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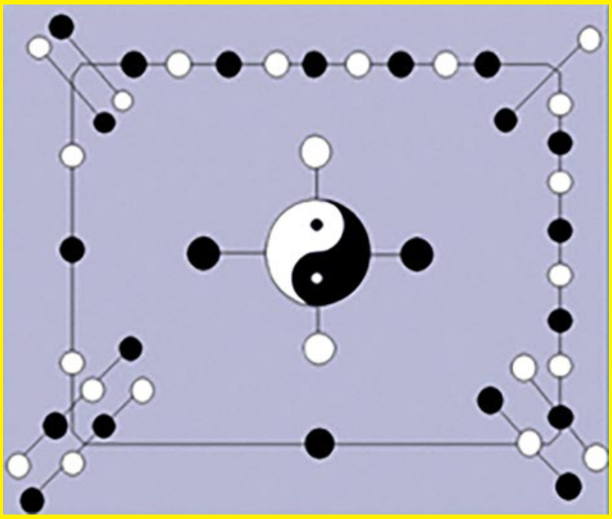

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PBIB-Designs and Association Schemes from Minimum Neighborhood Sets of Certain Jump Sizes of Circulant Graphs

B. Chaluvaraju, Vidya T. and Shaikh Ameer Basha

(Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bengaluru -560 056, India)

E-mail: bchaluvaraju@gmail.com, vidyaraghut@gmail.com, shaikhameerbasha@gmail.com

Abstract: A set $S \subseteq V$ of a graph $G = (V, E)$ is a Neighborhood set of G if $G = \bigcup_{v \in S} \langle N(v) \rangle$, where $\langle N(v) \rangle$ is the subgraph induced by v and all vertices adjacent to v. The neighborhood number, $\eta(G)$ is the minimum cardinality of a neighborhood set of G. The minimum neighborhood set S with $|S| = \eta(G)$ is called η-set. Generally, the partially balanced incomplete block(PBIB)-Designs are obtained from the family of strongly regular graphs. Surprisingly, in this paper we obtain the PBIB-Designs and m-association schemes for $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$ arising from η -sets of certain jump sizes of circulant graphs.

Key Words: Association schemes, PBIB-designs, neighborhood sets, circulant graph.

AMS(2010): 05B05, 05C51, 05C69, 05E30, 51E05.

§1. Introduction

Let $G = (V, E)$ be a finite and undirected graph with no loops and multiple edges of vertex set V and edge set E. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. For graph-theoretical terminologies which are not defined here, we follow [15].

For a given positive integer p, let $s_1, s_2, \cdots, s_t$ be a sequence of integers with $0 < s_1 < s_2 < \cdots < s_t < \frac{p+1}{2}$. The Circulant graph $C_p(s_1, s_2, \cdots, s_t)$ is the graph on p vertices labeled as $v_1, v_2, \cdots, v_p$ with vertex v_i adjacent to each vertex $v_{(i \pm s_j)(mod\ p)}$ and the values s_j; $1 \leq j \leq t$ are called jump sizes.

The applications are mainly in pure mathematics and technology which mysteriously reflects the abstract concrete dichotomy of the theory of Circulant. Also, which are important in digital encoding; this is a wondrous technology it enables devices ranging from computers to music players to recover from errors in transmission and storage of data and restore the original data. For more details, we refer to [17].

Bose and Nair [3] introduced a class of binary, equi-replicate and proper designs, which are called partially balanced incomplete block (PBIB)-Designs. In these designs, all the elementary contrasts are not estimated with the same variance. The variances depend on the type of association between the treatments. There are many applications of PBIB-Designs in cluster

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sampling, digital fingerprint codes and architecture of web solution. For more details on PBIB-Designs one can refer to [1], [6], [11], [12] and [13].

Given ν elements (objects or vertices), a relation satisfying the following conditions is said to be an association scheme with m classes:

(i) Any two elements are either first associates, or second associates, $\cdots$, or m^{th} associates, the relation of association being symmetric.

(ii) Each object x has n_k k^{th} associates, the number n_k being independent of x.

(iii) If two objects x and y are k^{th} associates, then the number of objects which are i^{th} associates of x and j^{th} associates of y is p^k_{ij} and is independent of the k^{th} associates x and y. Also $p^k_{ij} = p^k_{ji}$.

With the association scheme on ν objects, a PBIB-Design is an arrangement of ν objects into b sets (blocks) of size g where $g < \nu$ such that

(i) Every element is contained in exactly r blocks.

(ii) Each block contains g distinct elements.

(iii) Any two elements which are m^{th} associates occur together in exactly λ_m blocks.

The numbers ν, b, g, r, λ_1, λ_2, $\cdots$, λ_m are called the parameters of the first kind, whereas the numbers n_1, n_2, $\cdots$, n_m, p^k_{ij} $(i,\ j,\ k = 1,\ 2,\ \cdots,\ m)$ are called the parameters of the second kind.

Bose [2] has initiated the study of strongly regular graph with parameters (p, l, σ, μ) of a finite simple graph on p vertices, regular of degree l (with $0 < l < p-1$, so that there are both edges and nonedges), such that any two distinct vertices have σ common neighbors when they are adjacent, and μ common neighbors when they are nonadjacent. For more details on strongly regular graph and its related concepts, we refer to [4] and [14].

A set $S \subseteq V$ of a graph is a neighborhood set of G if $G = \bigcup_{v\in S}\langle N(v)\rangle$, where $\langle N(v)\rangle$ is the subgraph induced by v and all vertices adjacent to v. The neighborhood number $\eta(G)$ is the minimum cardinality of a neighborhood set of G. The minimum neighborhood set of S with $|S| = \eta(G)$ is called η-set. This concept was first introduced by Sampathkumar and Neeralagi [16]. For more details on neighborhood number and its related parameters, we refer to [7].

Slater [18] has introduced the concept of the number of dominating sets of G, which he denoted by $HED(G)$ in honor of Steve Hedetniemi. In this article, we will use $\tau_\eta(G)$ to denote the total number of η-set of a graph G. PBIB-Design associated with graph theoretic parameters are studied by [8], [9], [10] and [19].

§2. $\eta(G)$ and $\tau_\eta(G)$ for Different Jump Sizes of Circulant Graphs

2.1 Circulant Graph $C_p(1)$

The circulant graph with jump size one is also known as cycle C_p for $p \geq 3$, that is, $C_p(1) \cong C_p$ with $p \geq 3$.

Remark 2.1 The circulant graphs $C_4(1)$ and $C_5(1)$ are the strongly regular graphs.

Theorem 2.1 *For any circulant graph $G_1 = C_p(1)$ with $p \geq 4$ vertices*

(*i*) $\eta(G_1) = \left\lceil \frac{p}{2} \right\rceil$;

(*ii*) $\tau_\eta(G_1) = \begin{cases} 2 & \text{if } p \text{ is even}, \\ p & \text{if } p \text{ is odd.} \end{cases}$

Proof (*i*) The proof is due to [16].

(*ii*) Let $G_1 = C_p(1)$ be a circulant graph with $p \geq 4$ vertices labeled as $v_1, v_2, \cdots, v_p$. We have two cases for discussion:

Case 1. If p is even then by (i), we have $\eta(G_1) = \frac{p}{2}$ with two disjoint η-sets of G_1 as $\{v_1, v_3, v_5, \cdots, v_{(p-1)}\}$ and $\{v_2, v_4, v_6, \cdots, v_p\}$. Hence $\tau_\eta(G_1) = 2$.

Case 2. If p is odd then by (i), we have $\eta(G_1) = \left\lceil \frac{p}{2} \right\rceil$ with p number of η-sets of G_1 as $\{v_i, v_{(i+1)(mod\ p)}, v_{(i+3)(mod\ p)}, \cdots, v_{(i+\frac{p-1}{2})(mod\ p)}\}$; $1 \leq i \leq p$. Hence $\tau_\eta(G_1) = p$. □

2.2 Circulant Graph $C_p(\lfloor \frac{p}{2} \rfloor)$

The circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$, $p \geq 3$ is a circulant graph with jump size $\lfloor \frac{p}{2} \rfloor$.

Remark 2.2 The following results hold by definition of circulant graph:

(*i*) $C_p(\lfloor \frac{p}{2} \rfloor) \cong C_p(1); p = 2n-1, n \geq 2$;

(*ii*) The circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$ with $p = 2n$, $n \geq 1$ vertices contain n times of K_2's and they are disconnected, which are not strongly regular.

Theorem 2.2 *For any circulant graph $G_2 = C_p(\lfloor \frac{p}{2} \rfloor)$ with $p \geq 4$ vertices,*

(*i*) $\eta(G_2) = \left\lceil \frac{p}{2} \right\rceil$;

(*ii*) $\tau_\eta(G_2) = \begin{cases} 2^p & \text{if } n \text{ is even} \\ p & \text{if } p \text{ is odd.} \end{cases}$

Proof (*i*) Let $G_2 = C_p(\lfloor \frac{p}{2} \rfloor)$ be any circulant graph with $p \geq 4$ vertices labeled as $v_1, v_2, \cdots, v_p$. We have the following two cases:

Case 1. If $p = 2n; n \geq 2$, then $C_p(\lfloor \frac{p}{2} \rfloor)$ is bipartite graph with n number of $K_2's$. Hence

$$\eta(C_p(\lfloor \frac{p}{2} \rfloor)) = \frac{p}{2}.$$

Case 2. If $p = 2n-1; n \geq 2$, then by Theorem 2.1(i), neighborhood number of $C_p(1)$ is $\lceil \frac{p}{2} \rceil$ and by Remark 2.2(*i*), $C_p(\lfloor \frac{p}{2} \rfloor) \cong C_p(1)$;
$p = 2n-1, n \geq 2$. Therefore, we have

$$\eta(C_p(\lfloor \frac{p}{2} \rfloor)) = \lceil \frac{p}{2} \rceil, \ p \geq 4.$$

(*ii*) For the values of $\tau_\eta(G_2)$, we have

Case 1. If $p = 2n; n \geq 2$ then by (i), we have $\eta(C_p(\lfloor \frac{p}{2} \rfloor)) = \dfrac{p}{2}$ with 2^p disjoint η-sets. Hence

$$\tau_\eta(G_1) = 2^p.$$

Case 2. If $p = 2n+1; n \geq 2$, then the proof follows from Theorem 2.1(ii). □

2.3 Circulant Graph with Odd Jump Sizes

The circulant graph $C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$, $p \geq 6$ is a circulant graph with odd jump sizes.

Remark 2.3 If the sequence is of an odd jump size from 1 to $\lfloor \frac{p}{2} \rfloor$, then $C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$ is strongly regular graph.

Theorem 2.3 *For any circulant graph $G_3 = C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$ with $p = 4n-2$ or $4n-1$, $n \geq 2$,*

(i) $\eta(G_3) = \lfloor \frac{p}{2} \rfloor$;

(ii) $\tau_\eta(G_3) = \begin{cases} 2 & \text{if } p = 4n-2;\ n \geq 2 \\ 2p & \text{if } p = 4n-1;\ n \geq 2. \end{cases}$

Proof (i) Let $G_3 = C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$ be a circulant graph with vertices labeled as $v_1, v_2, \cdots, v_p$, where $p = 4n-2$ or $4n-1$, $n \geq 2$. We have

Case 1. If $p = 4n-2; n \geq 2$, then $C_p(1, 3, 5, \cdots, \lfloor \frac{p}{2} \rfloor) \cong K_{\frac{p}{2}, \frac{p}{2}}$. Hence

$$\eta(C_p(1, 3, 5, \cdots, \left\lfloor \frac{p}{2} \right\rfloor)) = \frac{p}{2}.$$

Case 2. If $p = 4n-1, n \geq 2$, then there are

$$S = \{v_i, v_{(i+1)(mod\ p)}, v_{(i+2)(mod\ p)} \cdots, v_{(i+\lfloor \frac{p}{2} \rfloor - 1)(mod\ p)}\}$$

and

$$S = \{v_i, v_{(i+2)(mod\ p)}, v_{(i+4)(mod\ p)} \cdots, v_{(i+2\lfloor \frac{p}{2} \rfloor - 2)(mod\ p)}\},$$

where $1 \leq i \leq p$ are the minimum neighborhood sets, containing $\lfloor \frac{p}{2} \rfloor$ elements. Therefore,

$$\eta(C_p(1, 3, 5, \cdots, \left\lfloor \frac{p}{2} \right\rfloor)) = \left\lfloor \frac{p}{2} \right\rfloor.$$

(ii) For the values of $\tau_\eta(G_3)$, we have

Case 1. If $p = 4n-2; n \geq 2$, then by (i), we have $\eta(G_3) = \dfrac{p}{2}$ with two disjoint η-sets of G_3 as $\{v_1, v_3, v_5, \cdots, v_{(p-1)}\}$ and $\{v_2, v_4, v_6, \cdots, v_p\}$. Hence,

$$\tau_\eta(G_3) = 2.$$

Case 2. If $p = 4n-1, n \geq 2$ then by (i), we have $\eta(G_3) = \left\lceil \dfrac{p}{2} \right\rceil$ with $2p$ number of η-sets. Hence

$$\tau_\eta(G_3) = 2p.$$ □

2.4 Circulant Graph with Even Jump Sizes

The circulant graph $C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$; $p \geq 6$ is a circulant graph with even jump sizes.

Remark 2.4 The circulant graphs $C_5(2)$, $C_6(2)$, $C_8(2, 4)$, $C_{10}(2, 4)$, $C_{12}(2, 4, 6)$ are few examples of strongly regular graphs.

Theorem 2.4 *For any circulant graph $G_4 = C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$ with $p = 4n$ or $4n + 1$, $n \geq 2$,*

$$(i)\quad \eta(G_4) = \begin{cases} 2 & \text{if } p = 4n;\ n \geq 2 \\ 4 & \text{if } p = 4n + 1;\ n \geq 2. \end{cases}$$

$$(ii)\quad \tau_\eta(G_4) = \begin{cases} (\frac{p}{2})^2 & \text{if } p = 4n;\ n \geq 2 \\ 4p & \text{if } p = 4n + 1;\ n \geq 2. \end{cases}.$$

Proof (i) Let $G_4 = C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$ be a circulant graph with $p = 4n$ or $4n + 1$, $n \geq 2$. We have the following two cases for discussion:

Case 1. If $p = 4n$; $n \geq 2$, the greatest common divisor of $2, 4, 6 \cdots, \lfloor \frac{p}{2} \rfloor = 2 \neq 1$. Hence G_4 has two disconnected blocks $B_1 = \{v_1, v_3, \cdots, v_{p-1}\}$ and $B_2 = \{v_2, v_4, \cdots, v_p\}$ for $1 \leq i \leq p$ and each block is complete graph $K_{p/2}$. This implies, $\eta(G_4) = 2$.

Case 2. If $p = 4n + 1$, $n \geq 2$, then the result follows from Theorem 2.3(i).

(ii) For the values of $\tau_\eta(G_4)$, we have the following two cases:

Case 1. If $p = 4n$, $n \geq 2$, then by (i), there exists η-sets $\{v_i, v_j\}$ of G_4, such that $v_i \in B_1$ and $v_j \in B_2$. Thus, it follows that

$$\tau_\eta(G_4) = (\frac{p}{2})^2.$$

Case 2. If $p = 4n + 1$, $n \geq 2$, then the result is similar to Theorem 2.1(ii). □

2.5 Circulant Graph $C_p(1, 2, \cdots, \lfloor \frac{p}{2} \rfloor)$

The circulant graph with $p \geq 3$ vertices and having jump size $1, 2, \cdots, \lfloor \frac{p}{2} \rfloor$ is known as the complete graph K_p, that is,

$$C_p\left(1, 2, \cdots, \left\lfloor \frac{p}{2} \right\rfloor\right) \cong K_p.$$

Remark 2.5 The complete graph K_p is strongly regular for all $p \geq 3$. The status of the trivial singleton graph K_1 is unclear. Opinions differ on if K_2 is a strongly regular graph, since it has no well-defined μ parameter, it is preferable to consider as not to be a strongly regular.

Theorem 2.5 *For any circulant graph $G_5 = C_p(1, 2, \cdots, \lfloor \frac{p}{2} \rfloor)$ with $p \geq 3$ vertices,*

(i) $\eta(G_5) = 1$;
(ii) $\tau_\eta(G_5) = p$.

Proof Let $G_5 = C_p(1, 2, \cdots, \lfloor \frac{p}{2} \rfloor)$ be a circulant graph with $p \geq 3$ vertices. Then,

(i) The proof is due to [16].

(ii) By Theorem 2.1(i), we have $\eta(G_5) = 1$ and η-sets of G_5 are $\{v_i\}; 1 \leq i \leq p$. Thus

$$\tau_\eta(G_5) = p. \qquad \square$$

§3. Matrix Representation of Circulant Graphs via Association Schemes

The matrix representations of certain classes of circulant graphs are shown as in the following table:

Circulant Graph $(n \geq 2)$		**Relations for Association Scheme**	**Matrix Respresentation.**
G_1	$p = 2n$	Two distinct vertices are said to be first associates, if their jump size is one as well as adjacent and k^{th} associates where $(2 \leq k \leq \lfloor \frac{p}{2} \rfloor)$, if their jump size is k as well as non adjacent.	Type 1
	$p = 2n+1$		Type 2
G_2	$p = 2n$	Two distinct vertices are said to be k^{th} associates, where $(1 \leq k \leq \lfloor \frac{p-2}{2} \rfloor)$, if their jump size is k as well as non adjacent and are $\lfloor \frac{p}{2} \rfloor^{th}$ associates if their jump size $\lfloor \frac{p}{2} \rfloor$ as well as adjacent.	Type 1
	$p = 2n+1$		Type 2
G_3	$p = 4n-2$	Two distinct vertices are said to be odd associates, if their jump size are odd as well as adjacent and even associates if their jump size are even as well as non adjacent	Type 1
	$p = 4n-1$		Type 2
G_4	$p = 4n$	Two distinct vertices are said to be even associates, if their jump size is even as well as adjacent and are odd associates, if their jump size are odd as well as non-adjacent	Type 1
	$p = 4n+1$		Type 2
G_5		Two distinct vertices v_i and v_j of V are k^{th} associates, $1 \leq k \leq \lfloor \frac{p}{2} \rfloor$, if $\lvert x-y \rvert = k$ and are adjacent.	Type 1 Type 2

Table 1. Relation defining association schemes with matrix representation.

In continuation of the relation from the above table, the following types of tables can be constructed for the association schemes and they are:

Type 1. The matrix representation of circulant graph for $p(\geq 2)$ even, with an association scheme is as follows:

Elements	**Association scheme**					
	First	Second	$\cdots$	k	$\cdots$	$\frac{p}{2}$
v_1	v_p, v_2	v_{p-1}, v_3	$\cdots$	$v_{(p-(k-1))(mod\ p)}$, $v_{(1+k)(mod\ p)}$	$\cdots$	$v_{1+\frac{p}{2}}$
v_2	v_1, v_3	v_p, v_4	$\cdots$	$v_{(p-(k-2))(mod\ p)}$, $v_{(2+k)(mod\ p)}$	$\cdots$	$v_{2+\frac{p}{2}}$
v_3	v_2, v_4	v_1, v_5	$\cdots$	$v_{(p-(k-3))(mod\ p)}$, $v_{(3+k)(mod\ p)}$	$\cdots$	$v_{3+\frac{p}{2}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_i	$v_{(i-1)(mod\ p)}$, $v_{(i+1)(mod\, p)}$	$v_{(i-2)(mod\ p)}$, $v_{(i+2)(mod\, p)}$	$\cdots$	$v_{(p-(k-i))(mod\, p)}$, $v_{(i+k)(mod\, p)}$	$\cdots$	$v_{(i+\frac{p}{2})(mod\, p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	v_{p-1}, v_1	v_{p-2}, v_2	$\cdots$	v_{p-k}, v_k	$\cdots$	$v_{\frac{p}{2}}$

Table 2. Association schemes of circulant graphs for $p\ (\geq 2)$ is even.

