International J.Math. Combin. Vol.1(2021), 85-90

Graphs with Large Semitotal Domination Number

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Abstract: Let $G = (V, E)$ be a graph without isolated vertex. A set $D \subseteq V$ is a semitotal dominating set of G if it is a dominating set and every vertex in D is within distance 2 of another vertex in D. The minimaum cardinality of a semitotal dominating set is called the semitotal domination number of G and is denoted by $\gamma_{t2}(G)$. In this paper we obtain an upper bound of this parameter and characterize the corresponding extremal graphs.

Key Words: Domination number, total domination number and semitotal domination number.

AMS(2010): 05C69.

§1. Introduction

The graph $G = (V, E)$ we mean a finite, undirected, graph with neither loops nor multiple edges and without isolated vertex. The order and size of G are denoted by n and m respectively. The degree of a vertex u in G is the number of edges incident with u and is denoted by $d(u)$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [3].

Let $v \in V$. The open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$. If $S \subseteq V$ then $N(S) = \bigcup_{v \in S} N(v)$ for all $v \in S$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbor set of u with respect to S is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. For any set $S \subseteq V$, the subgraph induced by S is the maximal subgraph of G with vertex set S and is denoted by $\langle S \rangle$. The vertex has degree one is called a pendant vertex. A support is a vertex which is adjacent to a pendant vertex. A weak support is a vertex which is adjacent to exactly one pendant vertex. A strong support is a vertex which is adjacent to at least two pendant vertices. An unicyclic graph is a graph with exactly one cycle. A graph without cycle is called acyclic graph and a connected acyclic graph is called a tree. $H(m_1, m_2, \cdots, m_n)$ denotes the graph obtained from the regular graph H by attaching m_i pendant edges to the vertex $v_i \in V(H), 1 \leq i \leq n$. The graph $K_2(m_1, m_2)$ is called bistar and it is also denoted by $B(m_1, m_2)$.

A subset D of V is called a dominating set of G if every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D of a graph G is called a total

[1]Received September 27, 2020, Accepted March 12, 2021.

dominating set of G if the graph induced by D has no isolated vertex.The minimum cardinality of a total dominating set is called the total domination number of G and is denoted by $\gamma_t(G)$. W.Goddard.et.al [2] introduced the concept of semitotal domination in graphs. A set D of vertices in a graph G with no isolated vertices to be a semitotal dominating set(STD-set) of G if it is a dominating set of G and every vertex in D is within distance 2 of another vertex of D. The minimum cardinality of a semitotal dominating set is called the semitotal domination number of G and is denoted by $\gamma_{t2}(G)$

Theorem 1.1 *For a cycle* $C_n, \gamma_{t2}(C_n) = \left\lceil \frac{2n}{5} \right\rceil$.

Observation 1.1 *Since any STD-set of a spanning subgraph H of a graph G is a STD-set of G, we have* $\gamma_{t2}(G) \leq \gamma_{t2}(H)$.

Observation 1.2 *If G is a disconnected graph with k components* $G_1, G_2, \cdots, G_k$ *then* $\gamma_{t2}(G) = \gamma_{t2}(G_1) + \gamma_{t2}(G_2) + \cdots + \gamma_{t2}(G_k)$.

§2. Main Results

Theorem 2.1 *For any graph G,* $\gamma_{t2}(G) \leq n-\Delta+1$. *Further,* $\gamma_{t2}(G) = n-\Delta+1$ *if and only if G is isomorphic to H or $sK_2 \cup H$ where H is any graph having a vertex v with* $d(v) = |V(H)| - 1$.

Proof Let $v \in V(H)$ and $d(v) = \Delta$. Let $S = N(v) - \{u\}$ where $u \in N(v)$. Then $V - S$ is a STD-set of G and hence $\gamma_{t2}(G) \leq n - \Delta + 1$. Now, let G be a graph with $\gamma_{t2} = n - \Delta + 1$.

Case 1. G is connected.

Let $v \in V(G)$ such that $d(v) = \Delta$. If $\Delta < n - 1$ then $V - N(v)$ is a STD-set of G with $|V - N(v)| = n - \Delta$, which is a contradiction. Hence $\Delta = n - 1$ and $d(v) = n - 1$. Thus $G = H$.

Case 2. G is disconnected.

