists two vertices H_i and H_j in $CG(G)$ such that (H_i, H_j) and (H_j, H_i) are not comparable graphs. This implies that either $o(H_i) \nmid o(H_j)$ and $o(H_j) \nmid o(H_i)$, or $o(H_i)|o(H_j)$ and $o(H_j)|o(H_i)$.

Case 1. Suppose $o(H_i) \nmid o(H_j)$ and $o(H_j) \nmid o(H_i)$. Then, clearly $gcd(o(H_i), o(H_j)) = d, d \geq 1$. But by the hypothesis, $gcd(o(H_i), o(H_j)) \neq 1$. Therefore, $gcd(o(H_i), o(H_j)) = d, d > 1 \Rightarrow gcd(\frac{o(H_i)}{d}, \frac{o(H_j)}{d})) = 1 \Rightarrow \frac{o(H_i)}{d}|n$ and $\frac{o(H_j)}{d}|n \Rightarrow$ There exists other subgroups $H_i^{'}$ and $H_j^{'}$ of a cyclic group G with distinct orders $\frac{o(H_i)}{d}$ and $\frac{o(H_j)}{d}$, respectively, which is a contradiction to the fact that no two non trivial proper subgroups of G be of relatively prime orders. So, in this case, $CG(G)$ is complete.

Case 2. Suppose $o(H_i)|o(H_j)$ and $o(H_j)|o(H_i)$. Then, clearly $o(H_i) = o(H_j)$, which is also a contradiction to our assumption that H_i and H_j are distinct proper subgroups of a cyclic group G of relatively prime order.

From the above two cases, it is clear that $CG(G)$ is complete.

Generally speaking, the collection $Sub(G) = \{(0), (1), (p), (p^2), \cdots, (p^{k-1})\}$ be the vertex set of the graph $CG(Z_{p^k})$ and thus no two proper subgroups of Z_{p^k} are of relatively prime orders, so the Theorem 3.7 shows that the graph $CG(Z_{p^k})$ is complete. Thus, the comparable graph of a cyclic p-group is isomorphic to the group Z_{p^k}. Further, if G is a finite p-group which is not a cyclic group, then G is either Abelian or non-Abelian. So there exists at least one non comparable pair (H, K) in G if and only if G is a non cyclic p-group. For instance, $((a), (b)), ((b), (c))$ and $((c), (a))$ are non-comparable pairs in the group K_4 of order p^2, where $p = 2$. So, the following results are immediate consequences of this information. □

Corollary 3.8 *Let G be a non cyclic p-group. Then $CG(G)$ is not a complete graph.*

Corollary 3.9 *The graphs $CG(Z_p)$ and $CG(Z_p)$ are respectively*

(1) $CG(Z_p) \cong K_2$;

(2) $CG(Z_{p^k}) \cong K_{n+1}$.

Now we state the following equivalent theorem due to connectedness of comparable graphs. The proof of the following theorem is essentially cleared the comparable pair $((e), G)$ for any group G with $|G| > 1$.

Theorem 3.10 *The comparable graph $CG(G)$ of G is always connected with diameter at most 2.*

By the above theorem, for given two positive integers $m, n > 1$, the comparable graph $CG(Z_m \times Z_n)$ of the group $Z_m \times Z_n$ is always connected but not complete. For this connection, the comparable graph $CG(G \times G)$ is connected but not complete for any finite group G, because $G \times G$ is not a cyclic group. For instance, $CG(Z_4)$ is a 2-regular graph and hence complete, but $CG(Z_2 \times Z_2)$ is connected and not 2-regular graph. Further, we observe that the complement of $CG(G)$, for any finite group G is not connected because $deg((e)) = |G|-1$ and $deg(G) = |G|-1$, and thus the complement of $CG(G)$ contains at least two disconnected components.

We have the following illustrations, where the comparable graph of a group G is connected but its corresponding complement graph is not connected.

Example 3.11 The comparable graphs of groups $Z_2 \times Z_2$ and K_4 are both connected, but it is simple to see that their complements are not connected.

In the modern mathematical field of graph theory, the most important graph is a bipartite graph. It is an undirected simple graph G whose vertices can be divided into two disjoint non empty subsets A and B such that every edge adjacent a vertex in A to one vertex in B, and the pair (A, B) is called bipartitions of the graph G. However, a graph G is called bipartite if G does not contain any cycle of odd length. In recent years, bipartite graphs are extensively applied in algebraic coding theory, cloud computing and used in biological systems [12], [13] and [14].

Theorem 3.12 *The comparable graph $CG(G)$ is bipartite if and only if $|G| = p$, a prime.*

Proof It is clear from the well known result that $|G| = p$ if and only if $CG(G) \cong K_2$. □

Theorem 3.13 *Let G be a group of Composite order. Then $CG(G)$ is not bipartite.*

Proof Consider $|G|$ is a composite order. Suppose $CG(G)$ is a bipartite graph. Then there is a bipartition (A, B). Without loss of generality we may assume that $(e) \in A$ and $G \in B$. Since $|G|$ is composite, by the Cauchys theorem [11] for finite groups, there exists a non trivial proper subgroup H of G such that $((e), H)$ and (H, G) are both comparable pairs in G. This implies that H is lies in both A and B. Thus, there exists an odd cycle $(e) - H - G - (e)$ in $CG(G)$. This violates the bipartite graph. Hence $CG(G)$ is not a bipartite graph. □

Corollary 3.14 *Let H be any vertex of a comparable graph $CG(G)$. Then $deg(H) \geq 1$ in $CG(G)$ if and only if $|G| > 1$.*

The following theorem is follows from the Theorem 2.13.

Theorem 3.15 *For any vertex (x) of a comparable graph $CG(Z_n), n > 0$ and x is a divisor of n, then $deg((x)) = |d(x)| + |M_n(x)|$.*

Theorem 3.16([16]) *For any Graph G, $\Sigma_{v \in V(G)} deg(v) = 2E(G)$, where v is a vertex and $E(G)$ is the total number of vertices.*

Theorem 3.17 *The size of the comparable graph $CG(Z_n), n > 0$ is $\frac{1}{2}\Sigma_{y|n}(|d(y)| + |M_n(y)|)$.*

Proof Proof follows from Theorems 3.15 and 3.16. □

§4. Traversability Properties of Comparable Graphs

Now we turn to study the characterization of Eulerian comparable graphs. First, we recall that an undirected simple Eulerian graph has two common arrangements in Graph theory. One is a connected graph with an Eulerian cycle, and the other one is a connected graph with every vertex of even degree. These two concepts coincide for connected graphs, see [17] and this equivalent concept is known as Eulers Theory. It is called the characterization of Euler graph. Hence, a connected graph is called Eulerian if and only if its every vertex has even degree.

In light of the argument above, the following theorem is particular for the comparable graphs of groups.

Theorem 4.1 *The comparable graph $CG(G)$ is an Euler graph if and if the graph G has odd number of subgraphs.*

Proof Suppose that the comparable graph $CG(G)$ is an Euler graph whose order is $r > 2$. Then each vertex H in $CG(G)$ has even degree. In particular, the vertex (e) has the degree $r-1$ since the vertex (e) is adjacent to remaining all vertices of $CG(G)$ and thus $r-1$ is even. Consequently, r must be odd. This shows that $CG(G)$ has odd number of vertices, and hence the group G has odd number of subgroups.

Consequently, assume that G has odd number of subgroups. Then $CG(G)$ is either complete or not complete. Consider the following two cases.

Case 1. Let $CG(G)$ be a complete graph with r vertices. Then $CG(G)$ is $(r-1)$- regular graph. This implies that $r-1$ must be even and thus $CG(G)$ is an Euler graph.

Case 2. Let $CG(G)$ be not a complete graph. By the Theorem 3.2, it is a connected graph with $deg((e)) = r-1$ and $deg(G) = r-1$. Further, to claim that $deg(H) = r-1$, where H is a non-trivial proper subgroup of G. Assume that $deg(H)$ is odd. Without loss of generality, we may assume that $deg(H) = 3k$ for some positive integer k. This implies that the vertex H is adjacent to the vertices $(e), G$ and K, where K is another non-trivial proper subgroups of G. Therefore, $|Sub(G)|$ is k, and thus G has even number of subgroups, which is a contradiction to our hypothesis that G has an odd number of subgroups. This completes the proof. □

The following result associates the set $Sub(G)$ of subgroups of G. The proof of the following result is clear from $deg((e)) = |Sub(G)| - 1 = deg(G)$, where (e) and G are two vertices of the graph $CG(G)$.

Theorem 4.2 *Let $|G| \neq 1$. Then $CG(G)$ is never a totally disconnected graph.*

Now we give some examples of Eulerian comparable graphs.

Example 4.3 The comparable graph $CG(Z_4)$ is a 2- regular connected graph, and hence $CG(Z_4)$ is an Eulerian comparable graph.

Example 4.4 The comparable graph $CG(K_4)$ is a non-regular connected graph but it is also an Eulerian graph because $Sub(K_4)$ contains odd number of subgroups.

Example 4.5 The comparable graph $CG(Q_8)$ is not Eulerian because $|Sub(Q_8)| = 6$.

In graph theory, there is another class of connected graphs, called Hamilton graphs. These graphs characterized by a cycle called Hamilton cycle, that is, a cycle containing each vertex of the graph. For this reason, we must clear that every Hamilton graph is traceble, and it was natural and particular that these traceble graphs would again similar attention. These attentions support the following result for comparable graphs.

Theorem 4.6 *If* $|Sub(G)| \geq 3$ *then the comparable graph* $CG(G)$ *is Hamiltonian.*

Proof By characterization of Hamiltonian graphs [18], it is enough to prove that for any two vertices H and K in the comparable graph $CG(G)$, the following inequality holds:

$$deg(H) + deg(K) \geq |Sub(G)|.$$

In anticipation of a contradiction, let us assume that there exist at least two vertices H and K in $CG(G)$ such that $deg(H) + deg(K) < |Sub(G)| \Rightarrow deg(H) + deg(K) - |Sub(G)| < 0$. Taking $deg(H) = |Sub(G)| - 1 = deg(K)$ in $CG(G)$.

We now pause to look at two concrete cases on $|Sub(G)|$.

Case 1. Suppose $|Sub(G)| = 2k$ for some positive integer $k > 1$. Then the above inequality reduces to $(2k-1) + (2k-1) - 2k < 0 \Rightarrow k < 1$, which is not true.

Case 2. Suppose $|Sub(G)| = 2k+1$ for some positive integer k. Then $2((2k-1)-1)-(2k-1) < 0$ $\Rightarrow k < \frac{1}{2}$, which is also not true.

From the above two cases, the only possible conclusion is that the characterization of Hamiltonian graphs holds good. Hence, the comparable graph $CG(G)$ is Hamiltonian. □

The proof of above theorem is all some what vague, of course, so let us look at a concrete example.

Example 4.7 Consider the cyclic group Z_{30} and the Figure 2 shown it corresponding comparable graph $CG(Z_{30})$. Because both $|Sub(Z_{30})| = 8$ and $30 = 2 \times 3 \times 5$, we see that $deg(H) + deg(K) \geq 8$ for all vertices H and K in $CG(Z_8)$. With reference to Theorem 4.6, the sequence of vertices (0) – (5) – (10) – (2) – (6) – (3) – (15) – (1) – (0) form a Hamilton cycle in $CG(Z_{30})$.

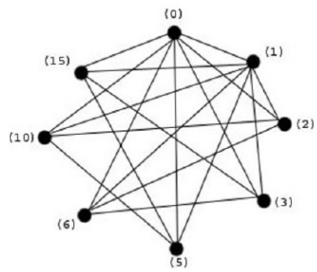

Figure 2. The Comparable Graph $CG(Z_{30})$

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Reciprocal Status-Distance Index of Mycielskian and its Complement

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Abstract: The reciprocal status-distance (RSD) index of a connected graph G is defined as

$$RSD(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{\sigma(u)+\sigma(v)}{d_G(u,v)},$$

where, $\sigma(u) = \sum_{v\in V(G)} d_G(u,v)$ is the status of a vertex u in $V(G)$. In this paper, we find RSD index of Mycielskian graphs and its complement in terms of Zagreb indices.

Key Words: Distance, status of a vertex, reciprocal status-distance index, Mycielskian graph.

AMS(2010): 05C12, 05C76, 05C90.

§1. Introduction

Consider the graph G, that has n vertices and m edges. Let's call its vertex and edge sets $V(G)$ and $E(G)$, respectively. The number of edges joining a vertex u in a graph G is indicated by $deg_G(u)$, which represents the degree of that vertex. The distance between the vertex u and v is given by $d_G(u,v)$, which is the length of the shortest path connecting u and v. The diameter of G is the largest distance between any two vertices in G and is denoted by $diam(G)$. For a graph theoretic terminology, we refer the books [4, 18].

A chemical graph is a graph in which the vertices represent atoms and the edges represent bonds between those atoms in a chemical structure. A topological index for a (*chemical*) graph G is a numerical quantity invariant under automorphisms of G and it does not depend on the labeling or pictorial representation of the graph. Topological indices and graph invariants based on the distances between vertices of a graph or vertex degrees are widely used for characterizing molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds and making their chemical applications [3].

The status [16] of a vertex $u \in V(G)$ is defined as the sum of its distance from every other

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vertex in $V(G)$ and is denoted by $\sigma(u)$. That is,

$$\sigma(u) = \sum_{v \in V(G)} d_G(u, v).$$

More results and applications on status related indices can be found in [19, 20, 25, 29, 16].

