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Degree Affinity Number of Graphs

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Abstract: This paper initiates a study on a new graph parameter called the *degree affinity number* of a graph. The degree affinity number of a graph G is obtained by iteratively constructing graphs, $G_1, G_2, \cdots, G_k$ of increased size by adding a maximal number of edges between absolute distinct pairs of distinct, non-adjacent vertices of equal degree. Results for cycle and path graphs and certain general results are presented.

Key Words: Degree affinity number, degree affinity edge, partition.

AMS(2010): 05C15, 05C38, 05C75, 05C85.

§1. Introduction

For general notation and concepts in graphs see [1, 3, 6]. Throughout the study only finite, simple and undirected graphs will be considered. Recall that an edge $uv \in E(G)$ may also be denoted by, $\{u, v\}$. The latter notation emphasizes that an edge may be considered to be a 2-element set of vertices joined by an edge. It is indeed the properties of sets which clarify notions such as, edges $\{u, v\}, \{x, z\} \in E(G)$ are identical if and only if, $|\{u, v\} \cap \{x, z\}| = 2$, and the edges are incident if and only if, $|\{u, v\} \cap \{x, z\}| = 1$. A pair of distinct vertices which are not necessarily joined by an edge is denoted by, $\{u, v\}^{\pm}$. Two pairs of distinct vertices say, $\{u, v\}^{\pm}$ and $\{x, z\}^{\pm}$ are said to be *distinct pairs* if and only if, $|\{u, v\}^{\pm} \cap \{x, z\}^{\pm}| \leq 1$. Two pairs of distinct vertices are said to be *absolutely distinct* if and only if, $|\{u, v\}^{\pm} \cap \{x, z\}^{\pm}| = 0$. A pair of distinct vertices say, $u, v \in V(G)$ which are not joined by an edge (non-adjacent) is denoted by, $\{u, v\}^{-}$. In similar fashion, two pairs of distinct non-adjacent vertices $\{u, v\}^{-}$ and $\{x, z\}^{-}$ are said to be absolute distinct if and only if $|\{u, v\}^{-} \cap \{x, z\}^{-}| = 0$. This absolute distinct relation between two pairs of distinct vertices is denoted by, $\{u, v\}^{\pm} \nmid \{x, z\}^{\pm}$.

Various studies with regards to the addition or deletion of edges in a graph G and the effect thereof on parameters of graphs appear in the literature. To mention a few with corresponding references, see [2, 4, 5, 7]. The wide research interest in the addition or deletion of edges and the principle that, researching *mathematics for the sake of mathematics* is acceptable, motivate the study of a new graph parameter called the *degree affinity number* of a graph. In a more general sense the idea of *degree affinity* can be conceptualized to be chemical affinity between atoms or molecular affinity in molecular structures or social affinity between the members in

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social networks. Such notions could lead to real world applications with regards to the changes in graph theoretical properties of these structures. The recent *contact tracing models* utilised by countries in managing SARS-CoV-2 virus spread through communities suggest real world informatica application of this new notion. This new parameter is introduced in the next section.

§2. Degree Affinity Number of Cycle Graphs

It is known that a graph of order $n \geq 2$ has at least two vertices of equal degree. If two non-adjacent vertices $u, v \in V(G)$ with $deg_G(u) = deg_G(v)$ exist then the added edge uv to obtain G' is called a *degree affinity edge*.

Maximal Degree Affinity Convention (MDAC): For a graph G the 1^{st}-iteration is a maximal addition of degree affinity edges in respect of absolute distinct pairs of distinct non-adjacent vertices of equal degree. The graph obtained is labeled G_1. Hence, by the same convention it is possible to construct G_i from G_{i-1} provided that at least one (absolute distinct) pair of distinct non-adjacent vertices of equal degree exists in G_{i-1}. When no further edges can be added on the k^{th}-iteration the MDAC terminates.