By Table 2, the parameters of second kind are given by $n_i = 2$ for $1 \leq i \leq \frac{p}{2} - 1$ and $n_{\frac{p}{2}} = 1$.

With the association scheme for the Table 2, we have the matrix representation of the circulant graph $C_p(s_1, s_2, \cdots, s_t)$; $p\ (\geq 2)$ vertices is

$$P^k = \begin{pmatrix} p^k_{11} & p^k_{12} & \cdots & p^k_{1\ \frac{p}{2}} \\ p^k_{21} & p^k_{22} & \cdots & p^k_{2\ \frac{p}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ p^k_{\frac{p}{2}1} & p^k_{\frac{p}{2}2} & \cdots & p^k_{\frac{p}{2}\ \frac{p}{2}} \end{pmatrix}.$$

Therefore, the possible values of k in the matrix P^k are given below:

If $k = 1$, then

(i) $p^1_{ij} = 1$ for $1 \leq i \leq \frac{p}{2} - 1$, $j = i + 1$;

(ii) $p^1_{ij} = 1$ for $1 \leq j \leq \frac{p}{2} - 1$, $i = 1 + j$.

If $2 \leq k \leq \frac{p}{2} - 1$, then

(i) $p^k_{ij} = 1$ for $1 \leq j \leq \frac{p}{2} - 1$ as well as $i + j = k$, $j = k + i$ and $i + j = p - k$;

(ii) $p^k_{ij} = 1$ for $1 \leq j \leq \frac{p}{2} - 1$, $i = k + j$ and $i + j = p - k$.

If $k = \frac{p}{2}$, then $p^k_{ij} = 2$ for $1 \leq i \leq \frac{p}{2} - 1$ and $j = k - i$ with the remaining entries zero.

Type 2. The matrix representation of circulant graph for $p (\geq 2)$ odd, with an association scheme is as follows:

Elements	Association scheme					
	First	Second	$\cdots$	k	$\cdots$	$\frac{p-1}{2}$
v_1	v_p, v_2	v_{p-1}, v_3	$\cdots$	$v_{(p-(k-1))(mod\ p)}$, $v_{(1+k)(mod\ p)}$	$\cdots$	$v_{1+\frac{p-1}{2}}$, $v_{1+\frac{p-1}{2}+1}$
v_2	v_1, v_3	v_p, v_4	$\cdots$	$v_{(p-(k-2))(mod\ p)}$, $v_{(2+k)(mod\ p)}$	$\cdots$	$v_{2+\frac{p-1}{2}}$, $v_{2+\frac{p-1}{2}+1}$
v_3	v_2, v_4	v_1, v_5	$\cdots$	$v_{(p-(k-3))(mod\ p)}$, $v_{(3+k)(mod\ p)}$	$\cdots$	$v_{3+\frac{p-1}{2}}$, $v_{3+\frac{p-1}{2}+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_i	$v_{(i-1)(mod\ p)}$, $v_{(i+1)(mod\ p)}$	$v_{(i-2)(mod\ p)}$, $v_{(i+2)(mod\ p)}$	$\cdots$	$v_{(p-(k-i))(mod\ p)}$, $v_{(i+k)(mod\ p)}$	$\cdots$	$v_{(i+\frac{p-1}{2})(mod\ p)}$, $v_{(i+\frac{p-1}{2}+1)(mod\ p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	v_{p-1}, v_1	v_{p-2}, v_2	$\cdots$	v_{p-k}, v_k	$\cdots$	$v_{\frac{p-1}{2}}$, $v_{\frac{p-1}{2}+1}$

Table 3. Association schemes of circulant graph with $p(\geq 3)$ is odd.

By Table 3 the parameters of second kind are given by $n_i = 2$ for $1 \leq i \leq \frac{p-1}{2}$ and $n_{\frac{p}{2}} = 1$.

With the association scheme for the Table 3, we have the matrix representation of the Circulant graph $C_p(s_1, s_2, \cdots, s_t)$; $p\ (\geq 3)$ vertices is

$$P^k = \begin{pmatrix} p^k_{11} & p^k_{12} & \cdots & p^k_{1\ \frac{p-1}{2}} \\ p^k_{21} & p^k_{22} & \cdots & p^k_{2\ \frac{p-1}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ p^k_{(\frac{p-1}{2})1} & p^k_{(\frac{p-1}{2})2} & \cdots & p^k_{(\frac{p-1}{2})\ (\frac{p-1}{2})} \end{pmatrix}.$$

Therefore, the possible values of k in the matrix P^k are given below:

If $k = 1$, then

(i) $p^1_{ij} = 1$ for $1 \leq i \leq \frac{p-1}{2} - 1$ $j = i + 1$;
(ii) $p^1_{ij} = 1$ for $1 \leq j \leq \frac{p-1}{2} - 1$, $i = 1 + j$;
(iii) $p^1_{ij} = 1$ for $i = \frac{p-1}{2}$, $j = \frac{p-1}{2}$.

If $2 \leq k \leq \frac{p-3}{2}$, then

(i) $p^k_{ij} = 1$ for $1 \leq i \leq \frac{p-3}{2}$ as well as $i + j = k$, $j = k + i$ and $i + j = p - k$;
(ii) $p^k_{ij} = 1$ for $1 \leq j \leq \frac{p-3}{2}$ as well as $i = k + j$ and $i + j = p - k$.

If $k = \frac{p-1}{2}$, then

(i) $p^k_{ij} = 1$, for $1 \leq i \leq \frac{p-3}{2}$, $j = \frac{p-1}{2} - i$;
(ii) $p^k_{ij} = 1$, for $1 \leq i \leq \frac{p-1}{2}$, $j = \frac{p+1}{2} - i$ with remaining entries are all zero.

§4. The Parameters of PBIB-Designs

By considering Theorems 2.1 to 2.5, Tables 1 and 2, the possible values of k in the matrix P^k using two different types, we have the parameters of PBIB-Design as follows:

Circuant Graph		**Parameters of PBIB-Designs**					
		p	b	g	r	λ_m	
G_1	$p = 2n,\ n \geq 2$	p	2	$\frac{p}{2}$	1	1 if m is even	0 if m is odd
	$p = 2n + 1,\ n \geq 2$	p	p	$\lceil \frac{p}{2} \rceil$	$\lceil \frac{p}{2} \rceil$	$\lceil \frac{p}{2} \rceil - \lceil \frac{m-2}{2} \rceil$ if m is even	$\lceil \frac{m}{2} \rceil$ if m is odd
G_2	$p = 2n,\ n \geq 2$	p	2^p	$\frac{p}{2}$	2^{p-1}	2^{p-2} if $1 \leq m < \frac{p}{2}$	0 if m $= \frac{p}{2}$
	$p = 2n + 1,\ n \geq 2$	p	p	$\lceil \frac{p}{2} \rceil$	$\lceil \frac{p}{2} \rceil$	$\lceil \frac{p}{2} \rceil - \lceil \frac{m-2}{2} \rceil$ if m is even	$\lceil \frac{m}{2} \rceil$ if m is odd
G_3	$p = 4n - 2,\ n \geq 2$	p	2	$\frac{p}{2}$	1	1 if m is even	0 if m is odd
	$p = 4n - 1,\ n \geq 2$	p	$2p$	$\lfloor \frac{p}{2} \rfloor$	$p - 1$	$p + \frac{m}{2} - 5$ if m is even	$\lfloor \frac{p}{2} \rfloor - \lceil \frac{m}{2} \rceil$ if m is odd
G_4	$p = 4n,\ n \geq 2$	p	$\frac{p^2}{4}$	2	$\frac{p}{2}$	0 if m is even	0 if m is odd
	$p = 4n + 1,\ n \geq 2$	p	$4p$	4	16	Problem 5.1	
G_5		p	p	1	1	0	

Table 4. Parameters of PBIB-designs.

§5. Conclusion and Open Problems

Generally, the PBIB-Designs are obtained from the families of strongly regular graphs. Interestingly, in this paper we determined the total number of η - sets, the PBIB-Designs and its association schemes arising from the η - sets of certain circulant graphs. Finally, we pose some open problems as follows:

Problem 5.1 *Generalize the* λ_m*, using PBIB-Designs associated with* $G_6 = C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$*;* $p = 4n + 1,\ n \geq 2$.

Problem 5.2 *Find all the strongly regular graphs for even and odd jump sizes of the circulant graphs.*

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Different Solution Method for Fuzzy Boundary Value Problem with Fuzzy Parameter

Tahir Ceylan

(Department of Mathematic, University of Sinop, Turkey)

Nihat Altinişik

(Department of Mathematics, University of Ondokuz Mays, Turkey)

E-mail: tceylan@sinop.edu.tr, anihat@omu.edu.tr

Abstract: In this paper Titchmark's method is applied for finding the fuzzy eigenvalues and fuzzy eigenfunctions of a fuzzy boundary value problem under generalized Hukuhara differentiability (gH-differentiability). The states of the obtained fuzzy eigenvalues are examined. In addition, this method is illustrated by solving fuzzy problem.

Key Words: gH-derivative, Fuzzy parameter, Fuzzy boundary value problem.

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§1. Introduction

Eigenvalue problems that are formulated by modeling the real case into mathematical models are very important in many fields such as quantum mechanics, control theory, vibration, heat problems. Eigenvalues and eigenfunctions play an important role in the study of ordinary and partial equations. However, in many real world problems, at least some of the parameters are represented by fuzzy rather than crisp numbers. In this case, it is so important to develop classical procedure to fuzzy procedure that would properly treat and find fuzzy eigenvalues and fuzzy differential equations.

The methods used in the solution of fuzzy differential equations have been examined with various studies (([1], [2], [5], [6], [7], [8], [9], [10], [11], [12], [13], [16]). One of the most well-known definitions of difference and derivative for fuzzy set value functions was given by Hukuhara in [10]. By using the H-derivative, Kaleva in [3] started to develop a theory for fuzzy differential equations. Many works have been done by several authers in theoretical and applied fields for fuzzy differential equations with the Hukuhara derivative ((([6], [10], [11])). But in some cases this approach suffers certain disadvantages since the diameter of the solutions is unbounded as time t increases (([3], [12]). So here we use gH-difference and gH-derivative to solve FDE under much less restrictive conditions [10]. Solution method of fuzzy boundary value problem with crisp eigenvalue parameter has been studied in (([11], [16]). At first, Buckley found fuzzy

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eigenvalues for matrix containing fuzzy numbers [17]. After, Chiao has studied generalized fuzzy eigenvalues [18].

In this paper we consider the two point fuzzy boundary value problem

$$L = -\frac{d^2}{dx^2}$$

$$L\widehat{u} = \widehat{\lambda}\widehat{u}, \qquad x \in [a,b] \tag{1.1}$$

which satisfy the conditions

$$\widehat{a}_1\widehat{u}(a) = \widehat{a}_2\widehat{u}'(a) \tag{1.2}$$

$$\widehat{b}_1\widehat{u}(b) = \widehat{b}_2\widehat{u}'(b) \tag{1.3}$$

where $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2$ are pozitive fuzzy numbers, $\widehat{\lambda} > 0$ fuzzy parameter and $\widehat{u}(x)$ positive fuzzy function.

§2. Notation and Preliminaries

In this section, we give some concepts and results besides the essential notations which will be used throughout the paper.

Let $\widehat{u}$ be a fuzzy subset on $\mathbb{R}$, i.e. a mapping $\widehat{u} : \mathbb{R} \to [0,1]$ associating with each real number t its grade of membership $\widehat{u}(t)$.

In this paper, the concept of fuzzy real numbers(fuzzy intervals) is considered in the sense of Xiao and Zhu which is defined below:

Definition 2.1([14]) *A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy intervals), whose $\alpha-$cut set is denoted by $[\widehat{u}]_\alpha$, i.e.,$[\widehat{u}]_\alpha = \{t : \widehat{u}(t) \geq \alpha\}$, if it satisfies two axioms:*

$(N1)$ There exists $r \in \mathbb{R}$ such that $\widehat{u}(r) = 1$.

$(N2)$ For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u_\alpha^- \leq u_\alpha^+ < +\infty$ such that $[\widehat{u}]_\alpha$ is equal to the closed interval $[u_\alpha^-, u_\alpha^+]$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $F(\mathbb{R})$. $F_K(\mathbb{R})$, the family of fuzzy sets of $\mathbb{R}$ whose $\alpha-$cuts are nonempty compact subsets of $\mathbb{R}$. If $\widehat{u} \in F(\mathbb{R})$ and $\widehat{u}(t) = 0$ whenever $t < 0$, then $\widehat{u}$ is called a non-negative fuzzy real number and $F^+(\mathbb{R})$ denotes the set of all non- negative fuzzy real numbers. For all $\widehat{u} \in F^+(\mathbb{R})$ and each $\alpha \in (0,1]$, real number u_α^- is positive.

The fuzzy real number $\widetilde{r} \in F(\mathbb{R})$ defined by

$$\widetilde{r} = \begin{cases} 1, & t = r \\ 0, & t \neq r, \end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$, that is if $\widetilde{r} \in (-\infty, \infty)$, then $\widetilde{r} \in F(\mathbb{R})$ satisfies $\widetilde{r}(t) = \widetilde{0}(t-r)$ and $\alpha-$cut of $\widetilde{r}$ is given by $[\widetilde{r}]_\alpha = [r,r], \alpha \in (0,1]$.

Definition 2.2([2]) *An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions* (u_α^-, u_α^+), $0 \le \alpha \le 1$, *which satisfy the following requirements:*

(i) u_α^- *is bounded non-decreasing left continuous function on* $(0, 1]$ *and right- continuous for* $\alpha = 0$*;*

(ii) u_α^+ *is bounded non-increasing left continuous function on* $(0, 1]$ *and right-continuous for* $\alpha = 0$*;*

(iii) $u_\alpha^- \le u_\alpha^+$, $0 \le \alpha \le 1$.

Definition 2.3([2]) *For* $\widehat{u}, \widehat{v} \in F(\mathbb{R})$, *and* $\lambda \in \mathbb{R}$, *the sum* $\widehat{u} \oplus \widehat{v}$ *and the product* $\lambda \odot \widehat{u}$ *are defined by*

$$\begin{aligned} [\widehat{u} \oplus \widehat{v}]^\alpha &= [\widehat{u}]^\alpha + [\widehat{v}]^\alpha = \{x + y : x \in [\widehat{u}]^\alpha, y \in [\widehat{v}]^\alpha\}, \qquad \forall \alpha \in [0, 1], \\ [\lambda \odot \widehat{u}]^\alpha &= \lambda \odot [\widehat{u}]^\alpha = \{\lambda x : x \in [\widehat{u}]^\alpha\}, \qquad \forall \alpha \in [0, 1]. \end{aligned}$$

Define $D : F(\mathbb{R}) \times F(\mathbb{R}) \to \mathbb{R}^+ \cup \{0\}$ by the equation

$$D(\widehat{u}, \widehat{v}) = \sup_{0<\alpha\le 1} \{max[|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|]\}$$

where $[\widehat{u}]^\alpha = [u_\alpha^-, u_\alpha^+]$, $[\widehat{v}]^\alpha = [v_\alpha^-, v_\alpha^+]$. Then it is easy to show that D is a metric in $F(\mathbb{R})$.

Definition 2.4([9]) *Let* $\widehat{u}, \widehat{v} \in F(\mathbb{R})$. *If there exist* $\widehat{w} \in F(\mathbb{R})$ *such that* $\widehat{u} = \widehat{v} \oplus \widehat{w}$, *then* $\widehat{w}$ *is called the Hukuhara difference of* $\widehat{u}$ *and* $\widehat{v}$ *and it is denoted by* $\widehat{u} \ominus_h \widehat{v}$. *If* $\widehat{u} \ominus_h \widehat{v}$ *exists, its* $\alpha-$*cuts are*

$$[\widehat{u} \ominus_h \widehat{v}]^\alpha = \left[u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\right]$$

for $\alpha \in [0, 1]$.

Definition 2.5([10]) *The generalized Hukuhara difference of two fuzzy numbers* $\widehat{u}, \widehat{v} \in F(\mathbb{R})$ *is defined as follows*

$$[\widehat{u} \ominus_{gH} \widehat{v}] = \widehat{w} \Leftrightarrow \begin{cases} (i) & \widehat{u} = \widehat{v} \oplus \widehat{w} \\ or\ (ii) & \widehat{v} = \widehat{u} \oplus (-1)\widehat{w}. \end{cases}$$

In terms of α*-cuts we have*

$$[\widehat{u} \ominus_{gH} \widehat{v}]^\alpha = \left[\min\left\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\right\}, \max\left\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\right\}\right]$$

and if the H -difference exists, then $\widehat{u} \ominus_h \widehat{v} = \widehat{u} \ominus_{gH} \widehat{v}$; *the conditions for the existence of* $\widehat{w} = \widehat{u} \ominus_{gH} \widehat{v} \in F(\mathbb{R})$ *are*

$$(i) \begin{cases} w_\alpha^- = u_\alpha^- - v_\alpha^- \text{ and } w_\alpha^+ = u_\alpha^+ - v_\alpha^+, \\ \text{with } w_\alpha^- \text{ increasing, } w_\alpha^+ \text{ decreasing,} w_\alpha^- \le w_\alpha^+ \end{cases}$$

$$(ii) \begin{cases} w_\alpha^- = u_\alpha^+ - v_\alpha^+ \text{ and } w_\alpha^+ = u_\alpha^- - v_\alpha^-, \\ \text{with } w_\alpha^- \text{ increasing, } w_\alpha^+ \text{ decreasing,} w_\alpha^- \le w_\alpha^+ \end{cases}$$

for all $\alpha \in [0,1]$.

Remark 2.1 Throughout the rest of this paper, we assume that $\widehat{u} \ominus_{gH} \widehat{v} \in F(\mathbb{R})$ and $\alpha-$cut representation of fuzzy-valued function $f:(a,b) \to F(\mathbb{R})$ is expressed by $[f(x)]^{\alpha} = [(f_{\alpha}^{-})(x), (f_{\alpha}^{+})(x)]$, $x \in (a,b)$ for each $\alpha \in [0,1]$.

Definition 2.6([10]) *Let* $x_0 \in (a,b)$ *and* h *be such that* $x_0 + h \in (a,b)$, *then the gH-derivative of a function* $f:(a,b) \to F(\mathbb{R})$ *at* x_0 *is defined as*

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{f(x_0+h) \ominus_{gH} f(x_0)}{h}. \tag{2.1}$$

If $f'_{gH}(x_0) \in F(\mathbb{R})$ *satisfying* (2.1) *exists, we say that* f *is generalized Hukuhara differentiable (gH-differentiable for short) at* x_0.

Definition 2.7([10]) *Let* $f:[a,b] \to F(\mathbb{R})$ *and* $x_0 \in (a,b)$ *with* $f_{\alpha}^{-}(x)$ *and* $f_{\alpha}^{+}(x)$ *both differentiable at* x_0. *We say that* f *is* $[(i)-gH]$*-differentiable at* x_0 *if*

(i) $\left[f'_{gH}(x_0)\right]^{\alpha} = \left[\left\{(f_{\alpha}^{-})'(x), (f_{\alpha}^{+})'(x)\right\}\right]$, $\forall\alpha \in [0,1]$ *and* f *is* $[(ii)-gH]$*-differentiable at* x_0 *if*

(ii) $\left[f'_{gH}(x_0)\right]^{\alpha} = \left[\left\{(f_{\alpha}^{+})'(x), (f_{\alpha}^{-})'(x)\right\}\right]$, $\forall\alpha \in [0,1]$.

