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$(1, N)$ - Arithmetic Labelling of $C_n \otimes S_m$ and $S_{m,n}$

S.Anubala

Research Scholar, Department of Mathematics and Research Centre
Mannar Thirumalai Naicker College, Madurai, Tamil Nadu, India

V.Ramachandran

Department of Mathematics
Mannar Thirumalai Naicker College, Madurai, Tamil Nadu, India

E-mail: anubala.ias@gmail.com, me.ram111@gmail.com

Abstract: A (p, q) - graph G is said to have $(1, N)$ - arithmetic labelling if there is an injection ϕ from the vertex set $V(G)$ to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, (q-1)N, (q-1)N+1\}$ such that the values of the edges, obtained as the sums of the labelling assigned to their end vertices can be arranged in the arithmetic progression $1, (N+1), (2N+1), \cdots, (q-1)N+1$. In this paper we prove that the $C_n \otimes S_m, S_{m,n}$ have $(1, N)$ - arithmetic labelling for every positive integer $N > 1$.

Key Words: $(1, N)$ - arithmetic labelling, $C_n \otimes S_m$, $S_{m,n}$.

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§1. Introduction

B.D. Acharya and S.M. Hedge [1], [2] introduced (k, d) - arithmetic graphs and certain vertex valuations of a graph. A (p, q) -graph is said to be (k, d) - arithmetic if its vertices can be assigned distinct non -negative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression $k, k+d, k+2d, \cdots, k+(q-1)d$.

Joseph A. Gallian [3] surveyed numerous graph labelling methods. V. Ramachandran and C.Sekar [4] introduced $(1, N)$ arithmetic labelling. They proved that stars, paths, complete bipartite graph $K_{m,n}$, highly irregular graph $Hi(m, m)$,Cycle C_{4k}, ladder and subdivision of ladder have $(1, N)$ - arithmetic labelling. They also proved that C_{4k+2} does not have $(1, N)$ - arithmetic labelling and no graph G containing an odd cycle has $(1, N)$ - arithmetic labelling for any integer N.

In this paper we prove that the $C_n \otimes S_m$ and $S_{m,n}$ have $(1, N)$ - arithmetic labelling.

§2. Main Results

Definition 2.1([5]) *Let G be any graph and S_m be a star with m spokes. We denote it by*

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$G \otimes S_m$ the graph obtained from G by identifying one vertex of G with any vertex of S_m other than the centre of S_m.

Definition 2.2([5]) *Let $S_{m,n}$ stand for a star with n spokes in which each spoke is a path of length m.*

Theorem 2.3 $\mathbf{C_n} \otimes \mathbf{S_m}$ *is $(1, N)$-arithmetic for $n = 4k$, $k \geq 1$,$n = 4k + 2$, $k \geq 1$ and $m \geq 1$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of C_n and $v_1, v_2, \cdots, v_m$ be the vertices of the star S_m where v_0 is the centre of the star. The graph$G = \mathbf{C_n} \otimes \mathbf{S_m}$ has $n + m$ vertices and $n + m$ edges such as those shown in Figure 1.

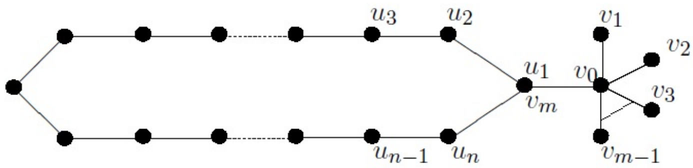

Figure 1

Case 1. $n = 4k$, $k \geq 1$, $m \geq 1$.

Define the vertex labeling mapping $\Phi : V(G) \longrightarrow \{0, 1, N, N+1, \cdots, (q-1)N, (q-1)N+1\}$ as follows:

$\Phi(v_0) = 0$,
$\Phi(v_i) = N(i - 1) + 1$ for $i = 1, 2, \cdots, m$,
$\Phi(u_1) = \Phi(v_m) = N(m - 1) + 1$,
$\Phi(u_{2i}) = Ni$ for $i = 1, 2, \cdots, 2k$,
$\Phi(u_{2i+1}) = N(m - 1) + 1 + Ni$ for $i = 1, 2, \cdots, k - 1$,
$\Phi(u_{2i+1}) = N(m - 1) + 1 + N(i + 1)$ for $i = k, k + 1, \cdots, 2k - 1$.

Clearly Φ is one-one. Also,

$\Phi^*(v_0v_i) = N(i - 1) + 1$ for $i = 1, 2, \cdots, m$,
$\Phi^*(u_iu_{i+1}) = N(m - 1) + 1 + Ni$ for $i = 1, 2, \cdots, 2k - 1$,
$\Phi^*(u_iu_{i+1}) = N(m - 1) + 1 + N(i + 1)$ for $i = 2k, 2k + 1, \cdots, 4k - 1$ and
$\Phi^*(u_{4k}u_1) = N(m-1)+1+2kN$. Thus, the edge labellings are $1, N+1, 2N+1, \cdots, (4k+m-1)N+1 = (q-1)N+1$, where q denotes the number of edges.