Remark 2.1 A subtle feature of the MDAC is that all degrees are considered per iteration. Hence, if in the i^{th}-iteration say, $\{u, v\}^- \nsim \{x, z\}^-$ exist such that, $deg_{G_{i-1}}(u)[= deg_{G_{i-1}}(v)] \neq deg_{G_{i-1}}(x)[= deg_{G_{i-1}}(z)]$ then both the degree affinity edges uv and xz must be added during the i^{th}-iteration. The procedure can be viewed as a graph operation denoted by say, $\Lambda(G)$ whereby:

(*i*) $V(G)$ is partitioned into sets, $\{X_i : u, v \in X_i \Leftrightarrow deg_G(u) = deg_G(v)\}$;

(*ii*) For each X_i the maximum partition Y_i of random unordered pairs of distinct non-adjacent vertices is considered;

(*iii*) For each such pair $\{u, v\}$ the edge uv is added to G.

It is obvious that in applying the MDAC a finite number of iterations are possible. Let $\eta(k)$ be the number of degree affinity edges added when applying the MDAC has exhausted (say at k^{th}-iteration).

Definition 2.1 *The degree affinity number of a graph G is given by*

$\eta(G) = max\{\eta(k)$: *over all choices of maximal number of absolute distinct pairs of distinct non-adjacent vertices of equal degree, through exhaustive application of the MDAC*$\}$.

An upperbound on the *exhaustive iteration count* follows immediately.

Corollary 2.1 *For any graph G of order $n \geq 2$, the exhaustive iteration count for obtaining $\eta(G)$ is*

$$k \leq \frac{n(n-1)}{2} - \varepsilon(G).$$

Proof Observe that any degree affinity edge e of G is an edge in $\overline{G}$. Hence, if a graph exists which permits the addition of only one degree affinity edge per iteration and on exhaustion a complete graph is obtained. Then,

$$k = \frac{n(n-1)}{2} - \varepsilon(G).$$

Therefore, for graphs in general

$$k \leq \frac{n(n-1)}{2} - \varepsilon(G).$$ □

Corollary 2.1 implies that $0 \leq k \leq \eta(G) \leq \varepsilon(\overline{G})$. This observation can be illustrated by adding the maximum number of degree affinity edges to the cycle C_6. See Figures 1 and 2.

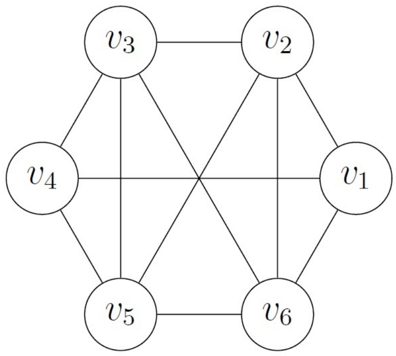

Figure 1 Graph C_6 for which a non-optimal application of the MDAC yields $k = 2$ and $\eta(k) = 5 < 9 = \varepsilon(\overline{C_6})$.

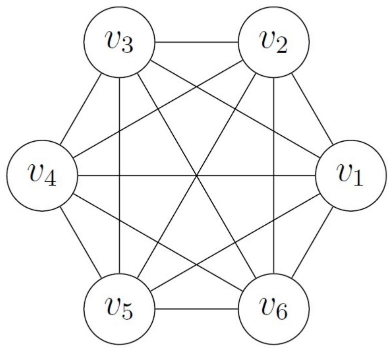

Figure 2 Graph C_6 for which an optimal application of the MDAC yields $k = 3$ and $\eta(3) = 9 = \varepsilon(\overline{C_6})$.

Figure 1 can be explained by: (i) begin with $G = C_6$ and (ii) add the degree affinity edges v_1v_4, v_2v_5, v_3v_6. Note this is a maximal addition of degree affinity edges during the 1^{st}-iteration to obtain G_1. Proceed with the 2^{nd}-iteration by considering G_1 and adding a maximal (the maximum in this case) degree affinity edges say, v_2v_6 and v_3v_5 to obtain G_2. Applying the MDAC is exhausted since no $\{u,v\}^-$, $u, v \in V(G_2)$ with $deg_{G_2}(u) = deg_{G_2}(v)$ exists. Note that $G_2 \ncong K_6$.

