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Comparable Graphs of Finite Groups

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Abstract: Let H and K be two subgroups of a finite group G. Then the pair (H, K) is called comparable in G if either H is a subgroup of K or K is a subgroup of H. For any finite group G, there is a comparable graph $CG(G)$ of G whose vertices are all subgroups $Sub(G)$ of G and in which two distinct vertices H and K are adjacent if and only if the pair (H, K) is comparable in G. The purpose of this paper is to give a general and a simple approach to describe comparable pairs in a finite group and structural properties of comparable graphs.

Key Words: Finite group, comparable pair, comparable graphs.

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§1. Introduction

It is well known that a logic in studying any algebraic structure is to consider substructures with the same structure. The strategy is that small structures should be easier to study than large ones and that by understanding enough parts of the whole structure, so we can questions and about it more easily. For this reason, the basic inter relation between the structure of the group and the corresponding structure of its subgroups constitutes at most important field of research in both modern algebra and algebraic graph theory. Many researchers generalized the graphical and design problems by defining the concept of the various algebraic graphs. It is a main research object in algebraic theory and the topological graph theory, and further it has important applications to design and network theory, see [1] and [2].

Associating algebraic graphs to subgroup structures and establishing their algebraic concepts and properties implying the algebraic methods in graph theory has been a fascinating field for modern and discrete mathematics in the last decades and consequently arousing researchers wide attention. For many group theoretic graphs, some are play most important role in the theory of codes, securities and designs. For example, directed Cayley graphs of groups [3], power graphs of groups [4], the cyclic graph of a finite group [5], the graph of subgroups of a finite group [6], inclusion graph of subgroups of a group [7], the subgroup graph of a group [8], order divisor graphs of finite groups [9], some metrical properties of lattice graphs of finite

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groups [10].

We have the algebraic system of integers modulo n, Z_n is partitioned into two disjoint non-empty subsets, in which one is $U(Z_n)$, consists only of multiplicative inverse elements called units, that is, $a \in U(Z_n)$ implies that there exists $b \in U(Z_n)$ such that $ab = ba = 1$. Other than $U(Z_n)$, there is another non-empty subset $Z(Z_n)$, that is $a \in Z(Z_n)$ means that there exists $b \in Z(Z_n)$ such that $ab = ba = 0$. These two concepts shows that $Z_n = U(Z_n) \cup Z(Z_n)$.

Multiples and divisors are two focal classes of positive integers which have appreciated incredible regard in the hypothesis of numbers. Now we turn our attention to the elements in the finite group Z_n, where $Z_n = \{0, 1, 2, \cdots, n-1\}$ and generalize the enumeration process of finding comparable pairs in Z_n. Further obtain a formula for enumerating the number of comparable pairs in Z_n where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 1$.

The main objective of this paper is to study and enumerate the combinatorial facts of the comparable pairs of a finite group. Every finite or infinite group has a corresponding comparable graph depending upon the order $|G|$ of a graph G. This algebraic graphical study helps to illuminate the comparable collection structure $Sub(G)$ of the subgroups of G.

In this paper G denotes a nontrivial ($|G| \geq 1$) finite group and $H < K$ denotes H is a subgroup of K. Our aim is to investigate the simple undirected graph $CG(G)$ which is associated with the subgroups of G. The vertices of $CG(G)$ are the subgroups of G, and we join two distinct vertices H, K, whenever either $H < K$ or $K < H$. This algebraic graph will be called the comparable graph. In this paper p and q are distinct primes, and n will always denote positive integer.

Our main aim in this paper is three fold. First, we classify all finite groups whose comparable pairs are finite and enumerated. Our second main aim is to study structural properties and to determine the diameter of $CG(G)$, denoted by $diam(CG(G))$, is bounded. Our bound 2, for $diam(CG(G))$ in the finite simple case. Finally, we describe and illustrate traversability properties of the graph $CG(G)$.

Let us consider some basic notations and definitions in the graph theory. Suppose X is a graph with vertex set $V(X)$ and edge set $E(X)$, and all graphs are simple and undirected, that are contains no loops and no multiple edges. We use the symbol K_n for the complete graph on n vertices with $\frac{n(n-1)}{2}$ edges. The number of vertices incident to the vertex x in X is called degree of x, and is denoted by $deg(x)$. Specifically, if $deg(x) = r$ for every vertex x in X, then X is called r - regular graph. Graph coloring is a simple way of labelling graph vertices with different colors. In a simple graph, no two adjacent vertices are colored with minimum number of colors, and this minimum number is called the chromatic number and the corresponding graph is called a properly colored graph. A graph X is called Eulerian if there exists a Eulerian path in which we can start at a vertex, traverse through every edge only once, and return to the same vertex where we started. A connected simple graph X is Eulerian if each vertex has even degree.

§2. Comparable Pairs in Groups

This section is concerned with the combinatorial facts of the comparable pairs in various finite groups. First we have studied subgroups and seen how to determine such pairs of subgroups when they comparable in Z_n.