Let $G_1, G_2, \cdots, G_k$ be the components of G and let $|V(G_i)| = n_i, 1 \leq i \leq k$. If $\Delta = 1$, then $\gamma_{t2}(G) = n$ and each G_i is isomorphic to K_2. Suppose $\Delta \geq 2$. Let $v \in V(G_1)$ be such that $d(v) = \Delta$. Since $\gamma_{t2}(G) = n-\Delta+1$, we have $\gamma_{t2}(G_1) = n_1-\Delta+1$ and $\gamma_{t2}(G_i) = n_i, i \geq 2$. Hence by case 1, G_1 is isomorphic to H where H is any graph contains a vertex v with $d(v) = |V(H)|-1$ and G_i is isomorphic to $K_2, i \geq 2$. The converse is obvious. □

Theorem 2.2 *Let G be a connected graph with $\Delta < n - 1$. Then*

$$\gamma_{t2}(G) \leq n - \Delta.$$

Proof Let $v \in V(G), d(v) = \Delta$. Clearly, $V - N(v)$ is a STD-set of G, which implies that

$$\gamma_{t2}(G) \leq n - \Delta.$$

□

Theorem 2.3 *Let T be a tree with $n \geq 3$. Then $\gamma_{t2}(T) = n - \Delta$ if and only if T can be obtained from a star by subdividing k of its edges, $1 \leq k \leq \Delta - 1$.*

Proof Let T be a tree with $\gamma_{t2}(T) = n - \Delta$. Let $u \in V(T)$ and $d(u) = \Delta$. It is clear that T is not a star graph and hence $\Delta < n - 1$. Let $N(u) = \{u_1, u_2, \cdots, u_\Delta\}, X = V(T) - N[v] = \{x_1, x_2, \cdots, x_k\}$ and let $T_1 = \langle X \rangle$.

Suppose $E(T_1) \neq \phi$. Let G_1 be a nontrivial component of T_1 and we assume that $u_1 \in N(x_1)$, where $x_1 \in V(G_1)$. Since G_1 is nontrivial, there exists a vertex $x_2 \in V(G_1)$ such that $x_1x_2 \in E(G_1)$. Then $D = V - (N(u) \cup \{x_2\})$ is a STD-set of cardinality $n - \Delta - 1$, which is a contradiction. Hence G_1 is trivial and hence $E(T_1) = \phi$.

If $d(u_i) \geq 3$ for some $u_i \in N(u)$, then $D = \{u, u_i\} \cup [X - (N(u_i) \cap X)]$ is a STD-set of T and $|D| \leq n - \Delta - 1$, which is a contradiction. Hence $d(u_i) \leq 2$. Suppose $d(u_i) = 2$ for all $i, 1 \leq i \leq \Delta$. Then $D = N(u)$ is a STD-set of G and $|D| = |N(u)| = \Delta = n - (n - \Delta) = n - (2\Delta + 1 - \Delta) = n - (\Delta + 1) = n - \Delta - 1$, which is a contradiction. Hence $d(u_i) = 1$ for some i. Thus T is obtained from $K_{1.\Delta}$ by subdividing k of its edges, $1 \leq k \leq \Delta - 1$. The converse is obvious. □

Notation 2.1 We define the graphs $G_i, 1 \leq i \leq 7$ as follows:

(1) $G_1 = C_3(m_1, 1, 0), m_1 \geq 1$;

(2) $G_2 = C_3(m_1, 1, 1), m_1 \geq 1$;

(3) G_3 is a graph obtained from $C_3(m_1, 0, 0), m_1 \geq 1$, by subdividing at least one pendant edge once;

(4) G_4 is a graph obtained from $C_3(m_1, 1, 0), m_1 \geq 2$, by subdividing t pendant edges which are incident with a vertex of degree $\Delta, 1 \leq t \leq m_1 - 1$;

(5) G_5 is a graph obtained from $C_3(m_1, 1, 1), m_1 \geq 2$, by subdividing t pendant edges which are incident with a vertex of degree $\Delta, 1 \leq t \leq m_1 - 1$;

(6) $G_6 = C_4(m_1, 0, 0, 0), m_1 \geq 1$;

(7) G_7 is a graph obtained from $C_4(m_1, 0, 0, 0), m_1 \geq 1$, by subdividing at least one pendant edge once.