Figure 2 can be explained by: (i) begin with $G = C_6$, (ii) add the degree affinity edges, v_1v_4, v_2v_6, v_3v_5 to obtain G_1 and (iii) add the degree affinity edges v_1v_5, v_2v_4, v_3v_6 to obtain G_2. Proceed with the 3^{rd}-iteration by considering G_2 and add degree affinity edges v_1v_3, v_2v_5, v_4v_6 to obtain G_3. Since no $\{u,v\}^-$, $u, v \in V(G_3)$ with $deg_{G_3}(u) = deg_{G_3}(v)$ exists, applying the MDAC is exhausted. Note that $G_3 \cong K_6$. From Definition 2.1 it follows that, $\eta(C_6) = max\{5, 9\} = 9$.

Theorem 2.2 *For an even cycle C_n, $n \geq 4$ the MDAC exhausts after $k = n - 3$ iterations and*

$$\eta(C_n)_{n,even} = \frac{n(n-3)}{2} \quad and \quad G_{n-3} \cong K_n.$$

Proof We describe an imbedded induction procedure for even cycles (EIPC). Construct $G = C_4$ as follows; take the disjoint union of paths P and Q on the vertices v_1, v_2 and u_1, u_2, respectively and add the edges v_1u_1, v_2u_2. For the exhaustive 1^{st}-iteration of the MDAC add the edges v_1u_2 and u_1v_2 to obtain K_4. Clearly, $k = 1 = 4 - 3$ and $\eta(C_4) = 2 = \frac{4(4-3)}{2}$. Also, $G_1 \cong K_4$.

Similarly, construct the cycle $G = C_6$. For the 1^{st}-iteration of MDAC add the edges v_1u_3 and u_1v_2, u_2v_3. For the 2^{nd}-iteration of the MDAC add the edges u_1v_3 and v_1u_2, v_2u_3. For the exhaustive 3^{rd}-iteration of the MDAC add the edges v_1v_3, u_1u_3 and v_2u_2 to obtain K_6. Clearly, $k = 3 = 6 - 3$,

$$\eta(C_6) = 9 = \frac{6(6-3)}{2} \text{ and } G_3 \cong K_6.$$

For a similar construction of $G = C_8$ add the degree affinity edges v_1u_4, u_1v_2, u_2v_3, and u_3v_4 on the 1^{st}-iteration. For the 2^{nd}-iteration add u_1v_4, v_1u_2, v_2u_3, v_3u_4. For the 3^{rd}-iteration add v_1u_3, u_1v_3, u_2v_4, v_2u_4. For the 4^{th}-iteration add v_2u_2, v_3u_3, v_1v_4, u_1u_4. For the exhaustive 5^{th}-iteration add v_1v_3, v_2v_4, u_1u_3, u_2u_4. During all iterations the maximal degree affinity edges was a maximum. Therefore, $k = 5 = 2 \times 4 - 3$, $\eta(C_8) = 20 = \frac{(2\times4)((2\times4)-3)}{2}$ and $G_{k=5} \cong K_{2\times4}$. Through immediate induction it follows that the EIPC can be utilized to add all degree affinity edges of the cycle $C_{2\ell}$ to obtain $K_{2\ell}$ through, $k = 2\ell - 3$ iterations. Finally,

$$\eta(C_{2\ell}) = \frac{2\ell(2\ell-1)}{2} - 2\ell = \frac{2\ell(\ell-3)}{2}.$$

This settles the result. □

An immediate corollary follows.

Corollary 2.3 *For an odd cycle C_n, $n \geq 5$ the MDAC exhausts after $k = n - 3$ iterations and*

$$\eta(C_n)_{n,odd} = \frac{(n-2)(n-3)}{2}.$$

Proof Let $\ell = \lfloor \frac{n}{2} \rfloor$. Construct an odd cycle C_n, $n \geq 5$ as follows; take the disjoint union of paths P and Q on the vertices $v_1, v_2, v_3, \cdots, v_\ell$ and $u_1, u_2, u_3, \cdots, u_{\ell+1}$, respectively. Add the edges v_1u_1 and $v_\ell u_{\ell+1}$. Consider the path $v_\ell u_{\ell+1}u_\ell$ equivalent to the *ghost edge* $v_\ell u_\ell$ and apply the EIPC to add degree affinity edges to $v_1, v_2, v_3, \cdots, v_\ell$ and $u_1, u_2, u_3, \cdots, u_\ell$ similar to that for, C_{n-1}. The aforesaid step minimized the number of vertices which could not be paired to permit additional degree affinity edges. The exhaustive iteration of the MDAC is the addition of the single edge $v_\ell u_\ell$. Hence, a total of $\frac{(n-1)(n-4)}{2} + 1 = \frac{(n-2)(n-3)}{2}$ degree affinity edges have been added.