Definition 2.8([4]) *The second generalized Hukuhara derivative of a fuzzy function* $f:[a,b] \to F(\mathbb{R})$ *at* x_0 *is defined as*

$$f''_{gH}(x_0) = \lim_{h \to 0} \frac{f'(x_0+h) \ominus_{gH} f'(x_0)}{h},$$

if $f''_{gH}(x_0) \in F(\mathbb{R})$, *we say that* $f'_{gH}(x)$ *is generalized Hukuhara derivative at* x_0.

Also we say that $f'_{gH}(x)$ *is* $[(i)-gH]-$*differentiable at* x_0 *if*

$$f'_{i.gH}(x_0,\alpha) = \begin{cases} \left[(f_{\alpha}^{-})''(x_0), (f_{\alpha}^{+})''(x_0)\right], & \text{if } f \text{ is } [(i)-gH]-\text{differentiable on } (a,b) \\ \left[(f_{\alpha}^{+})''(x_0), (f_{\alpha}^{-})''(x_0)\right], & \text{if } f \text{ is } [(ii)-gH]-\text{differentiable on } (a,b) \end{cases},$$

for all $\alpha \in [0,1]$ *and that* $f'_{gH}(x)$ *is* $[(ii)-gH]-$*differentiable at* x_0 *if*

$$f'_{ii.gH}(x_0,\alpha) = \begin{cases} \left[(f_{\alpha}^{+})''(x_0), (f_{\alpha}^{-})''(x_0)\right], & \text{if } f \text{ is } [(i)-gH]-\text{differentiable on } (a,b) \\ \left[(f_{\alpha}^{-})''(x_0), (f_{\alpha}^{+})''(x_0)\right], & \text{if } f \text{ is } [(ii)-gH]-\text{differentiable on } (a,b) \end{cases},$$

for all $\alpha \in [0,1]$.

§3. Solution Method of the Two Point Boundary Problem

In this section we concern with fuzzy eigenvalues and fuzzy eigenfunctions of two-point fuzzy boundary value problems. To do this, at first we need to use fuzzy derivatives. So here we use gH-difference and gH-derivative [10]. For the solution of the problem, we apply the solution method of Titchmark using gH- derivative [15]. Here we accept pozitive fuzzy functions.

Definition 3.1 *Let $\widehat{u}:[a,b]\subset\mathbb{R}\to F(\mathbb{R})$ be a fuzzy function and*

$$\left[\widehat{u}\left(x,\widehat{\lambda}\right)\right]^{\alpha}=\left[u_{\alpha}^{-}\left(x,\lambda_{\alpha}^{-},\lambda_{\alpha}^{+}\right),u_{\alpha}^{+}\left(x,\lambda_{\alpha}^{-},\lambda_{\alpha}^{+}\right)\right]$$

be the $\alpha-$cut representation of the fuzzy function $\widehat{u}(x)$ for all $x\in[a,b]$ and $\alpha\in[0,1]$. If the fuzzy differential equation (1.1) *has the nontrivial solutions such that $u_{\alpha}^{-}(x,\lambda_{\alpha}^{-},\lambda_{\alpha}^{+})\neq 0$, $u_{\alpha}^{+}(x,\lambda_{\alpha}^{-},\lambda_{\alpha}^{+})\neq 0$ and u_{α}^{-} and u_{α}^{+} satisfy Definition* 2.1 *conditions, then $\widehat{\lambda}$ is fuzzy eigenvalue of* (1.1).

Consider the fuzzy eigenvalues of the fuzzy boundary value problem (1.1) – (1.3). Four different solutions are obtained from this problem with generalized Hukuhara derivative as follows.

— If $\widehat{u}$ is $(i)-gH$ differentiable and $\widehat{u}'$ is $(ii)-gH$ differentiable then we say problem (1.1) – (1.3) has (1) – solution.

— If $\widehat{u}$ is $(ii)-gH$ differentiable and $\widehat{u}'$ is $(i)-gH$ differentiable then we say problem (1.1) – (1.3) has (2) – solution.

— If $\widehat{u}$ and $\widehat{u}'$ are $(i)-gH$ differentiable then we say problem (1.1) – (1.3) has (3) – solution.

— If $\widehat{u}$ and $\widehat{u}'$ are $(ii)-gH$ differentiable then we say problem (1.1) – (1.3) has (4) – solution.

For (1)- solution, using the $\alpha-cut$ sets and fuzzy arithmetic we get from (1.1) – (1.3) for $\left[\widehat{\lambda}\right]^{\alpha}=[\lambda_{\alpha}^{-},\lambda_{\alpha}^{+}]=\left[(k_{\alpha}^{-})^{2},(k_{\alpha}^{+})^{2}\right]$, $k_{\alpha}^{-},k_{\alpha}^{+}>0$ that

$$\left[-\left(u_{\alpha}^{-}\right)''(x),-\left(u_{\alpha}^{+}\right)''(x)\right]=\left[\left(k_{\alpha}^{-}\right)^{2},\left(k_{\alpha}^{+}\right)^{2}\right]\left[u_{\alpha}^{-}(x),u_{\alpha}^{+}(x)\right] \tag{3.1}$$

$$\left[(a_1)_{\alpha}^{-},(a_1)_{\alpha}^{+}\right]\left[u_{\alpha}^{-}(a),u_{\alpha}^{+}(a)\right]=\left[(a_2)_{\alpha}^{-},(a_2)_{\alpha}^{+}\right]\left[\left(u_{\alpha}^{-}\right)'(a),\left(u_{\alpha}^{+}\right)'(a)\right] \tag{3.2}$$

$$\left[(b_1)_{\alpha}^{-},(b_1)_{\alpha}^{+}\right]\left[u_{\alpha}^{-}(b),u_{\alpha}^{+}(b)\right]=\left[(b_2)_{\alpha}^{-},(b_2)_{\alpha}^{+}\right]\left[\left(u_{\alpha}^{-}\right)'(b),\left(u_{\alpha}^{+}\right)'(b)\right]. \tag{3.3}$$

The general solution of the fuzzy differential equation (3.1) is

$$\left[\widehat{u}\left(x,\widehat{\lambda}\right)\right]^{\alpha}=\left[u_{\alpha}^{-}\left(x,k_{\alpha}^{-}\right),u_{\alpha}^{+}\left(x,k_{\alpha}^{+}\right)\right]$$

where

$$u_{\alpha}^{-}\left(x,k_{\alpha}^{-}\right)=C_1\left(\alpha,k_{\alpha}^{-}\right)u_1\left(x,k_{\alpha}^{-}\right)+C_2\left(\alpha,k_{\alpha}^{-}\right)u_2\left(x,k_{\alpha}^{-}\right)$$

$$u_{\alpha}^{+}\left(x,k_{\alpha}^{+}\right)=C_3\left(\alpha,k_{\alpha}^{+}\right)u_1\left(x,k_{\alpha}^{+}\right)+C_4\left(\alpha,k_{\alpha}^{+}\right)u_2\left(x,k_{\alpha}^{+}\right).$$

In view of the [19] the fuzzy problem

$$\left[-\left(u_{\alpha}^{-}\right)''(x),-\left(u_{\alpha}^{+}\right)''(x)\right]=\left[\left(k_{\alpha}^{-}\right)^{2},\left(k_{\alpha}^{+}\right)^{2}\right]\left[u_{\alpha}^{-}(x),u_{\alpha}^{+}(x)\right] \tag{3.4}$$

$$\begin{aligned} \left[u_\alpha^-(a), u_\alpha^+(a)\right] &= \left[(a_2)_\alpha^-, (a_2)_\alpha^+\right] \\ \left[(u_\alpha^-)'(a), (u_\alpha^+)'(a)\right] &= \left[(a_1)_\alpha^-, (a_1)_\alpha^+\right] \end{aligned} \tag{3.5}$$

has an unique fuzzy solution $\left[\widehat{u}\left(x,\widehat{\lambda}\right)\right]^\alpha = \left[\widehat{\Phi}\left(x,\widehat{\lambda}\right)\right]^\alpha$ for each $x \in [a,b]$ [20]. This solution can be expressed as

$$\begin{aligned} \Phi_\alpha^-\left(x,k_\alpha^-\right) &= C_{11}\cos\left(\left(k_\alpha^-\right)x\right) + C_{12}\sin\left(\left(k_\alpha^-\right)x\right) \\ \Phi_\alpha^+\left(x,k_\alpha^+\right) &= C_{13}\cos\left(\left(k_\alpha^+\right)x\right) + C_{14}\sin\left(\left(k_\alpha^+\right)x\right). \end{aligned} \tag{3.6}$$

and we write (3.5) initial conditions in (3.6) such that

$$\begin{aligned} \Phi_\alpha^-\left(a,k_\alpha^-\right) &= C_{11}cos\left(\left(k_\alpha^-\right)a\right) + C_{12}sin\left(\left(k_\alpha^-\right)a\right) = (a_2)_\alpha^- \\ \left(\Phi_\alpha^-\right)'\left(a,k_\alpha^+\right) &= -\left(k_\alpha^-\right)C_{11}sin\left(\left(k_\alpha^-\right)a\right) + \left(k_\alpha^-\right)C_{12}cos\left(\left(k_\alpha^-\right)a\right) = (a_1)_\alpha^-. \end{aligned}$$

From the determinant of the coefficients matrix of the above linear system, we get C_{11} and C_{12} such that

$$\begin{aligned} C_{11} &= \frac{\begin{vmatrix} (a_2)_\alpha^- & \sin\left(\left(k_\alpha^-\right)a\right) \\ (a_1)_\alpha^- & \left(k_\alpha^-\right)\cos\left(\left(k_\alpha^-\right)a\right) \end{vmatrix}}{\begin{vmatrix} \cos\left(\left(k_\alpha^-\right)a\right) & \sin\left(\left(k_\alpha^-\right)a\right) \\ -\left(k_\alpha^-\right)\sin\left(\left(k_\alpha^-\right)a\right) & \left(k_\alpha^-\right)\cos\left(\left(k_\alpha^-\right)a\right) \end{vmatrix}} \\ &= (a_2)_\alpha^-\cos\left(\left(k_\alpha^-\right)a\right) - (a_1)_\alpha^- . \frac{\sin\left(\left(k_\alpha^-\right)a\right)}{\left(k_\alpha^-\right)}, \\ C_{12} &= \frac{\begin{vmatrix} \cos\left(\left(k_\alpha^-\right)a\right) & (a_2)_\alpha^- \\ -\left(k_\alpha^-\right)sin\left(\left(k_\alpha^-\right)a\right) & (a_1)_\alpha^- \end{vmatrix}}{\begin{vmatrix} \cos\left(\left(k_\alpha^-\right)a\right) & \sin\left(\left(k_\alpha^-\right)a\right) \\ -\left(k_\alpha^-\right)\sin\left(\left(k_\alpha^-\right)a\right) & \left(k_\alpha^-\right)\cos\left(\left(k_\alpha^-\right)a\right) \end{vmatrix}} \\ &= (a_2)_\alpha^-\sin\left(\left(k_\alpha^-\right)a\right) - (a_1)_\alpha^- . \frac{\cos\left(\left(k_\alpha^-\right)a\right)}{\left(k_\alpha^-\right)}. \end{aligned} \tag{3.7}$$

Substituting this (3.7) coefficients the above equations in (3.6), the general solution is obtained as

$$\begin{aligned} \Phi_\alpha^-\left(x,k_\alpha^-\right) &= \left((a_2)_\alpha^-\cos\left(\left(k_\alpha^-\right)a\right) - (a_1)_\alpha^- . \frac{\sin\left(\left(k_\alpha^-\right)a\right)}{\left(k_\alpha^-\right)}\right)\cos\left(\left(k_\alpha^-\right)x\right) \\ &\quad + \left((a_2)_\alpha^-\sin\left(\left(k_\alpha^-\right)a\right) + (a_1)_\alpha^- \frac{\cos\left(\left(k_\alpha^-\right)a\right)}{\left(k_\alpha^-\right)}\right)\sin\left(\left(k_\alpha^-\right)x\right). \end{aligned} \tag{3.8}$$

Similarly we find $\Phi_\alpha^+ (x, k_\alpha^+)$ as

$$\begin{aligned}\Phi_\alpha^+ (x, k_\alpha^+) &= \left((a_2)_\alpha^+ \cos\left((k_\alpha^+)\, a \right) - (a_1)_\alpha^+ . \frac{\sin\left((k_\alpha^+)\, a \right)}{(k_\alpha^+)} \right) \cos\left((k_\alpha^+)\, x \right) \\ &+ \left((a_2)_\alpha^+ \sin\left((k_\alpha^+)\, a \right) + (a_1)_\alpha^+ \frac{\cos\left((k_\alpha^+)\, a \right)}{(k_\alpha^+)} \right) \sin\left((k_\alpha^+)\, x \right). \end{aligned} \tag{3.9}$$

Again in view of the [19] the fuzzy differential equation (3.4) has an unique fuzzy solution $\left[\widehat{u}\left(x, \widehat{\lambda} \right) \right]^\alpha = \left[\widehat{\chi}\left(x, \widehat{\lambda} \right) \right]^\alpha$ satisfying the initial conditions

$$\begin{aligned} \left[u_\alpha^- (b), u_\alpha^+ (b) \right] &= \left[(b_2)_\alpha^-, (b_2)_\alpha^+ \right] \\ \left[(u_\alpha^-)' (b), (u_\alpha^+)' (b) \right] &= \left[(b_1)_\alpha^-, (b_1)_\alpha^+ \right] \end{aligned} \tag{3.10}$$

for each $x \in [a, b]$ [20]. This solution can be expressed as

$$\begin{aligned} \chi_\alpha^- (x, k_\alpha^-) &= C_{21} \cos\left((k_\alpha^-)\, x \right) + C_{22} \sin\left((k_\alpha^-)\, x \right) \\ \chi_\alpha^+ (x, k_\alpha^+) &= C_{23} \cos\left((k_\alpha^+)\, x \right) + C_{24} \sin\left((k_\alpha^+)\, x \right). \end{aligned} \tag{3.11}$$

and we write (3.10) initial conditions in (3.11) such that

$$\begin{aligned} \chi_\alpha^- (b, k_\alpha^-, k_\alpha^+) &= C_{21} cos\left((k_\alpha^-)\, b \right) + C_{22} sin\left((k_\alpha^-)\, b \right) = (b_2)_\alpha^- \\ (\chi_\alpha^-)' (b, k_\alpha^-, k_\alpha^+) &= -(k_\alpha^-)\, C_{21} sin\left((k_\alpha^-)\, b \right) + (k_\alpha^-)\, C_{22} cos\left((k_\alpha^-)\, a \right) = (b_1)_\alpha^- . \end{aligned}$$

From the determinant of the coefficients matrix of the above linear system, we get C_{21} and C_{22} such that

$$\begin{aligned} C_{21} &= \frac{\begin{vmatrix} (b_2)_\alpha^- & \sin((k_\alpha^-)\, b) \\ (b_1)_\alpha^- & (k_\alpha^-) \cos((k_\alpha^-)\, b) \end{vmatrix}}{\begin{vmatrix} cos((k_\alpha^-)\, b) & \sin((k_\alpha^-)\, b) \\ -(k_\alpha^-)\, sin((k_\alpha^-)\, b) & (k_\alpha^-) \cos((k_\alpha^-)\, b) \end{vmatrix}} \\ &= (b_2)_\alpha^- \cos\left((k_\alpha^-)\, b \right) - (b_1)_\alpha^- \frac{\sin((k_\alpha^-)\, b)}{(k_\alpha^-)}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} C_{22} &= \frac{\begin{vmatrix} \cos((k_\alpha^-)\, b) & (b_2)_\alpha^- \\ -(k_\alpha^-) \sin((k_\alpha^-)\, b) & (b_1)_\alpha^- \end{vmatrix}}{\begin{vmatrix} \cos((k_\alpha^-)\, b) & \sin((k_\alpha^-)\, b) \\ -(k_\alpha^-) \sin((k_\alpha^-)\, b) & (k_\alpha^-) \cos((k_\alpha^-)\, b) \end{vmatrix}} \\ &= (a_2)_\alpha^- \sin\left((k_\alpha^-)\, a \right) - (a_1)_\alpha^- \frac{\cos((k_\alpha^-)\, a)}{(k_\alpha^-)}. \end{aligned}$$

Substituting this (3.12) coefficients the above equations in (3.11), the general solution is obtained as

$$\begin{aligned}\chi_\alpha^-\left(x,k_\alpha^-\right) &= \left((b_2)_\alpha^- \cos\left(\left(k_\alpha^-\right)b\right) - (b_1)_\alpha^- \frac{\sin\left(\left(k_\alpha^-\right)b\right)}{\left(k_\alpha^-\right)}\right)\cos\left(\left(k_\alpha^-\right)x\right) \\ &\quad + \left((b_2)_\alpha^- \sin\left(\left(k_\alpha^-\right)b\right) + (b_1)_\alpha^- \frac{\cos\left(\left(k_\alpha^-\right)b\right)}{\left(k_\alpha^-\right)}\right)\sin\left(\left(k_\alpha^-\right)x\right). \qquad (3.13)\end{aligned}$$

Similarly we find $\chi_\alpha^+\left(x,k_\alpha^-\right)$

$$\begin{aligned}\chi_\alpha^+\left(x,k_\alpha^+\right) &= \left((b_2)_\alpha^+ \cos\left(\left(k_\alpha^+\right)b\right) - (b_1)_\alpha^+ \frac{\sin\left(\left(k_\alpha^+\right)b\right)}{\left(k_\alpha^+\right)}\right)\cos\left(\left(k_\alpha^+\right)x\right) \\ &\quad + \left((b_2)_\alpha^+ \sin\left(\left(k_\alpha^+\right)b\right) + (b_1)_\alpha^+ \frac{\cos\left(\left(k_\alpha^+\right)b\right)}{\left(k_\alpha^+\right)}\right)\sin\left(\left(k_\alpha^+\right)x\right). \qquad (3.14)\end{aligned}$$

Then, from (3.8) – (3.9) and (3.13) – (3.14) we find Wronskian function as

$$\begin{aligned}W\left(\Phi_\alpha^-,\chi_\alpha^-\right)\left(x,k_\alpha^-\right) &= \Phi_\alpha^-\left(x,k_\alpha^-\right)\left(\chi_\alpha^-\right)'\left(x,k_\alpha^-\right) - \chi_\alpha^-\left(x,k_\alpha^-\right)\left(\Phi_\alpha^-\right)'\left(x,k_\alpha^-\right) \\ &= \left(\left((a_2)_\alpha^-(b_1)_\alpha^- - (a_1)_\alpha^-(b_2)_\alpha^-\right)\cos\left(\left(k_\alpha^-\right)(a-b)\right)\right) \\ &\quad - \left(\left(k_\alpha^-\right)(a_2)_\alpha^-(b_2)_\alpha^- + \frac{(a_1)_\alpha^-(b_1)_\alpha^-}{\left(k_\alpha^-\right)}\right)\sin\left(\left(k_\alpha^-\right)(a-b)\right)\end{aligned}$$

and similarly, we have

$$\begin{aligned}W\left(\Phi_\alpha^+,\chi_\alpha^+\right)\left(x,k_\alpha^+\right) &= \Phi_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right)\left(\chi_\alpha^+\right)'\left(x,k_\alpha^-,k_\alpha^+\right) - \chi_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right)\left(\Phi_\alpha^+\right)'\left(x,k_\alpha^-,k_\alpha^+\right) \\ &= \left(\left((a_2)_\alpha^+(b_1)_\alpha^+ - (a_1)_\alpha^+(b_2)_\alpha^+\right)\cos\left(\left(k_\alpha^+\right)(a-b)\right)\right) \\ &\quad - \left(\left(k_\alpha^+\right)(a_2)_\alpha^+(b_2)_\alpha^+ + \frac{(a_1)_\alpha^+(b_1)_\alpha^+}{\left(k_\alpha^+\right)}\right)\sin\left(\left(k_\alpha^+\right)(a-b)\right)\end{aligned}$$