Let us recall some basic definitions and notations in subgroups of a group. A nonempty subset H of a group G is called a subgroup of G if H is also a group under the same binary operation defined on G. Particularly, if $|G| = 1$ then $H = \{e\} = G$, where e is the identity element in G, is called trivial subgroup of G. If $|G| = 2$ then G has exactly two subgroups, namely $\{e\}$ and G itself.

Further, a subgroup H of G is called a non-trivial proper subgroup of G if $H \neq \{e\}$ and $H \neq G$. So, generally $H < G$ is denoted as H is a subgroup of G.

We now turn to define comparable pair in a finite group.

Definition 2.1 *Let H and K be two subgroups of a finite group G. Then the pair (H, K) is called comparable in G if either H is a subgroup of K or K is a subgroup of H.*

The set of all subgroups in a group G is denoted by $Sub(G)$ and the set of all comparable pairs in G is denoted by $\varsigma(G)$.

The following example illustrates the above definition.

Example 2.2 For the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k : i^2 = j^2 = k^2 = -1,\ ij = k, ji = -k, etc.\}$, $Sub(Q_8) = \{(1), (-1), (i), (j), (k), Q_8\}$ where $(1) = \{1\}, (-1) = \{1, -1\}, (i) = \{1, -1, i, -i\}, (j) = \{1, -1, j, -j\}, (k) = \{1, -1, k, -k\}$ and Q_8 are the subgroups of Q_8. So, we have $\varsigma(Q_8) = \{((1), (-1)),\ ((1), (i)), ((1), (j)), ((1), (k)),\ ((1), Q_8), ((-1), (i)), ((-1), (j)), ((-1), (k)), ((-1), Q_8),\ ((i), Q_8), ((j), Q_8), ((k), Q_8)\}$ and $|\ \varsigma(Q_8)\ | = 12$. Because $(1) < (-1), (1) < (i), (1) < (j), (1) < (k),\ (1) < Q_8, (-1) < (i), (-1) < (j),\ (-1) < (k), (1) < Q_8, (i) < Q_8, (j) < Q_8$ and $(k) < Q_8$.

Now we wish to find the comparable pairs in the group Z_n. We know that (u) is a subgroup of Z_n for every u in Z_n.

Lemma 2.3 *If u and v are two distinct units of the group Z_n, then $((u), (v))$ is not a comparable pair in Z_n.*

Proof Suppose $((u), (v))$ is a comparable pair in Z_n. Then $(u) < (v)$ or $(v) < (u)$. This is contradiction, because $(u) = (v)$ for any two distinct units of the group Z_n. So, our assumption is not true and hence $((u), (v))$ is not a comparable pair in Z_n. □

By Lemma 2.3 we conclude that every pair of two distinct elements u and v in $U(Z_n)$, $((u), (v))$ does not form a comparable pair. So, our required comparable pair $((u), (v))$ exists for u and v in $Z(Z_n)$ only.

Remark 2.4 *For every element $u \in U(Z_n)$, $(u) = Z_n$ and $v \in Z(Z_n)$, $(v) \subset Z_n$.*

Recently, the authors Sajana and Bharathi explored many results in [15]. The set of all elements in Z_n can be written as the disjoint union of the sets $S_d^{'}$ for all d in D, where $S_d = \{x \in Z_n : (x) = (d)\}$and the set D denotes the set of all divisors of the positive integer n. So for every non unit element in Z_n is an element in some S_d, where $d \neq 1, d \in D$.

For every positive integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 1$, the set of divisors of n is denoted by $D(n)$ and is its cardinality defined as $|D(n)| = d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_m + 1)$. Also the number of subgroups of Z_n is $d(n)$. For instance, if $n = 6 = 2^1 3^1$, then $d(6) = (1+1)(1+1) = 4$, since $D(6) = \{1, 2, 3, 6\}$.

Remark 2.5 *For any divisor d of n, (d) is a subgroup of Z_n. If x is a proper divisor of d, then (d) is a subgroup of (x) under modulo n. Similarly, if y is a proper multiple of d which is a divisor of n, then (y) is a subgroup of (d) under modulo n. This shows that for any divisor d of n, $((d), (x))$ is a comparable pair, where x is a proper divisor of d or proper multiple of d which is a divisor of n.*

Definition 2.6 *Let n be a positive integer and d be a divisor of n. Then the set of all proper multiples of d which is a divisor of n under modulo n is denoted by $M_n(d)$ with cardinality $|M_n(d)|$.*

Now, first we wish to find the number of proper multiples of a divisor d of n, which is divisor of n. This clearly gives the number of proper subgroups of the subgroup (d) in the group Z_n. Now first we can find these proper multiples by the method of induction.