Clearly the procedure yields $\eta(C_n)_{n,odd} = max\{\eta(k)$: over all choices of maximal number of absolute distinct pairs of distinct non-adjacent vertices of equal degree through exhaustive application of the MDAC$\} = \frac{(n-2)(n-3)}{2}$. The result $k = n - 3$ follows by similar reasoning. □

Theorem 2.4 *If a non-complete r-regular graph G of even order n reaches completeness on exhaustion of the MDAC then,*

$$k = n - (r+1) \quad and \quad \eta(G) = \frac{n(n-(1+r))}{2}.$$

Proof Let G be a non-complete regular graph of even order n. Since G reaches completeness on exhaustion of the MDAC it is possible to find $\frac{n}{2}$ absolutely distinct pairs of distinct, non-adjacent vertices for the 1^{st}-iteration. If the aforesaid is not possible it implies that some vertex $v \in V(G)$ exists which is adjacent to all vertices in $V(G)\backslash v$. Hence, $deg_G(v) = n - 1$ implying that G is complete. This is a contradiction.

Add the $\frac{n}{2}$ degree affinity edges to obtain G_1. Clearly, G_1 is regular. If G_1 is complete then the MDAC has exhausted. Else, since G reaches completeness on exhaustion of the MDAC, so does G_1. Hence, repeat the reasoning iteratively.

Since, the number of degree affinity edges which must be added to any vertex $v \in V(G)$ is, $(n-1) - r = n - (r+1)$ the result, $k = n - (r+1)$ is settled. The aforesaid implies that

$$\eta(G) = \frac{n(n-(1+r))}{2}.$$ □

§3. Degree Affinity Number of Path Graphs

It is assumed that the reader is familiar with a path graph (simply, a path) P_n, $n \geq 1$ on the vertices, $v_1v_2, v_3, \cdots, v_n$. Both paths P_1, P_2 are complete and path P_3 requires one iteration to exhaust the MDAC by adding one edge and is thereafter complete. Path P_4 requires two iterations to add three degree affinity edges to reach completeness. For the path $G = P_6$ consider the disjoint union on the paths $v_1v_2v_3$, $u_1u_2u_3$ and add the edge v_1u_1. During the 1^{st}-iteration of the MDAC the maximal (not maximum) degree affinity edges v_2u_2 and v_3u_3 are added. Then add the edges, v_1v_3 and u_1u_3 where-after, the application of the EIPC is followed to reach completeness. This procedure renders, $k = 4$, $\eta(P_6) = 10$ and $G_4 \cong K_6$.

Next we present a lemma for which the proof is left for the reader. Thereafter, the result for paths of even order $n \geq 8$ follows.

Lemma 3.1 *For the positive integers, n_1, m_1 and n_2, m_2 with, $n_1 \geq n_2$; $m_1 \leq m_2$ and $n_1 + m_1 = n_2 + m_2$ we have that,*

$$\frac{n_1(n_1-1)}{2} + \frac{m_1(m_1-1)}{2} \geq \frac{n_2(n_2-1)}{2} + \frac{m_2(m_2-1)}{2}.$$

From a graph theoretical perspective, Lemma 3.1 states that under the stated conditions,

$$\varepsilon(K_{n_1}) + \varepsilon(K_{m_1}) \geq \varepsilon(K_{n_2}) + \varepsilon(K_{m_2}).$$