For (2)- solution, we get from (1.1) – (1.3) that

$$\left[-\left(u_\alpha^-\right)''(x), -\left(u_\alpha^+\right)''(x)\right] = \left[\left(k_\alpha^-\right)^2, \left(k_\alpha^+\right)^2\right]\left[u_\alpha^-(x), u_\alpha^+(x)\right]$$

$$\left[(a_1)_\alpha^-, (a_1)_\alpha^+\right]\left[u_\alpha^-(a), u_\alpha^+(a)\right] = \left[(a_2)_\alpha^-, (a_2)_\alpha^+\right]\left[\left(u_\alpha^+\right)'(a), \left(u_\alpha^-\right)'(a)\right]$$

$$\left[(b_1)_\alpha^-, (b_1)_\alpha^+\right]\left[u_\alpha^-(b), u_\alpha^+(b)\right] = \left[(b_2)_\alpha^-, (b_2)_\alpha^+\right]\left[\left(u_\alpha^+\right)'(b), \left(u_\alpha^-\right)'(b)\right].$$

From (1) – solution similarly we find $\left[\widehat{\Phi}\left(x,\widehat{\lambda}\right)\right]^\alpha$ and $\left[\widehat{\chi}\left(x,\widehat{\lambda}\right)\right]^\alpha$ solution functions such that

$$\begin{aligned}
\Phi_\alpha^-\left(x,k_\alpha^-\right) &= \left((a_2)_\alpha^-\cos\left(\left(k_\alpha^-\right)a\right)-(a_1)_\alpha^+.\frac{\sin\left(\left(k_\alpha^-\right)a\right)}{\left(k_\alpha^-\right)}\right)\cos\left(\left(k_\alpha^-\right)x\right)\\
&+\left((a_2)_\alpha^-\sin\left(\left(k_\alpha^-\right)a\right)+(a_1)_\alpha^+\frac{\cos\left(\left(k_\alpha^-\right)a\right)}{\left(k_\alpha^-\right)}\right)\sin\left(\left(k_\alpha^-\right)x\right)\\
\Phi_\alpha^+\left(x,k_\alpha^+\right) &= \left((a_2)_\alpha^+\cos\left(\left(k_\alpha^+\right)a\right)-(a_1)_\alpha^-.\frac{sin\left(\left(k_\alpha^+\right)a\right)}{\left(k_\alpha^+\right)}\right)\cos\left(\left(k_\alpha^+\right)x\right)\\
&+\left((a_2)_\alpha^+\sin\left(\left(k_\alpha^+\right)a\right)+(a_1)_\alpha^-\frac{\cos\left(\left(k_\alpha^+\right)a\right)}{\left(k_\alpha^+\right)}\right)\sin\left(\left(k_\alpha^+\right)x\right). \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
\chi_\alpha^-\left(x,k_\alpha^-\right) &= \left((b_2)_\alpha^-\cos\left(\left(k_\alpha^-\right)b\right)-(b_1)_\alpha^+\frac{\sin\left(\left(k_\alpha^-\right)b\right)}{\left(k_\alpha^-\right)}\right)\cos\left(\left(k_\alpha^-\right)x\right)\\
&+\left((b_2)_\alpha^-\sin\left(\left(k_\alpha^-\right)b\right)+(b_1)_\alpha^+\frac{\cos\left(\left(k_\alpha^-\right)b\right)}{\left(k_\alpha^-\right)}\right)\sin\left(\left(k_\alpha^-\right)x\right)\\
\chi_\alpha^+\left(x,k_\alpha^+\right) &= \left((b_2)_\alpha^+ cos\left(\left(k_\alpha^+\right)b\right)-(b_1)_\alpha^-\frac{\sin\left(\left(k_\alpha^+\right)b\right)}{\left(k_\alpha^+\right)}\right) cos\left(\left(k_\alpha^+\right)x\right)\\
&+\left((b_2)_\alpha^+\sin\left(\left(k_\alpha^+\right)b\right)+(b_1)_\alpha^-\frac{\cos\left(\left(k_\alpha^+\right)b\right)}{\left(k_\alpha^+\right)}\right)\sin\left(\left(k_\alpha^+\right)x\right) \qquad (3.16)
\end{aligned}$$

Then, from (3.15) and (3.16) we find Wronskian function as

$$\begin{aligned}
W\left(\Phi_\alpha^-,\chi_\alpha^-\right)\left(x,k_\alpha^-\right) &= \Phi_\alpha^-\left(x,k_\alpha^-\right)\left(\chi_\alpha^-\right)'\left(x,k_\alpha^-\right)-\chi_\alpha^-\left(x,k_\alpha^-\right)\left(\Phi_\alpha^-\right)'\left(x,k_\alpha^-\right)\\
&= \left((a_2)_\alpha^-(b_1)_\alpha^+-(a_1)_\alpha^+(b_2)_\alpha^-\right)\cos\left(\left(k_\alpha^-\right)(a-b)\right)\\
&-\left(\left(k_\alpha^-\right)(a_2)_\alpha^-(b_2)_\alpha^-+\frac{(a_1)_\alpha^+(b_1)_\alpha^+}{\left(k_\alpha^-\right)}\right)\sin\left(\left(k_\alpha^-\right)(a-b)\right)
\end{aligned}$$

and

$$\begin{aligned}
W\left(\Phi_\alpha^+,\chi_\alpha^+\right)\left(x,k_\alpha^+\right) &= \Phi_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right)\left(\chi_\alpha^+\right)'\left(x,k_\alpha^-,k_\alpha^+\right)-\chi_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right)\left(\Phi_\alpha^+\right)'\left(x,k_\alpha^-,k_\alpha^+\right)\\
&= \left(\left((a_2)_\alpha^+(b_1)_\alpha^--(a_1)_\alpha^-(b_2)_\alpha^+\right)\cos\left(\left(k_\alpha^+\right)(a-b)\right)\right)\\
&-\left(\left(k_\alpha^+\right)(a_2)_\alpha^+(b_2)_\alpha^++\frac{(a_1)_\alpha^-(b_1)_\alpha^-}{\left(k_\alpha^+\right)}\right)\sin\left(\left(k_\alpha^+\right)(a-b)\right)
\end{aligned}$$

For (3)- solution, we get

$$\left[-\left(u_\alpha^+\right)''(x),-\left(u_\alpha^-\right)''(x)\right]=\left[\left(k_\alpha^-\right)^2,\left(k_\alpha^+\right)^2\right]\left[u_\alpha^-(x),u_\alpha^+(x)\right] \qquad (3.17)$$

$$\left[(a_1)_\alpha^-,(a_1)_\alpha^+\right]\left[u_\alpha^-(a),u_\alpha^+(a)\right]=\left[(a_2)_\alpha^-,(a_2)_\alpha^+\right]\left[\left(u_\alpha^-\right)'(a),\left(u_\alpha^+\right)'(a)\right]$$

$$\left[(b_1)_\alpha^-, (b_1)_\alpha^+\right]\left[u_\alpha^-(b), u_\alpha^+(b)\right] = \left[(b_2)_\alpha^-, (b_2)_\alpha^+\right]\left[\left(u_\alpha^-\right)'(b), \left(u_\alpha^+\right)'(b)\right].$$

Using fuzzy arithmetic, fuzzy differential equation (3.17) conversions to the linear system of differential equations such that

$$\begin{aligned} -\left(u_\alpha^+\right)''(x) - \left(k_\alpha^-\right)^2 u_\alpha^-(x) &= 0 \\ -\left(u_\alpha^-\right)''(x) - \left(k_\alpha^+\right)^2 u_\alpha^+(x) &= 0. \end{aligned}$$

The general solution of this linear system is

$$\left[\widehat{u}\left(x,\widehat{\lambda}\right)\right]^\alpha = \left[u_\alpha^-\left(x,k_\alpha^-,k_\alpha^+\right), u_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right)\right]$$

where

$$\begin{aligned} u_\alpha^-\left(x,k_\alpha^-,k_\alpha^+\right) &= -C_1\cosh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x - C_2\sinh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\ &\quad +C_3\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2} + xC_4\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}x, \\ u_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right) &= C_1\cosh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x + C_2\sinh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\ &\quad +C_3\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}x + C_4\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}x. \end{aligned}$$

From (1)− solution similarly we find $\left[\widehat{\Phi}\left(x,\widehat{\lambda}\right)\right]^\alpha$ and $\left[\widehat{\chi}\left(x,\widehat{\lambda}\right)\right]^\alpha$ solution functions such that

$$\begin{aligned} \Phi_\alpha^-\left(x,k_\alpha^-,k_\alpha^+\right) &= -\left(\frac{A_\alpha^+}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\cosh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\ &\quad -\left(\frac{B_\alpha^+}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\sinh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\ &\quad +\left(\frac{A_\alpha^-}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\ &\quad +\left(\frac{B_\alpha^-}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}x, \\ \Phi_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right) &= \left(\frac{A_\alpha^+}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\cosh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\ &\quad +\left(\frac{B_\alpha^+}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\sinh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\ &\quad +\left(\frac{A_\alpha^-}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\ &\quad +\left(\frac{B_\alpha^-}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
\chi_\alpha^-\left(x,k_\alpha^-,k_\alpha^+\right) &= -\left(\frac{C_\alpha^+}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\cosh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\
&\quad -\left(\frac{D_\alpha^+}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\sinh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\
&\quad +\left(\frac{C_\alpha^-}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\
&\quad +\left(\frac{D_\alpha^-}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}x, \\
\chi_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right) &= \left(\frac{C_\alpha^+}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\cosh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\
&\quad +\left(\frac{D_\alpha^+}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\sinh\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\
&\quad +\left(\frac{C_\alpha^-}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}x \\
&\quad +\left(\frac{D_\alpha^-}{2\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\right)\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}x, \qquad (3.19)
\end{aligned}$$

where,

$$\begin{aligned}
A_\alpha^+ &= \left((a_2)_\alpha^+-(a_2)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\cos\text{h}\left(k_\alpha^-k_\alpha^+\right)^{1/2}a \\
&\quad -\left((a_1)_\alpha^+-(a_1)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\sin\text{h}\left(k_\alpha^-k_\alpha^+\right)^{1/2}a, \\
A_\alpha^- &= \left((a_2)_\alpha^++(a_2)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}a \\
&\quad -\left((a_1)_\alpha^++(a_1)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}a, \\
B_\alpha^+ &= \left((a_2)_\alpha^+-(a_2)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\sin\text{h}\left(k_\alpha^-k_\alpha^+\right)^{1/2}a \\
&\quad -\left((a_1)_\alpha^+-(a_1)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\cos\text{h}\left(k_\alpha^-k_\alpha^+\right)^{1/2}a, \\
B_\alpha^- &= \left((a_2)_\alpha^++(a_2)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}a \\
&\quad -\left((a_1)_\alpha^++(a_1)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}a, \\
C_\alpha^+ &= \left((b_2)_\alpha^+-(b_2)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\cos\text{h}\left(k_\alpha^-k_\alpha^+\right)^{1/2}b \\
&\quad -\left((b_1)_\alpha^+-(b_1)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\sin\text{h}\left(k_\alpha^-k_\alpha^+\right)^{1/2}b, \\
C_\alpha^- &= \left((b_2)_\alpha^++(b_2)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}b \\
&\quad -\left((b_1)_\alpha^++(b_1)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}b, \\
D_\alpha^+ &= \left((b_2)_\alpha^+-(b_2)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\sin\text{h}\left(k_\alpha^-k_\alpha^+\right)^{1/2}b \\
&\quad -\left((b_1)_\alpha^+-(b_1)_\alpha^-\right)\left(k_\alpha^-k_\alpha^+\right)^{1/2}\cos\text{h}\left(k_\alpha^-k_\alpha^+\right)^{1/2}b,
\end{aligned}$$

$$\begin{aligned} D_\alpha^- &= \left((b_2)_\alpha^+ + (b_2)_\alpha^-\right)\left(k_\alpha^- k_\alpha^+\right)^{1/2} \sin\left(k_\alpha^- k_\alpha^+\right)^{1/2} b \\ &\quad - \left((b_1)_\alpha^+ + (b_1)_\alpha^-\right)\left(k_\alpha^- k_\alpha^+\right)^{1/2} \cos\left(k_\alpha^- k_\alpha^+\right)^{1/2} b. \end{aligned}$$

Then, from (3.18) and (3.19) we find Wronskian function as

$$\begin{aligned} W\left(\Phi_\alpha^-, \chi_\alpha^-\right)\left(x, k_\alpha^-, k_\alpha^+\right) &= \Phi_\alpha^-\left(x, k_\alpha^-, k_\alpha^+\right)\left(\chi_\alpha^-\right)'\left(x, k_\alpha^-, k_\alpha^+\right) - \chi_\alpha^-\left(x, k_\alpha^-, k_\alpha^+\right)\left(\Phi_\alpha^-\right)'\left(x, k_\alpha^-, k_\alpha^+\right), \\ W\left(\Phi_\alpha^+, \chi_\alpha^+\right)\left(x, k_\alpha^-, k_\alpha^+\right) &= \Phi_\alpha^+\left(x, k_\alpha^-, k_\alpha^+\right)\left(\chi_\alpha^+\right)'\left(x, k_\alpha^-, k_\alpha^+\right) - \chi_\alpha^+\left(x, k_\alpha^-, k_\alpha^+\right)\left(\Phi_\alpha^+\right)'\left(x, k_\alpha^-, k_\alpha^+\right). \end{aligned}$$

In these equations k_α^- and k_α^+ are in the product case. So we can't find k_α^- and k_α^+ values separately satisfying the equations such that

$$\begin{aligned} W\left(\Phi_\alpha^-, \chi_\alpha^-\right)\left(x, k_\alpha^-, k_\alpha^+\right) &= 0, \\ W\left(\Phi_\alpha^+, \chi_\alpha^+\right)\left(x, k_\alpha^-, k_\alpha^+\right) &= 0. \end{aligned} \tag{3.20}$$

Since we can't find $\widehat{\lambda}$ eigenvalue of (3.17) problem, there is no fuzzy solution for 3-solution.

Similarly there is no fuzzy solution for (4)-solution.

In this case for (1.1) − (1.3) fuzzy problem (1) − solution and (2) − solution methods can be applied but (3) −solution and (4) −solution methods cannot be applied.

Theorem 3.1([11]) *The Wronskian functions* $W\left(\Phi_\alpha^-, \chi_\alpha^-\right)\left(x, k_\alpha^-, k_\alpha^+\right)$ *and* $W\left(\Phi_\alpha^+, \chi_\alpha^+\right)\left(x, k_\alpha^-, k_\alpha^+\right)$ *are independent of variable* x *for* $x \in (a, b)$*, where functions* $\Phi_\alpha^-, \chi_\alpha^-, \Phi_\alpha^+, \chi_\alpha^+$ *are the solution of the fuzzy boundary value problem* (1.1) − (1.3) *and it is show that*

$$\left[\widehat{W}\left(\widehat{\lambda}\right)\right]^\alpha = \left[W_\alpha{}^-\left(k_\alpha^-, k_\alpha^+\right), W_\alpha{}^+\left(k_\alpha^-, k_\alpha^+\right)\right] \tag{3.21}$$

for all $\alpha \in [0, 1]$.

Theorem 3.2([11]) *The fuzzy eigenvalues of the fuzzy boundary value problem (1.1)-(1.3) if and only if consist of the zeros of functions* $W_\alpha{}^-\left(k_\alpha^-, k_\alpha^+\right)$ *and* $W_\alpha{}^+\left(k_\alpha^-, k_\alpha^+\right)$.

Example 3.1 Consider the two point fuzzy boundary problem

$$-\widehat{u}'' = \lambda \widehat{u}, \quad x \in [0, 1], \tag{3.22}$$

$$\widehat{1}\widehat{u}(0) = \widehat{2}\widehat{u}'(0), \tag{3.23}$$

$$\widehat{u}(1) = 0, \tag{3.24}$$

where $[1]^\alpha = [\alpha, 2-\alpha], [2]^\alpha = [\alpha+1, 3-\alpha]$, $\left[\widehat{\lambda}\right]^\alpha = [\lambda_\alpha^-, \lambda_\alpha^+] = \left[\left(k_\alpha^-\right)^2, \left(k_\alpha^+\right)^2\right]$, $k_\alpha^-, k_\alpha^+ > 0$ and $\widehat{u}(x)$ is positive solution functions. For (1)−solution ,using the $\alpha-$cut sets and fuzzy arithmetic we get from (3.22) − (3.24)

$$\left[-\left(u_\alpha^-\right)''(x), -\left(u_\alpha^+\right)''(x)\right] = \left[\left(k_\alpha^-\right)^2, \left(k_\alpha^+\right)^2\right]\left[u_\alpha^-(x), u_\alpha^+(x)\right] \tag{3.25}$$

$$[\alpha, 2-\alpha]\left[u_\alpha^-(0), u_\alpha^+(0)\right] = [\alpha+1, 3-\alpha]\left[\left(u_\alpha^-\right)'(0), \left(u_\alpha^+\right)'(0)\right] \tag{3.26}$$

$$\left[u_\alpha^-(1), u_\alpha^+(1)\right] = 0. \tag{3.27}$$

Let $\left[\widehat{\Phi}\left(x, \widehat{\lambda}\right)\right]^\alpha$ be a solution which is satisfying

$$\left[u_\alpha^-(0), u_\alpha^+(0)\right] = [\alpha + 1, 3 - \alpha], \quad \left[\left(u_\alpha^-\right)'(0), \left(u_\alpha^+\right)'(0)\right] = [\alpha, 2 - \alpha].$$

initial conditions of fuzzy differential equations (3.25). Then we find $\left[\widehat{\Phi}\left(x, \widehat{\lambda}\right)\right]^\alpha$ as

$$\begin{aligned} \Phi_\alpha^-\left(x, k_\alpha^-\right) &= (\alpha + 1)\, cos\left(\left(k_\alpha^-\right)x\right) + \frac{\alpha}{\left(k_\alpha^-\right)} sin\left(\left(k_\alpha^-\right)x\right) \\ \Phi_\alpha^+\left(x, k_\alpha^+\right) &= (3 - \alpha)\, cos\left(\left(k_\alpha^+\right)x\right) + \frac{2 - \alpha}{\left(k_\alpha^+\right)} sin\left(\left(k_\alpha^+\right)x\right) \end{aligned} \tag{3.28}$$

Similarly $\left[\widehat{\chi}\left(x, \widehat{\lambda}\right)\right]^\alpha$ be a solution which is satisfying

$$\left[u_\alpha^-(1), u_\alpha^+(1)\right] = 0, \quad \left[\left(u_\alpha^-\right)'(1), \left(u_\alpha^+\right)'(1)\right] = 1$$

initial conditions of fuzzy differential equations (3.25). Then we find $\left[\widehat{\chi}\left(x, \widehat{\lambda}\right)\right]^\alpha$ as

$$\begin{aligned} \chi_\alpha^-\left(x, k_\alpha^-\right) &= \frac{cos\left(k_\alpha^-\right)}{\left(k_\alpha^-\right)} sin\left(\left(k_\alpha^-\right)x\right) - \frac{sin\left(k_\alpha^-\right)}{\left(k_\alpha^-\right)} cos\left(\left(k_\alpha^-\right)x\right) \\ \chi_\alpha^+\left(x, k_\alpha^-\right) &= \frac{cos\left(k_\alpha^+\right)}{\left(k_\alpha^+\right)} sin\left(\left(k_\alpha^+\right)x\right) - \frac{sin\left(k_\alpha^+\right)}{\left(k_\alpha^-\right)} cos\left(\left(k_\alpha^-\right)x\right) \end{aligned} \tag{3.29}$$