Therefore, $\mathbf{C_n} \otimes \mathbf{S_m}$ is $(1, N)$-arithmetic in this case.

Case 2. $n = 4k + 2, k \geq 1, m \geq 1$.

In this case, define the vertex labeling mapping $\Phi : V(G) \longrightarrow \{0, 1, N, N+1, \cdots, (q-1)N, (q-1)N+1\}$ as follows:

$\Phi(v_0) = 0$,
$\Phi(v_1) = (4k + 2)N + 1$,
$\Phi(v_i) = (4k + 2)N + 1 + iN$ for $i = 2, \cdots, m - 1$,
$\Phi(u_1) = \Phi(v_m) = 1$,

$\Phi(u_{2i}) = Ni$ for $i = 1, 2, \cdots, 2k$,
$\Phi(u_{4k+2}) = (2k + 2)N$,
$\Phi(u_{2i+1}) = Ni + 1$ for $i = 1, 2, \cdots, k$,
$\Phi(u_{2i+1}) = N(i + 1) + 1$ for $i = k + 1, k + 2, \cdots, 2k$.

Clearly Φ is one-one and also

$\Phi^*(v_0v_1) = (4k + 2)N + 1$,
$\Phi^*(v_0v_i) = (4k + 2)N + 1 + iN$ for $i = 2, 3, \cdots, m - 1$,
$\Phi^*(v_0v_m) = 1$,
$\Phi^*(u_iu_{i+1}) = 1 + Ni$ for $i = 1, 2, \cdots, 2k + 1$,
$\Phi^*(u_iu_{i+1}) = 1 + N(i + 1)$ for $i = 2k + 2, \cdots, 4k$,
$\Phi^*(u_{4k+1}u_{4k+2}) = (4k + 3)N + 1$,
$\Phi^*(u_{4k+2}u_1) = 1 + N(2k + 2)$.

Thus, the edge labellings are $1, N + 1, 2N + 1, \cdots, (4k + 2 + m - 1)N + 1 = (q - 1)N + 1$. Therefore $\mathbf{C_n} \otimes \mathbf{S_m}$ is $(1, N)$-arithmetic in this case also. □

Example 2.4 A $(1, 4)$-arithmetic labelling of $C_{16} \otimes S_{10}$ is shown in Figure 2.

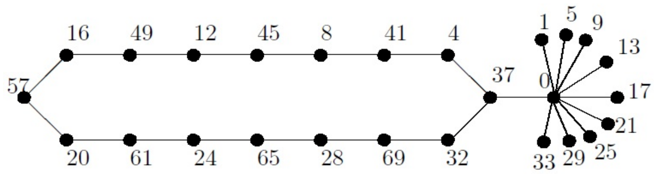

Figure 2

Example 2.5 A $(1, 7)$-arithmetic labelling of $C_{14} \otimes S_8$ is shown in Figure 3.

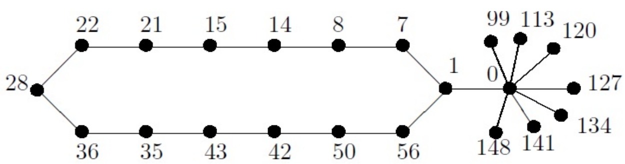

Figure 3

Theorem 2.6 *The graph* $S_{m.n}$ *is* $(1, N)$*-arithmetic for all* m *and* n*.*

Proof Let v_0 be the centre of the star. Let $v_i^{(j)}$, $1 \leq i \leq m$, $j = 1, 2, \cdots, n$ be the other vertices of the j^{th} spoke of length m. The graph $S_{m,n}$ has $mn + 1$ vertices and mn edges such as those shown in Figure 4.

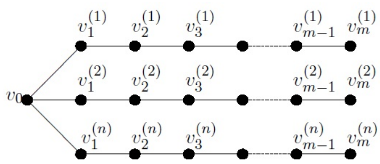

Figure 4

The proof is divided into two cases following.

Case 1. n is odd.

Let $n = 2r+1$. In this case, define

$\Phi(v_0) = 0$,
$\Phi(v_{2i-1}^{(j)}) = (2r+1)N(i-1) + (j-1)N + 1$ for $i = 1, 2, 3, \cdots$ and $j = 1, 2, \cdots, 2r+1$,
$\Phi(v_{2i}^{(j)}) = (2r+1)Ni + 2rN - (j-1)N$ for $i = 1, 2, 3, \cdots$ and $j = 1, 2, \cdots, 2r+1$.

Clearly Φ is one-one and

$\Phi^*(v_0 v_1^{(j)}) = (j-1)N + 1$ for $j = 1, 2, \cdots, 2r+1$,
$\Phi^*(v_i^{(j)} v_{i+1}^{(j)}) = (2r+1)Ni + 2rN - (j-1)N + 1$ for $i = 1, 2, 3, \cdots, m-1$ and $j = 1, 2, 3, \cdots, 2r+1$.

Thus, the edge labellings are $1, N+1, \cdots, ((2r+1)m-1)N+1 = (q-1)N+1$, where q denotes the number of edges. Therefore $S_{m,n}$ is $(1, N)$-arithmetic in this case.