Theorem 3.2 *For an even path P_n, $n \geq 8$ the MDAC exhausts after $k = n - 4$ iterations and*

$$\eta(P_n)_{n,even} = \frac{n^2 - 7n + 14}{2}.$$

Proof Consider the path $P_{2\ell}$, $\ell = 4, 5, \cdots$. Construct the path as follows; take the disjoint union of paths P and Q on the vertices $v_1, v_2, v_3, \cdots, v_\ell$ and $u_1, u_2, u_3, \ldots, u_\ell$, respectively and add the edge v_1u_1. From Lemma 3.1 it follows that for the 1^{st}-iteration add the degree affinity edges $v_\ell u_\ell$. Consider the path $v_{\ell-1}v_\ell u_\ell u_{\ell-1}$ equivalent to the *ghost edge* $v_{\ell-1}u_{\ell-1}$ and add those degree affinity edges to $C_{2\ell-2}$ for vertices v_i, u_i, $1 \leq i \leq (\ell - 1)$ by the EIPC in Theorem 2.3. The aforesaid iteration ensures the maximum number of vertices of degree 3. Hence, it minimizes the number of vertices i.e. two vertices, which exhaust on 1^{st}-iteration. Finally, on exhaustion of the EIPC add the degree affinity edge, $v_{\ell-1}u_{\ell-1}$. Relying on the induction reasoning of Theorem 2.3 the results, $k = (n-2) - 3 + 1 = n - 4$ and follows

$$\eta(P_n)_{n,even} = \frac{(n-2)((n-2)-3)}{2} + 2 = \frac{n^2 - 7n + 14}{2}. \qquad \square$$

Corollary 3.3 *An even path P_n, $n \geq 8$ does not reach completeness with the application of the MDAC.*

Proof The proof is a direct consequence of the proof of Theorem 3.2. □

A path P_2 i.e. v_1v_2 and path Q_3 i.e. $u_1u_2u_3$ with the added edge v_1u_1 yield the odd path P_5. It is easy to see that three iterations are needed to exhaust the MDAC. For path P_5 we have, $k = 3$, $\eta(P_5) = 4$.

Theorem 3.4 *For an odd path P_n, $n \geq 7$ the MDAC exhausts after $k = n - 5$ iterations and*

$$\eta(P_n)_{n,odd} = \frac{n^2 - 9n + 24}{2}.$$

Proof Let path $P_{2\ell+1}$ be constructed from paths P and Q on the vertices $v_1, v_2, v_3, \cdots, v_\ell$ and $u_1, u_2, u_3, \cdots, u_{\ell+1}$ by adding edge v_1u_1.

Case 1. From Lemma 3.1 it follows that, for P_7 add the degree affinity edge, v_3u_4 and consider the path $v_2v_3u_4u_3u_2$ equivalent to the *ghost edge* v_2u_2 to add those degree affinity edges to C_4. In the 2^{nd} iteration which exhausts the MDAC, add degree affinity edges, v_2u_2 and v_3u_3.

Case 2. From Lemma 3.1 it follows that for P_n, $n \geq 9$ add the degree affinity edges, $v_\ell u_{\ell+1}$ and consider the path $v_{\ell-1}v_\ell u_{\ell+1}u_\ell u_{\ell-1}$ equivalent to the *ghost edge* $v_{\ell-1}u_{\ell-1}$ to add those degree affinity edges to $C_{\ell-1}$. During the 2^{nd}-iteration include the degree affinity edge $v_\ell u_\ell$. For the exhaustive iteration add the edge $v_{\ell-1}u_{\ell-1}$.

In both cases the maximum number of degree affinity edges were obtained in that, the minimum number of vertices exhausted respectively at one vertex of degree 1 and two vertices of degree 3. The parameters $k = n - 5$ and

$$\eta(P_n)_{n,odd} = \frac{n^2 - 9n + 24}{2}$$

follow easily through similar reasoning found for even paths. □

§4. Certain General Results

Denote a null graph (edgeless) of order n by, $\mathfrak{N}_n$. Consider two null graphs i.e. $\mathfrak{N}_{n_1}$, $\mathfrak{N}_{n_2}$ on the vertices v_i,$i = 1, 2, 3, \cdots, n_1$ and u_j, $j = 1, 2, 3, \ldots, n_2$, respectively. Partial MDAC is performed by considering only distinct pairs of vertices $\{v_i, u_j\}$. This partial MDAC operation between the null graphs are denoted by, $\mathfrak{N}_{n_1} \uplus \mathfrak{N}_{n_2}$. The next lemma has a trivial proof.

Lemma 4.1 *For the null graphs, $\mathfrak{N}_{n_1}$, $\mathfrak{N}_{n_2}$ with $n_1 = n_2$, graph $\mathfrak{N}_{n_1} \uplus \mathfrak{N}_{n_2}$ is a complete bipartite graph.*

A direct consequence of Lemma 4.1 is given as a corollary.