The Wiener index $W(G)$ of a connected graph G is defined as the sum of the distances between all pairs of vertices of G [35]. That is,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{u \in V(G)} \sigma(u).$$

The Wiener index is also called as gross status [16] and total status [4]. For more about the Wiener index one can refer [5, 8, 15, 28, 30, 31, 35].

The first and second Zagreb indices of a graph G are defined as [38]

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} [d_G(u) d_G(v)].$$

The Zagreb indices were used in the structure property model [13, 33]. Recent results on the Zagreb indices can be found in [6, 11, 12, 23, 27, 36].

One of the well known index called as degree distance was introduced by Dobrynin and Kochetova [2] and is defined as,

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)(d_G(u) + d_G(v)).$$

More on degree distance can be found in [2, 37].

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [10], developed an interesting graph transformation [32] as follows. For a graph $G = (V, E)$, the Mycielskian of G is the graph $\psi(G)$ (or simply, ψ) with the disjoint union $V \cup Y \cup \{y\}$ as its vertex set and $E \cup \{v_p y_q : v_q v_p \in E\} \cup \{y y_p : 1 \leq p \leq n\}$ as its edge set, where $V = \{v_1, v_2, v_3, ...v_n\}$ and $Y = \{y_1, y_2, y_3, \cdots, y_n\}$ [22]. The Mycielskian and generalized Mycielskians have fascinated graph theorists a great deal. This has resulted in studying several graph parameters of these graphs. Fisher et al. [9] determine the domination number of the Mycielskian in 1998, Taeri et al. [26] determine the Wiener index of the Mycielskian in 2012, and Ashrafi et al. [17] determine Zagreb coindices of the Mycielskian in 2012 [3].

Recently, Kishori P. N et al. introduced reciprocal status-distance index of a graph in [24] and is defined as

$$RSD(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{\sigma(u) + \sigma(v)}{d_G(u, v)},$$

where $\sigma(u)$ and $\sigma(v)$ are the status of the vertex u and v, respectively.

In this paper, determined the reciprocal status-distance index of the Mycielskian of each graph with diameter two. Also, computed the reciprocal status-distance index of the complement of Mycielskian of arbitrary graphs in terms of Zagreb indices.

§2. Reciprocal Status-Distance Index of the Mycielskian Graph

To determine the reciprocal status-distance index of Mycielskian graphs,we need following observations. Here on wards we will treat that G is a connected graph, for any vertex u of G there are $deg_G(u)$ which are at distance at 1 from u and the remaining $(n-1-deg_G(u))$ vertices are at distance at least 2. Therefore, $\sigma_u \leq deg_G(u) + (n-1-deg_G(u))$.

Let $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$, $Y = \{y_1, y_2, y_3, \cdots, y_n\}$, $V(G) \cap Y = \phi$, $y \notin V(G) \cup Y$ and ψ is the Mycielskain of G, where $V(\psi) = \{v_1, v_2, v_3, \cdots, v_n, y_1, y_2, y_3, \cdots, y_n, \{y\}\}$ and $E(\psi) = E(G) \cup \{v_p y_q : v_p v_q \in E(G)\} \cup \{y y_p : 1 \leq p \leq n\}$.

We begin with the following straight forward, previously known auxiliary result.

Observation 2.1([3]) Let ψ be the Mycielskian of G. Then for each $v \in V(\psi)$ we have

$$deg_\psi(v) = \begin{cases} n, & v = y \\ 1 + deg_G(v_p), & v = y_p \\ 2deg_G(v_p), & v = v_p \end{cases}$$

Observation 2.2([3]) In the Mycielskian ψ of G, the distance between two vertices $u, v \in V(\psi)$ are given as follows

$$d_\psi(u,v) = \begin{cases} 1, & u = y, v = y_p \\ 2, & u = y, v = y_p \\ 2, & u = y_p, v = y_q \\ d_G(v_p, v_q), & u = y_p, v = y_q, d_G(v_p, v_q) \leq 3 \\ 4, & u = v_p, v = v_q, d_G(v_p, v_q) \geq 4 \\ 2, & u = v_p, v = y_q, p = q \\ d_G(v_p, v_q), & u = y_p, v = y_q, p \neq q, d_G(v_p, v_q) \leq 2 \\ 3, & u = v_p, v = y_q, p \neq q, d_G(v_p, v_q) \geq 3 \end{cases}$$

Specially, the diameter of the Mycielskian graph is at most four [3]. There are m unordered pairs of vertices in V whose distance is one and

$$\sum_{(u,v) \in V \times V, d_G(u,v)=1} (deg_G(u) + deg_G(v)) = 2 \sum_{uv \in E(G)} (deg_G(u) + deg_G(v)) = 2M_1(G).$$

lemma 2.1([3]) *Let G be a graph of size m whose vertex set is $V = \{v_1, v_2, v_3, \cdots, v_n\}$. Then,*

$$\sum_{\{v_p,v_q\}\subseteq V} (deg_G(u) + deg_G(v)) = 2m(n-1)$$

lemma 2.2([3]) *Let G be a graph of size m. Then,*

$$\sum_{\{v_p,v_q\}\notin E(G)} (deg_G(u_p) + deg_G(v_q)) = 2m(n-1) - M_1(G).$$

Theorem 2.1 *Let G be a graph of order n and size m whose $diam(G) = 2$. If ψ is the Mycielskain of G, Then,*

$$\begin{aligned} RSD(\psi(G)) &= \frac{13}{2}n^2 - \frac{17}{2}n - 15m + 6mn \\ &+ 4(n-1)\left[H(G) + (\binom{n}{2} - m)\right] \\ &+ \binom{n}{2}(2n-3) - \left[M_1(G) + \frac{\overline{M_1}(G)}{2}\right] \end{aligned}$$

Proof By the definition of reciprocal status-distance index, we have

$$RSD(\psi(G)) = \sum_{\{u,v\}\subseteq V(\psi)} \frac{\sigma_\psi(u) + \sigma_\psi(v)}{d_\psi(u,v)}.$$

In respect of various viable cases which u and v can be taken from the set $V(\psi)$, considered the following cases. In what follows, the symbols are as before and two observations 1 and 2 are applied to determine the reciprocal status-distance of the $\psi(G)$.

Case 1. $u = y$ and $v \in Y$:

$$\begin{aligned} \sum_{p=1}^{n} \frac{\sigma_\psi(y) + \sigma_\psi(y_p)}{d_\psi(y, y_p)} &= \sum_{p=1}^{n} \sigma_\psi(y) + \sigma_\psi(y_p) \\ &= \sum_{p=1}^{n} 4(n-1) - (deg_\psi(y) + deg_\psi(y_p)) \\ &= \sum_{p=1}^{n} 4(n-1) - \sum_{p=1}^{n}(deg_\psi(y) + deg_\psi(y_p)) \\ &= 4n(n-1) - \sum_{p=1}^{n}(1 + n + deg_G(v_p)) \\ &= 4n(n-1) - (n(n+1) + 2m) \\ &= 3n^2 - 5n - 2m \end{aligned}$$

Case 2. $u = y$ and $v \in V(G)$:

$$\sum_{p=1}^{n} \frac{\sigma_\psi(y) + \sigma_\psi(v_p)}{d_\psi(y, v_p)} = \sum_{p=1}^{n} \frac{4(n-1) - (deg_\psi(y) + deg_\psi(v_p))}{d_\psi(y, v_p)}$$
$$= \sum_{p=1}^{n} \frac{4(n-1) - (n + 2deg_G(v_p))}{2}$$

$$\sum_{p=1}^{n} \frac{\sigma_\psi(y) + \sigma_\psi(v_p)}{d_\psi(y, v_p)} = \sum_{p=1}^{n} 2(n-1) - \frac{1}{2}\sum_{p=1}^{n}(n + 2deg_G(v_p))$$
$$= 2n(n-1) - \frac{1}{2}(\sum_{p=1}^{n} n + 2\sum_{p=1}^{n} deg_G(v_p))$$
$$= 2n(n-1) - \frac{1}{2}(n^2 + 2.2m)$$
$$= 2n(n-1) - \frac{1}{2}n^2 - 2m.$$

Case 3. $\{u, v\} \subseteq Y$:

Using lemma 2.1, we see that

$$\sum_{\{y_p, y_q\} \subseteq Y} \frac{4(n-1) - (deg_\psi(y_p) + deg_\psi(y_q))}{d_\psi(y_p, y_q)}$$
$$= \sum_{\{y_p, y_q\} \subseteq Y} \frac{4(n-1) - (1 + deg_G(v_p) + 1 + deg_G(v_q))}{2}$$
$$= \sum_{\{y_p, y_q\} \subseteq Y} \frac{4(n-1) - (2 + deg_G(v_p) + deg_G(v_q))}{2}$$
$$= 2\binom{n}{2}(2n-3) + \sum_{\{y_p, y_q\} \subseteq [n]} (d_G(v_p) + d_G(v_q))$$
$$= \binom{n}{2}(2n-3) + \sum_{p=1}^{n}(n-1)deg_G(v_p)$$
$$= \binom{n}{2}(2n-3) - 2m(n-1)$$

Case 4. $\{u, v\} \subseteq V(G)$. Since the diameter of G is two, Observation 2.2 implies that $d_\psi(v_p, v_q) = d_G(v_p, v_q)$. Hence,

$$\sum_{\{v_p, v_q\} \subseteq V(G)} \frac{4(n-1) - (deg_\psi(v_p) + deg_\psi(v_q))}{d_\psi(v_p, v_q)}$$
$$= \sum_{\{v_p, v_q\} \subseteq V(G)} \frac{4(n-1) - (2deg_G(v_p) + 2deg_G(v_q))}{d_G(v_p, v_q)}$$
$$= 4(n-1)H(G) - 2RSD(G).$$

Case 5. $u = v_p$ and $v = y_p$, $1 \leq p \leq n$

$$\begin{aligned}\sum_{p=1}^{n} \frac{4(n-1) - (d_\psi(v_p) + deg_\psi(y_p)}{d_\psi(v_p, y_p)} &= 2n(n-1) - \frac{1}{2}\sum_{p=1}^{n}(deg_\psi(v_p) + deg_\psi(y_p)) \\ &= 2n(n-1) - \frac{1}{2}\sum_{p=1}^{n}(2deg_G(v_p) + deg_G(v_p) + 1) \\ &= 2n(n-1) - \frac{1}{2}\sum_{p=1}^{n}(3deg_G(v_p) + 1) \\ &= 2n(n-1) - \frac{1}{2}n - 3m = 2n^2 - \frac{3}{2}n - 3m\end{aligned}$$

Case 6. $u = v_p$ and $v = y_q$, $p \neq q$

$$\begin{aligned}&\sum_{\{v_p, y_q\}_{p\neq q} \subseteq V(\psi)} \frac{4(n-1) - (deg_\psi(v_p)) + deg_\psi(y_q)}{d_\psi(v_p, y_q)} \\ &= \sum_{\{v_p, y_q, p\neq q\} \subseteq V(\psi)} \frac{4(n-1) - (2deg_G(v_p) + deg_G(v_p) + 1)}{d_\psi(v_p, y_q)} \\ &= \sum_{\{v_p, y_q\} \subseteq V(\psi)} \frac{4(n-1) - (deg_G(v_p) + 1)}{d_\psi(v_p, y_q)} \\ &- \sum_{\{v_p, y_q\} \subseteq V(\psi)} \frac{deg_G(v_p) + deg_G(y_q)}{d_\psi(v_p, y_q)}\end{aligned}$$

Since $d_\psi(v_p, y_q) = d_\psi(v_q, y_p), d_\psi(v_p, v_p) = 0$, using observation 2.2, we have

$$\sum_{\{v_p, y_q, p\neq q\} \subseteq V(\psi)} \frac{deg_G(v_p) + deg_G(v_q)}{d_\psi(v_p, y_q)} = 2 \sum_{\{v_p, y_q,\} \subseteq V(G)} \frac{deg_G(v_p) + deg_G(v_q)}{d_G(v_p, v_q)} = 2RSD(G)$$

Each edge $v_pv_q = v_qv_p \in E(G)$ corresponds to two pairs $\{v_p, y_q\}$ and $\{v_q, y_p\}$ of distance 1 in the Mycielskian graph ψ. Since the diameter of G is 2 and using Lemma 2.2, we obtain [3]

$$\begin{aligned}&\sum_{\{v_p, y_q, p\neq q\} \subseteq V(\psi)} \frac{4(n-1) - (deg_G(v_p) + 1)}{d_\psi(v_p, y_q)} \\ &= \sum_{\{v_p, y_q\} \subseteq V(\psi), v_p vq \in E(G)} (4(n-1) - (1 + deg_G(v_p))) \\ &+ \sum_{\{v_p, y_q\} \subseteq V(\psi), v_p v_q \notin E(G)} \frac{(4(n-1) - (deg_G(v_p) + 1))}{2} \\ &= \sum_{\{v_p, y_q\} \subseteq V(\psi), v_p vq \in E(G)} ((4n-5) - deg_G(v_p)) \\ &+ \sum_{\{v_p, y_q\} \subseteq V(\psi), v_p v_q \notin E(G)} \frac{(4(n-1) - (deg_G(v_p) + 1))}{2}\end{aligned}$$

$$= (4n-5)2m - \sum_{v_pv_q \in E(G)} (deg_G(v_p) + deg_G(v_q))$$

$$+ 4(n-1)(\binom{n}{2} - m) - \sum_{v_pvq \notin E(G)} \frac{deg_G(v_p) + deg_G(v_q)}{2}$$

$$= 2m(4n-5) + 4(n-1)(\binom{n}{2} - m) - M_1(G) - \frac{\overline{M_1}(G)}{2}$$

$$\sum_{\{v_p, y_q\}_{p \neq q} \subseteq V(\psi)} \frac{4(n-1) - (deg_\psi(v_p)) + deg_\psi(v_q)}{d_\psi(v_p, y_q)}$$

$$= 2m(4n-5) + 4(n-1)(\binom{n}{2} - m) - M_1(G) - \frac{\overline{M_1}(G)}{2}$$

$$+ 2RSD(G)$$

$$RSD(\psi(G)) = 3n^2 - 5n - 2m + 2n(n-1) - \frac{1}{2}n^2 - 2m + \binom{n}{2}(2n-3) - 2m(n-1)$$

$$+ 4(n-1)H(G) - 2RSD(G) + 2n^2 - \frac{3}{2}n - 3m + 2m(4n-5)$$

$$+ 4(n-1)(\binom{n}{2} - m) - M_1(G) - \frac{\overline{M_1}(G)}{2} + 2RSD(G)$$

$$RSD(\psi(G)) = \frac{13}{2}n^2 - \frac{17}{2}n - 15m + 6mn + 4(n-1)[H(G) + (\binom{n}{2} - m)] + \binom{n}{2}(2n-3)$$

$$- [M_1(G) + \frac{\overline{M_1}(G)}{2}].$$

This completes the proof. □

§3. Reciprocal Status-Distance Index of the Complement of Mycielskian

In order to determine the reciprocal status-distance index of the complement of Mycielskian graphs, we required the following two observations [3].