From Theorem 3.2, fuzzy eigenvalues of the fuzzy problem (3.25) – (3.27) are zeros of the functions ${W_\alpha}^-\left(k_\alpha^-, k_\alpha^+\right)$ and ${W_\alpha}^+\left(k_\alpha^-, k_\alpha^+\right)$. So we get

$${W_\alpha}^-\left(k_\alpha^-\right) = (\alpha + 1)\, cos\left(\left(k_\alpha^-\right)\right) + \frac{\alpha}{\left(k_\alpha^-\right)} sin\left(\left(k_\alpha^-\right)\right) = 0 \tag{3.30}$$

$${W_\alpha}^+\left(k_\alpha^+\right) = (3 - \alpha)\, cos\left(\left(k_\alpha^+\right)\right) + \frac{(2 - \alpha)}{\left(k_\alpha^+\right)} sin\left(\left(k_\alpha^+\right)\right) = 0 \tag{3.31}$$

For each $\alpha \in [0, 1]$, if the $\left(k_\alpha^-\right)$ and $\left(k_\alpha^+\right)$values satisfying (3.30) and (3.31) equations compute with Matlab Program ,then eigenvalues of the fuzzy problem (3.25) – (3.27) are obtained. So we show $\left(k_\alpha^-\right)$ values of (3.30) equation with $\left(k_n\right)_\alpha^-, n = 1, 2..$ in Table 1 such that

Table 1. $\left(k_\alpha^-\right)$ eigenvalues corresponding to α

$\alpha \in [0, 1]$	k_0^-	k_1^-	k_2^-	k_3^-	k_4^-
$\alpha = 0$	1.5708	4.7124	7.8540	10.9956	14.1372
$\alpha = 0.2$	1.6703	4.7475	7.8751	11.0107	14.1489
$\alpha = 0.5$	1.7582	4.7820	7.8962	11.0258	14.1607
$\alpha = 0.8$	1.8114	4.8046	7.9101	11.0358	14.1685
$\alpha = 1$	1.8366	4.8158	7.9171	11.0408	14.1724

and (k_{α}^{+}) values of (3.31) with $(k_n)_{\alpha}^{+}, n = 1, 2\cdots$ in Table 2 such that

Table 2. (k_{α}^{+}) eigenvalues corresponding to α

$\alpha \in [0,1]$	k_0^+	k_1^+	k_2^+	k_3^+	k_4^+
$\alpha = 0$	1.9071	4.8490	7.9378	11.0558	14.1841
$\alpha = 0.2$	1.8975	4.8443	7.9348	11.0537	14.1825
$\alpha = 0.5$	1.8798	4.8358	7.9295	11.0498	14.1795
$\alpha = 0.8$	1.8566	4.8250	7.9227	11.0449	14.1756
$\alpha = 1$	1.8366	4.8158	7.9171	11.0408	14.1724

When Table 1 and Table 2 are examined, it is seen that the eigenvalues provided the fuzzy conditions from Definition 3.2. In this case $\left[\widehat{\lambda}\right]^{\alpha} = [\lambda_{\alpha}^{-}, \lambda_{\alpha}^{+}] = \left[(k_{\alpha}^{-})^2, (k_{\alpha}^{+})^2\right]$ are fuzzy eigenvalues of fuzzy problem (3.25) − (3.27) for (1)−solution. So if we write $(k_n)_{\alpha}^{-}$ and $(k_n)_{\alpha}^{+}$ eigenvalues in (3.28) and (3.29), then $\left[\widehat{\Phi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha}$ and $\left[\widehat{\chi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha}$ are

$$\begin{aligned}\left[\widehat{\Phi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha} &= \left[(\Phi_n)_{\alpha}^{-}\left(x,(k_n)_{\alpha}^{-}\right), (\Phi_n)_{\alpha}^{+}\left(x,(k_n)_{\alpha}^{+}\right)\right] \\ &= \Bigg[(\alpha+1)\, cos\left((k_n)_{\alpha}^{-} x\right) + \frac{\alpha}{(k_{\alpha}^{-})} sin\left((k_n)_{\alpha}^{-} x\right), \\ &\qquad (3-\alpha)\, cos\left((k_n)_{\alpha}^{+} x\right) + \frac{(2-\alpha)}{(k_{\alpha}^{+})} sin\left((k_n)_{\alpha}^{+} x\right)\Bigg]\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\left[\widehat{\chi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha} &= \left[(\chi_n)_{\alpha}^{-}\left(x,(k_n)_{\alpha}^{-}\right), (\chi_n)_{\alpha}^{+}\left(x,(k_n)_{\alpha}^{+}\right)\right] \\ &= \Bigg[\frac{cos\left((k_n)_{\alpha}^{-}\right)}{(k_n)_{\alpha}^{-}} sin\left((k_n)_{\alpha}^{-} x\right) - \frac{sin\left((k_n)_{\alpha}^{-}\right)}{(k_n)_{\alpha}^{-}} cos\left((k_n)_{\alpha}^{-} x\right), \\ &\qquad \frac{cos\left((k_n)_{\alpha}^{+}\right)}{(k_n)_{\alpha}^{+}} sin\left((k_n)_{\alpha}^{+} x\right) - \frac{sin\left((k_n)_{\alpha}^{+}\right)}{(k_n)_{\alpha}^{+}} cos\left((k_n)_{\alpha}^{+} x\right)\Bigg]\end{aligned} \tag{3.33}$$

To define a valid $\alpha-$cut set of $\left[\widehat{\Phi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha}$ and $\left[\widehat{\chi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha}$ functions, the conditions

$$\frac{\partial (\Phi_n)_{\alpha}^{-}}{\partial\alpha} \geq 0, \frac{\partial (\Phi_n)_{\alpha}^{+}}{\partial\alpha} \leq 0 \quad \text{and} \quad (\Phi_n)_{\alpha}^{-} \leq (\Phi_n)_{\alpha}^{+} \tag{3.34}$$

$$\frac{\partial (\chi_n)_{\alpha}^{-}}{\partial\alpha} \geq 0, \frac{\partial (\chi_n)_{\alpha}^{+}}{\partial\alpha} \leq 0 \quad \text{and} \quad (\chi_n)_{\alpha}^{-} \leq (\chi_n)_{\alpha}^{+} \tag{3.35}$$

must be satisfied for all $\alpha \in [0,1]$.

Then for all $\alpha \in [0,1]$, (3.32) and (3.33) are the eigenfunctions corresponding to $\left[\widehat{\lambda}_n\right]^{\alpha} = \left[\widehat{k}_n\right]^{\alpha}$ eigenvalues and they must satisfy (3.34) and (3.35) equations. Consider that eigenvalues of $\left[\widehat{\Phi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha}$ and $\left[\widehat{\chi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha}$ eigenfunctions depend on $\alpha-$ cut set. So if we change α, then this eigenvalues change and eigenfunctions corresponding to $\widehat{\lambda}$ change.

In particular, we select $(k_2)^-_{0.2} = 7.8751$ in Table 1 and $(k_2)^+_{0.2} = 7.9348$ in Table 1. for $\alpha = 0.2$. If we substitute this values respectively in (3.32) and (3.33), we have the following figures.

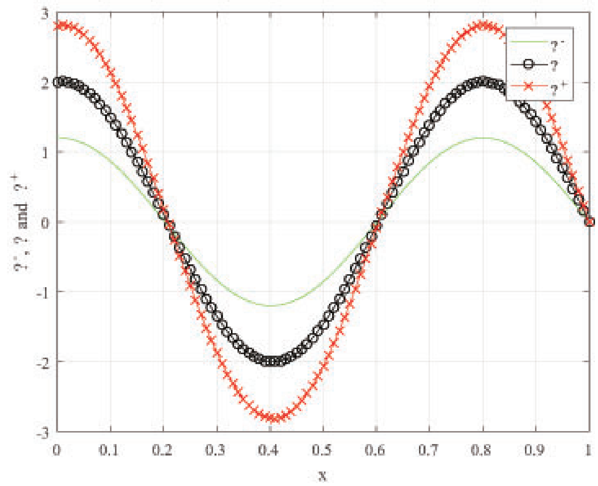

Figure 1 $\widehat{\Phi}(x,k)$ and $\widehat{\Psi}(x,k)$.

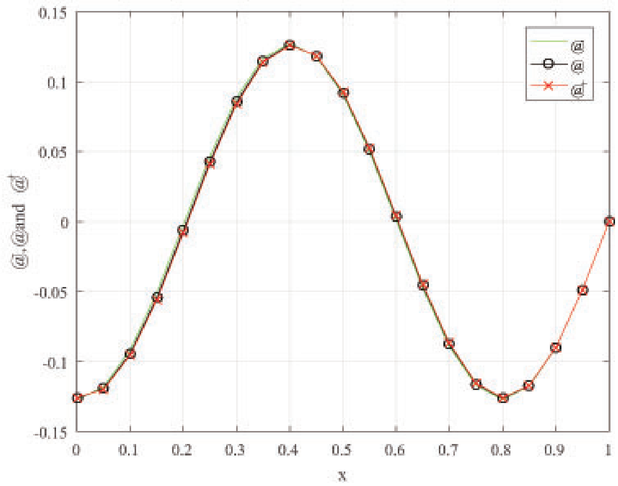

Figure 2 $\widehat{\Phi}(x,k)$ and $\widehat{\Psi}(x,k)$.

From Definition 3.2, we see $\left[\widehat{\Phi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha}$ represent a valid fuzzy number for $x \in [0, 0.202]$ and $x \in [0.6063, 1]$ in Figure 1 and we see $\left[\widehat{\chi}_n\left(x,\widehat{\lambda}\right)\right]^{\alpha}$ represent a valid fuzzy number for $x \in [0.434, 0.601]$ in Figure 2.

For (2)−solution, we get from (3.22)-(3.24)

$$\left[-\left(u_\alpha^-\right)''(x), -\left(u_\alpha^+\right)''(x)\right] = \left[\left(k_\alpha^-\right)^2, \left(k_\alpha^+\right)^2\right]\left[u_\alpha^-(x), u_\alpha^+(x)\right], \tag{3.36}$$

$$[\alpha, 2-\alpha]\left[u_\alpha^-(0), u_\alpha^+(0)\right] = [\alpha+1, 3-\alpha]\left[\left(u_\alpha^+\right)'(0), \left(u_\alpha^-\right)'(0)\right], \tag{3.37}$$

$$\left[u_\alpha^-(1), u_\alpha^+(1)\right] = 0. \tag{3.38}$$

Let $\left[\widehat{\Phi}\left(x,\widehat{\lambda}\right)\right]^\alpha$ be a solution which is satisfying

$$\left[u_\alpha^-(0), u_\alpha^+(0)\right] = [\alpha+1, 3-\alpha]$$

$$\left[\left(u_\alpha^+\right)'(0), \left(u_\alpha^-\right)'(0)\right] = [\alpha, 2-\alpha].$$

initial conditions of fuzzy differential equations (3.36). Then we find $\left[\widehat{\Phi}\left(x,\widehat{\lambda}\right)\right]^\alpha$ as

$$\begin{aligned} \Phi_\alpha^-\left(x,k_\alpha^-\right) &= (\alpha+1)\cos\left(\left(k_\alpha^-\right)x\right) + \frac{2-\alpha}{\left(k_\alpha^-\right)}\sin\left(\left(k_\alpha^-\right)x\right) \\ \Phi_\alpha^+\left(x,k_\alpha^+\right) &= (3-\alpha)\cos\left(\left(k_\alpha^+\right)x\right) + \frac{\alpha}{\left(k_\alpha^+\right)}\sin\left(\left(k_\alpha^+\right)x\right) \end{aligned} \tag{3.39}$$

Similarly $\left[\widehat{\chi}\left(x,\widehat{\lambda}\right)\right]^\alpha$ be a solution which is satisfying

$$\begin{aligned} \left[u_\alpha^-(1), u_\alpha^+(1)\right] &= 0 \\ \left[\left(u_\alpha^+\right)'(1), \left(u_\alpha^-\right)'(1)\right] &= 1 \end{aligned}$$

initial conditions of fuzzy differential equations (3.36). Then we find $\left[\widehat{\chi}\left(x,\widehat{\lambda}\right)\right]^\alpha$ as

$$\begin{aligned} \chi_\alpha^-\left(x,k_\alpha^-\right) &= \frac{\cos\left(k_\alpha^-\right)}{\left(k_\alpha^-\right)}\sin\left(\left(k_\alpha^-\right)x\right) - \frac{\sin\left(k_\alpha^-\right)}{\left(k_\alpha^-\right)}\cos\left(\left(k_\alpha^-\right)x\right) \\ \chi_\alpha^+\left(x,k_\alpha^-\right) &= \frac{\cos\left(k_\alpha^+\right)}{\left(k_\alpha^+\right)}\sin\left(\left(k_\alpha^+\right)x\right) - \frac{\sin\left(k_\alpha^+\right)}{\left(k_\alpha^-\right)}\cos\left(\left(k_\alpha^-\right)x\right) \end{aligned} \tag{3.40}$$

From Theorem 3.2, fuzzy eigenvalues of the fuzzy problem (3.36) – (3.38) are zeros of the functions ${W_\alpha}^-\left(k_\alpha^-, k_\alpha^+\right)$ and ${W_\alpha}^+\left(k_\alpha^-, k_\alpha^+\right)$. So we get

$${W_\alpha}^-\left(k_\alpha^-\right) = (\alpha+1)\cos\left(\left(k_\alpha^-\right)\right) + \frac{(2-\alpha)}{\left(k_\alpha^-\right)}\sin\left(\left(k_\alpha^-\right)\right) = 0 \tag{3.41}$$

$${W_\alpha}^+\left(k_\alpha^+\right) = (3-\alpha)\cos\left(\left(k_\alpha^+\right)\right) + \frac{\alpha}{\left(k_\alpha^+\right)}\sin\left(\left(k_\alpha^+\right)\right) = 0 \tag{3.42}$$

For each $\alpha \in [0,1]$, if the (k_α^-) and (k_α^+)values satisfying (3.41) and (3.42) equations compute with Matlab Program ,then eigenvalues of the fuzzy problem (3.36) – (3.38) are obtained. So we show (k_α^-) values of (3.41) equation with $(k_n)_\alpha^-$, $n = 1, 2\cdots$ in Table 3 such that

Table 3. (k_α^-) eigenvalues corresponding to α

$\alpha \in [0,1]$	k_0^-	k_1^-	k_2^-	k_3^-	k_4^-
$\alpha = 0$	2.2889	5.0870	8.0962	11.1727	14.2764
$\alpha = 0.2$	2.1746	5.0036	8.0385	11.1295	14.2421
$\alpha = 0.5$	2.0288	4.9131	7.9787	11.0855	14.2074
$\alpha = 0.8$	1.9071	4.8490	7.9378	11.0558	14.1841
$\alpha = 1$	1.8366	4.8158	7.9171	11.0408	14.1724

and (k_α^+) values of (3.31) with $(k_n)_\alpha^+, n = 1, 2\cdots$ in Table 4 such that

Table 4. (k_α^+) eigenvalues corresponding to α

$\alpha \in [0,1]$	k_0^+	k_1^+	k_2^+	k_3^+	k_4^+
$\alpha = 0$	1.5708	4.7124	7.8540	10.9956	14.1372
$\alpha = 0.2$	1.6150	4.7275	7.8631	11.0021	14.1422
$\alpha = 0.5$	1.6887	4.7544	7.8794	11.0137	14.1513
$\alpha = 0.8$	1.7731	4.7882	7.9000	11.0285	14.1628
$\alpha = 1$	1.8366	4.8158	7.9171	11.0408	14.1724

When Table 3 and Table 4 are examined, it is seen that the eigenvalues didn't provide the fuzzy condition which is $(k_n)_\alpha^- \leq (k_n)_\alpha^+$ from Defination 3.2. In this case $\left[\widehat{\lambda}\right]^\alpha = [\lambda_\alpha^-, \lambda_\alpha^+] = \left[(k_\alpha^-)^2, (k_\alpha^+)^2\right]$ are not fuzzy eigenvalues of fuzzy problem $(3.36) - (3.38)$ for (2)-solution.So we can not write $(k_n)_\alpha^-$ and $(k_n)_\alpha^+$ eigenvalues for $\left[\widehat{\Phi}\left(x, \widehat{\lambda}\right)\right]^\alpha$ and $\left[\widehat{\chi}\left(x, \widehat{\lambda}\right)\right]^\alpha$ functions.

For $(3)-$solution, we get from (3.22)-(3.24)

$$\left[-\left(u_\alpha^+\right)''(x), -\left(u_\alpha^-\right)''(x)\right] = \left[\left(k_\alpha^-\right)^2, \left(k_\alpha^+\right)^2\right]\left[u_\alpha^-(x), u_\alpha^+(x)\right] \tag{3.43}$$

$$[\alpha, 2-\alpha]\left[u_\alpha^-(0), u_\alpha^+(0)\right] = [\alpha+1, 3-\alpha]\left[\left(u_\alpha^+\right)'(0), \left(u_\alpha^-\right)'(0)\right] \tag{3.44}$$

$$\left[u_\alpha^-(1), u_\alpha^+(1)\right] = 0. \tag{3.45}$$

Using fuzzy arithmetic, fuzzy differential equation (3.43) conversions to the linear system of differential equations such that

$$\begin{aligned} -\left(u_\alpha^+\right)''(x) - \left(k_\alpha^-\right)^2 u_\alpha^-(x) &= 0 \\ -\left(u_\alpha^-\right)''(x) - \left(k_\alpha^+\right)^2 u_\alpha^+(x) &= 0. \end{aligned}$$

If we solve this linear system we find $\left[\widehat{\Phi}\left(x, \widehat{\lambda}\right)\right]^\alpha$ and $\left[\widehat{\chi}\left(x, \widehat{\lambda}\right)\right]^\alpha$ solution functions such that

$$\begin{aligned}\Phi_\alpha^-\left(x,k_\alpha^-,k_\alpha^+\right) &= -(1-\alpha)\cosh\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)-\frac{(\alpha-1)}{\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\sinh\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)\\ &\quad+2\cos\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)+\frac{1}{\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\sin\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)\\ \Phi_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right) &= (1-\alpha)\cosh\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)+\frac{(\alpha-1)}{\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\sinh\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)\\ &\quad+2\cos\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)+\frac{1}{\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\sin\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right) \qquad (3.46)\end{aligned}$$

and

$$\begin{aligned}\chi_\alpha^-\left(x,k_\alpha^-,k_\alpha^+\right) &= -\frac{\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}}{\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\cos\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)\\ &\quad+\frac{\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}}{\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\sin\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)\\ \chi_\alpha^+\left(x,k_\alpha^-,k_\alpha^+\right) &= \frac{\sin\left(k_\alpha^-k_\alpha^+\right)^{1/2}}{\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\cos\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right)\\ &\quad+\frac{\cos\left(k_\alpha^-k_\alpha^+\right)^{1/2}}{\left(k_\alpha^-k_\alpha^+\right)^{1/2}}\sin\left(\left(k_\alpha^-k_\alpha^+\right)^{1/2}x\right). \qquad (3.47)\end{aligned}$$

When we write (3.46) and (3.47) functions in the Wronskian functions, we see that k_α^- and k_α^+ are in the product case. Since we can't find eigenvalue of fuzzy problem(3.43)-(3.45), there is no fuzzy solution for 3-solution.

Similarly there is no fuzzy solution for 4-solution.

§4. Conclusions

The eigenvalue parameter $\widehat{\lambda}$ in the fuzzy boundary value problem was considered fuzzy parameter. Solution of the problem was examined by using generalized Hukuhara differentiability concept. The type of [gH]- differentiablity of the solution were discussed.

When solutions were analyzed by derivative type, there were (1)-solution and (2)-solution, but (3)-solution and (4)-solution could not be obtained due to lack of fuzzy eigenvalues.