Theorem 2.7 *If $n = p^\alpha, \alpha \geq 1$, then the number of proper multiples of the divisor $d = p^\beta, 0 \leq \beta \leq \alpha$ of n under modulo n is $((\alpha - \beta) + 1) - 1$.*

Proof The set of all proper multiples of the divisor d of n under modulo n is $M_n(d) = \{p^{\beta+1}, p^{\beta+2}, \cdots, p^\alpha\} = 0$. This implies that the number of proper multiples of d under modulo n is $|M_n(d)| = (\alpha - \beta) = ((\alpha - \beta) + 1) - 1$. □

Example 2.8 For $n = 8, 8 = 2^3$ and $d = 2 = 2^1$. Then the number of proper multiples of the divisor 2 of 8 under modulo 8 is $|M_8(2)| = ((3 - 1) + 1) - 1 = 2$, because $M_8(2) = \{2, 4 = 2^2, 8 = 2^3 = 0\}$.

Remark 2.9 *For $\beta = \alpha$, the number of proper multiples of the divisor $d = p^\beta, 0 \leq \beta \leq \alpha$ of $n = p^\alpha, \alpha \geq 1$ under modulo n is 0.*

For example, if $n = 8, 8 = 2^3$ and $d = 8, 8 = 2^3$, then d has no proper multiples under modulo 8.

Theorem 2.10 *If $n = p_1^{\alpha_1} p_2^{\alpha_2}, \alpha_i \geq 1, 1 \leq i \leq 2$, then the number of proper multiples of the divisor $d = p_1^{\beta_1} p_2^{\beta_2}, 0 \leq \beta_i \leq \alpha_i, 1 \leq i \leq 2$ of n under modulo n is $\prod_{i=1}^{2}((\alpha_i - \beta_i) + 1) - 1$.*

Proof The set of all proper multiples of the divisor d of n under modulo n is $M_n(d) = \{p_1^{\beta_1+1} p_2^{\beta_2}, p_1^{\beta_1+2} p_2^{\beta_2}, \cdots, p_1^{\alpha_1} p_2^{\beta_2}$ ($\alpha_1 - \beta_1$ terms), $p_1^{\beta_1} p_2^{\beta_2+1}, p_1^{\beta_1+1} p_2^{\beta_2+1}, \cdots, p_1^{\alpha_1} p_2^{\beta_2+1}$ ($\alpha_1 - \beta_1 + 1$ terms), $p_1^{\beta_1} p_2^{\beta_2+2}, p_1^{\beta_1+1} p_2^{\beta_2+2}, \cdots, p_1^{\alpha_1} p_2^{\beta_2+2}$ ($\alpha_1 - \beta_1 + 1$ terms),$\cdots$, $p_1^{\beta_1} p_2^{\alpha_2}, p_1^{\beta_1+1} p_2^{\alpha 2+2}, \cdots, p_1^{\alpha_1} p_2^{\alpha_2}$ ($\alpha_1 - \beta_1 + 1$ terms)$\}$.

Now the total number of number of proper multiples of the divisor $d = p_1^{\beta_1} p_2^{\beta_2}$ of n under modulo n is, $|M_n(d)| = (\alpha_1 - \beta_1) + (\alpha_1 - \beta_1 + 1)(\alpha_2 - \beta_2) = (\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2)$ $= ((\alpha_1 - \beta_1) + 1)((\alpha_2 - \beta_2) + 1) - 1 = \prod_{i=1}^{2}((\alpha_i - \beta_i) + 1) - 1$. □

Example 2.11 For $n = 36, 36 = 2^2 \cdot 3^2$ and $d = 6, 6 = 2^1 \cdot 3^1$. Then the number of proper multiples of the divisor 6 of 36 under modulo 36 is $|M_{36}(6)| = \prod_{i=1}^{2}((\alpha_i - \beta_i) + 1) - 1 = ((2-1)+1)((2-1)+1) - 1 = 3$ because $M_{36}(6) = \{12 = 2^2 \cdot 3,\ 18 = 2 \cdot 3^2, 2^2 \cdot 3^2 = 0\}$.

Clearly, Theorem 2.12 follows by Theorem 2.10.

Theorem 2.12 *If* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 1$, *then the number of proper multiples of the divisor* $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}, m \geq 1, 0 \leq \beta_i \leq \alpha_i, 1 \leq i \leq m$ *of* n *under modulo* n *is* $\prod_{i=1}^{m}((\alpha_i - \beta_i) + 1) - 1$.

Theorem 2.13 *In the group* $Z_n, n > 0$ *and* y *is a divisor of* n, *then the number of subgroups which are comparable to the subgroup* (y) *in* Z_n *is* $|D(y)| + |M_n(y)| - 1$.