Theorem 2.4 *Let G be a connected unicyclic graph with cycle $C = (v_1, v_2, \cdots, v_r, v_1)$. Then $\gamma_{t2}(G) = n - \Delta$ if and only if G is isomorphic to either C_4 or $G_i, 1 \leq i \leq 7$.*

Proof Let G be an unicyclic graph with cycle C and $\gamma_{t2} = n - \Delta$. If $G = C$ then it follows from Theorem 1.1 that $n \leq 4$ and hence G is isomorphic to C_4.

Suppose $G \neq C$. Let X denote the set of all pendant vertices in G and let $|X| = k$. Clearly,

$$\Delta - 2 \leq k \leq \Delta. \tag{1}$$

Claim 1. If $v \in V(G)$ and $d(v) = \Delta$ then v lies on C.

Suppose v is not on C. Then $k = \Delta - 1$ or Δ. Let $v_1 \in V(C)$ such that $d(v, v_1) = d(v, C)$. Then $D = V - (X \cup \{v_2, v_3\})$ is a STD-set with $|D| \leq n - \Delta - 1$, which is a contradiction. Hence v lies on C. Let $C = (v_1, v_2, \cdots, v_r, v_1)$ and let $d(v_1) = \Delta$.

Claim 2. $d(x) = 1$ or 2 for all $x \in V(G) - V(C)$.

Suppose there exists a vertex $x \in V(G) - V(C)$ with $d(x) \geq 3$. Then $k = \Delta - 1$ or Δ. If $k = \Delta - 1$ then all the vertices of $V(C) - \{v_1\}$ have degree 2 and hence $D = V(G) - [X \cup \{v_2, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$.

If $k = \Delta$ then at least one vertex v_i on C has degree 2. Then $D = V(G) - [X \cup \{v_i\}]$ is a STD-set of G with $|D| < n - \Delta$. Hence $d(x) = 1$ or 2 for all $x \in V(G) - V(C)$.

Claim 3. Every vertex of $V(C) - \{v_1\}$ has degree 2 or 3.

Inequality (1) gives that $d(v_i) \leq 4$ for all $i \neq 1$. Suppose that $v_i \in V(C)$ with $d(v_i) = 4$ for some i. Then $k = \Delta$ and $d(v_j) = 2$ for all $j \neq 1, i$. Hence $D = V(G) - [X \cup \{v_i\}]$ is a STD-set of G with $|D| < n - \Delta$. This proves claim 3.

Claim 4. $r \leq 4$.

Suppose $r \geq 5$. If $k = \Delta$, then there exists a vertex v_i such that $d(v_i) = 2$ and $D = V(G) - [X \cup \{v_i\}]$ is a STD-set of G with $|D| = n - \Delta - 1$. If $k = \Delta - 1$, then there exist two adjacent vertices v_i and v_j such that $d(v_i) = d(v_j) = 2$. Hence $D = V(G) - [X \cup \{v_i, v_j\}]$ is a STD-set of G with $|D| = n - \Delta - 2$.. If $k = \Delta - 2$, then every vertex of $V(C) - \{v_1\}$ has degree 2 and hence $D = V(G) - [X \cup \{v_2, v_3, v_5\}]$ is a STD-set of G with $|D| < n - \Delta$. Thus $r \leq 4$.

Now, we only need to consider two cases following.

Case 1. $r = 3$

Suppose, there exists a vertex $x_1 \in X$ such that $d(x_1, C) \geq 3$. Let $(x_1, x_2, \cdots, x_s, v_i), s \geq 3$ be the unique $x_1 - v_i$ path. If $k = \Delta - 2$, then $D = V(G) - [X \cup \{x_s, v_2, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$. Let $k = \Delta - 1$. We assume $d(v_2) = 3$. If $i = 1$ then $D = V(G) - [X \cup \{x_s, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$. If $i = 2$ then $D = V(G) - [X \cup \{v_2, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$, which is a contradiction. If $k = \Delta$ then $D = V(G) - [X \cup \{x_s\}]$ is a STD-set of G with $|D| < n - \Delta$, which is a contradiction. Hence every $x \in X, d(x, C) \leq 2$.