Observations 3.1 Let $\overline{\psi}$ be the complement of Mycielskian ψ of G, Then for each $v \in V(\overline{\psi})$ we have [3]

$$deg_{\overline{\psi}}(v) = \begin{cases} n, & v = y \\ 2n - (1 + deg_G(v_p), & v = y_p \\ 2n - 2deg_G(v_p), & v = v_p \end{cases}$$

Observations 3.2 In the complement of Mycielskian ψ of G, the distance between two vertices

$u, v \in V(\overline{\psi})$ are given as follows [3]

$$d_{\overline{\psi}}(u,v) = \begin{cases} 2, & u = y, v = y_p \\ 1, & u = y, v = v_p \\ 1, & u = y_p, v = y_q \\ 1, & u = y_p, v = y_q, d_G(v_p, v_q) > 1 \\ 2, & u = v_p, v = v_q, d_G(v_p, v_q) = 1 \\ 1, & u = v_p, v = y_q, p = q \\ 1, & u = v_p, v = y_q, p \neq q, d_G(v_p, v_q) > 1 \\ 2, & u = v_p, v = y_q, p \neq q, d_G(v_p, v_q) = 1. \end{cases}$$

Specially, the diameter of $\overline{\psi}$ is exactly 2 [3].

Theorem 3.1 *Let G be a graph of order m and size n and let $\overline{\psi}$ be the complement of the Mycielskian ψ of G. Then the reciprocal status -distance index of $\overline{\psi}$ is given by*

$$RSD(\overline{\psi}(G)) = \frac{n^2}{2} - \frac{15}{2}n + 7m + 2mn + 4(n-1)(2\binom{n}{2} - m) - 3\binom{n}{2} + 2M_1(G).$$

Proof By the definition of reciprocal status-distance index, we have

$$RSD(\overline{\psi}) = \sum_{\{u,v\} \subseteq V(\overline{\psi})} \frac{4(n-1) - (deg_{\overline{\psi}}(u) + deg_{\overline{\psi}}(v))}{d_{\overline{\psi}}(u,v)}.$$

Case 1. $u = y$ and $v \in Y$. In this case,

$$\begin{aligned} \sum_{p=1}^{n} \frac{4(n-1) - (deg_{\overline{\psi}}(y) + deg_{\overline{\psi}}(y_p))}{d_{\overline{\psi}}(y, y_p)} &= 2n(n-1) - \frac{1}{2}\sum_{p=1}^{n}(3n - deg_G(v_p) - 1) \\ &= 2n^2 - 2n - \frac{3}{2}n^2 + m - \frac{n}{2} \\ &= \frac{7}{2}n^2 - \frac{5}{2}n + m. \end{aligned}$$

Case 2. $u = y$ and $v \in V(G)$. In this case,

$$\begin{aligned} &\sum_{p=1}^{n} \frac{4(n-1) - (deg_{\overline{\psi}}(y) + deg_{\overline{\psi}}(v_p))}{d_{\overline{\psi}}(y, y_p)} \\ &= 4n(n-1) - \sum_{p=1}^{n}(3n - 2deg_G(v_p)) \\ &= 4n^2 - 4n - 3n^2 + 4m = n^2 - 4n + 4m. \end{aligned}$$

Case 3. $\{u, v\} \subseteq Y$. using Lemma 2.1 we see that

$$\sum_{\{y_p,y_q\}\subseteq V(Y)} \frac{4(n-1)-(deg_{\overline{\psi}}(y_p)+deg_{\overline{\psi}}(y_q))}{d_{\overline{\psi}}(y_p,y_q)}$$
$$= \sum_{\{y_p,y_q\}\subseteq V(Y)} 4(n-1)-(deg_{\overline{\psi}}(y_p)+deg_{\overline{\psi}}(y_q))$$
$$= 4\binom{n}{2}(n-1) - \sum_{\{p,q\}\subseteq[n]} (2n-(1+deg_G(v_p))$$
$$- \sum_{\{p,q\}\subseteq[n]} (2n-(1+deg_G(v_p)))$$
$$= 4\binom{n}{2}(n-1) - \sum_{\{p,q\}\subseteq[n]} (4n-2)$$
$$+ \sum_{\{p,q\}\subseteq[n]} (deg_G(v_p)+deg_G(v_q))$$
$$= 4\binom{n}{2}(n-1) - 4n^2 + 2n + 2m(n-1).$$

Case 4. $\{u, v\} \subseteq V(G)$. Using Lemma 2.2 we have

$$\sum_{\{v_p,v_q\}\subseteq V(G)} \frac{4(n-1)-(deg_{\overline{\psi}}(v_p)+deg_{\o\psi}(v_q))}{d_{\overline{\psi}}(v_p,v_q)}$$
$$= \sum_{v_pv_q\notin E(G)} 4(n-1)-2(deg_G(v_p)+deg_G(v_q))$$
$$+ \sum_{v_p,v_q\in E(G)} \frac{4(n-1)-(4n-2(deg_G(v_p)+deg_G(v_q)))}{2}$$
$$= 4(n-1)(\binom{n}{2} - m) - 2\overline{M_1}(G)$$
$$+ 2(n-1)m - 2mn + M_1(G).$$

Case 5. $u = v_p$ and $v = y_p$, $1 \leq p \leq n$.

$$\sum_{p=1}^{n} \frac{4(n-1)-(deg_{\overline{\psi}}(v_p)+deg_{\overline{\psi}}(y_p))}{d_{\overline{\psi}}(v_p,y_p)}$$
$$= \sum_{p=1}^{n} 4(n-1)-(4n-3deg_G(v_p)-1)$$
$$= 4n(n-1) - 4n^2 + 6m + n$$
$$= 6m - 3n$$

Case 6. $u = v_p$ and $v = y_q$, $p \neq q$. by Observation 3.2 [3], $d_{\overline{\psi}}(v_p, y_q) = d_{\overline{\psi}}(v_q, y_p)$ is 1 when

$v_pv_q \notin E(G)$,

$$\sum_{\{v_p,y_q\}\subseteq V(\psi)} \frac{4(n-1)-(deg_{\overline{\psi}}(v_p)+deg_{\overline{\psi}}(y_q))}{d_{\overline{\psi}}(v_p,y_q)}$$
$$= \sum_{(v_p,v_q),v_pv_q\notin E(G)} 4(n-1)-(4n-1-2deg_G(v_p)-deg_G(v_q))$$
$$+ \sum_{(v_p,v_q),v_p,v_q\in E(G)} \frac{4(n-1)-(4n-1-2deg_G(v_p)-deg_G(v_q))}{2}$$

and each vertex v_p can be paired with $n-1-deg_G(v_p)$ vertices as (v_p,v_q) with the condition $v_pv_q \notin E(G)$. Also note that $\sum_{(v_p,v_q)}(deg_G(v_p)+deg_G(v_q))$ is equal to $2\sum_{\{v_pv_q\}}(deg_G(v_p)+deg_G(v_q))$. Hence, using Lemma 2.2 we obtain

$$\sum_{\{v_p,y_q\}\subseteq V(\psi)} \frac{4(n-1)-(deg_{\psi}(v_p)+deg_{\psi}(y_q))}{d_{\psi}(v_p,y_q)} = a+\frac{1}{2}b,$$

where,

$$a = \sum_{(v_p,v_q),v_pv_q\notin E(G)} 4(n-1)-(4n-1-2deg_G(v_p)-deg_G(v_q))$$
$$= 4(n-1)\left(\binom{n}{2}-m\right)-(4n-1)\left(\binom{n}{2}-m\right)$$
$$+\{\sum_{v_pv_q\notin E(G)}(deg_G(v_p)+deg_G(v_q))+\sum_{(v_p,v_q)\in E(G)} deg_G(v_p)\}$$
$$= \left(\binom{n}{2}-m\right)[4n-4-4n+1]+øM_1(G)+\sum_{p=1}^{n}(deg_G(v_p))^2$$
$$= 3\left(m-\binom{n}{2}\right)+2\overline{M_1}(G)$$

$$b = \sum_{(v_p,v_q),v_pv_q\in E(G)} 4(n-1)-(4n-1-2deg_G(v_p)-deg_G(v_q))$$
$$= 4(2m(n-1))-2m(4n-1)+\sum_{v_pv_q\in E(G)}(deg_G(v_p)+deg_G(v_q))+\sum_{p=1}^{n}(deg_G(v_p))^2$$
$$= 2m(4n-4-4n+1)+2M_1(G) = 2m(-3)+2M_1(G)$$
$$= 2M_1(G)-6m$$

$$\sum_{\{v_p,y_q\}\subseteq V(\overline{\psi})} \frac{4(n-1)-(deg_{\overline{\psi}}(v_p)+deg_{\overline{\psi}}(y_q))}{d_{\overline{\psi}}(v_p,y_q)}$$

$$= 3(m - \binom{n}{2}) + 2\overline{M_1}(G) + M_1(G) - 3m$$
$$= 3m - 3\binom{n}{2} + 2\overline{M_1}(G) + M_1(G) - 3m$$
$$= 2\overline{M_1}(G) + M_1(G) - 3\binom{n}{2}$$

$$RSD(\overline{\psi}(G)) = \frac{7}{2}n^2 - \frac{5}{2}n + m + n^2 - 4n + 4m + 4\binom{n}{2}(n-1) - 4n^2 + 2n + 2m(n-1)$$
$$+ 4(n-1)(\binom{n}{2} - m) - 2\overline{M_1}(G) + 2(n-1)m - 2mn + M_1(G) + 6m - 3n$$
$$+ 2\overline{M_1}(G) + M_1(G) - 3\binom{n}{2}$$
$$RSD(\overline{\psi}(G)) = \frac{n^2}{2} - \frac{15}{2}n + 7m + 2mn + 4(n-1)(2\binom{n}{2} - m) - 3\binom{n}{2} + 2M_1(G),$$

This completes the proof. □

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Odd Vertex Equitable Even Labeling of Duplication and Product Graphs

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Abstract: Let G be a graph with p vertices and q edges and $A = \{1, 3, \cdots, q\}$ if q is odd or $A = \{1, 3, \cdots, q + 1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \cdots, 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph. In this paper, we find some new results on odd vertex equitable even labeling and establish that some standard graphs admit odd vertex equitable even labeling.

Key Words: Vertex equitable labeling, odd vertex equitable even labeling, odd vertex equitable even graph, Smarandachely k-vertex equitable labeling, Smarandachely odd k-vertex equitable even labeling.

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§1. Introduction

All graphs considered here are simple, finite and undirected. Let $G(V, E)$ be a graph with p vertices and q edges. We follow the basic notations and terminology of graph theory as in [2]. We denote the vertex set and edge set of a graph by $V(G)$ and $E(G)$ respectively. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. A pioneering paper on graph labeling problems was published in 1967 by Rosa [9]. Over the last five decades, many types of graph labeling techniques have been introduced and studied by several authors. All these graph labeling techniques are beautifully classified and updated in his survey by Gallian [1]. Vertex equitable labeling, introduced by Lourdusamy and Seenivasan [7] is one among the labelings and it is classified under miscellaneous labelings in Gallian survey.

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Motivated by the concept of vertex equitable labeling Jeyanthi, Maheswari and Vijayalakshmi [3] extended this concept and introduced a new labeling called odd vertex equitable even labeling and proved that the graphs path, $P_n \odot P_m(n, m \geq 1)$, $K_{1,n} \cup K_{1,n-2}(n \geq 3)$, $K_{2,n}$, Tp-tree, cycle $C_n(n \equiv 0 \text{ or } 1(\bmod 4))$, quadrilateral snake Q_n, ladder L_n, $L_n \odot K_1$, arbitrary super subdivision of any path P_n, are odd vertex equitable even graphs. In addition, they established that if every edge of a graph G is an edge of a triangle, then G is not an odd vertex equitable even graph. Further more, the same authors gave odd vertex equitable even labelings for cycle snake related families of graphs in [4], ladder related families of graphs in [5] and cycle related families of graphs in [6]. Lourdusamy et al. gave odd vertex equitable even labelings for quadrilateral snake related families of graphs in [8].