Proof By the Remark 2.5, clearly for any divisor y of n, (y) is a subgroup of Z_n. If x is a proper divisor of y, then (y) is a subgroup of (x) under modulo n. Similarly, if x is a proper multiple of y which is a divisor of n, then (x) is a subgroup of (y) under modulo n. This shows that for any divisor y of n, $((y), (x))$ is a comparable pair, where x is a proper divisor of y or proper multiple of y which is a divisor of n. Therefore the number of subgroups in Z_n which are comparable to the subgroup (y) is $|D(y)| - 1 + |M_n(y)| = |D(y)| + |M_n(y)| - 1$. □

Example 2.14 For $n = 6, 6 = 2.3$ and $d = 2$, then the number of subgroups which are comparable to the subgroup (2) in Z_6 is $|D(2)| + |M_6(2)| - 1 = 2 + 1 - 1 = 2$, where $D(2) = \{1\}$, $M_6(2) = \{6 \equiv 0 (mod 6)\}$. Clearly, $(1) = Z_6$ and $(0) = \{0\}$ are two subgroups which are comparable to the subgroup $(2) = \{0, 2\}$ in Z_6, because $(2) < (1)$ and $(0) < (2)$.

Theorem 2.15 *The number of comparable pairs of subgroups of a finite group* $Z_n, n > 0$ *is* $\frac{1}{2}\Sigma_{y|n}(|D(y)| + |M_n(y)| - 1)$.

Proof By the Theorem 2.13, the number of subgroups which are comparable to the subgroup (y) in Z_n is $|D(y)| + |M_n(y)| - 1$. So twice the number of comparable pairs of subgroups of a finite group Z_n is equal to the sum of the number of subgroups which are comparable to every subgroup (y) in Z_n. Therefore, the proof follows. □

Example 2.16 For $n = 6, 6 = 2.3$, the number of comparable pairs of subgroups of a finite group Z_6 is $\frac{1}{2}\Sigma_{y|6}(|D(y)| + |M_6(y)| - 1) = \frac{1}{2}[(|D(1)| + |M_6(1)| - 1) + (|D(2)| + |M_6(2)| - 1) + (|D(3)| + |M_6(3)| - 1) + (|D(6)| + |M_6(6)| - 1)] = \frac{1}{2}[(1+3-1) + (2+1-1) + (2+1-1) + (4+0-1)]$ $= \frac{1}{2}[10] = 5$. These comparable pairs are $((0), (2)), ((0), (3)), ((0), (1)), ((2), (1))$ and $((3), (1))$, where $(1) = Z_6$.

§3. Structural Properties of Comparable Graphs of Groups

In this section G denotes a non-trivial finite group and $H < K$ denotes H is a subgroup of K. Our aim is to consider and we study their structural properties of the simple undirected graph $CG(G)$ which is associated with the subgroups of G. In this paper p and q are distinct primes, and n will always denote positive integer.

Definition 3.1 *For a finite group* G, *the comparable graph* $CG(G)$ *is a simple undirected graph* *whose vertex set is $Sub(G)$, subgroups of G and we join two distinct vertices H_i, H_j in $CG(G)$, whenever $H_i < H_j$ or $H_j < H_i$.*

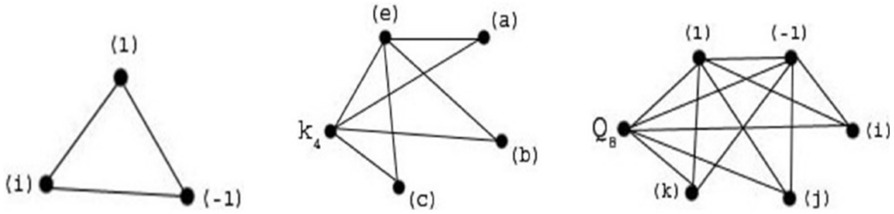

Figure 1. The comparable graph of fourth roots of unity $\{1, i\}$ Kleins four group K_4 and quaternion group Q_8

If G is a finite cyclic group, then it is easy to check that $CG(G)$ has exactly two vertices if and only if G is isomorphic to cyclic group of order p. Another specific trivial result is that the comparable graph of a group G is connected if and only if $|G| > 1$. Hence, for all non elementary finite cyclic p-group, graph $CG(G)$ is complete.

If $|G|$ is not a power of prime and if G is a non-Abelian group then every comparable graph of G is not complete. For instance, $CG(Q_8)$ is not complete. Therefore, we conclude that $CG(G)$ is complete if and only if G is any finite cyclic group of prime power order.

Before proving structured properties of comparable graphs, we introduce the appropriate distance notion of these graphs. Let H_i and H_j be two distinct vertices of the graph $CG(G)$. If either $H_i < H_j$ or $H_j < H_i$ then we say that H_i and H_j are always adjacent in $CG(G)$ and we write $H_i - H_j$ or equivalently, (H_i, H_j). Further, we say that H_i and H_j are connected; and we write $H_i - H_j$, if there exists a finite path in $CG(G)$ between them; otherwise they will be called disconnected Vertices. A comparable graph will be called disconnected if it contains at least two disconnected vertices. Otherwise it will be called connected. If $H_i = H_j$ then we define $d(H_i, H_j) = 0$. Thus $d(H_i, H_j) \leq n - 1$ for some positive integer n will mean that either $H_i = H_j$ or $H_i \neq H_j$ and they can be connected by a path of length up to $n - 1$. Clearly, $d(H_i, H_j) = 1$ if and only if (H_i, H_j) is an edge in $CG(G)$. If H_i and H_j are disconnected then $d(H_i, H_j) = \infty > n - 1$ for all n. If $CG(G)$ is connected then $diam(CG(G))$ denotes its diameter.