Suppose $d(x, C) = 1$ for all $x \in X$. If $k = \Delta - 2$ then $d(v_1) = n - 1$ and hence $\gamma_{t2}(G) = n - \Delta + 1$,which is a contradiction. If $k = \Delta - 1$ then G is isomorphic to G_1. If $k = \Delta$ then G is isomorphic to G_2.

Suppose $d(x, C) = 2$ for some $x \in X$. If $k = \Delta - 2$ then G is isomorphic to G_3 If $k = \Delta - 1$ then G is isomorphic to G_4. If $k = \Delta$ then G is isomorphic to G_5.

Case 2. $r = 4$.

Suppose, there exists a vertex $x_1 \in X$ such that $d(x_1, C) \geq 3$. Let $(x_1, x_2, \cdots, x_s, v_i), s \geq 3$ be the unique $x_1 - v_i$ path. If $k = \Delta - 2$, then $D = V(G) - [X \cup \{x_s, v_2, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$. If $k = \Delta - 1$ or Δ, then there exists a vertex in C, say v_2 with $d(v_2) = 2$. Then $D = V(G) - [X \cup \{x_s, v_2\}]$ is a STD-set of G with $|D| < n - \Delta$. Hence $d(x, C) \leq 2$ for all $x \in X$. Now, if $k = \Delta - 2$, then G is isomorphic to $G_i, 6 \leq i \leq 7$. If $k = \Delta - 1$ or Δ, then there is no graphs satisfy $\gamma_{t2}(G) = n - \Delta$. The converse is obvious. □

Theorem 2.5 *Let G be a connected graph with $\gamma_{t2}(G) = n - \Delta$ and let v be a vertex of G with $d(v) = \Delta$. Then, each vertex $u \in N(v), |N(u) \cap V(G - N[v])| \leq 1$.*

Proof Suppose there exists a vertex $u \in N(v)$ such that u is adjacent to k vertices in $G - N[v], k \geq 2$. Let $X = \{x_1, x_2, \cdots, x_k\}$ be the set of vertices in $G - N[v]$ such that $ux_i \in E(G), 1 \leq i \leq k$. Then $D = [V - (N(v) \cup X)] \cup \{u\}$ is a STD-set of G and $|D| = n - (\Delta + k) + 1 \leq n - \Delta - 1$, which is a contradiction. Hence the result follows. □

Theorem 2.6 *Let G be a graph with $\Delta(G) = 2$. Then $\gamma_{t2}(G) = n - \Delta$ if and only if G is isomorphic to one of the following graphs:*

(1) $C_3 \cup P_3 \cup sK_2, 6 + 2s = n$;

(2) $C_4 \cup sK_2, 4 + 2s = n$;

(3) $2C_3 \cup sK_2, 6 + 2s = n$;

(4) $P_4 \cup sK_2, 4 + 2s = n$;

(5) $2P_3 \cup sK_2, 6 + 2s = n$.

Proof Let G be a graph with $\Delta = 2$ and $\gamma_{t2} = n - \Delta$. It is clear that every component of G is either a path or a cycle. If there exists a component G_1 of G with $|V(G_1)| = n_1 \geq 5$, then $\gamma_{t2}(G_1) \leq n_1 - 3$ and hence $\gamma_{t2}(G) \leq n - 3$, which is a contradiction. Thus the order of each component of G is at most 4.

Further, if G has three components which are cycles of order 3 or 4 then $\gamma_{t2} \leq n-3 < n-\Delta$. Hence at most two components of G are a cycle of order 3 or 4. Suppose two components of G be cycles. If G contains cycles C_3 and C_4 then $\gamma_{t2} \leq n - 3 < n - \Delta$, which is a contradiction. If G contains $2C_4$ then $\gamma_{t2} \leq n - 4 < n - \Delta$, which is a contradiction. Thus G contains $2C_3$ and hence $G = 2C_3 \cup sK_2$ where $6 + 2s = n$.

Suppose exactly one component of G be a cycle. Let it be C. Suppose $C = C_4$. If G contains a path of order 3 or 4, then $\gamma_{t2} \leq n-3$, which is a contradiction. Hence $G = C_4 \cup sK_2$ where $4 + 2s = n$. Let $C = C_3$. If G contains a path of order 4, then $\gamma_{t2} \leq n - 3$, which is a contradiction. If G contains $2P_3$, then $\gamma_{t2} \leq n-3$, which is a contradiction. If G contains no P_3 then $\gamma_{t2} = n-1$ which is a contradiction. Thus G has exactly one P_3 Hence $G = C_3 \cup P_3 \cup sK_2$ where $6 + 2s = n$.