Corollary 4.2 *A null graph $\mathfrak{N}_{n_1}$, n_1 is even, reaches completeness on exhaustion of the MDAC.*

It is obvious that a null graph of odd order will have an isolated vertex on exhaustion of the MDAC. The general result implied by Lemma 4.1 is stated below.

Theorem 4.3 *Consider two graphs G and H of equal order say, n, which reach completeness on exhaustion of the MDAC, respectively. Then the disjoint union $G \cup H$ reaches completeness on exhaustion of the MDAC.*

Proof Since both G and H reach completeness independently in compliance to MDAC the first phase is to apply the MDAC independently to G and H. Thereafter, apply the MDAC to the distinct pairs $\{v, u\}$, $v \in V(G)$, $u \in V(H)$ as required. Lemma 4.1 guarantees that completeness is reached. □

Example 4.1 Let G and H be two copies of $K_1 + C_6$. Label the two K_1-vertices as v_0 and u_0, respectively. During the 1^{st}-iteration of the MDAC add the appropriate edges to the $C_6^{'s}$ (see figure 2) as well as the edge v_0u_0. Thereafter, apply the MDAC independently until both copies reach completeness and apply the MDAC to the two distinct complete subgraphs, $K_6^{'s}$, finally.

In respect of the join of two graphs a straight forward result is presented.

Theorem 4.4 *For the join of any two graphs G and H it follows that, $\eta(G+H) = \eta(G)+\eta(H)$.*

Proof Let graphs G and H be of order n and m, respectively. In the join $G+H$ each vertex $v \in V(G)$ has $deg_{G+H}(v) = deg_G(v) + m$. Similarly, $\forall\ u \in V(H)$, $deg_{G+H}(u) = deg_H(u) + n$. Furthermore, the edge vu exists for all pairs $\{v, u\}$, $v \in V(G)$, $u \in V(H)$. Note that the join operation did not change the adjacency property within G and H *per se*. Therefore, the prescriptions of the MDAC applies to G and H as before. Hence, applying the MDAC to $G+H$ is equivalent to applying the MADC independently to G and H simultaneously. Therefore the result. □

Theorem 4.4 yields a useful corollary.

Corollary 4.5 *Let G of order n have a single vertex v with $deg_G(v) = n-1$ then, $\eta(G) = \eta(G-v)$.*

Proof Since $v \in V(G)$ is the only vertex with $deg_G(v) = n-1 = \Delta(G)$ the graph can be denoted by, $K_{1_{(=v)}} + (G-v)$. By Theorem 4.4, $\eta(G) = \eta(K_1) + \eta(G-v) = 0 + \eta(G-v)$. □

Corollary 4.5 permits immediate results for specialized classes of graphs such as wheels, fans, stars, windmill graphs and others. In the next result Nordhaus-Gaddum type bounds are presented.

Proposition 4.6 *For any graph G of order n,*

$$\max\{\eta(G), \eta(\overline{G})\} \leq \eta(G) + \eta(\overline{G}) \leq \varepsilon(G),$$

$$0 \leq \eta(G)\cdot\eta(\overline{G}) \leq \left\lceil \frac{\varepsilon(K_n)}{2} \right\rceil \left\lfloor \frac{\varepsilon(K_n)}{2} \right\rfloor.$$

Proof The proof is respectively on the two inequalities following.

(i) If both G and $\overline{G}$ reach completeness on exhaustion of the MDAC then $\eta(G) + \eta(\overline{G}) = \varepsilon(G)$. Note that equally does not always hold as can be seen by say, $G = \mathfrak{N}_5$ since $\overline{\mathfrak{N}_5} = K_5$. Since $\eta(G) + \eta(\overline{G}) > \varepsilon(G)$ is impossible for simple graphs it follows that, $\eta(G) + \eta(\overline{G}) \leq \varepsilon(G)$. The lower bound is trivial. Hence, $max\{\eta(G), \eta(\overline{G})\} \leq \eta(G) + \eta(\overline{G}) \leq \varepsilon(G)$.