We wish to show that if G is a finite group, then $CG(G)$ is connected and the $diam(CG(G))$ is bounded, and its bound is 2. First we prove the following theorem.

Theorem 3.2 *Let G be a finite group. Then $|G| > 1$ if and only if $CG(G)$ is connected.*

Proof If G is a group of prime order, then the theorem certainly holds. Similarly, by the definition of comparable graph, the theorem also holds if $|G| = p^n$. So we may assume that G is a finite group and $|G| \neq p^n$. The distance in $CG(G)$ will be denoted by d. We shall prove each item separately.

(i) It is obvious since (e) is a subgroup of each non-trivial subgroup H of G, so the vertex (e) is adjacent with remaining all the vertices of $CG(G)$.

(ii) Suppose that the result is false, and the subgroups (e) and G itself of G satisfying

$d((e), G) > 2$. Then there exists non-trivial proper subgroups $H_1, H_2, \cdots, H_k$ of G such that either each H_i is a subgroup of H_j or each H_j is a subgroup of H_i or each H_i is not a subgroup of H_j for $1 \leq i \neq j \leq k$.

Let H_i is not a subgroup of H_j (or H_j is not a subgroup of H_i) for each $i \neq j$. Then $d((e), G) \leq d((e), H_i) + d(H_j, G) \leq 1 + 1 = 2$, a contradiction. Therefore, H_i is adjacent to H_j. So $(e) - H_1 - H_2 - \cdots - H_k - G$ in $CG(G)$. This implies that $o(e) \mid o(H_1)$, $o(H_1)|o(H_2)$,$\cdots$, $o(H_{k-1})|o(H_k)$ and $o(H_k)|o(G)$. Therefore, order of G is a power of a single prime, which is a contradiction to the fact that $|G| \neq p^n$. The proof of the theorem is complete. □

For example, Figure 1 shows that the Comparable graphs of Fourth roots of unity $\{\pm 1, \pm i\}$, Kleins four group K_4 and Quaternion group Q_8 are connected.

The following results are immediate consequences of the above theorem.

Corollary 3.3 *Let G be a finite group of $|G| > 1$. Then there are at least two vertices of $CG(G)$ which are adjacent to each and every other vertex.*

Obviously, for any finite group G the vertices (e) and G of $Sub(G)$ are adjacent to remaining all the vertices in the comparable graph $CG(G)$.

Corollary 3.4 *The comparable graph $CG(G) \cong K_2$ if and only if G is a group of order p.*

Remark 3.5 *If $|G| \neq p$ then $|Sub(G)| > 2$, where p is a prime. This implies that*

$$|E(CG(G))| > 2.$$

The following result associates the set of comparable pairs in G and cycles of length three in $CG(G)$.

Theorem 3.6 *For any finite group G with $|G| \neq p$, the comparable graph $CG(G)$ has at least one cycle of length* 3.

Proof Suppose $|G| \neq p$. Then there exists a non trivial subgroup H of G such that the vertex H is adjacent with (e) and G in $CG(G)$. Hence we have the cycle $(e) - H - G - (e)$ of length 3 in $CG(G)$. □

In the light of the above Theorem 3.2 the following result is clear. For finite cyclic groups we have the following necessary and sufficient condition for completeness of comparable graphs.

Theorem 3.7 *For any finite cyclic group G of order n, the comparable graph $CG(G)$ is complete if and only if no two non trivial proper subgroup of G are of relatively prime orders.*

Proof Suppose that $CG(G)$ is a complete graph of a cyclic group G of order n. Then any two vertices H_i and H_j are adjacent in $CG(G)$, $i \neq j$. Consequently, either $H_i < H_j$ or $H_j < H_i$. This implies that, by the Lagranges theorem [11] for finite groups, either $o(H_i)$ divides $o(H_j)$, or $o(H_j)$ divides $o(H_i)$. Hence no two non trivial proper subgroups of G are of relatively prime orders.

Conversely, suppose no two non trivial proper subgroups of a cyclic group G are of relatively prime orders, that is, $gcd(o(H_i), o(H_j)) \neq 1$ for $i \neq j$. Then we claim that $CG(G)$ is complete. Assume that $CG(G)$ is not complete. There exists two vertices H_i and H_j in $CG(G)$ such that (H_i, H_j) and (H_j, H_i) are not comparable graphs. This implies that either $o(H_i) \nmid o(H_j)$ and $o(H_j) \nmid o(H_i)$, or $o(H_i)|o(H_j)$ and $o(H_j)|o(H_i)$.