It is yet another study on odd vertex equitable even labeling with the objective to find some new results on odd vertex equitable even labeling. The following definitions are useful for the present study.

Definition 1.1([7]) *Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \cdots, \lceil \frac{q}{2} \rceil\}$. A graph G is said to be vertex equitable if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \cdots, q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. The vertex labeling f is known as vertex equitable labeling. A graph G is said to be a vertex equitable if it admits vertex equitable labeling.*

Generally, a vertex equitable labeling f is said to be Smarandachely k-vertex equitable labeling for any integer $k \leq |G|$ if there is a vertex subset $V' \subset V(G)$ with $|V'| = k$ with $|v_f(a) - v_f(b)| \leq 1$ in G.

Definition 1.2([3]) *A graph G with p vertices and q a edges and $A = \{1, 3, \cdots, q\}$ if q is odd or $A = \{1, 3, \cdots, q + 1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \cdots, 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph.*

Generally, an odd vertex equitable even labeling f is said to be Smarandachely odd k-vertex equitable even labeling for an integer $k \leq |G|$ if there is a vertex subset $V' \subset V(G)$ with $|V'| = k$ with $|v_f(a) - v_f(b)| \leq 1$ in G.

Clearly, if $k = |V(G)|$, a Smarandachely k-vertex equitable labeling or a Smarandachely odd k-vertex equitable even labeling f is noting else but the vertex equitable labeling of G, a Smarandachely 0-vertex equitable labeling and a Smarandachely odd 0-vertex equitable even labeling f are respectively the vertex equitable labeling or odd vertex equitable even labeling f with $|v_f(a) - v_f(b)| \geq 2$, and for integer $k' \leq k$, a Smarandache k-vertex equitable labeling f is also a Smarandache k'-vertex equitable labeling, a Smarandachely odd k-vertex equitable even labeling of G is odd vertex equitable even labeling by definition.

Definition 1.3 *The direct product of G and H is the graph denoted $G \times H$ whose vertex* *set is* $V(G) \times V(H)$ *and for which vertices* (g, h) *and* (g', h') *are adjacent precisely if* $gg' \in E(G)$ *and* $hh' \in E(H)$. *Then* $V(G \times H) = \{(g, h)/g \in V(G)$ *and* $h \in V(H)\}$, $E(G \times H) = \{(g, h), (g', h')/gg' \in E(G)$ *and* $hh' \in E(H)\}$.

Notice that the graph $P_m \times P_n$ is a disconnected graph with two components.

Definition 1.4 *The graph* $P_n \times P_2$ *is called a ladder graph.*

Definition 1.5 *Let* G *be a graph and* v *be any vertex of* G. *A new vertex* v' *is said to be duplication of* v *if all the vertices which are adjacent to* v *are adjacent to* v'. *The graph obtained by duplication* v *is denoted by* $D(G, v')$.

Definition 1.6 *A quadrilateral snake* Q_n *is a graph obtained from a path* P_n *with vertices* $u_1, u_2, \cdots, u_n$ *by joining* u_i, u_{i+1} *to the new vertices* v_i, w_i *respectively and then joining* v_i *and* w_i. *That is, every edge of the path is replaced by a cycle* C_4.

Definition 1.7 *The double quadrilateral snake* $D(Q_n)$ *is a graph obtained from a path* P_n *with vertices* $u_1, u_2, \cdots, u_n$ *by joining* u_i *and* u_{i+1} *to the new vertices* v_i, x_i *and* w_i, y_i *respectively and then joining* v_i, w_i *and* x_i, y_i *for* $i = 1, 2, \cdots, n-1$.

Definition 1.8 *The subdivision graph S(G) is obtained from G by subdividing each edge of G with a vertex.*

§2. Main Results

In this section, we prove that the graphs $D(L_n, v_1')$, $D(SL_n, v_1')$, $D(Q_n, u_1')$, $D(D(Q_n), v_1')$, $D(S(Q_n), u_1'')$ and $P_3 \times P_n$ $(n$ *is odd*, $n \geq 3)$ are odd vertex equitable even graphs.

Theorem 2.1 *The duplicate graph* $D(L_n, v_1')$ *of a ladder* L_n *is an odd vertex equitable even graph.*

Proof Let the vertex set of $D(L_n, v_1')$ be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, v_1'\}$ and the edge set of $D(L_n, v_1')$ be $\{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i/1 \leq i \leq n\} \cup \{v_1'v_2, v_1'u_1\}$. Clearly, $D(L_n, v_1')$ has $2n+1$ vertices and $3n$ edges.

Let $A = \begin{cases} 1, 3, 5, \cdots, 3n \text{ if n is odd} \\ 1, 3, 5, \cdots, 3n+1 \text{ if n is even.} \end{cases}$

Define a vertex labeling $f : V(D(L_n, v_1')) \to A$ as follows:

$f(u_1) = 1, \ f(v_1') = 1,$

$f(u_i) = \begin{cases} 3i - 1 & ; 1 \leq i \leq n, \quad i \text{ is even} \\ 3i & ; 3 \leq i \leq n, \quad i \text{ is odd} \end{cases}$ and

$f(v_i) = \begin{cases} 3i + 1 & ; 1 \leq i \leq n, \quad i \text{ is even} \\ 3i & ; 1 \leq i \leq n, \quad i \text{ is odd.} \end{cases}$

It can be verified that the induced edge labels of $D(L_n, v_1^{'})$ are $2, 4, \cdots, 6n$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(L_n, v_1^{'})$. Thus $D(L_n, v_1^{'})$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(L_4, v_1^{'})$ is shown in Figure 1.

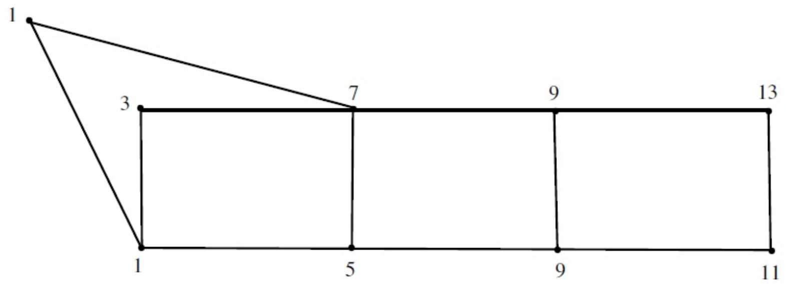

Figure 1. Odd vertex equitable even labeling of $D(L_4, v_1^{'})$

Theorem 2.2 *The duplicate graph $D(SL_n, v_1^{'})$ of a subdivision SL_n is an odd vertex equitable even graph.*

Proof Let the vertex set of $D(SL_n, v_1^{'})$ be $\{u_1, u_2, \cdots, u_n, v_1, v_2 \cdots, v_n\} \cup \{u_1^{'}, u_2^{'}, \cdots, u_{n-1}^{'}\} \cup \{w_1^{'}, w_2^{'}, \cdots, w_n^{'}\} \bigcup \{v_1^{''}, v_2^{''}, \cdots, v_{n-1}^{''}\} \cup \{v_1^{'}\}$ and the edge set of $D(SL_n, v_1^{'})$ be $\{u_i u_i^{'}, v_i v_i^{''}/1 \leq i \leq n-1\} \cup \{u_i^{'} u_{i+1}, v_i^{''} v_{i+1}/1 \leq i \leq n-1\} \cup \{u_i w_i^{'}, v_i w_i^{'}/1 \leq i \leq n\} \cup \{v_1^{'} w_1^{'}, v_1^{'} v_1^{''}\}$. Clearly, $D(SL_n, v_1^{'})$has $5n - 1$ vertices and $6n - 2$ edges.

Let $A = \{1, 3, 5, \cdots, 6n - 1\}$. Define a vertex labeling $f : V(D(SL_n, v_1^{'})) \to A$ as follows:

$f(u_1) = 5, f(v_1^{'}) = 7, f(u_2) = 9, f(v_2) = f(v_2^{''}) = 11, f(w_1) = 1, f(v_1^{''}) = 3,$

$f(u_{2i+1}) = f(w_{2i+1}) = 12i + 3$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(u_{2i+2}) = 12i + 7$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$

$f(w_{2i}) = 12i - 3$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor,$

$f(u_{2i-1}^{'}) = 12i - 5$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor,$

$f(u_{2i}^{'}) = 12i + 5$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(v_{2i-1}) = 12i - 11$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil,$

$f(v_{2i+2}) = 12i + 9$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$

$f(v_{2i+1}^{''}) = 12i + 11$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$

$f(v_{2i+2}^{''}) = 12i + 13$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1.$

It can be verified that the induced edge labels of $D(SL_n, v_1^{'})$ are $2, 4, \cdots, 12n - 4$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(SL_n, v_1^{'})$. Thus, $D(SL_n, v_1^{'})$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(SL_4, v_1^{'})$ is shown in Figure 2.

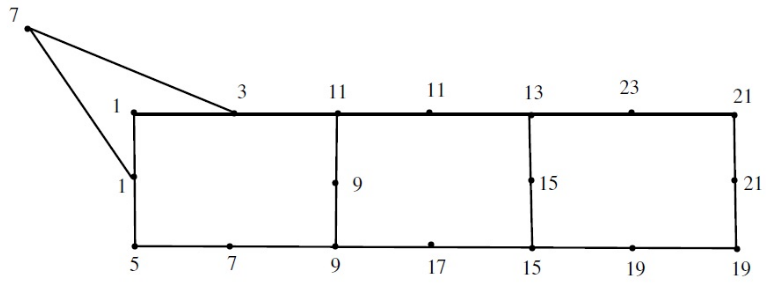

Figure 2. Odd vertex equitable even labeling of $D(SL_4, v_1^{'})$

Theorem 2.3 *The duplicate graph $D(Q_n, u_1^{'})$ of a quadrilateral Q_n is an odd vertex equitable even graph.*

Proof Let the vertex set of $D(Q_n, u_1^{'})$ be $\{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq n-1\} \cup \{u_1^{'}\}$ and the edge set of $D(Q_n, u_1^{'})$ be $\{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i, u_{i-1}w_i, v_iw_i/1 \leq i \leq n-1\} \cup \{u_1^{'}v_1, u_1^{'}u_2\}$. Clearly, $D(Q_n, u_1^{'})$has $3n-1$ vertices and $4n-2$ edges.

Let $A = \{1, 3, 5, \cdots, 4n-1\}$. Define a vertex labeling $f : V(D(Q_n, u_1^{'})) \to A$ as follows:

$f(u_1^{'}) = 3, f(v_1) = f(w_1) = 1, f(u_1) = 5,$

$f(u_{2i}) = 8i - 1$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor,$

$f(u_{2i+1}) = 8i + 3$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(v_{2i}) = 8i - 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(v_{2i+1}) = 8i + 3$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$

$f(w_{2i}) = 8i + 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(w_{2i-1}) = 8i + 5$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1.$

It can be verified that the induced edge labels of $D(Q_n, u_1^{'})$ are $2, 4, \cdots, 8n-4$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(Q_n, u_1^{'})$. Thus, $D(Q_n, u_1^{'})$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(Q_5, u_1^{'})$ is shown in Figure 3.

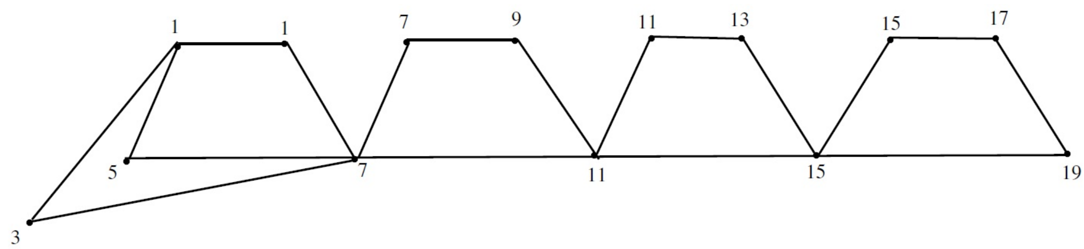

Figure 3. Odd vertex equitable even labeling of $D(Q_5, u_1^{'})$

Theorem 2.4 *The duplicate graph $D(D(Q_n), v_1^{'})$ of a quadrilateral $D(Q_n)$ is an odd vertex equitable even graph.*

Proof Let the vertex set of $D(D(Q_n), v_1^{'})$ be $\{u_i : 1 \leq i \leq n\} \cup \{v_i, x_i, w_iy_i : 1 \leq i \leq$

$n-1\} \cup \{v_1^{'}\}$ and the edge set of $D(D(Q_n), v_1^{'})$ be $\{u_iu_{i+1}/1 \le i \le n-1\} \cup \{u_{i+1}x_i, u_{i+1}y_i : 1 \le i \le n-1\} \cup \{w_ix_i, v_iy_i/1 \le i \le n-1\} \cup \{u_iv_i, u_iw_i/1 \le i \le n-1\} \cup \{v_1^{'}y_1, v_1^{'}u_1\}$. Clearly, $D(D(Q_n), v_1^{'})$has $5n-3$ vertices and $7n-5$ edges.