(ii) It is known that for any two non-negative integers a, b the product,

$$ab \leq \left\lceil \frac{a+b}{2} \right\rceil \left\lfloor \frac{a+b}{2} \right\rfloor.$$

Hence,

$$\eta(G)\cdot\eta(\overline{G}) \leq \left\lceil \frac{\varepsilon(K_n)}{2} \right\rceil \left\lfloor \frac{\varepsilon(K_n)}{2} \right\rfloor$$

is immediate. The lower bound is trivial. Therefore,

$$0 \leq \eta(G) \cdot \eta(\overline{G}) \leq \left\lceil \frac{\varepsilon(K_n)}{2} \right\rceil \left\lfloor \frac{\varepsilon(K_n)}{2} \right\rfloor . \qquad \square$$

The number of vertices of graph G which has degree equal to $\delta(G)$ is denoted by, $\theta(G)$.

Theorem 4.7 *If an incomplete graph G or an iterative graph G_i has odd value $\theta(G)$ or, odd value $\theta(G_i)$ then G cannot reach completeness on exhaustion of the MDAC.*

Proof It is sufficient to proof the result for G only. Since G is incomplete it has order $n \geq 2$. Let G have odd value $\theta(G) \geq 3$. Since exactly one vertex say, u, $deg_G(u) = \delta(G)$ cannot be paired it follows that $deg_{G_1}(u) = deg_G(u)$ after the first iteration. Clearly, $\delta(G_1) = \delta(G)$, $\theta(G_1) = 1$ and $\delta(G_i) = \delta(G)$, $\theta(G_i) = 1$ for all G_i through to exhaustion of the MDAC. Therefore, the results holds. $\square$

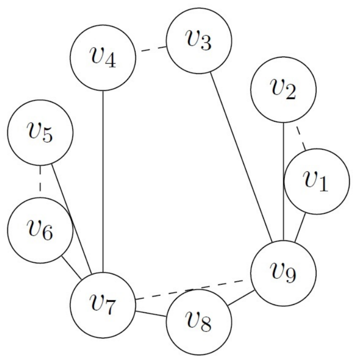

Figure 1 Graph G for which
$\delta(G) = 1$, $\theta(G) = 6$ and $\delta(G_1) = 2$, $\theta(G_1) = 7$.

Figure 3 illustrates the applicability of Theorem 4.7. Note that G is a tree on solid lines and G_1 has the dotted degree affinity edges added on 1^{st}-iteration.

§5. Conclusion

The numerous well-defined classes of graphs indicate that a wide scope for further research exists. It is conjectured that a salient dual problem has been solved without explicit mentioning or proof thereof. The claim is that the number of MDAC iterations k required to yield $\eta(G)$, is a minimum.

Problem 5.1 *Prove or disprove the claim that in applying the MDAC exhaustively, the number of iterations k to yield $\eta(G)$ is, $k = min\{k' : \eta(k') = \eta(G)\}$.*

Finding an efficient algorithm to ensure that during each iteration of the MDAC, the maximum (rather than maximal) number of degree affinity edges is added, is a worthy research problem.

Theorem 4.4 provides a result for the join of graphs. This prompts the next problems.

Problem 5.2 *Prove or disprove that, for the disjoint union $G \cup H$ of any two graphs G, H we have:* $\eta(G) + \eta(H) \leq \eta(G \cup H)$.

Problem 5.3 *Prove or disprove that, for the corona of any two graphs G and, H of order m we have:* $\eta(G) + m\eta(H) \leq \eta(G \circ H)$.

Theorem 4.7 as illustrated in Figure 3 suggests that it could be hard or impossible to characterize graphs which reach completeness on exhaustion of the MDAC.

Problem 5.4 *If possible, characterize graphs which reach completeness on exhaustion of the MDAC.*

The graph K_1 has the property that, $\eta(K_1) = 0 = \eta(\overline{K_1})$.

Problem 5.5 *Do graphs $G \ncong K_1$ exist for which, $\eta(G) = 0 = \eta(\overline{G})$?*

Figures 1 and 2 suggest that there is a *flawed strategy* in application of the MDAC in that it may result in not yielding $\eta(G)$. Therefore, the problem could possibly be approached as a two person zero-sum game.

Problem 5.6 *If possible, analize the optimal application of the MDAC as a two person zero-sum game.*

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