Case 1. Suppose $o(H_i) \nmid o(H_j)$ and $o(H_j) \nmid o(H_i)$. Then, clearly $gcd(o(H_i), o(H_j)) = d, d \geq 1$. But by the hypothesis, $gcd(o(H_i), o(H_j)) \neq 1$. Therefore, $gcd(o(H_i), o(H_j)) = d, d > 1 \Rightarrow gcd(\frac{o(H_i)}{d}, \frac{o(H_j)}{d})) = 1 \Rightarrow \frac{o(H_i)}{d}|n$ and $\frac{o(H_j)}{d}|n \Rightarrow$ There exists other subgroups $H_i^{'}$ and $H_j^{'}$ of a cyclic group G with distinct orders $\frac{o(H_i)}{d}$ and $\frac{o(H_j)}{d}$, respectively, which is a contradiction to the fact that no two non trivial proper subgroups of G be of relatively prime orders. So, in this case, $CG(G)$ is complete.

Case 2. Suppose $o(H_i)|o(H_j)$ and $o(H_j)|o(H_i)$. Then, clearly $o(H_i) = o(H_j)$, which is also a contradiction to our assumption that H_i and H_j are distinct proper subgroups of a cyclic group G of relatively prime order.

From the above two cases, it is clear that $CG(G)$ is complete.

Generally speaking, the collection $Sub(G) = \{(0), (1), (p), (p^2), \cdots, (p^{k-1})\}$ be the vertex set of the graph $CG(Z_{p^k})$ and thus no two proper subgroups of Z_{p^k} are of relatively prime orders, so the Theorem 3.7 shows that the graph $CG(Z_{p^k})$ is complete. Thus, the comparable graph of a cyclic p-group is isomorphic to the group Z_{p^k}. Further, if G is a finite p-group which is not a cyclic group, then G is either Abelian or non-Abelian. So there exists at least one non comparable pair (H, K) in G if and only if G is a non cyclic p-group. For instance, $((a), (b)), ((b), (c))$ and $((c), (a))$ are non-comparable pairs in the group K_4 of order p^2, where $p = 2$. So, the following results are immediate consequences of this information. □

Corollary 3.8 *Let G be a non cyclic p-group. Then $CG(G)$ is not a complete graph.*

Corollary 3.9 *The graphs $CG(Z_p)$ and $CG(Z_p)$ are respectively*

(1) $CG(Z_p) \cong K_2$;
(2) $CG(Z_{p^k}) \cong K_{n+1}$.

Now we state the following equivalent theorem due to connectedness of comparable graphs. The proof of the following theorem is essentially cleared the comparable pair $((e), G)$ for any group G with $|G| > 1$.

Theorem 3.10 *The comparable graph $CG(G)$ of G is always connected with diameter at most 2.*

By the above theorem, for given two positive integers $m, n > 1$, the comparable graph $CG(Z_m \times Z_n)$ of the group $Z_m \times Z_n$ is always connected but not complete. For this connection, the comparable graph $CG(G \times G)$ is connected but not complete for any finite group G, because $G \times G$ is not a cyclic group. For instance, $CG(Z_4)$ is a 2-regular graph and hence complete, but $CG(Z_2 \times Z_2)$ is connected and not 2-regular graph. Further, we observe that the complement of $CG(G)$, for any finite group G is not connected because $deg((e)) = |G| - 1$ and $deg(G) = |G| - 1$, and thus the complement of $CG(G)$ contains at least two disconnected components.

We have the following illustrations, where the comparable graph of a group G is connected but its corresponding complement graph is not connected.

Example 3.11 The comparable graphs of groups $Z_2 \times Z_2$ and K_4 are both connected, but it is simple to see that their complements are not connected.

In the modern mathematical field of graph theory, the most important graph is a bipartite graph. It is an undirected simple graph G whose vertices can be divided into two disjoint non empty subsets A and B such that every edge adjacent a vertex in A to one vertex in B, and the pair (A, B) is called bipartitions of the graph G. However, a graph G is called bipartite if G does not contain any cycle of odd length. In recent years, bipartite graphs are extensively applied in algebraic coding theory, cloud computing and used in biological systems [12], [13] and [14].

Theorem 3.12 *The comparable graph $CG(G)$ is bipartite if and only if $|G| = p$, a prime.*

Proof It is clear from the well known result that $|G| = p$ if and only if $CG(G) \cong K_2$. □

Theorem 3.13 *Let G be a group of Composite order. Then $CG(G)$ is not bipartite.*

Proof Consider $|G|$ is a composite order. Suppose $CG(G)$ is a bipartite graph. Then there is a bipartition (A, B). Without loss of generality we may assume that $(e) \in A$ and $G \in B$. Since $|G|$ is composite, by the Cauchys theorem [11] for finite groups, there exists a non trivial proper subgroup H of G such that $((e), H)$ and (H, G) are both comparable pairs in G. This implies that H is lies in both A and B. Thus, there exists an odd cycle $(e) - H - G - (e)$ in $CG(G)$. This violates the bipartite graph. Hence $CG(G)$ is not a bipartite graph. □

Corollary 3.14 *Let H be any vertex of a comparable graph $CG(G)$. Then $deg(H) \geq 1$ in $CG(G)$ if and only if $|G| > 1$.*

The following theorem is follows from the Theorem 2.13.