Suppose no components of G is a cycle. Then all the components of G are paths. If G has three components which are paths of order 3 or 4, then $\gamma_{t2} \leq n - 3 < n - \Delta$. Hence at most two components of G are paths of order 3 or 4. If G contains a P_3 and a P_4, then $\gamma_{t2} \leq n - 3 < n - \Delta$, which is a contradiction. If G has $2P_4$ then $\gamma_{t2} \leq n - 4 < n - \Delta$, which is a contradiction.Hence G contains $2P_3$ or one P_4. Hence G is isomorphic to $2P_3 \cup sK_2$ where $6 + 2s = n$ or $P_4 \cup sK_2$ where $4 + 2s = n$. The converse is obvious. □

Theorem 2.7 *Let G be a connected graph and let v be a vertex of degree Δ . If $V - N[v]$ is an independent set and every vertex in $N(v)$ is adjacent to at most one vertex in $V - N[v]$, then $\gamma_{t2}(G) = n - \Delta$ or $n - \Delta - 1$*

Proof Let D be a $\gamma_{t2}-$set. Since every vertex of $N(v)$ is adjacent to at most one vertex in $V - N[v]$, it follows that $|D| \geq |V - N[v]|$. Hence $\gamma_{t2} \geq n - (\Delta + 1)$. Also $V - (N(v))$ is a STD-set and hence $\gamma_{t2}(G) \leq n - \Delta$. Thus $\gamma_{t2} = n - \Delta$ or $n - \Delta - 1$. □

Theorem 2.8 *Let G be a connected graph and let v be a vertex of degree Δ. If*

(1) $V - N[v]$ is an independent set;

(2) Every vertex in $N(v)$ is adjacent to at most one vertex in $V - N[v]$;

(3) $N(v)$ Contains a vertex of degree one,

then, $\gamma_{t2}(G) = n - \Delta$.

Proof Let D be a γ_{t2}-set of G. Let $u \in N(v)$ be a vertex of degree 1. It follows from Theorem 2.7 that $|D| = n - \Delta$ or $n - \Delta - 1$. Since u is a pendent vertex of $G, v \in D$. Also it follows from $i)$ and $ii)$ that D contains $n - \Delta - 1$ vertices for dominating the vertices of $V - N[v]$. Hence $\gamma_{t2} = |D| = n - \Delta$. □

Theorem 2.9 *Let G be a connected graph with bipartition $\{V_1, V_2\}$ and let $v \in V_1$ with $d(v) = \Delta$. Suppose $\gamma_{t2} = n - \Delta$. Then the following conditions are satisfied. $i)|V_2| = \Delta(G)$. $ii)$ Every pair of vertices $u, w \in V_1, u \neq v, w \neq v$ such that $N(u) \cap N(w) = \phi$. $iii)$ Each vertex in V_2 has degree at most 2 and at least one vertex of V_2 has degree 2.*

Proof Let $N(v) = \{v_1, v_2, \cdots, v_\Delta\}$. Since $N(v) \subseteq V_2, \Delta(G) \leq |V_2|$. If there exists a vertex $x \in V_2 - N(v)$, then since G is connected, $D = [V - (N(v) \cup \{x\})]$ is a STD-set of G with $|D| = n - \Delta - 1$ which is a contradiction. Hence $|V_2| = \Delta$.

Suppose $N(u) \cap N(w) \neq \phi$. Let $y \in N(u) \cap N(w)$. It is clear that $y \in N(v)$. Then $D = [V - (N(v) \cup \{u, w\})] \cup \{y\}$ is a STD-set of G with $|D| = n - \Delta - 1$ which is a contradiction. Hence $N(u) \cap N(w) \neq \phi$.

Suppose there exists a vertex $z \in V_2$ with $d(z) \geq 3$. Let $u, w \in V_1$ be such that $uz, wz \in E$. Then $N(u) \cap N(w) \neq \phi$ which is a contradiction. Hence each vertex in V_2 has degree at most 2. Also if all the vertices of V_2 have degree 1, then G is a star and $\gamma_{t2} \neq n - \Delta$. Hence at least one vertex of V_2 has degree 2. □

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