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Neighbourly and Highly Irregular Neutrosophic Fuzzy Graphs

S. Sivabala

PG and Research Department of Mathematics
G.Venkataswamy Naidu College, Kovilpatti - 628501, Tamil Nadu, India

N.R. Santhi Maheswari

Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India

E-mail: sivamaths13@gmail.com, nrsmaths@yahoo.com

Abstract: In this paper, the concepts of neighbourly irregular neutrosophic fuzzy graphs, highly irregular neutrosophic fuzzy graphs, neighbourly totally irregular and highly totally irregular neutrosophic fuzzy graphs are introduced. Also, we proved some theorems and results of these graphs.

Key Words: Neighbourly irregular neutrosophic fuzzy graphs, highly irregular neutrosophic fuzzy graphs, neighbourly totally irregular and highly totally irregular neutrosophic fuzzy graphs.

AMS(2010): 05C12, 03E72, 05C72.

§1. Introduction

F.Smarandache [13] introduced notion of neutrosophic set which is useful for dealing real life problems having imprecise, indeterminacy and inconsistent data. They are generalization of the theory of fuzzy sets, intuitionistics fuzzy set, interval valued fuzzy set, and interval valued intuitionistic fuzzy sets.

N. Shah and Hussain [11, 14] introduced the notion of soft neutrosophic graphs. N. Shah [12] introduces the notion of neutrosophic graphs and different operations like union, intersection and complement in his work. A neutrosophic set is characterized by a truth membership degree (t), an indeterminacy membership degree (i), falsity membership degree (f) independently, which are with in the real standard or non standard unit interval $]^{-}0, 1^{+}[$.

N. R. Santhi Maheswari and C. Sekar [7] introduced the notion of Neighbourly irregular graphs and semi neighbourly irregular graphs [8] on m - neighbourly irregular Fuzzy graphs, on neighbourly edge irregular fuzzy graphs [9]. N. R. Santhi Maheswari, R. Muneeswari and S. Ravi Narayanan [10] introduced the notion of 2 - highly irregular fuzzy graphs.

Divya and Dr. J. Malarvizhi [1] introduced the notion of neutrosophic fuzzy graph and few fundamental operation on neutrosophic fuzzy graph. This idea motivate us to introduce regular and irregular neutrosophic fuzzy graphs.

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§2. Preliminaries

In this section, we recall the notions related to Neutrosophic set, fuzzy graph, Neutrosophic fuzzy set and neturosophic fuzzy graph.

Definition 2.1([13]) *Let X be a space of points with generic elements in X denoted by x. A neutrosophic set $A(NSA)$ is an object having the form*

$$A = \{< x : T_A(x), I_A(x), F_A(x) >, x \in X\},$$

where the functions $T, I, F \rightarrow]^{-}0, 1^{+}[$ define respectively a truth membership function, an indeterminacy membership function and a falsity membership function of the element $x \in X$ to the set A with the condition

$$^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}.$$

The functions $T_A(x), I_A(x), F_A(x)$ are real standard or non standard subsets of $]^{-}0, 1^{+}[$.

Definition 2.2([5]) *A fuzzy graph is a pair of functions $G = (\sigma, \mu)$, where σ is a fuzzy subset of a non-empty set V and is a symmetric fuzzy relation of σ i.e $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$ such that $\mu(uv) \leq \sigma(u)\Lambda\sigma(v)$ for $\forall u, v \in V$ where uv denote the edge between u and v and $\sigma(u)\Lambda\sigma(v)$ denotes the minimum of $\sigma(u)$ and $\sigma(v)$, σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.*

Definition 2.3([1]) *Let X be a space of points with generic elements in X denoted by x. A neutrosophic fuzzy set $A(NFSA)$ is characterized by truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity membership function $F_A(x)$.*

For each point $x \in X$, $T_A(x), I_A(x), F_A(x) \in [0, 1]$. A $(NFSA)$ can be written as $A = \{< x : T_A(x), I_A(x), F_A(x) >, x \in X\}$.

Definition 2.4([1]) *Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be neutrosophic fuzzy sets on a set X. If $A = (T_A, I_A, F_A)$ is a neutrosophic fuzzy relation on a set X, then $A = (T_A, I_A, F_A)$ is called a neutrosophic fuzzy relation on $B = (T_B, I_B, F_B)$ if*

$$\begin{aligned} T_B(x, y) &\leq T_A(x).T_A(y), \\ I_B(x, y) &\leq I_A(x).I_A(y), \\ F_B(x, y) &\leq F_A(x).F_A(y) \end{aligned}$$

for all $x, y \in X$, where . means the ordinary multiplication.

Definition 2.5([1]) *A neutrosophic fuzzy graph $(NFgraph)$ with underlying set V is defined to be a pair $N_G = (A, B)$, where*

(i) the functions $T_A, I_A, F_A : V \rightarrow [0, 1]$ denote the degree of truth membership, degree of indeterminacy membership and the degree of falsity membership of the element $v_i \in V$ respectively and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$;

(ii) $E \subseteq V \times V$ *where the functions* $T_B, I_B, F_B : V \times V \to [0,1]$ *are defined by*

$$\begin{aligned} T_B(v_i, v_j) &\leq T_A(v_i).T_A(v_j), \\ I_B(v_i, v_j) &\leq I_A(v_i).I_A(v_j), \\ F_B(v_i, v_j) &\leq F_A(v_i).F_A(v_j) \end{aligned}$$

for all $v_i, v_j \in V$*, where . means ordinary multiplication denotes the degrees of truth membership, indeterminacy membership and falsity membership of the edge* $(v_i, v_j) \in E$ *respectively, where*

$$0 \leq T_B(x) + I_B(x) + F_B(x) \leq 3$$

for all $(v_i, v_j) \in E$ $(j = 1, 2, \cdots, n)$.

§3. Degree of Vertex in Neutrosophic Fuzzy Graphs

Throughout this paper, we denote $G = (V, E)$ a crisp graph, $N_G = (A, B)$ a neutrosophic fuzzy graph of graph G.

Definition 3.1 *Let* $N_G = (A, B)$ *be a neutrosophic fuzzy graph. The neighbourhood degree of a vertex* x *in* N_G *defined by*

$$d_{N_G}(x) = (deg_T(x), deg_I(x), deg_F(x)),$$

where

$$\begin{aligned} deg_T(x) &= \sum_{xy \in E} T_B(xy), \\ deg_I(x) &= \sum_{xy \in E} I_B(xy), \\ deg_F(x) &= \sum_{xy \in E} F_B(xy). \end{aligned}$$

Example 3.2 Let N_G be the neutrosophic fuzzy graph shown in Fig.1.

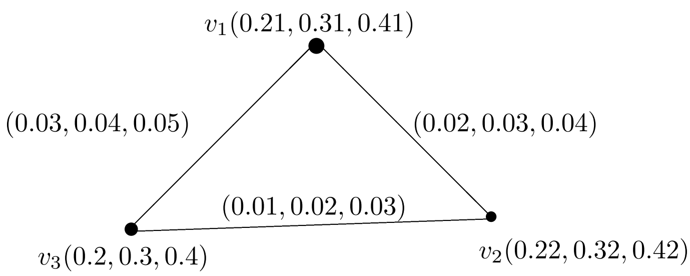

Fig.1

In this graph,

$$d_{N_G}(v_1) = (0.05, 0.07, 0.09),$$
$$d_{N_G}(v_2) = (0.03, 0.05, 0.07),$$
$$d_{N_G}(v_3) = (0.04, 0.06, 0.08).$$

Definition 3.3 *Let $N_G = (A, B)$ be a neutrosophic fuzzy graph. The closed neighbourhood degree of a vertex x in N_G defined by*

$$d_{N_G}[x] = (deg_T[x], deg_I[x], deg_F[x]),$$

where

$$\begin{aligned} deg_T(x) &= \sum_{xy \in E} T_B(xy) + T_A(x), \\ deg_I(x) &= \sum_{xy \in E} I_B(xy) + I_B(x), \\ deg_F(x) &= \sum_{xy \in E} F_B(xy) + F_B(x). \end{aligned}$$

Example 3.4 Consider the neutrosophic fuzzy graph N_G in Fig.2.

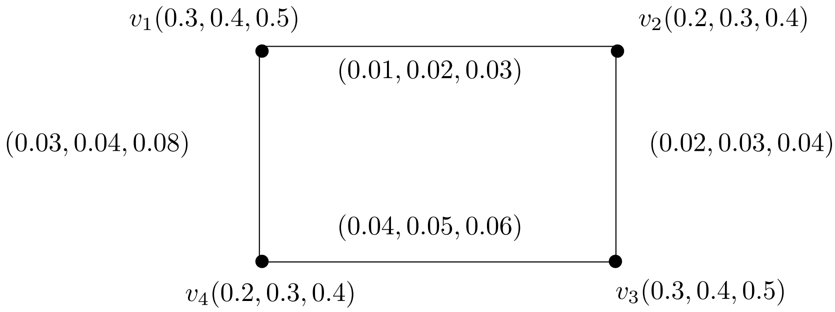

Fig.2

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.04, 0.06, 0.08),\ d_{N_G}(v_2) = (0.03, 0.05, 0.07), \\ d_{N_G}(v_3) &= (0.06, 0.08, 0.10),\ d_{N_G}(v_4) = (0.07, 0.09, 0.14), \\ d_{N_G}[v_1] &= (0.34, 0.46, 0.58),\ d_{N_G}[v_2] = (0.23, 0.35, 0.47), \\ d_{N_G}[v_3] &= (0.36, 0.48, 0.6),\ d_{N_G}[v_4] = (0.27, 0.39, 0.54). \end{aligned}$$

§4. Regular and Irregular Neutrosophic Fuzzy Graphs

Definition 4.1 *A neutrosophic fuzzy graph is called regular if all the vertices of N_G have the same open neighbourhood degree.*

Example 4.2 Consider the neutrosophic fuzzy graph N_G shown in Fig.3.

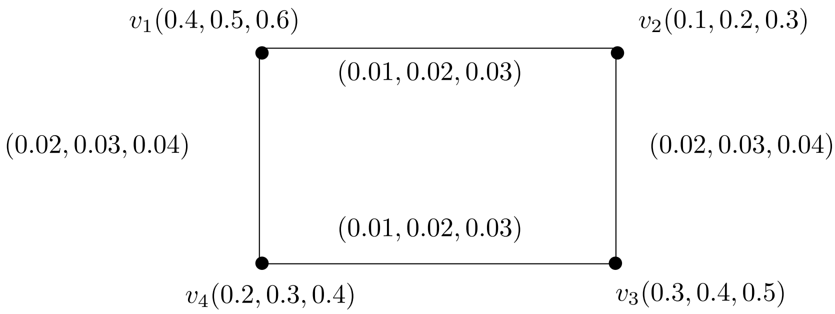

Fig. 3

In this graph,

$$d_{N_G}(v_1) = (0.03, 0.05, 0.07),\ d_{N_G}(v_2) = (0.03, 0.05, 0.07),$$
$$d_{N_G}(v_3) = (0.03, 0.05, 0.07),\ d_{N_G}(v_4) = (0.03, 0.05, 0.07).$$

Here all the vertices having same open neighbourhood degree. Hence this N_G is regular neutrosophic fuzzy graph.

Definition 4.3 *A neutrosophic fuzzy graph is said to be irregular if there is a vertex which is adjacent to vertices with distinct open neighbourhood degrees.*

Example 4.4 Let N_G be the neutrosophic fuzzy graph shown in Fig.4.

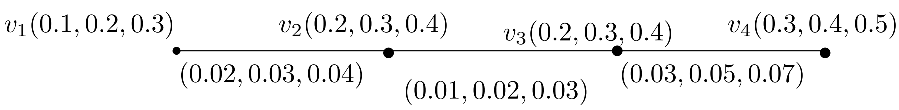

Fig.4

In this graph,

$$d_{N_G}(v_1) = (0.02, 0.03, 0.04),\ d_{N_G}(v_2) = (0.03, 0.05, 0.07),$$
$$d_{N_G}(v_3) = (0.04, 0.07, 0.10),\ d_{N_G}(v_4) = (0.03, 0.05, 0.07).$$

Here there is a vertex v_2 adjacent to the vertices v_1 and v_3 which are having distinct open neighbourhood degrees. Hence this graph is irregular neutrosophic fuzzy graph.

Definition 4.5 *A neutrosophic fuzzy graph N_G is called totally irregular neutrosophic fuzzy graphs if there is a vertex which is adjacent to the vertices with distinct closed neighbourhood degrees.*

Example 4.6 Let N_G be the neutrosophic fuzzy graph shown in Fig.5.

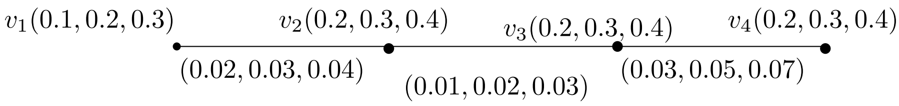

Fig.5

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.02, 0.03, 0.04),\ d_{N_G}(v_2) = (0.03, 0.05, 0.07), \\ d_{N_G}(v_3) &= (0.04, 0.07, 0.10),\ d_{N_G}(v_4) = (0.03, 0.05, 0.07), \\ d_{N_G}[v_1] &= (0.12, 0.23, 0.34),\ d_{N_G}[v_2] = (0.23, 0.35, 0.47), \\ d_{N_G}[v_3] &= (0.24, 0.37, 0.5),\ d_{N_G}[v_4] = (0.23, 0.35, 0.47). \end{aligned}$$

Here there is a vertex v_2 adjacent to the vertices v_1 and v_3 which are having distinct closed neighbourhood degrees. Hence this graph is totally irregular neutrosophic fuzzy graph.

§5. Neighbourly Irregular Neutrosophic Fuzzy Graphs

Definition 5.1 *Let N_G be a neutrosophic fuzzy graph. If every two adjacent vertices of N_G have distinct open neighbourhood degrees, then it is referred as a neighbourly irregular neutrosophic fuzzy graph.*

Example 5.2 Let N_G be the neutrosophic fuzzy graph shown in Fig.6.

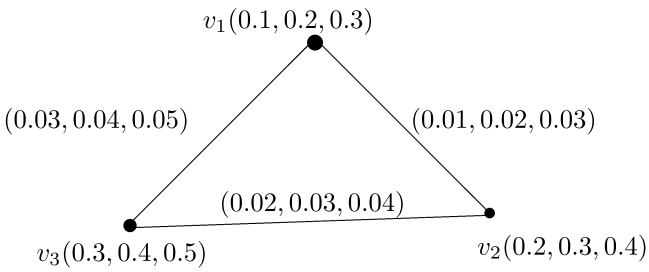

Fig.6

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.04, 0.06, 0.08),\ d_{N_G}(v_2) = (0.03, 0.05, 0.07), \\ d_{N_G}(v_3) &= (0.05, 0.07, 0.09). \end{aligned}$$

Here every two adjacent vertices having distinct open neighbourhood degrees. Hence this graph is neighbourly irregular neutrosophic fuzzy graph.

Definition 5.3 *A neutrosophic fuzzy graph is said to be neighbourly totally irregular neutrosophic fuzzy graph if every two adjacent vertices in N_G have distinct closed neighbourhood degrees.*

Example 5.3 Let N_G be the neutrosophic fuzzy graph shown in Fig.7.

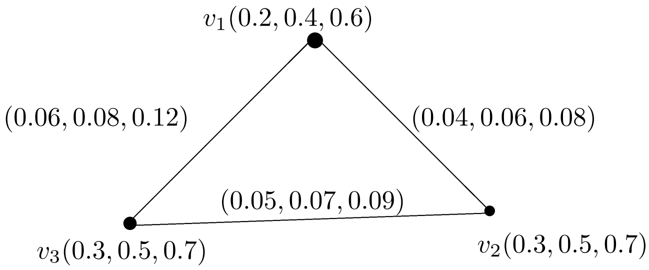

Fig.7

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.10, 0.14, 0.20),\ d_{N_G}(v_2) = (0.09, 0.13, 0.17), \\ d_{N_G}(v_3) &= (0.11, 0.15, 0.21),\ d_{N_G}[v_1] = (0.20, 0.44, 0.80), \\ d_{N_G}[v_2] &= (0.39, 0.63, 0.87),\ d_{N_G}[v_3] = (0.41, 0.65, 0.91). \end{aligned}$$

Here every two adjacent vertices having distinct closed neighbourhood degrees. Hence this graph is neighbourly totally irregular neutrosophic fuzzy graph.

Observation 5.5 Every neighbourly irregular neutrosophic fuzzy graphs need not be a neighbourly totally irregular neutrosophic fuzzy graph.

Example 5.6 Let N_G be the neutrosophic fuzzy graph shown in Fig.8.

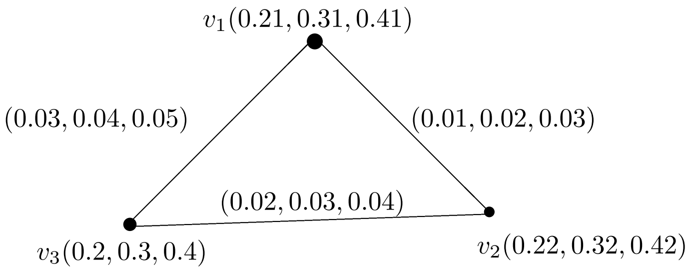

Fig.8

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.04, 0.06, 0.08),\ d_{N_G}(v_2) = (0.03, 0.05, 0.07), \\ d_{N_G}(v_3) &= (0.05, 0.07, 0.09),\ d_{N_G}[v_1] = (0.25, 0.37, 0.49), \\ d_{N_G}[v_2] &= (0.25, 0.37, 0.49),\ d_{N_G}[v_3] = (0.25, 0.37, 0.49). \end{aligned}$$

Here, every pair of adjacent vertices having distinct open neighbourhood degrees but every pair adjacent vertices having same closed neighbourhood degrees.

Hence N_G is neighbourly irregular neutrosophic fuzzy graph. But N_G is not neighbourly totally irregular neutrosophic fuzzy graph.

Observation 5.7 Every neighbourly totally irregular neutrosophic fuzzy graphs need not be a neighbourly irregular neutrosophic fuzzy graph.

Example 5.8 Consider the neutrosophic fuzzy graph N_G in Fig.9, we get that

$$\begin{aligned} d_{N_G}(v_1) &= (0.04, 0.12, 0.24),\ d_{N_G}(v_2) = (0.04, 0.12, 0.24), \\ d_{N_G}(v_3) &= (0.04, 0.12, 0.24),\ d_{N_G}(v_4) = (0.04, 0.12, 0.24), \\ d_{N_G}[v_1] &= (0.24, 0.42, 0.64),\ d_{N_G}[v_2] = (0.14, 0.22, 0.54), \\ d_{N_G}[v_3] &= (0.24, 0.42, 0.64),\ d_{N_G}[v_4] = (0.14, 0.22, 0.54). \end{aligned}$$

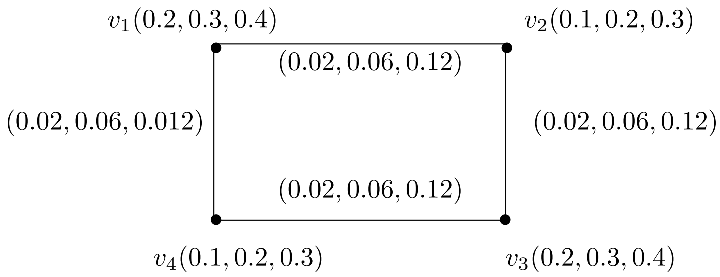

Fig.9

Here every pair of adjacent vertices having distinct closed neighbourhood degrees. But every pair adjacent vertices having same open neighbourhood degrees. Hence N_G is neighbourly totally irregular neutrosophic fuzzy graph. But N_G is not neighbourly irregular neutrosophic fuzzy graph.