Let $A = \begin{cases} 1, 3, 5, \cdots, 7n-4 \text{ if n is odd} \\ 1, 3, 5, \cdots, 7n-5 \text{ if n is even.} \end{cases}$

Define a vertex labeling $f : V(D(D(Q_n), v_1^{'})) \to A$ as follows:

$f(u_1) = f(w_1) = 1, f(v_1^{'}) = 5, f(v_1) = 7, f(x_1) = 3,$

$f(u_{2i}) = 14i-5$ if $1 \le i \le \lfloor \frac{n}{2} \rfloor,$

$f(u_{2i+}) = 14i+3$ if $1 \le i \le \lceil \frac{n}{2} \rceil - 1,$

$f(v_{2i}) = 14i-1$ if $1 \le i \le \lceil \frac{n}{2} \rceil - 1,$

$f(v_{2i+1}) = 14i+5$ if $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1,$

$f(y_{2i-1}) = 14i-5$ if $1 \le i \le \lfloor \frac{n}{2} \rfloor,$

$f(y_{2i}) = 14i+1$ if $1 \le i \le \lceil \frac{n}{2} \rceil - 1,$

$f(w_{2i+1}) = 14i+3$ if $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1,$

$f(w_{2i}) = 14i-3$ if $1 \le i \le \lceil \frac{n}{2} \rceil - 1,$

$f(x_{2i}) = 14i-1$ if $1 \le i \le \lceil \frac{n}{2} \rceil - 1,$

$f(x_{2i+1}) = 14i+7$ if $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1.$

It can be verified that the induced edge labels of $D(D(Q_n), v_1^{'})$ are $2, 4, \cdots, 14n-10$ and $|v_f(i) - v_f(j)| \le 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(D(Q_n), v_1^{'})$. Thus, $D(D(Q_n), v_1^{'})$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(D(Q_4), v_1^{'})$ is shown in Figure 4.

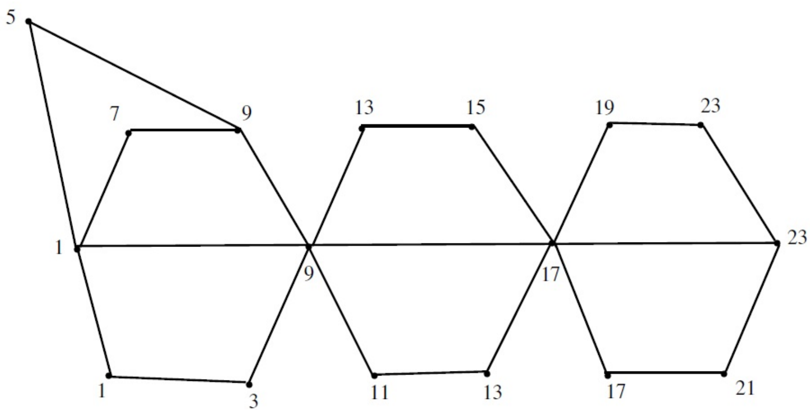

Figure 4 Odd vertex equitable even labeling of $D(D(Q_4), v_1^{'})$

Theorem 2.5 *The duplicate graph $D(S(Q_n), u_1^{''})$ of a quadrilateral $S(Q_n)$ is an odd vertex equitable even graph.*

Proof Let P_n be the path $u_1, u_2, \cdots, u_n$ and let $V(Q_n) = \{v_i, w_i : 1 \le i \le n-1\} \bigcup \{u_i : 1 \le i \le n\}$. Let the vertex set of $D(S(Q_n), u_1^{''})$ be $\{x_i, y_i, z_i, u_i^{'} : 1 \le i \le$

$n-1\} \cup \{u_1^{''}\} \bigcup V(Q_n)$ and the edge set of $D(S(Q_n), u_1^{''})$ be $\{u_iu_i^{'}, u_ix_i, u_i^{'}u_{i+1}, y_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{x_iv_i, v_iz_i, z_iw_i, w_iy_i / 1 \leq i \leq n-1\} \bigcup \{x_1u_1^{''}, u_1^{'}u_1^{''}\}$. Clearly, $D(S(Q_n), u_1^{''})$has $7n-5$ vertices and $8n-6$ edges.

Let $A = \{1, 3, 5, \cdots, 8n-1\}$. Define a vertex labeling $f : V(D(S(Q_n), u_1^{''})) \to A$ as follows:

$f(u_1) = f(x_1) = 1, f(u_2) = 11, f(u_1^{'}) = 3, f(x_2) = 11, f(v_1) = 5, f(v_2) = 13,$

$f(w_1) = 9, f(z_1) = 7, f(z_2) = 13,$

when $n > 3$, $f(u_{2i-1}) = 16i - 15$ if $2 \leq i \leq \lceil \frac{n}{2} \rceil$,

$f(u_{2i}) = 16i - 17$ if $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$f(u_{2i+1}^{'}) = 16i + 11$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(u_{2i}^{'}) = 16i + 3$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(x_{2i+1}) = 16i + 5$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(x_{2i}) = 16i - 3$ if $n > 4, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(y_{2i-1}) = 16i - 7$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$f(y_{2i}) = 16i + 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(v_{2i+1}) = 16i + 3$ if $n > 3, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(v_{2i}) = 16i - 5$ if $n > 4, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(w_{2i+1}) = 16i + 7$ if $n > 3, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(w_{2i}) = 16i - 1$ if $n \geq 3, 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(z_{2i+1}) = 16i + 7$ if $n > 3, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(z_{2i}) = 16i - 1$ if $n > 4, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1$.

It can be verified that the induced edge labels of $D(S(Q_n), u_1^{''})$ are $2, 4, \cdots, 16n - 12$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(S(Q_n), u_1^{''})$. Thus, $D(S(Q_n), u_1^{''})$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(S(Q_4), u_1^{''})$ is shown in Figure 5.

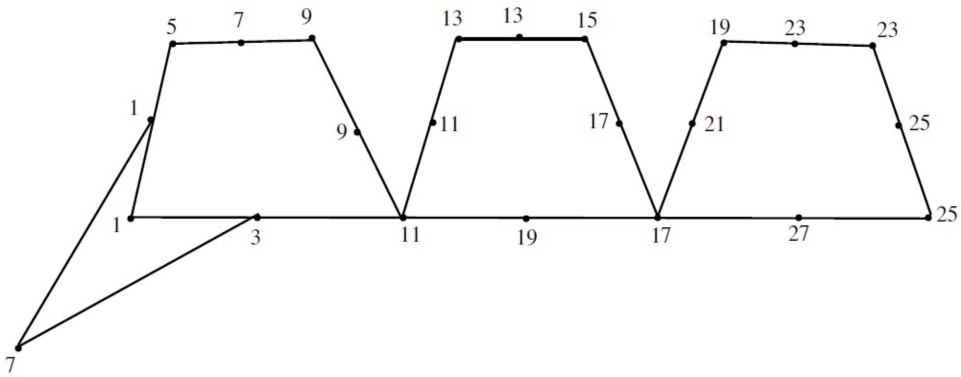

Figure 5. Odd vertex equitable even labeling of $D(S(Q_4), u_1^{''})$

Theorem 2.6 *The graph $P_3 \times P_n$ is an odd vertex equitable even graph for any $n \geq 3$ and n is odd.*

Proof Let the vertex set of $P_3 \times P_n$ be $\{u_{ij} : 1 \le i \le 3, 1 \le j \le n\}$ and the edge set of $P_3 \times P_n$ be $\{u_{1(2j)}u_{2(2j-1)}, u_{1(2j)}u_{2(2j+1)}, u_{3(2j)}u_{2(2j-1)}, u_{3(2j)}u_{2(2j+1)} : 1 \le j \le \lfloor \frac{n}{2} \rfloor\} \cup \{u_{11}u_{22}, u_{31}u_{22}\} \cup \{u_{1n}u_{2(n-1)}, u_{3n}u_{2(n-1)}\} \cup \{u_{1(2j+1)}u_{2(2j)}, u_{1(2j+1)}u_{2(2j+2)} : 1 \le j \le \lfloor \frac{n}{2} \rfloor - 1\} \cup \{u_{3(2j+1)}u_{2(2j)}, u_{3(2j+1)}u_{2(2j+2)} : 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1\}$. Clearly $P_3 \times P_n$ has $3n$ vertices and $4(n-1)$ edges.

Let $A = \{1, 3, 5, \cdots, 4n-3\}$. Define a vertex labeling $f : V(P_3 \times P_n) \to A$ as follows:

$$f(u_{1j}) = \begin{cases} 2j-1 & if \quad j \text{ is even} \\ 2n+2j-5 & if \quad j \text{ is odd} \end{cases},$$

$$f(u_{2j}) = \begin{cases} 2j-1 & if \quad j \text{ is odd} \\ 2n+2j-3 & if \quad j \text{ is even} \end{cases},$$

$$f(u_{3j}) = \begin{cases} 2j-3 & if \quad j \text{ is even} \\ 2n+2j-3 & if \quad j \text{ is odd} \end{cases}.$$

It can be verified that the induced edge labels of $P_3 \times P_n$ are $2, 4, \cdots, 8n-8$ and $|v_f(i) - v_f(j)| \le 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of$P_3 \times P_n$. Thus, $P_3 \times P_n$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $P_3 \times P_7$ is shown in Figure 6.

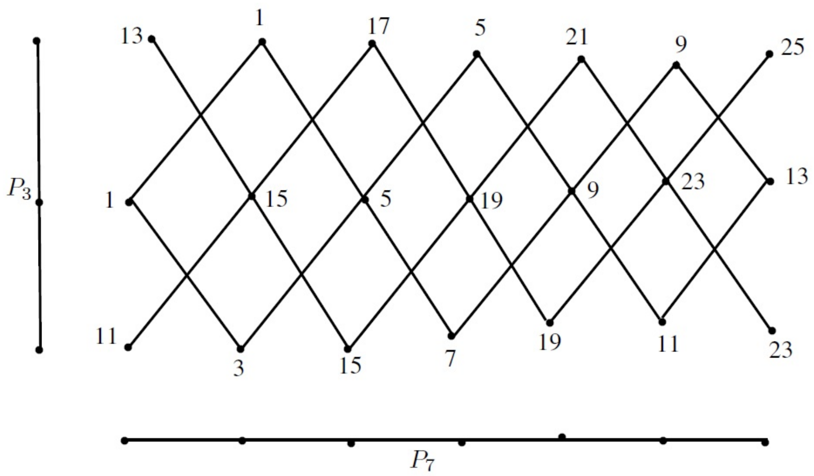

Figure 6. Odd vertex equitable even labeling of $P_3 \times P_7$

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On Pathos Block Vertex Graph of a Tree

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Abstract: A pathos block vertex graph of a tree T, written $PBV(T)$, is a graph whose vertices are the vertices, blocks (edges), and paths of a pathos of T, with two vertices of $PBV(T)$ adjacent whenever one corresponds to a block B_i of T and the other to a vertex v_j of T such that B_i is incident with v_j or the block lies on the corresponding path of the pathos; two distinct pathos vertices P_m and P_n of $PBV(T)$ are adjacent whenever the corresponding paths of the pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex in T. We study the properties of $PBV(T)$; and present the characterization of graphs whose $PBV(T)$ are planar; outerplanar; and crossing number one. We further show that for any tree T, $PBV(T)$ is not maximal outerplanar and not minimally nonouterplanar.

Key Words: Crossing number, inner vertex number, path, cycle.

AMS(2010): 05C05, 05C45.

§1. Introduction

Notations and definitions not introduced here can be found in [1]. There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graph, the total graph, and their generalizations.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.

A graph G is connected if between any two distinct vertices there is a path. A maximal connected subgraph of G is called a *component* of G. A *cut-vertex* of a graph is one whose removal increases the number of components. A *non-separable graph* is connected, non-trivial, and has no cut-vertices. A *block* of a graph is a maximal non-separable subgraph. If two distinct blocks B_1 and B_2 are incident with a common cut-vertex, then they are called *adjacent blocks*.

The *block graph* of a graph G, written $B(G)$, is the graph whose vertices are the blocks of

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G and in which two vertices are adjacent whenever the corresponding blocks have a cut-vertex in common.

The *cut-vertex graph* $C(G)$ of a graph G is the graph whose vertices are the cut-vertices of G and in which two vertices are adjacent whenever the corresponding cut-vertices lie on a common block of G.

Harary et al. [3] introduced the concept of block cut-vertex graph of a graph as follows. For a connected graph G with blocks $\{B_i\}$ and cut-vertices $\{c_j\}$, the *block cut-vertex graph* of G, denoted by $bc(G)$, is defined as the graph having vertex set $\{B_i\} \cup \{c_j\}$, with two vertices adjacent if one corresponds to a block B_i and other corresponds to a cut-vertex c_j and c_j is in B_i.

Kulli [5] introduced the concept of block-vertex tree of a graph as follows. The *block-vertex tree BV(G)* of a graph G is the graph whose vertices can be put in one-to-one correspondence with the set of vertices and blocks of G in such a way that two vertices of $BV(G)$ are adjacent if and only if one corresponds to a block B of G and the other to a vertex v of G and v is in B. Clearly, if G_1 is the graph obtained from $BV(G)$ by deleting its end vertices, then $G_1 = bc(G)$.

The following characterization of the block cut-vertex graphs is well known.

Theorem 1.1 (F. Harary and G. Prins, [3]) *A graph G is the block cut-vertex graph of some graph H if and only if it is a tree in which the distance between any two end vertices is even.*

In view of Theorem 1.1, the author in [5] will speak of the block vertex tree of a graph.

If a path P_n of order n $(n \geq 2)$ starts at one vertex and ends at a different vertex, then P_n is called an *open path*. The concept of *pathos* of a graph G was introduced by Harary [2] as a collection of minimum number of edge disjoint open paths whose union is G. The path number of a graph G is the number of paths in any pathos. The path number of a tree T equals k, where $2k$ is the number of odd degree vertices of T. A *pathos vertex* is a vertex corresponding to a path of the pathos of T.

Motivated by the studies above, we now define a new graph operator called a pathos block vertex graph of a tree.