Theorem 3.15 *For any vertex (x) of a comparable graph $CG(Z_n), n > 0$ and x is a divisor of n, then $deg((x)) = |d(x)| + |M_n(x)|$.*

Theorem 3.16([16]) *For any Graph G, $\Sigma_{v \in V(G)} deg(v) = 2E(G)$, where v is a vertex and $E(G)$ is the total number of vertices.*

Theorem 3.17 *The size of the comparable graph $CG(Z_n), n > 0$ is $\frac{1}{2}\Sigma_{y|n}(|d(y)| + |M_n(y)|)$.*

Proof Proof follows from Theorems 3.15 and 3.16. □

§4. Traversability Properties of Comparable Graphs

Now we turn to study the characterization of Eulerian comparable graphs. First, we recall that an undirected simple Eulerian graph has two common arrangements in Graph theory. One is a connected graph with an Eulerian cycle, and the other one is a connected graph with every vertex of even degree. These two concepts coincide for connected graphs, see [17] and this equivalent concept is known as Eulers Theory. It is called the characterization of Euler graph. Hence, a connected graph is called Eulerian if and only if its every vertex has even degree.

In light of the argument above, the following theorem is particular for the comparable graphs of groups.

Theorem 4.1 *The comparable graph $CG(G)$ is an Euler graph if and if the graph G has odd number of subgraphs.*

Proof Suppose that the comparable graph $CG(G)$ is an Euler graph whose order is $r > 2$. Then each vertex H in $CG(G)$ has even degree. In particular, the vertex (e) has the degree $r-1$ since the vertex (e) is adjacent to remaining all vertices of $CG(G)$ and thus $r-1$ is even. Consequently, r must be odd. This shows that $CG(G)$ has odd number of vertices, and hence the group G has odd number of subgroups.

Consequently, assume that G has odd number of subgroups. Then $CG(G)$ is either complete or not complete. Consider the following two cases.

Case 1. Let $CG(G)$ be a complete graph with r vertices. Then $CG(G)$ is $(r-1)$- regular graph. This implies that $r-1$ must be even and thus $CG(G)$ is an Euler graph.

Case 2. Let $CG(G)$ be not a complete graph. By the Theorem 3.2, it is a connected graph with $deg((e)) = r-1$ and $deg(G) = r-1$. Further, to claim that $deg(H) = r-1$, where H is a non-trivial proper subgroup of G. Assume that $deg(H)$ is odd. Without loss of generality, we may assume that $deg(H) = 3k$ for some positive integer k. This implies that the vertex H is adjacent to the vertices $(e), G$ and K, where K is another non-trivial proper subgroups of G. Therefore, $|Sub(G)|$ is k, and thus G has even number of subgroups, which is a contradiction to our hypothesis that G has an odd number of subgroups. This completes the proof. □

The following result associates the set $Sub(G)$ of subgroups of G. The proof of the following result is clear from $deg((e)) = |Sub(G)| - 1 = deg(G)$, where (e) and G are two vertices of the graph $CG(G)$.

Theorem 4.2 *Let $|G| \neq 1$. Then $CG(G)$ is never a totally disconnected graph.*

Now we give some examples of Eulerian comparable graphs.

Example 4.3 The comparable graph $CG(Z_4)$ is a 2- regular connected graph, and hence $CG(Z_4)$ is an Eulerian comparable graph.

Example 4.4 The comparable graph $CG(K_4)$ is a non-regular connected graph but it is also an Eulerian graph because $Sub(K_4)$ contains odd number of subgroups.

Example 4.5 The comparable graph $CG(Q_8)$ is not Eulerian because $|Sub(Q_8)| = 6$.

In graph theory, there is another class of connected graphs, called Hamilton graphs. These graphs characterized by a cycle called Hamilton cycle, that is, a cycle containing each vertex of the graph. For this reason, we must clear that every Hamilton graph is traceble, and it was natural and particular that these traceble graphs would again similar attention. These attentions support the following result for comparable graphs.

Theorem 4.6 *If* $|Sub(G)| \geq 3$ *then the comparable graph* $CG(G)$ *is Hamiltonian.*

Proof By characterization of Hamiltonian graphs [18], it is enough to prove that for any two vertices H and K in the comparable graph $CG(G)$, the following inequality holds:

$$deg(H) + deg(K) \geq |Sub(G)|.$$

In anticipation of a contradiction, let us assume that there exist at least two vertices H and K in $CG(G)$ such that $deg(H) + deg(K) < |Sub(G)| \Rightarrow deg(H) + deg(K) - |Sub(G)| < 0$. Taking $deg(H) = |Sub(G)| - 1 = deg(K)$ in $CG(G)$.