Case 2. n is even.

Let $n = 2r$. We divide the discuss into two subcases.

Subcase 2.1 m is even.

Let $m = 2s$. In this case, define

$\Phi(v_0) = Ns$,
$\Phi(v_{2i-1}^{(1)}) = N(s-i) + 1$ for $i = 1, 2, \cdots, s$,
$\Phi(v_{2i}^{(1)}) = N(s-i)$ for $i = 1, 2, 3, \cdots, s$,
$\Phi(v_{2i-1}^{(j)}) = (2r-1)N(i-1) + Ns + (j-2)N + 1$ for$i = 1, 2, 3, \cdots$ and $j = 2, 3, \cdots, 2r$,
$\Phi(v_{2i}^{(j)}) = (2r-1)Ni + Ns + (2r-2)N - (j-2)2N$ for$i = 1, 2, 3, \cdots$ and $j = 2, 3, \cdots, 2r$.

Clearly Φ is one-one and

$\Phi^*(v_0 v_1^{(1)}) = (2s-1)N + 1$,
$\Phi^*(v_i^{(1)} v_{i+1}^{(1)}) = (2s-1-i)N + 1$ for$i = 1, 2, \cdots, 2s+1$,
$\Phi^*(v_0 v_1^{(j)}) = 2Ns + (j-1)N + 1$ for $j = 2, 3, \cdots, 2r$,
$\Phi^*(v_i^{(j)} v_{i+1}^{(j)}) = 2Ns + (2r-2)N + (2r-1)Ni - (j-2)N + 1$ for $i = 1, 2, \cdots, 2s-1$ and $j = 2, 3, \cdots, 2r$.

Thus, the edge labellings are $1, N+1, \cdots, (4rs-1)N+1 = (q-1)N+1$, where q denotes the number of edges. Therefore $S_{m,n}$ is $(1, N)$-arithmetic in this Case.

Subcase 2.2 m is odd.

Let $m = 2s+1$. In this case, define

$\Phi(v_0) = Ns + 1$,
$\Phi(v_{2i-1}^{(1)}) = N(s+1-i)$ for$i = 1, 2, \cdots, s+1$,
$\Phi(v_{2i}^{(1)}) = N(s-i) + 1$ for $i = 1, 2, 3, \cdots, s$,
$\Phi(v_{2i-1}^{(j)}) = (2r-1)N(i-1) + N(s+1) + (j-2)N$ for $i = 1, 2, 3, \cdots, s+1$ and $j = 2, 3, \cdots, 2r$,
$\Phi(v_{2i}^{(j)}) = (2r-1)Ni + Ns + (2r-2)N - (j-2)2N + 1$ for $i = 1, 2, 3, \cdots$ and $j = 2, 3, \cdots, 2r$.

Clearly Φ is one-one and

$\Phi^*(v_0 v_1^{(1)}) = 2Ns + 1$,

$\Phi^*(v_i^{(1)} v_{i+1}^{(1)}) = N(2s - i) + 1$ for $j = 1, 2, 3, \cdots, 2s$,

$\Phi^*(v_0 v_1^{(j)}) = N(2s + 1) + (j - 2)N + 1$ for $j = 2, 3, \cdots, 2r$,

$\Phi^*(v_i^{(j)} v_{i+1}^{(j)}) = (2s + 1)N + (2r - 2)N + (2r - 1)Ni - (j - 2)N + 1$ for $i = 1, 2, \cdots, 2s$ and $j = 2, 3, \cdots, 2r$.

Thus, the edge labellings are $1, N + 1, \cdots, (4rs + 2r - 1)N + 1$, where $q = 2r(2s + 1)$ is the number of edges. Therefore $S_{m,n}$ is $(1, N)$-arithmetic in this case also. □

Example 2.7 A $(1, 3)$-arithmetic labelling of $S_{6,7}$ is shown in Figure 5.

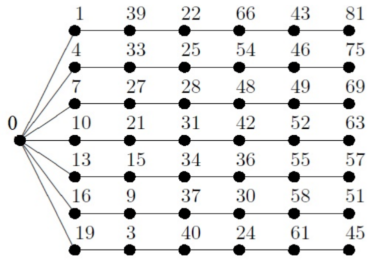

Figure 5

Example 2.8 A $(1, 5)$-arithmetic labelling of $S_{8,8}$ is shown in Figure 6.

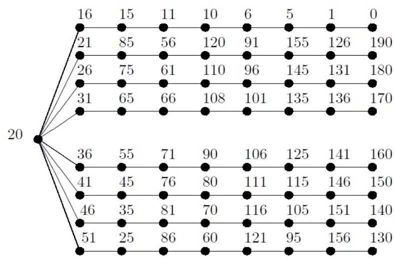

Figure 6

Example 2.9 A $(1, 6)$-arithmetic labelling of $S_{5,8}$ is shown in Figure 7.

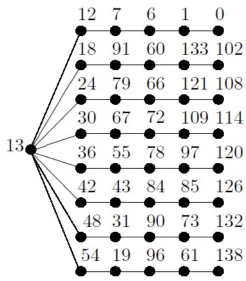

Figure 7

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