§2. Preliminaries

A *graph* $G = (V, E)$ is a pair, consisting of some set V, the so-called *vertex set*, and some subset E of the set of all 2-element subsets of V, the *edge set*. We write $x = (p, q)$ and say that p and q are *adjacent vertices* (sometimes denoted p adj q). A graph G is *connected* if between any two distinct vertices there is a path. A *maximal connected subgraph* of G is called a *component* of G. A *cut-vertex* of a graph is one whose removal increases the number of components. A *nonseperable* graph is connected, nontrivial, and has no cut-vertices. A *block* of a graph is a maximal nonseparable subgraph.

A graph G is *planar* if it has a drawing without crossings. For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane.

If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be *outerplanar*, i.e., $i(G) = 0$.

An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing outerplanarity. A graph G is said to be *minimally nonouterplanar* if $i(G)$=1 [4]. A minimally nonouterplanar graph G is said to be *maximal minimally nonouterplanar* if no edge can be added without losing minimally nonouterplanarity. The least number of edge crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by $\mathrm{cr}(G)$.

The *Dutch Windmill graph* $D_3^{(m)}$, also called a *friendship graph*, is the graph obtained by taking m copies of the cycle graph C^3 with a vertex in common and therefore corresponds to the usual *Windmill graph* $W_n^{(m)}$. It is therefore natural to extend the definition to $D_n^{(m)}$, consisting of m copies of C^n. The *Windmill graph* $W_n^{(m)}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common.

§3. **Definition of** $PBV(T)$

A *pathos block vertex graph* of a tree T, written $PBV(T)$, is a graph whose vertices are the vertices, blocks (edges), and paths of a pathos of T, with two vertices of $PBV(T)$ adjacent whenever one corresponds to a block B_i of T and the other to a vertex v_j of T such that B_i is incident with v_j or the block lies on the corresponding path of the pathos; two distinct pathos vertices P_m and P_n of $PBV(T)$ are adjacent whenever the corresponding paths of the pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex in T.

In Figure 1, a tree T and its different pathos block vertex graph $PBV(T)$ are shown.

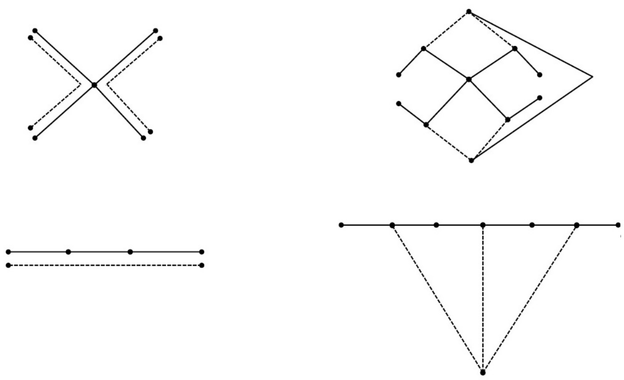

Figure 1

Note that there is freedom in marking the pathos of a tree T in different ways, provided that the path number k of T is fixed. For example, consider the marking of the pathos by dotted lines of a tree (on left) in Figure 1, where $k = 2$. Since the order of marking of the pathos of a tree is not unique, the corresponding pathos block vertex graph is also not unique. This obviously raises the question of the existence of "unique" pathos block vertex graph. One can easily check that if the path number of a tree is exactly one, i.e., k=1, then the corresponding pathos block vertex graph is unique. Since the path number of a path P_n on $n \geq 2$ vertices is one, it follows that pathos block vertex graph of a path is unique. Furthermore, for different ways of marking of pathos of a star graph $K_{1,n}$ on $n \geq 3$ vertices, the corresponding pathos block vertex graphs are isomorphic.

§4. Basic Properties of $PBV(T)$

In this section we present some of the properties of $PBV(T)$.

Property 4.1 *If v is a vertex of degree n in T, then the degree of v in $PBV(T)$ is also n. Consequently, if v is an end-vertex in T, then the corresponding vertex v in $PBV(T)$ is also an end-vertex. Therefore, $PBV(T)$ is non-eulerian and non-hamiltonian.*

Property 4.2 *The degree of every block vertex in $PBV(T)$ is three.*

Property 4.3 *Let T be a tree of order n $(n \geq 3)$. Then the number of edges whose end-vertices are the pathos vertices in $PBV(T)$ is at most $\frac{k(k-1)}{2} = \beta$ (say), where k is the path number of T. In particular, if T is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the number of edges whose end-vertices are the pathos vertices in $PBV(T)$ is exactly β, i.e., in a pathos block vertex graph of a star graph, the pathos vertices are pairwise adjacent.*

While defining any class of graphs, it is desirable to know the order and size of each; it is easy to determine for $PBV(T)$.

Proposition 4.4 *Let T be a tree with vertex set $V(T) = \{v_1, v_2, \cdots, v_n\}$ and edge (block) set $E(T) = \{e_1, e_2, \cdots, e_{n-1}\}$. Then the order and size of $PBV(T)$ are*

$$2n + k - 1 \quad and \quad 3(n-1) + \frac{k(k-1)}{2},$$

respectively, where k is the path number of T.

Proof Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$. Then the order of $PBV(T)$ equals the sum of order, size, and the path number of T. Thus $V(PBV(T)) = 2n + k - 1$. The size of $PBV(T)$ is equal to thrice the size of T and the number of edges whose end-vertices are the pathos vertices. By Property 4.3,

$$E(PBV(T)) = 3(n-1) + \frac{k(k-1)}{2}. \qquad \Box$$

§5. Characterization of $PBV(T)$

5.1 Planar Pathos Block Vertex Graphs

We now characterize the graphs whose $PBV(T)$ is planar.

Theorem 5.1 *A pathos block vertex graph $PBV(T)$ of a tree T is planar if and only if $\Delta(T) \leq 6$,* *for every vertex $v \in T$.*

Proof Suppose $PBV(T)$ is planar. Assume that $\Delta(T) > 6$, for every vertex $v \in T$. If there exists a vertex v of degree seven in T, i.e., $T = K_{1,7}$, where v is the central vertex. By definition, $BV(T)$ is a graph obtained by adjoining a pendant edge at each pendant vertex of the star graph $K_{1,7}$. Let $P(T) = \{P_1, P_2, P_3, P_4\}$ be a pathos set of T. Then $D_4^{(4)} - v_1$ is an induced subgraph of $PBV(T)$, where v_1 is a vertex at distance one from v. Clearly $\mathrm{cr}(PBV(T)) = 0$. Furthermore, the pathos vertices P_1, P_2, P_3, and P_4 of $PBV(T)$ are pairwise adjacent. This shows that $\mathrm{cr}(PBV(T)) = 1$, a contradiction.

For sufficiency, we consider the following cases.

Case 1. Suppose that T is a path of order n $(n \geq 2)$. Let $V(T) = \{v_1, v_2, \cdots, v_n\}$ and $E(T) = \{e_1, e_2, \cdots, e_{n-1}\}$ be the vertex set and edge set of T, respectively. Then $BV(T)$ is a path with edges (v_i, e_i) and (e_i, v_{i+1}) for $1 \leq i \leq n-1$. The path number of T is one, say P_1, and the corresponding pathos vertex P_1 is adjacent to every vertex e_i $(1 \leq i \leq n-1)$ of $BV(T)$. This shows that $\mathrm{cr}(PBV(T)) = 0$.

Case 2. Suppose that T is $K_{1,2}$ (or the path P_3). Then $BV(T)$ is the path P_5. The path number of T is one. Then $PBV(T)$ is a graph obtained by adjoining a pendant edge at any two consecutive vertices of the cycle C^4. Clearly $\mathrm{cr}(PBV(T)) = 0$.

Case 3. Suppose that T is a star graph $K_{1,n}$ $(3 \leq n \leq 6)$. Then $BV(T)$ is a graph obtained by adjoining a pendant edge at each pendant vertex of $K_{1,n}$. The path number of T is at most three. For $n = 3$ and 5, $D_4^{(2)} - v_1$ and $D_4^{(3)} - v_1$, respectively, are the induced subgraphs of $PBV(T)$, where v_1 is a vertex at distance one from the central vertex of $K_{1,n}$. Next, for $n = 4$ and 6, $D_4^{(2)}$ and $D_4^{(3)}$, respectively, are the induced subgraphs of $PBV(T)$. Clearly $\mathrm{cr}(PBV(T)) = 0$. Furthermore, the pathos vertices of these induced subgraphs are pairwise adjacent and does not increase the crossing number of $PBV(T)$. Thus $\mathrm{cr}(PBV(T)) = 0$.

Case 4. Suppose that the degree of each vertex of T is at most six. Then $BV(T)$ is a graph obtained by adjoining a pendant edge at each pendant vertex of T such that $\mathrm{cr}(BV(T)) = 0$. The path number of T is at least one. Then $PBV(T)$ contains either C^4 or P_2 or the product of P_2 and P_3 as subgraphs, which shows that $\mathrm{cr}(PBV(T)) = 0$. Finally, the edges joining pathos vertices of $PBV(T)$ does not increase crossing number of $PBV(T)$. Hence by all the cases above, $PBV(T)$ is planar. This completes the proof. □

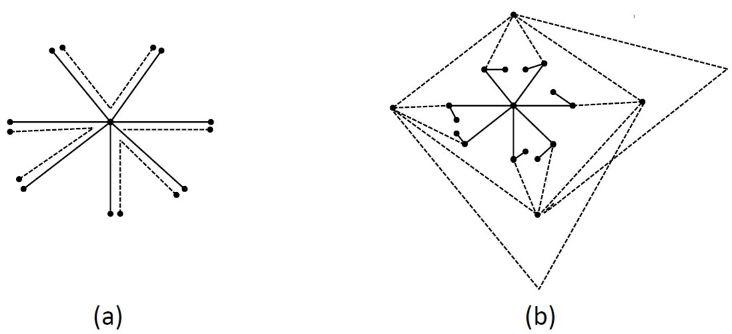

Figure 2 Star graph $K_{1,7}$ and $PBV(K_{1,7})$

Note that the path number of a star graph $T = K_{1,8}$ is four and the corresponding pathos vertices are pairwise adjacent in $PBV(T)$. This shows that the crossing number of $PBV(T)$ is one. Therefore, the necessity of Theorem 5.1 can also be proved by assuming $T = K_{1,8}$.

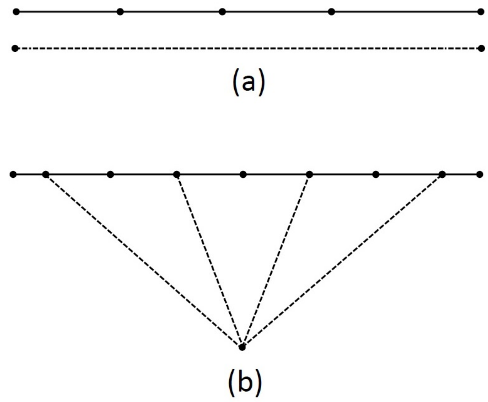

Figure 3 The path P_5 and $PBV(P_5)$

We now establish a characterization of graphs whose $PBV(T)$ are outerplanar, maximal outerplanar and minimally nonouterplanar.

Theorem 5.2 *A pathos block vertex graph $PBV(T)$ of a tree T is outerplanar if and only if T is a path of order n $(n \geq 2)$.*

Proof Suppose $PBV(T)$ is outerplanar. Assume that there exists a vertex of degree three in T, i.e., $T = K_{1,3}$. Let $P(T) = \{P_1, P_2\}$ be a pathos set of T. Then $PBV(T)$ contains $D_4^{(2)} - v_1$ as an induced subgraph. Furthermore, the pathos vertices P_1 and P_2 are adjacent. Clearly

$$i(PBV(T)) > 1,$$

a contradiction.

Conversely, suppose that T is a path of order n $(n \geq 2)$. We consider the following cases.

Case 1. Suppose that T is the path P_2. Then $PBV(T) = K_{1,3}$, which is outerplanar.

Case 2. Suppose that T is the path P_3. By Case 2 of sufficiency of Theorem 5.1, $PBV(T)$ is a graph obtained by adjoining a pendant edge at any two consecutive vertices of the cycle C^4. This shows that

$$i(PBV(T)) = 0.$$

Thus, $PBV(T)$ is outerplanar.