§6. Highly Irregular Neutrosophic Fuzzy graphs

Definition 6.1 *Let N_G be a neutrosophic fuzzy graph. If every vertex in N_G is adjacent to vertices with distinct open neighbourhood degrees, then it is called as highly irregular neutrosophic fuzzy graph.*

Example 6.2 Consider the neutrosophic fuzzy graph N_G shown in Fig.10.

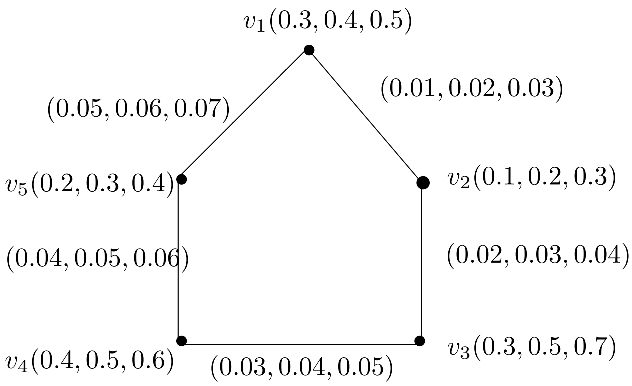

Fig.10

In this graph,

$$d_{N_G}(v_1) = (0.06, 0.08, 0.10),\ d_{N_G}(v_2) = (0.03, 0.05, 0.07),$$
$$d_{N_G}(v_3) = (0.05, 0.07, 0.09),\ d_{N_G}(v_4) = (0.07, 0.09, 0.11),\ d_{N_G}(v_4) = (0.09, 0.11, 0.13).$$

Here every vertex in N_G is adjacent to vertices with distinct open neighbourhood degrees. Hence this N_G is highly irregular neutrosophic fuzzy graph.

Definition 6.3 *Let N_G be a neutrosophic fuzzy graph. If every vertex in N_G is adjacent to vertices with distinct closed neighbourhood degrees, then it is called as highly totally irregular neutrosophic fuzzy graph.*

Example 6.4 Let N_G be the neutrosophic fuzzy graph shown in Fig.11.

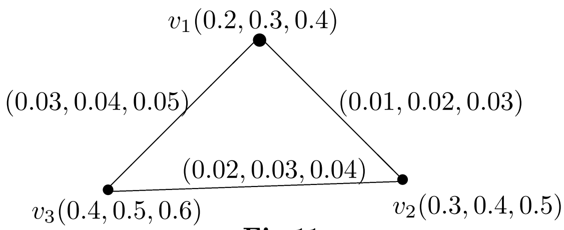

Fig.11

In this graph,

$$d_{N_G}(v_1) = (0.04, 0.06, 0.08),\ d_{N_G}(v_2) = (0.03, 0.05, 0.07),$$
$$d_{N_G}(v_3) = (0.05, 0.07, 0.09),\ d_{N_G}[v_1] = (0.24, 0.36, 0.48),$$
$$d_{N_G}[v_2] = (0.33, 0.45, 0.57),\ d_{N_G}[v_3] = (0.45, 0.57, 0.69).$$

Here every vertex in N_G is adjacent to vertices with distinct closed neighbourhood degrees. Hence this N_G is highly totally irregular neutrosophic fuzzy graph.

Observation 6.5 Every highly irregular neutrosophic fuzzy graphs need not be a highly totally irregular neutrosophic fuzzy graph.

Example 6.6 Consider the neutrosophic fuzzy graph N_G shown in Fig.12.

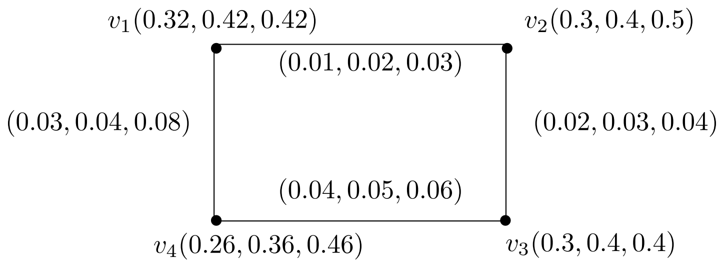

Fig. 12

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.04, 0.06, 0.08),\ d_{N_G}(v_2) = (0.03, 0.05, 0.07), \\ d_{N_G}(v_3) &= (0.06, 0.08, 0.10),\ d_{N_G}(v_4) = (0.07, 0.09, 0.11), \\ d_{N_G}[v_1] &= (0.36, 0.48, 0.5),\ d_{N_G}[v_2] = (0.33, 0.45, 0.57), \\ d_{N_G}[v_3] &= (0.36, 0.48, 0.5),\ d_{N_G}[v_4] = (0.33, 0.45, 0.57). \end{aligned}$$

Here, every vertex in N_G is adjacent to vertices with distinct open neighborhood degrees. But a vertex v_2 adjacent to the vertices v_1 and v_3 which are having same closed neighbourhood degrees. Hence N_G is highly irregular neutrosophic fuzzy graph. But N_G is not highly totally irregular neutrosophic fuzzy graph.

Observation 6.7 Every highly totally irregular neutrosophic fuzzy graph needs not be a highly irregular neutrosophic fuzzy graph.

Example 6.8 Consider the neutrosophic fuzzy graph N_G shown in Fig.13. We know that

$$\begin{aligned} d_{N_G}(v_1) &= (0.05, 0.07, 0.09),\ d_{N_G}(v_2) = (0.05, 0.07, 0.09), \\ d_{N_G}(v_3) &= (0.04, 0.06, 0.08),\ d_{N_G}(v_4) = (0.03, 0.05, 0.07), \\ d_{N_G}(v_5) &= (0.05, 0.07, 0.09),\ d_{N_G}[v_1] = (0.35, 0.47, 0.57), \\ d_{N_G}[v_2] &= (0.15, 0.27, 0.39),\ d_{N_G}[v_3] = (0.34, 0.36, 0.48), \\ d_{N_G}[v_4] &= (0.13, 0.25, 0.37),\ d_{N_G}[v_5] = (0.25, 0.37, 0.49). \end{aligned}$$

Here, every vertex in N_G is adjacent to the vertices with distinct closed neighbourhood degrees. But a vertex v_2 is adjacent to the vertices v_1 and v_5 which are having same neighbourhood degrees. Hence N_G is highly totally irregular neutrosophic fuzzy graph. But N_G is not highly irregular neutrosophic fuzzy graph.

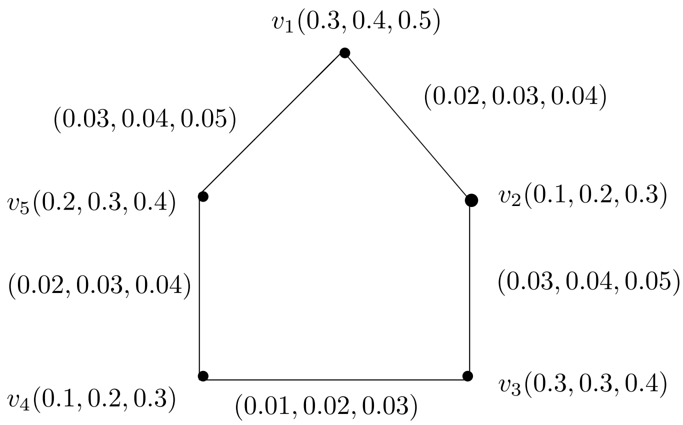

Fig.13

Theorem 6.9 *Let N_G be a neutrosophic fuzzy graph. Then N_G is highly irregular neutrosophic* *fuzzy graph and neighbourly irregular neutrosophic fuzzy graph iff the open neighbourhood degrees of all the vertices of N_G are distinct.*

Proof Let N_G be a neutrosophic fuzzy graph with n vertices $v_1, v_2, \cdots, v_n$. Suppose N_G is both highly and neighbourly irregular neutrosophic fuzzy graph. we shall show that the open neighbourhood degrees of all the vertices of N_G are distinct.

Let $d_{N_G}(v_i) = (deg_T(v_i), deg_I(v_i), deg_F(v_i))$, $i = 1, 2, \cdots, n$. Let the adjacent vertices of v_1 be $v_2, v_3, \cdots, v_n$ with open neighbourhood degrees

$$\begin{gathered}(deg_T(v_2), deg_I(v_2), deg_F(v_2)),\\ (deg_T(v_3), deg_I(v_3), deg_F(v_3)),\\ \cdots\cdots\cdots\cdots\cdots\cdots,\\ (deg_T(v_n), deg_I(v_n), deg_F(v_n)),\end{gathered}$$

respectively. Since N_G is highly irregular neutrosophic fuzzy graph, we have

$$\begin{gathered}deg_T(v_2) \neq deg_T(v_3) \neq \cdots \neq deg_T(v_n),\\ deg_I(v_2) \neq deg_I(v_3) \neq \cdots \neq deg_I(v_n),\\ deg_F(v_2) \neq deg_F(v_3) \neq \cdots \neq deg_F(v_n).\end{gathered}$$

Also since N_G is neighbourly irregular neutrosophic fuzzy graph, we have

$$\begin{gathered}deg_T(v_1) \neq deg_T(v_2) \neq deg_T(v_3) \neq \cdots \neq deg_T(v_n),\\ deg_I(v_1) \neq deg_I(v_2) \neq deg_I(v_3) \neq \cdots \neq deg_I(v_n),\\ deg_F(v_1) \neq deg_F(v_2) \neq deg_F(v_3) \neq \cdots \neq deg_F(v_n).\end{gathered}$$

Thus,

$$\begin{aligned}(deg_T(v_1), deg_I(v_1), deg_F(v_1)) &\neq (deg_T(v_2), deg_I(v_2), deg_F(v_2))\\ &\neq \cdots \neq (deg_T(v_n), deg_I(v_n), deg_F(v_n)).\end{aligned}$$

Hence the open neighbourhood degrees of all vertices are distinct.

Conversely, suppose we take the neighbourhood degrees of all vertices are distinct. To show that N_G is highly and neighbourly irregular neutrosophic fuzzy graphs. Let $d_{N_G}(v_i) = (deg_T(v_i), deg_I(v_i), deg_F(v_i))$, $i = 1, 2, \cdots, n$.

Given that

$$\begin{gathered}deg_T(v_1) \neq deg_T(v_2) \neq deg_T(v_3) \neq \cdots \neq deg_T(v_n),\\ deg_I(v_1) \neq deg_I(v_2) \neq deg_I(v_3) \neq \cdots \neq deg_I(v_n),\\ deg_F(v_1) \neq deg_F(v_2) \neq deg_F(v_3) \neq \cdots \neq deg_F(v_n).\end{gathered}$$

Therefore, any two adjacent vertices have distinct open neighbourhood degrees and also every vertex, which is adjacent to the vertices having distinct open neighbourhood degrees. Hence N_G is both highly and neighbourly irregular neutrosophic fuzzy graphs. □

Theorem 6.10 *Let N_G be neutrosophic fuzzy graph. If N_G is neighbourly irregular neutrosophic fuzzy graph and (T_A, I_A, F_A) is a constant function, then N_G is a neighbourly totally irregular neutrosophic fuzzy graph.*

Proof Let N_G be a neighbourly irregular neutrosophic fuzzy graph. Let $v_i, v_j \in V$, where v_i and v_j are adjacent vertices with distinct open neighbourhood degrees

$$(deg_T(v_1), deg_I(v_1), deg_F(v_1)) \text{ and } (deg_T(v_2), deg_I(v_2), deg_F(v_2))$$

respectively.

Suppose we assume that

$$(T_A(v_i), I_A(v_i), F_A(v_i)) = (T_A(v_j), I_A(v_j), F_A(v_j)) = (c_1, c_2, c_3),$$

where c_1, c_2, c_3 are constants and $c_1, c_2, c_3 \in [0, 1]$. Then,

$$\begin{aligned}
deg_T[v_i] &= deg_T(v_i) + T_A(v_i) = deg_T(v_i) + c_1, \\
deg_I[v_i] &= deg_I(v_i) + I_A(v_i) = deg_I(v_i) + c_2, \\
deg_F[v_i] &= deg_F(v_i) + F_A(v_i) = deg_F(v_i) + c_3, \\
deg_T[v_j] &= deg_T(v_j) + T_A(v_j) = deg_T(v_j) + c_1, \\
deg_I[v_j] &= deg_I(v_j) + I_A(v_j) = deg_I(v_j) + c_2, \\
deg_F[v_j] &= deg_F(v_j) + F_A(v_j) = deg_F(v_j) + c_3.
\end{aligned}$$

We need to show that $deg_T[v_i] \neq deg_T[v_j]$, $deg_I[v_i] \neq deg_I[v_j]$ and $deg_F[v_i] \neq deg_F[v_j]$. Suppose

$$\begin{aligned}
deg_T[v_i] = deg_T[v_j] &\implies deg_T(v_i) + c_1 = deg_T(v_j) + c_1 \\
&\implies deg_T(v_i) - deg_T(v_j) = c_1 - c_1 \\
&\implies deg_T(v_i) - deg_T(v_j) = 0 \\
&\implies deg_T(v_i) = deg_T(v_j),
\end{aligned}$$

which is a contradiction to the fact that $deg_T(v_i) \neq deg_T(v_j)$. Therefore, $deg_T[v_i] \neq deg_T[v_j]$.

Similarly, $deg_I[v_i] \neq deg_I[v_j]$, $deg_F[v_i] \neq deg_F[v_j]$. Hence N_G is neighbourly totally irregular neutrosophic fuzzy graph. □

Observation 6.11 Every neighbourly irregular neutrosophic fuzzy graph need not be a highly irregular neutrosophic fuzzy graph.

Example 6.12 Consider the neutrosophic fuzzy graph N_G shown in Fig.14.

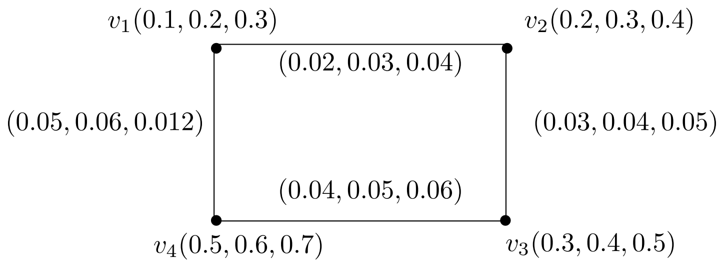

Fig. 14

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.07, 0.09, 0.1),\ d_{N_G}(v_2) = (0.05, 0.07, 0.09), \\ d_{N_G}(v_3) &= (0.07, 0.09, 0.11),\ d_{N_G}(v_4) = (0.09, 0.11, 0.17). \end{aligned}$$

Here, every two adjacent vertices having distinct open neighbourhood degrees. But a vertex v_2 adjacent to the vertices v_1 and v_3 which are having same open neighbourhood degree. Hence N_G is neighbourly irregular neutrosophic fuzzy graph. But N_G is not highly irregular neutrosophic fuzzy graph.

Observation 6.13 Every neighbourly totally irregular neutrosophic fuzzy graph need not be a highly totally irregular neutrosophic fuzzy graph.

Example 6.14 Consider the neutrosophic fuzzy graph N_G shown in Fig.15.

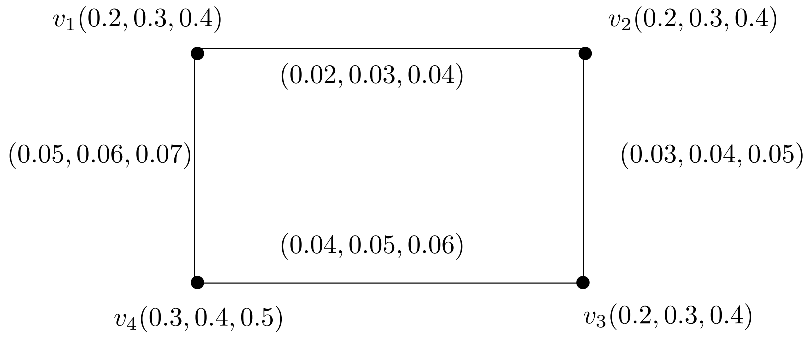

Fig. 15

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.07, 0.09, 0.11),\ d_{N_G}(v_2) = (0.05, 0.07, 0.09), \\ d_{N_G}(v_3) &= (0.07, 0.09, 0.11),\ d_{N_G}(v_4) = (0.09, 0.11, 0.13), \\ d_{N_G}[v_1] &= (0.27, 0.39, 0.51),\ d_{N_G}[v_2] = (0.25, 0.37, 0.49), \\ d_{N_G}[v_3] &= (0.27, 0.39, 0.51),\ d_{N_G}[v_4] = (0.39, 0.51, 0.63). \end{aligned}$$

Here every two adjacent vertices having distinct closed neighbourhood degrees but a vertex v_2 adjacent to the vertices v_1 and v_3 having same closed neighbourhood degree. Hence N_G is neighbourly totally irregular neutrosophic fuzzy graph. But N_G is not highly totally irregular neutrosophic fuzzy graph.

Observation 6.15 Every highly irregular neutrosophic fuzzy graph need not be a neighbourly irregular neutrosophic fuzzy graph.

Example 6.16 Consider the neutrosophic fuzzy graph N_G shown in Fig.16.

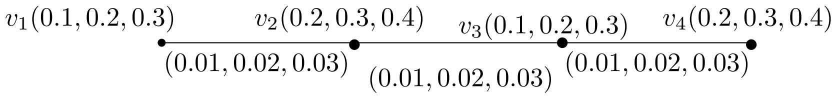

Fig.16

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.01, 0.02, 0.03),\ d_{N_G}(v_2) = (0.02, 0.04, 0.06), \\ d_{N_G}(v_3) &= (0.02, 0.04, 0.06),\ d_{N_G}(v_4) = (0.01, 0.02, 0.03). \end{aligned}$$

Here, every vertex is adjacent to the vertices with distinct degrees but the two adjacent vertices v_2 and v_3 have same degree. Hence N_G is highly irregular neutrosophic fuzzy graph. But N_G is not neighbourly irregular neutrosophic fuzzy graph.

Observation 6.17 Every highly totally irregular neutrosophic fuzzy graph need not be a neighbourly totally irregular neutrosophic fuzzy graph.

Example 6.18 Consider the neutrosophic fuzzy graph N_G shown in Fig.17.

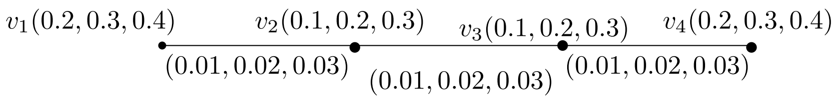

Fig.17

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.01, 0.02, 0.03),\ d_{N_G}(v_2) = (0.02, 0.04, 0.06), \\ d_{N_G}(v_3) &= (0.02, 0.04, 0.06),\ d_{N_G}(v_4) = (0.01, 0.02, 0.03), \\ d_{N_G}[v_1] &= (0.21, 0.32, 0.43),\ d_{N_G}[v_2] = (0.12, 0.24, 0.36), \\ d_{N_G}[v_3] &= (0.12, 0.24, 0.36),\ d_{N_G}[v_1] = (0.21, 0.32, 0.43). \end{aligned}$$

Here, every vertex is adjacent to the vertices with distinct closed neighborhood degrees but the two adjacent vertices v_2 and v_3 have same closed neighbourhood degree. Hence N_G is highly totally irregular neutrosophic fuzzy graph. But N_G is not neighbourly totally irregular neutrosophic fuzzy graph.