We now pause to look at two concrete cases on $|Sub(G)|$.

Case 1. Suppose $|Sub(G)| = 2k$ for some positive integer $k > 1$. Then the above inequality reduces to $(2k-1) + (2k-1) - 2k < 0 \Rightarrow k < 1$, which is not true.

Case 2. Suppose $|Sub(G)| = 2k+1$ for some positive integer k. Then $2((2k-1)-1)-(2k-1) < 0 \Rightarrow k < \frac{1}{2}$, which is also not true.

From the above two cases, the only possible conclusion is that the characterization of Hamiltonian graphs holds good. Hence, the comparable graph $CG(G)$ is Hamiltonian. □

The proof of above theorem is all some what vague, of course, so let us look at a concrete example.

Example 4.7 Consider the cyclic group Z_{30} and the Figure 2 shown it corresponding comparable graph $CG(Z_{30})$. Because both $|Sub(Z_{30})| = 8$ and $30 = 2 \times 3 \times 5$, we see that $deg(H) + deg(K) \geq 8$ for all vertices H and K in $CG(Z_8)$. With reference to Theorem 4.6, the sequence of vertices (0) – (5) – (10) – (2) – (6) – (3) – (15) – (1) – (0) form a Hamilton cycle in $CG(Z_{30})$.

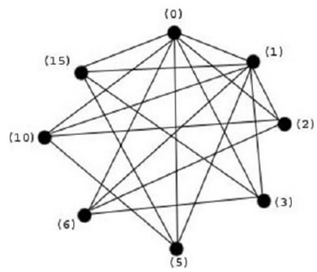

Figure 2. The Comparable Graph $CG(Z_{30})$

References

[1] Aggarwal A., Klawe M, Shor P., Multilayer grid embeddings for VLSI, *Algorithmica*, 6

(1991), 129 - 151.

[2] Ramanathan S., Lloyd E.L., Scheduling algorithms for multihop radio networks, *IEEE/ACM Transactions on Networking*, 1 (2) (1993), 166 - 177.

[3] Cayley P., Desiderata and suggestions - the theory of groups : graphical representation, *American Journal of Mathematics*, 1 (2) (1878), 174 - 176.

[4] Aalipour G., Akbari S., Cameron P. J., Nikandish R., Shaveisi F., On the structure of the power graph and the enhanced power graph of a group, *Electronic Journal of Combinatorics*, 24 (3) (2017), 1-22.

[5] Ma X.L., Wei H.Q., Zhong G., The cyclic graph of a finite group, *Algebra*, 2013 (2013), 1-7.

[6] Csâkâny B., Pollak G., The graph of subgroups of a finite group, *Czechoslovak Mathematical Journal*, 19 (2) (1969), 241-247.

[7] Devi P., Rajkumar R., Inclusion graph of subgroups of a group, *arXiv:1604.08259v1* [math.GR], (2016) 1-22.

[8] Anderson D.F., Fasteen J., LaGrange J.D., The subgroup graph of a group, *Arabian Journal of Mathematics*, 1 (2012), 17-27.

[9] Chalapathi T., Kumar R. VMSS., Order divisor graphs of finite groups, *Malaya Journal of Mathematik*, 5 (2) (2017) 464-474.

[10] Liu J., Munir M., Munir Q., Nizami A.R., Some metrical properties of lattice graphs of finite groups, *Mathematics*, 7(5) (2019).

[11] Gallian J.A., *Contemporary Abstract Algera*, Eighth Edition, Brooks/Cole Cengage Learning, USA, 2013.

[12] Pavlopoulos G.A., Kontou P.I. et al., Bipartite graphs in systems biology and medicine: a survey of methods and applications, *GigaScience*, 7 (2018), 1C31.

[13] Zhang X., Nadeem M. et al., On applications of bipartite graph associated with algebraic structures, *De Gruyter C Open Mathematics*, 18(1) (2020), 57-66.

[14] Arunkumar B.R., Komala R., Applications of bipartite graph in diverse fields including cloud computing, *International Journal of Modern Engineering Research*, 5 (7) (2015), 2015, 1-7.

[15] Sajana S., Bharathi D., Number theoretic properties of the commutative ring, *Int. J. Res. Ind. Eng.*, Vol. 8(1), 2019, 77-88.

[16] Bondy J.A., Mutry U.S.R., *Graph Theory with Application*, Springer India, 2013.

[17] Mallows C.L., Sloane N.J.A., Two-graphs, switching classes and Euler graphs are equal in numbers, *SIAM J. of Appl. Math.*, Vol. 28(4): 876-880.

[18] G. Chartrand P. Zhang, *Introduction to Graph Theory*, Tata McGraw-Hill, 2006.