Case 3. Suppose that T is a path of order n $(n \geq 4)$. By definition, $BV(T)$ is a path of order $2\alpha + 5$, where $\alpha = (n - 3)$, $n \geq 4$. The path number of T is one, say P_1. Then $PBV(T)$ is a graph obtained by taking the join of alternative vertices of the path (of order $2\alpha + 5$) and P_1. This shows that

$$i(PBV(T)) = 0.$$

This completes the proof. □

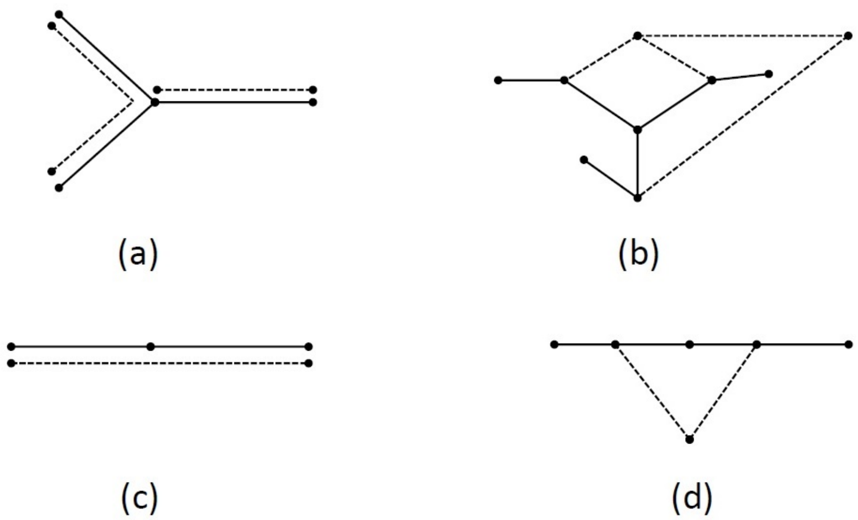

Figure 4

Theorem 5.3 (F. Harary, [1]) *Every maximal outerplanar graph G with n vertices has $2n-3$ edges.*

Theorem 5.4 *For any tree T, $PBV(T)$ is not maximal outerplanar.*

Proof We use contradiction. Suppose $PBV(T)$ is maximal outerplanar. Assume that T is a path of order n ($n \geq 2$). Then the order and size of $PBV(T)$ are $2\alpha+2$ and 3α, respectively, where $\alpha = (n-1)$, $n \geq 2$. But $3\alpha < 4\alpha + 1 = 2(2\alpha+2) - 3$. Since the size of $PBV(T)$ is 3α, Theorem 5.3 implies that $PBV(T)$ is not maximal outerplanar, a contradiction. This completes the proof. □

Theorem 5.5 *For any tree T, $PBV(T)$ is not minimally nonouterplanar.*

Proof We use contradiction. Suppose that $PBV(T)$ is minimally nonouterplanar. We consider the following three cases.

Case 1. Suppose that $\Delta(T) \geq 7$, for every vertex $v \in T$. By Theorem 5.1, $PBV(T)$ is planar, a contradiction.

Case 2. Suppose that $\Delta(T) \leq 2$, for every vertex $v \in T$. By Theorem 5.2, $PBV(T)$ is outerplanar, a contradiction.

Case 3. Suppose that $\Delta(T) \geq 3$. If there exists a vertex of degree three in T. By necessity of Theorem 5.2, $i(PBV(T)) > 1$, a contradiction. Consequently, if there exists a vertex of degree n ($4 \leq n \leq 6$), $i(PBV(T)) > 2$, again a contradiction. Hence by all the cases above, $PBV(T)$ is not minimally nonouterplanar. This completes the proof. □

Remark 5.6 *By Theorem 5.5, for any tree T, $PBV(T)$ is not minimally nonouterplanar.*

Therefore, $PBV(T)$ can never be maximal minimally nonouterplanar.

Theorem 5.7 *A pathos block vertex graph $PBV(T)$ of a tree T has crossing number one if and only if T is either $K_{1,7}$ or $K_{1,8}$.*

Proof Suppose that $PBV(T)$ has crossing number one. Assume that $T = K_{1,9}$, where v is the central vertex. By definition, $BV(T)$ is a graph obtained by adjoining a pendant edge at each pendant vertex of the star graph $K_{1,9}$. Let $P(T) = \{P_1, P_2, P_3, P_4, P_5\}$ be a pathos set of T. Then $D_4^{(5)} - v_1$ is an induced subgraph of $PBV(T)$, where v_1 is a vertex at distance one from v. Furthermore, since the pathos vertices P_1, P_2, P_3, P_4, and P_5 of $PBV(T)$ are pairwise adjacent, $\mathrm{cr}(PBV(T)) > 1$, a contradiction.

Conversely, suppose that T is either $K_{1,7}$ or $K_{1,8}$. By necessity of Theorem 5.1, the crossing number of $PBV(T)$ is one. This completes the proof. □

§6. Open Question

One can naturally extend these concepts to the directed graph version. What can one say about the properties of the directed version?

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$(1, N)$ - Arithmetic Labelling of $C_n \otimes S_m$ and $S_{m,n}$

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Abstract: A (p, q) - graph G is said to have $(1, N)$ - arithmetic labelling if there is an injection ϕ from the vertex set $V(G)$ to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, (q-1)N, (q-1)N+1\}$ such that the values of the edges, obtained as the sums of the labelling assigned to their end vertices can be arranged in the arithmetic progression $1, (N+1), (2N+1), \cdots, (q-1)N+1$. In this paper we prove that the $C_n \otimes S_m, S_{m,n}$ have $(1, N)$ - arithmetic labelling for every positive integer $N > 1$.

Key Words: $(1, N)$ - arithmetic labelling, $C_n \otimes S_m$, $S_{m,n}$.

AMS(2010): 05C78.

§1. Introduction

B.D. Acharya and S.M. Hedge [1], [2] introduced (k, d) - arithmetic graphs and certain vertex valuations of a graph. A (p, q) -graph is said to be (k, d) - arithmetic if its vertices can be assigned distinct non -negative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression $k, k+d, k+2d, \cdots, k+(q-1)d$.

Joseph A. Gallian [3] surveyed numerous graph labelling methods. V. Ramachandran and C.Sekar [4] introduced $(1, N)$ arithmetic labelling. They proved that stars, paths, complete bipartite graph $K_{m,n}$, highly irregular graph $Hi(m, m)$,Cycle C_{4k}, ladder and subdivision of ladder have $(1, N)$ - arithmetic labelling. They also proved that C_{4k+2} does not have $(1, N)$ - arithmetic labelling and no graph G containing an odd cycle has $(1, N)$ - arithmetic labelling for any integer N.

In this paper we prove that the $C_n \otimes S_m$ and $S_{m,n}$ have $(1, N)$ - arithmetic labelling.

§2. Main Results

Definition 2.1([5]) *Let G be any graph and S_m be a star with m spokes. We denote it by*

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$G \otimes S_m$ the graph obtained from G by identifying one vertex of G with any vertex of S_m other than the centre of S_m.

Definition 2.2([5]) *Let $S_{m,n}$ stand for a star with n spokes in which each spoke is a path of length m.*

Theorem 2.3 $\mathbf{C_n} \otimes \mathbf{S_m}$ *is $(1, N)$-arithmetic for $n = 4k$, $k \geq 1$,$n = 4k + 2$, $k \geq 1$ and $m \geq 1$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of C_n and $v_1, v_2, \cdots, v_m$ be the vertices of the star S_m where v_0 is the centre of the star. The graph$G = \mathbf{C_n} \otimes \mathbf{S_m}$ has $n + m$ vertices and $n + m$ edges such as those shown in Figure 1.

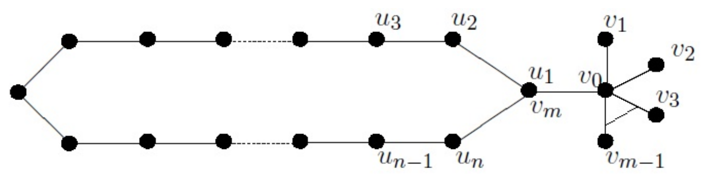

Figure 1

Case 1. $n = 4k$, $k \geq 1$, $m \geq 1$.

Define the vertex labeling mapping $\Phi : V(G) \longrightarrow \{0, 1, N, N+1, \cdots, (q-1)N, (q-1)N+1\}$ as follows:

$\Phi(v_0) = 0$,
$\Phi(v_i) = N(i-1) + 1$ for $i = 1, 2, \cdots, m$,
$\Phi(u_1) = \Phi(v_m) = N(m-1) + 1$,
$\Phi(u_{2i}) = Ni$ for $i = 1, 2, \cdots, 2k$,
$\Phi(u_{2i+1}) = N(m-1) + 1 + Ni$ for $i = 1, 2, \cdots, k-1$,
$\Phi(u_{2i+1}) = N(m-1) + 1 + N(i+1)$ for $i = k, k+1, \cdots, 2k-1$.

Clearly Φ is one-one. Also,

$\Phi^*(v_0 v_i) = N(i-1) + 1$ for $i = 1, 2, \cdots, m$,
$\Phi^*(u_i u_{i+1}) = N(m-1) + 1 + Ni$ for $i = 1, 2, \cdots, 2k-1$,
$\Phi^*(u_i u_{i+1}) = N(m-1) + 1 + N(i+1)$ for $i = 2k, 2k+1, \cdots, 4k-1$ and
$\Phi^*(u_{4k} u_1) = N(m-1) + 1 + 2kN$. Thus, the edge labellings are $1, N+1, 2N+1, \cdots, (4k + m-1)N + 1 = (q-1)N + 1$, where q denotes the number of edges.

Therefore, $\mathbf{C_n} \otimes \mathbf{S_m}$ is $(1, N)$-arithmetic in this case.

Case 2. $n = 4k + 2, k \geq 1, m \geq 1$.

In this case, define the vertex labeling mapping $\Phi : V(G) \longrightarrow \{0, 1, N, N+1, \cdots, (q-1)N, (q-1)N+1\}$ as follows:

$\Phi(v_0) = 0$,
$\Phi(v_1) = (4k+2)N + 1$,
$\Phi(v_i) = (4k+2)N + 1 + iN$ for $i = 2, \cdots, m-1$,
$\Phi(u_1) = \Phi(v_m) = 1$,

$\Phi(u_{2i}) = Ni$ for $i = 1, 2, \cdots, 2k$,
$\Phi(u_{4k+2}) = (2k+2)N$,
$\Phi(u_{2i+1}) = Ni + 1$ for $i = 1, 2, \cdots, k$,
$\Phi(u_{2i+1}) = N(i+1) + 1$ for $i = k+1, k+2, \cdots, 2k$.

Clearly Φ is one-one and also

$\Phi^*(v_0v_1) = (4k+2)N + 1$,
$\Phi^*(v_0v_i) = (4k+2)N + 1 + iN$ for $i = 2, 3, \cdots, m-1$,
$\Phi^*(v_0v_m) = 1$,
$\Phi^*(u_iu_{i+1}) = 1 + Ni$ for $i = 1, 2, \cdots, 2k+1$,
$\Phi^*(u_iu_{i+1}) = 1 + N(i+1)$ for $i = 2k+2, \cdots, 4k$,
$\Phi^*(u_{4k+1}u_{4k+2}) = (4k+3)N + 1$,
$\Phi^*(u_{4k+2}u_1) = 1 + N(2k+2)$.

Thus, the edge labellings are $1, N+1, 2N+1, \cdots, (4k+2+m-1)N+1 = (q-1)N+1$. Therefore $\mathbf{C_n} \otimes \mathbf{S_m}$ is $(1, N)$-arithmetic in this case also. □

Example 2.4 A $(1, 4)$-arithmetic labelling of $C_{16} \otimes S_{10}$ is shown in Figure 2.

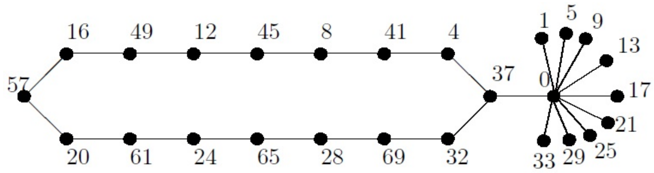

Figure 2

Example 2.5 A $(1, 7)$-arithmetic labelling of $C_{14} \otimes S_8$ is shown in Figure 3.

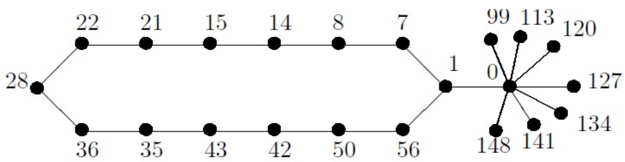

Figure 3

Theorem 2.6 *The graph $S_{m.n}$ is $(1, N)$-arithmetic for all m and n.*

Proof Let v_0 be the centre of the star. Let $v_i^{(j)}$, $1 \leq i \leq m$, $j = 1, 2, \cdots, n$ be the other vertices of the j^{th} spoke of length m. The graph $S_{m,n}$ has $mn+1$ vertices and mn edges such as those shown in Figure 4.

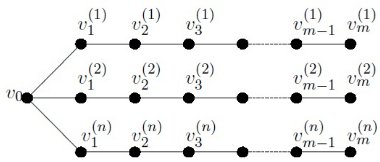

Figure 4

The proof is divided into two cases following.

Case 1. n is odd.

Let $n = 2r + 1$. In this case, define

$\Phi(v_0) = 0$,
$\Phi(v_{2i-1}^{(j)}) = (2r + 1)N(i - 1) + (j - 1)N + 1$ for $i = 1, 2, 3, \cdots$ and $j = 1, 2, \cdots, 2r + 1$,
$\Phi(v_{2i}^{(j)}) = (2r + 1)Ni + 2rN - (j - 1)N$ for $i = 1, 2, 3, \cdots$ and $j = 1, 2, \cdots, 2r + 1$.

Clearly Φ is one-one and

$\Phi^*(v_0v_1^{(j)}) = (j - 1)N + 1$ for $j = 1, 2, \cdots, 2r + 1$,
$\Phi^*(v_i^{(j)}v_{i+1}^{(j)}) = (2r + 1)Ni + 2rN - (j - 1)N + 1$ for $i = 1, 2, 3, \cdots, m - 1$ and $j = 1, 2, 3, \cdots, 2r + 1$.

Thus, the edge labellings are $1, N + 1, \cdots, ((2r + 1)m - 1)N + 1 = (q - 1)N + 1$, where q denotes the number of edges. Therefore $S_{m,n}$ is $(1, N)$-arithmetic in this case.

Case 2. n is even.

Let $n = 2r$. We divide the discuss into two subcases.

Subcase 2.1 m is even.

Let $m = 2s$. In this case, define

$\Phi(v_0) = Ns$,
$\Phi(v_{2i-1}^{(1)}) = N(s - i) + 1$ for $i = 1, 2, \cdots, s$,
$\Phi(v_{2i}^{(1)}) = N(s - i)$ for $i = 1, 2, 3, \cdots, s$,
$\Phi(v_{2i-1}^{(j)}) = (2r - 1)N(i - 1) + Ns + (j - 2)N + 1$ for$i = 1, 2, 3, \cdots$ and $j = 2, 3, \cdots, 2r$,
$\Phi(v_{2i}^{(j)}) = (2r - 1)Ni + Ns + (2r - 2)N - (j - 2)2N$ for$i = 1, 2, 3, \cdots$ and $j = 2, 3, \cdots, 2r$.

Clearly Φ is one-one and

$\Phi^*(v_0v_1^{(1)}) = (2s - 1)N + 1$,
$\Phi^*(v_i^{(1)}v_{i+1}^{(1)}) = (2s - 1 - i)N + 1$ for$i = 1, 2, \cdots, 2s + 1$,
$\Phi^*(v_0v_1^{(j)}) = 2Ns + (j - 1)N + 1$ for $j = 2, 3, \cdots, 2r$,
$\Phi^*(v_i^{(j)}v_{i+1}^{(j)}) = 2Ns + (2r - 2)N + (2r - 1)Ni - (j - 2)N + 1$ for $i = 1, 2, \cdots, 2s - 1$ and $j = 2, 3, \cdots, 2r$.

Thus, the edge labellings are $1, N + 1, \cdots, (4rs - 1)N + 1 = (q - 1)N + 1$, where q denotes the number of edges. Therefore $S_{m,n}$ is $(1, N)$-arithmetic in this Case.

Subcase 2.2 m is odd.

Let $m = 2s + 1$. In this case, define

$\Phi(v_0) = Ns + 1$,
$\Phi(v_{2i-1}^{(1)}) = N(s + 1 - i)$ for$i = 1, 2, \cdots, s + 1$,
$\Phi(v_{2i}^{(1)}) = N(s - i) + 1$ for $i = 1, 2, 3, \cdots, s$,
$\Phi(v_{2i-1}^{(j)}) = (2r-1)N(i-1)+N(s+1)+(j-2)N$ for $i = 1, 2, 3, \cdots, s+1$ and $j = 2, 3, \cdots, 2r$,
$\Phi(v_{2i}^{(j)}) = (2r-1)Ni+Ns+(2r-2)N-(j-2)2N+1$ for $i = 1, 2, 3, \cdots$ and $j = 2, 3, \cdots, 2r$.

Clearly Φ is one-one and

$\Phi^*(v_0 v_1^{(1)}) = 2Ns + 1$,

$\Phi^*(v_i^{(1)} v_{i+1}^{(1)}) = N(2s - i) + 1$ for $j = 1, 2, 3, \cdots, 2s$,

$\Phi^*(v_0 v_1^{(j)}) = N(2s + 1) + (j - 2)N + 1$ for $j = 2, 3, \cdots, 2r$,

$\Phi^*(v_i^{(j)} v_{i+1}^{(j)}) = (2s+1)N + (2r-2)N + (2r-1)Ni - (j-2)N + 1$ for $i = 1, 2, \cdots, 2s$ and $j = 2, 3, \cdots, 2r$.

Thus, the edge labellings are $1, N+1, \cdots, (4rs+2r-1)N+1$, where $q = 2r(2s+1)$ is the number of edges. Therefore $S_{m,n}$ is $(1, N)$-arithmetic in this case also. $\square$

Example 2.7 A $(1, 3)$-arithmetic labelling of $S_{6,7}$ is shown in Figure 5.

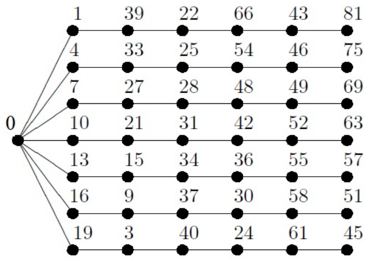

Figure 5

Example 2.8 A $(1, 5)$-arithmetic labelling of $S_{8,8}$ is shown in Figure 6.

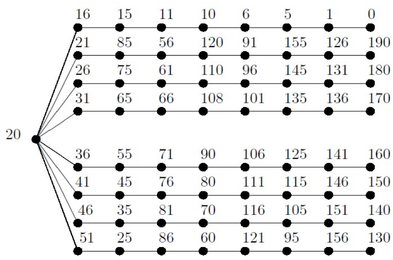

Figure 6

Example 2.9 A $(1, 6)$-arithmetic labelling of $S_{5,8}$ is shown in Figure 7.

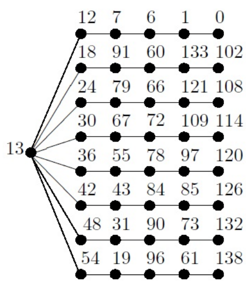

Figure 7

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AMCA's Science Manifesto

Foreword

In order to further promote the harmonious coexistence of humans with the nature, and promote the transformation of science from partial known to hold on the reality of objective things, upon the boards consideration, AMCA issues this manifesto to scientific communities for regulating scientific research, exploration and transformation of technological achievements, and prevent improper application or commercialization from harming humans.

Part 1. Nature of Science

Article 1. AMCA believes that science is the humans understanding on the state and development of objective things in the universe. It is a verifiable knowledge system constructed on the basis of local understanding or cognition based on the combination of things, including:

a. Science is the understanding on reality of objective things by humans ourselves;

b. All established science is the knowledge on one or several integral or partial characters of objective things, which is a kind of local or the global understanding, i.e., the reality of objective things;

c. Any objective thing inherits a combinatorial structure in the eyes of humans, accompanied with its development and change on time. To hold on the reality of things, it is necessary to combine the local understanding on the basis of the inherited combinatorial structure of objective things.

Article 2. AMCA believes that the purpose of science is to understanding the reality of objective things, promote the harmonious coexistence of humans with the nature and then, to serve the sustainable development of human beings, including:

a. The purpose of science is to hold on the reality or the truth of objective things. However, a lots of conclusion are the local or conditional truth on things provided by classical science.

b. The purpose of understanding the reality of objective things is to promote the sustainable development of humans with the nature, has nothing to do with the politics, religious beliefs and the social special groups, and its purpose is not for business or profit;

c. Science needs to uphold by the notion of sustainable development for recognizing the reality of objective things, namely humans and the nature constitute a 2-element system with interaction and symbiosis, and then serves humans ourselves.

[1]Received February 5, 2021, Accepted March 16, 2022.

Article 3. AMCA believes that the reality of objective things is their states of the global or local characters in the past, present or future existing objectively in the world, regardless of whether they are observed or understand by humans, namely the objective things is not subject to the willing of humans, including:

a. The objective things exist independently of human's cognition;

b. The knowledge on one or several, integral or partial characters is a kind of local or conditional understanding on objective things, which is the local truth of things under set conditions;

c. A scientific representation of reality of an objective thing is in a progressive form, which is the union of known local true, i.e, Smarandache multi-spaces or systems;

d. The development of an objective thing can be understood by humans, follows the time parameters set by humans but its change is not necessarily balanced and consistent.

Part 2. Scientific Research

Article 4. AMCA believes that all scientific researches are open and free, and is not restricted by the nature of the institution, the social status, political and religious beliefs of the researcher. At the same time, scientific research and exploration should follow the principle of non-interference with the nature, not disturb or damage to the ecology within scientific research or exploration, including:

a. Anyone can spontaneously carry out research and scientific exploration on a scientific problem;

b. Scientific research has nothing to do with social status, political and religious beliefs because it aims at understanding the reality of objective things;

c. All kinds of harmful substances and garbage produced in scientific research or exploration, including all kinds of waste gases and aircraft left over from space exploration, should be safely recycled or treated or sealed.

Article 5. AMCA believes that all scientific research are finished by those of humans dedicated to reveal the reality of objective things called scientists. Anyone who published scientific achievements in a public academic media or contribution to humans understanding on objective things is belong to this category whether he or she has an academic title, engaged in an academic organizations or institutions, including:

a. One engaged in a professional research in public or private scientific research institution or academic organization;

b. One teaching in public or private school;

c. One working in other social organizations, including government, business or public institutions as well as freelance workers.

Certainly, all scientists study the state and development of objective things, discover the reality of objective things under set conditions form science. They contribute to the coordination and symbiosis for humans with the nature.

Part 3. Scientific Achievements

Article 6. AMCA believes that science is the partial truth or reality on objective things, and the first discovery of a scientific achievement should be published in an acceptable with accessible language and form, including:

a. The widespread use or understanding of language in which the scientific achievement is recorded;

b. The data of demonstration or testing are true, and the conditions are open;

c. Others may re-demonstrate or test the published scientific achievement and confirm or accept a newly discovered scientific achievement.

The scientific achievement discovered for the first time should be disclosed to the public on paper or electronic professional media, including but not limited to professional books, monographs, academic journals or conference collections, professional newspapers,..., etc.

Article 7. AMCA believes that professional books, monographs, academic journals or conference proceedings and professional newspapers are carriers or platforms for the public disclosure of scientific achievements, and are of equal status with each other, including:

a. Do not reject a scientific achievement on the basis of peer review by an open media organization since such peer review is only the subjective judgment of a few humans;

b. The social value of a scientific achievement should not be judged by it is published in a scientific media, included in a database or a higher citation factor because its value needs to be tested by social practice, and the citation factor only partially reflects the status of a scientific research. At the same time, lots existing databases are dominated by the commercial value or profit.

Article 8. AMCA believes that an award on a scientific achievement by one or more institutions or organizations is the recognition and encouragement on a scientific achievement but does not implies also this achievement is the reality of an objective thing, including:

a. An award on a scientific achievement is the spiritual encouragement for the scientist who made this achievement;

b. The recognition of a scientific achievement by one or several institutions or organizations, including the recognition of a scientific achievement by the whole society, is still the recognition in the eyes of humans ourselves.

Article 9. AMCA believes that practice is the unique criterion for testing and accepting a scientific achievement, including:

a. The society would not accept a scientific achievement on the discoverer's family, interest group or his own academic or social status;

b. The practice is the only criterion by which scientific achievements can be tested and accepted by the society;

c. The practical tests on a scientific achievement can be repeated many times.

Article 10. AMCA believes that the evaluation of a scientific achievement is divided into the five grades following according to its contribution to the understanding on the reality of objective things:

G1. The scientific achievement is valuable to solving a problem or answer a question in a branch of a subject in science;

G2. The scientific achievement is valuable to advance a branch of a subject in science;

G3. The scientific achievement is valuable to advance a subject in science;

G4. The scientific achievement is valuable to advance the social development at a certain time;

G5. The scientific achievement is valuable for humans understanding of the nature, and the promotion of harmony and symbiosis for humans with the nature.

Part 4. Scientific Applications

Article 11. AMCA believes that the application of science should be follows a norm that "*no restricted on research fields but constraint on the application*", i.e., a scientific research can be carried out on the truth of any objective thing but the application of science should be on one criterion, i.e., benefiting humans but not harmful to the nature, including:

a. There are no restricted fields for scientific research, and a research can be carried out on any matter that reveals the truth of objective things;

b. The application of science is restricted to non-war situations that can bring benefits to humans, do not harmful to the nature also, i.e. peaceful applications.

For partial or conditionally real scientific achievements on objective things, the application norm of "*comply with all conditions*" is adhered to, namely, the application conditions should be consistent with all hypothetical conditions of a scientific achievement, and humans can control the application results, including:

a. The application conditions should consistent with those of that used for the scientific achievements;

b. The application results can be controlled by humans without bringing disasters to humans and the nature.

Article 12. AMCA believes that the scientific benefit of humans should not be at the expense of nature. All hazardous materials and wastes produced in the scientific application should be safely disposed or reused in a pollution-free manner, including:

a. The conduct harmless treatment of harmful substances accompanying with the application;

b. The safely dispose and reuse the waste produced in application;

c. Do not naturally consume harmful substances and wastes produced in scientific applications for the purpose of pursuing economic benefits.

Part 5. Code of Conduct

Article 13. AMCA believes that a scientific research and exploration should comply with the public order and good customs of society and should not be carried out in violation of social ethics or harmful to human development. At the same time, all scientists are equal in rights, regardless of their social status, institutional position, family or his or her previous contributions, including:

a. The equal right in the reveal the reality of objective things and are free to enter a certain field for research;

b. the equality in the disclosure of the first scientific achievement and freedom to choose the publishing media recognized by the society.

Article 14. AMCA believes that a scientist should be honest and trustworthy, and it is his duty to reveal the true nature of objective things. At the same time, they should respect the scientific achievements of others, and should not deceive the society by deliberately fabricating, piecing together, changing concepts, forms or copying other people's scientific achievements in violation of social norms. They would not personally attack, suppress or exclude those who hold different scientific views or similar to their own research, including:

a. Copying others' scientific achievements as his or her personal achievements;

b. Change the symbols, expressions or other formal features of others' or their own existing achievements as his new achievements;

c. Introducing the same concept from existing scientific achievements but with different disguise from those of others or individuals as his or her new achievements;

d. Steal others' scientific achievements as his own;

e. Use his personal prestige or status to suppress others in public, deliberately delay the review time or deliberately refuse to publish a achievement similar to his or her own research in the review processing.

Academy of Mathematical Combinatorics & Applications
(AMCA, USA)

February 4, 2022

Famous Words

What's in a name? That which we call a rose by any other name would smell as sweet.

By William Shakespeare, an English dramatist and writer

Author Information

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Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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