Theorem 6.19 *Let N_G be a neutrosophic fuzzy graph. If N_G is neighbourly totally irregular neutrosophic fuzzy graph and (T_A, I_A, F_A) is a constant function. Then N_G is neighbourly irregular neutrosophic fuzzy graph.*

Proof Let N_G be a neighbourly totally irregular neutrosophic fuzzy graph. Then by definition, the closed neighbourhood degree of every two adjacent vertices are distinct. Let

$v_i, v_j \in V$, where v_i, v_j are adjacent vertices with distinct closed neighbourhood degrees. To prove this N_G is neighbourly irregular neutrosophic fuzzy graph. Suppose we assume that

$$(T_A(v_i), I_A(v_i), F_A(v_i)) = (T_A(v_j), I_A(v_j), \quad F_A(v_j)) = (c_1, c_2, c_3),$$

where c_1, c_2, c_3 are constants and $c_1, c_2, c_3 \in [0,1]$ and $deg[v_i] \neq deg[v_j]$. We show that $deg(v_i) \neq deg(v_j)$. Since $deg[v_i] \neq deg[v_j]$, we have

$$deg_T[v_i] \neq deg_T[v_j], \; deg_I[v_i] \neq deg_I[v_j], \; deg_F[v_i] \neq deg_F[v_j].$$

Now take

$$\begin{aligned} deg_T[v_i] \neq deg_T[v_j] \quad & \Longrightarrow \quad deg_T(v_i) + c_1 \neq deg_T(v_j) + c_1 \\ & \Longrightarrow \quad deg_T(v_i) - deg_T(v_j) \neq c_1 - c_1 = 0 \\ & \Longrightarrow \quad deg_T(v_i) \neq deg_T(v_j). \end{aligned}$$

Similarly, $deg_I(v_i) \neq deg_I(v_j)$ and $deg_F(v_i) \neq deg_F(v_j)$. Therefore, the degree of $v_i, v_j \in V$ are distinct. This result is true for all pair of adjacent vertices in N_G. Hence N_G is neighbourly irregular neutrosophic fuzzy graph. □

Observation 6.20 Let N_G be a neutrosophic fuzzy graph, where N_G is a cycle with (T_B, I_B, F_B) is a constant function, Then N_G is neighbourly and highly regular neutrosophic fuzzy graph.

Example 6.21 Let N_G be the neutrosophic fuzzy graph shown in Fig.18. Then, we know that

$$\begin{aligned} d_{N_G}(v_1) &= (0.02, 0.04, 0.06), \; d_{N_G}(v_2) = (0.02, 0.04, 0.06), \\ d_{N_G}(v_3) &= (0.02, 0.04, 0.06). \end{aligned}$$

Here, every two adjacent vertices having same degree and every vertex adjacent to the vertices having same degree. Hence N_G is neighbourly and highly regular neutrosophic fuzzy graph.

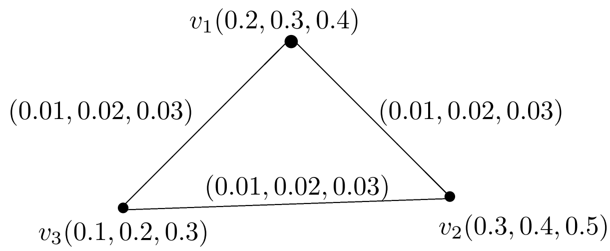

Fig.18

Observation 6.22 Let N_G be a neutrosophic fuzzy graph, where N_G is a cycle with (T_A, I_A, F_A) and (T_B, I_B, F_B) is a constant function, Then N_G is neighbourly and highly regular neutrosophic fuzzy graph.

Example 6.23 Consider the neutrosophic fuzzy graph N_G shown in Fig.19.

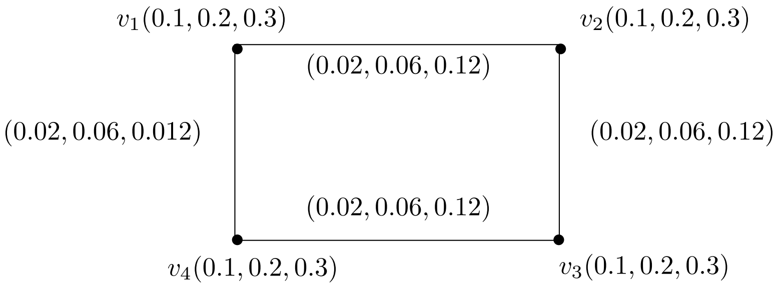

Fig. 19

In this graph,

$$\begin{aligned} d_{N_G}(v_1) &= (0.04, 0.12, 0.24),\ d_{N_G}(v_2) = (0.04, 0.12, 0.24), \\ d_{N_G}(v_3) &= (0.04, 0.12, 0.24),\ d_{N_G}(v_4) = (0.04, 0.12, 0.24), \\ d_{N_G}[v_1] &= (0.14, 0.22, 0.54),\ d_{N_G}[v_2] = (0.14, 0.22, 0.54), \\ d_{N_G}[v_3] &= (0.14, 0.22, 0.54),\ d_{N_G}[v_4] = (0.14, 0.22, 0.54). \end{aligned}$$

Here, every two adjacent vertices having same closed neighbourhood degree and every vertex adjacent to the vertices having same closed neighbourhood degree. Hence N_G is neighbourly and highly regular neutrosophic fuzzy graph.

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The Modular Nilpotent Group $M_{p^n} \times C_p$ for $p > 2$

S. A. Adebisi

(Department of Mathematics , Faculty of Science, University of Lagos, Nigeria)

M. EniOluwafe

(Department of mathematics,Faculty of Science, university of Ibadan, Nigiria)

E-mail: adesinasunday@yahoo.com, michael.enioluwafe@gmail.com

Abstract: In this paper, the classification of finite p-groups is extended to the modular nilpotent group of the form $M_{p^n} \times C_p$ in which, p is greater than 2.

Key Words: Finite p-groups, nilpotent group, fuzzy subgroups, dihedral group, inclusion-exclusion principle, maximal subgroups.

AMS(2010): 20D15, 20E28, 20F18, 20N25, 20K27.

§1. Introduction

The following properties for the fuzzy subgroups of G were known:

(1) The level sets of a fuzzy subset of a finite set form a chain;

(2) λ is a fuzzy subgroup of G iff its level sets are subgroups of G';

(3) The relation $\sim$ is an equivalence relation on fuzzy subgroups of G,where for fuzzy subgroups μ, ν of G , $\mu \sim \nu$ iff $\forall x, y \in G, (\mu(x) > \mu(y) iff \nu(x) > \nu(y))$.

§2. Preliminaries

Suppose that $(G, \cdot, e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is said to be a fuzzy subgroup of G if the following two conditions are satisfied:

(i) $\lambda(ab) \geq \in \{\lambda(a), \lambda(b)\}, \ \forall\, a, b \in G$;

(ii) $\lambda(a^{-1} \geq \lambda(a)$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$. Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$. [Marius [1]].

Now, concerning the subgroups, the set $FL(G)$ possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.

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In what follows, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \cdots, M_t$ are the maximal subgroups of G. Let $h(G)$ denote the number of chains of subgroups of G which ends in G. The method of computing $h(G)$ is based on the application of the inclusion-exclusion principle. If A is the set of chains in G of type

$$C_1 \subset C_2 \subset \cdots \subset C_r = G$$

and A' represents the set of chains of A' which are contained in M_r, $r = 1, \cdots, t$. Then, we have

$$\begin{aligned} |A| &= 1 + |A'| = \left| \bigcup_{r=1}^{t} A_r \right| \\ &= 1 + \sum_{r=1}^{t} |A_r| - \sum_{1 \leq r_1 \leq r_2 \leq t} |A_{r_1} \cap A_{r_2}| + \cdots + (-1)^{t-1} \left| \bigcap_{r=1}^{t} A_r \right| \end{aligned}$$

Observe that, for every $1 \leq w \leq t$ and $1 \leq r_1 < r_2 < \cdots < r_w \leq t$, the set $\bigcap_{i=1}^{w} A_{r_i}$ consists of all chains of A' which are included in $\bigcap_{i=1}^{w} M_{r_i}$. We have that

$$\left| \bigcap_{i=1}^{w} A_{r_i} \right| = 2h \left(\bigcap_{i=1}^{w} M_{r_i} \right)^{-1}$$

Therefore,

$$\begin{aligned} |A| &= 1 + \sum_{r=1}^{t} (2h(M_r) - 1) - \sum_{1 \leq r_1 < r_2 \leq t} (2h(M_{r_1} \cap M_{r_2}) - 1) \\ &\quad + \cdots + (-1)^{t-1} \left(2h \left(\bigcap_{r=1}^{t} M_r \right) - 1 \right) \\ &= 2 \left(\sum_{r=1}^{t} h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h \left(\bigcap_{r=1}^{t} M_r \right) \right) + C, \end{aligned}$$

where, the constant C can be determined by

$$\begin{aligned} C &= 1 + \sum_{r=1}^{t} (-1) - \sum_{1 \leq r_1 < r_2 \leq t} (-1) + \cdots + (-1)^{t-1}(-1) \\ &= (1-1)^t = 0 \end{aligned}$$

and we have that

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1<r_2\le t} h(M_{r_1}\cap M_{r_2}) + \cdots + (-1)^{t-1}h\left(\bigcap_{r=1}^{t} M_r\right)\right) \qquad (c)$$

In [2], the equality (c) was used to obtain the explicit formulas of $h(D_{2n})$ for some positive integers n.

Theorem 2.1 *The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:*

(1) $h(\mathbb{Z}_{p^n}) = 2^n$;

(2) $h(D_{2^n}) = 2^{2n-1}$;

(3) $h(\varphi_{2^n}) = 2^{2n-2}$;

(4) $h(S_{2^n}) = 3.2^{2n-3}$;

(5) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2+(n-1)p]$.

§3. The Fuzzy Subgroup for the Nilpotent Group of the Form: $M_{p^n} \times C_p$

Recall that the case for $p = 2$ was already handled in [3]. Now, for $p > 2$ and, of course, p is a prime, we consider this case as follows.

3.1 The Derivation of $h(M_{p^n} \times C_p)$ for $p > 2$

We begin with the case $p = 3$ and $n = 3$ where

$$\begin{aligned} M_{3^3} &= \langle x, y | x^9 = y^3 = 1,\ y^{-1}xy = x^4\rangle \\ &= \left\{\begin{array}{l} 1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, \\ y, y^2, xy, xy^2, x^2y, x^2y^2, x^3y, x^3y^2, x^4y, \\ x^4y^2, x^5y, x^5y^2, x^6y, x^6y^2, x^7y, x^7y^2, x^8y, x^8y^2 \end{array}\right\} \end{aligned}$$

and

$$M_{3^3} \times C_3 = \left\{\begin{array}{l} 1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, \\ y, y^2, xy, xy^2, x^2y, x^2y^2, x^3y, x^3y^2, x^4y, \\ x^4y^2, x^5y, x^5y^2, x^6y, x^6y^2, x^7y, x^7y^2, x^8y, x^8y^2 \end{array}\right\} \times \{1, a, a^2\}$$

$$= \left\{ \begin{array}{l} (1,1),(1,a),(1,a^2),(x,1),(x,a),(x,a^2),(x^2,1),(x^2,a),(x^2,a^2), \\ (x^3,1),(x^3,a),(x^3,a^2),(x^4,1),(x^4,a),(x^4,a^2),(x^5,1),(x^5,a), \\ (x^5,a^2),(x^6,1),(x^6,a),(x^6,a^2),(x^7,1),(x^7,a),(x^7,a^2), \\ (x^8,1),(x^8,a),(x^8,a^2),(y,1),(y,a),(y,a^2),(y^2,1),(y^2,a),(y^2,a^2) \\ (xy,1),(xy,a),(xy,a^2),(xy^2,1),(xy^2,a),(xy^2,a^2),(x^2y,1) \\ (x^2y,a),(x^2y,a^2),(x^2y^2,1),(x^2y^2,a),(x^2y^2,a^2),(x^3y,1) \\ (x^3y,a),(x^3y,a^2),(x^3y^2,1)(x^3y^2,a),(x^3y^2,a^2) \\ (x^4y,1),(x^4y,a),(x^4y,a^2),(x^4y^2,1),(x^4y^2,a) \\ (x^4y^2,a^2),(x^5y,1),(x^5y,a),(x^5y,a^2) \\ (x^5y^2,1),(x^5y^2,a),(x^5y^2,a^2),(x^6y,1), \\ (x^6y,a),(x^6y,a^2),(x^6y^2,1),(x^6y^2,a), \\ (x^6y^2,a^2),(x^7y,1),(x^7y,a),(x^7y,a^2) \\ (x^7y^2,1),(x^7y^2,a),(x^7y^2,a^2),(x^8y,1), \\ (x^8y,a),(x^8y,a^2),(x^8y^2,1),(x^8y^2,a),(x^8y^2,a^2) \end{array} \right\}$$

Lemma 3.1(Berkovich,[3]) *Let G be a group of order p^n.*

(i) If A is a subgroup of G of order p^k and $k < m < n$, then, the number of subgroups of G of order p^m containing $A \equiv 1(\mathrm{mod} p)$.

(ii) If G is a noncyclic group of order p^n, $1 < m < n-1$, then,

$$S_m(G) \in \{1+p, 1+p+p^2\},$$

where $S_m(G)$ is the number of subgroups of order p^m in G.

By Lemma 3.1, let $\mathcal{M}$ be the collection of all the maximal subgroups of G. Then, the set

$$|\mathcal{M}| = 1+p+p^2$$

and there exists 13 distinct maximal subgroups for $M_{3^3} \times C_3$. These maximal subgroups are generated by

$$\left\{ \begin{array}{l} \langle (1,a),(x,1) \rangle, \langle (y,1),(x,1) \rangle, \langle (y,a),(x,1) \rangle, \langle (y,a),(x,a) \rangle, \\ \langle (y,1),(x,a) \rangle, \langle (y,a^2),(x,a) \rangle, \langle (xy^2,1),(1,a) \rangle, \\ \langle (xy,1),(1,a) \rangle, \langle (y,1),(x,a^2) \rangle, \langle (y,a),(x,a^2) \rangle, \\ \langle (y,a^2),(x,1) \rangle, \langle (y,1),(1,a),(x^3,1) \rangle \text{ and } \langle (y,a^2),(x,a^2) \rangle \end{array} \right\}.$$

We therefore have

$$M_1 = \left\{ \begin{array}{l} (1,1),(1,a),(1,a^2),(x,1),(x^2,1),(x^3,1),(x^4,1),(x^5,1),(x^6,1), \\ (x^7,1),(x^8,1),(x,a),(x,a^2),(x^2,a),(x^2,a^2),(x^3,a),(x^3,a^2),(x^4,a), \\ (x^4,a^2),(x^5,a),(x^5,a^2),(x^6,a),(x^6,a^2),(x^7,a),(x^7,a^2),(x^8,a),(x^8,a^2) \end{array} \right\},$$

$$M_2 = \left\{ \begin{array}{l} (1,1),(x,1),(y,1),(x^2,1),(x^3,1),(x^4,1),(x^5,1),(x^6,1),(x^7,1), \\ (x^8,1),(y^2,1),(xy,1),(xy^2,1),(x^2y,1),(x^2y^2,1),(x^3y,1),(x^3y^2,1), \\ (x^4y,1),(x^4y^2,1),(x^5y,1),(x^5y^2,1),(x^6y,1),(x^6y^2,1),(x^7y,1), \\ (x^7y^2,1),(x^8y,1),(x^8y^2,1) \end{array} \right\},$$

$$M_3 = \left\{ \begin{array}{l} (1,1),(y,a),(x,1),(y^2,a^2),(x^2,1),(x^3,1),(x^4,1),(x^5,1),(x^6,1), \\ (x^7,1),(x^8,1),(xy,a),(xy^2,a^2),(x^2y,a),(x^2y^2,a^2),(x^3y,a), \\ (x^3y^2,a^2),(x^4y,a),(x^4y^2,a^2),(x^5y,a),(x^5y^2,a^2),(x^6y,a),(x^6y^2,a^2), \\ (x^7y,a),(x^7y^2,a^2),(x^8y,a),(x^8y^2,a^2) \end{array} \right\},$$

$$M_4 = \left\{ \begin{array}{l} (1,1),(x,a),(y,a),(x^2,a^2),(x^3,1),(x^4,a),(x^5,a^2),(x^6,1),(x^7,a), \\ (x^8,a^2),(y^2,a^2),(xy,a^2),(xy^2,1),(x^2y,1),(x^2y^2,a),(x^3y,a),(x^3y^2,a^2), \\ (x^4y,a^2),(x^4y^2,1),(x^5y,1),(x^5y^2,a),(x^6y,a),(x^6y^2,a^2),(x^7y,a^2), \\ (x^7y^2,1),(x^8y,1),(x^8y^2,a) \end{array} \right\},$$

$$M_5 = \left\{ \begin{array}{l} (1,1),(x,a),(y,1),(x^2,a^2),(x^3,1),(x^4,a),(x^5,a^2),(x^6,1),(x^7,a), \\ (x^8,a^2),(y^2,1),(xy,a),(xy^2,a),(x^2y,a^2),(x^2y^2,a^2),(x^3y,1),(x^3y^2,1), \\ (x^4y,a),(x^4y^2,a),(x^5y,a^2),(x^5y^2,a^2),(x^6y,1),(x^6y^2,1),(x^7y,a), \\ (x^7y^2,a),(x^8y,a^2),(x^8y^2,a^2) \end{array} \right\},$$

$$M_6 = \left\{ \begin{array}{l} (1,1),(1,a),(y,1),(x^3,1),(1,a^2),(y^2,1),(x^6,1),(y,a),(y^2,a),(y,a^2), \\ (y^2,a^2),(x^3y,1),(x^3y^2,1),(x^6y,1),(x^6y^2,1),(x^3,a),(x^3,a^2),(x^6,a), \\ (x^6,a^2),(x^3y^2,a^2),(x^3y^2,a),(x^3y,a^2),(x^3y,a),(x^6y,a^2),(x^6y,a), \\ (x^6y^2,a^2),(x^6y^2,a) \end{array} \right\},$$

$$M_7 = \left\{ \begin{array}{l} (1,1),(1,a),(1,a^2),(xy^2,1),(x^5y,1),(x^3,1),(x^4y^2,1),(x^8y,1),(x^6,1), \\ (x^7y^2,1),(x^2y,1),(xy^2,a),(x^5y,a),(x^3,a),(x^4y^2,a),(x^8y,a),(x^6,a), \\ (x^7y^2,a),(x^2y,a),(xy^2,a^2),(x^5y,a^2),(x^3,a^2),(x^4y^2,a^2),(x^8y,a^2), \\ (x^6,a^2),(x^7y^2,a^2),(x^2y,a^2) \end{array} \right\},$$

$$M_8 = \left\{ \begin{array}{l} (1,1),(1,a),(1,a^2),(xy,1),(x^8y^2,1),(x^3,1),(x^4y,1),(x^2y^2,1), \\ (x^6,1),(x^7y,1),(x^5y^2,1),(xy,a),(x^8y^2,a),(x^3,a),(x^4y,a), \\ (x^2y^2,a),(x^6,a),(x^7y,a),(x^5y^2,a),(xy,a^2),(x^8y^2,a^2),(x^3,a^2), \\ (x^4y,a^2),(x^2y^2,a^2),(x^6,a^2),(x^7y,a^2),(x^5y^2,a^2) \end{array} \right\},$$

$$M_9 = \left\{ \begin{array}{l} (1,1),(y,1),(y^2,1),(x,a^2),(x^2,a),(x^3,1),(x^4,a^2),(x^5,a),(x^6,1), \\ (x^7,a^2),(x^8,a),(xy,a^2),(x^2y,a),(x^3y,1),(x^4y,a^2),(x^5y,a),(x^6y,1), \\ (x^7y,a^2),(x^8y,a),(xy^2,a^2),(x^2y^2,a),(x^3y^2,1),(x^4y^2,a^2),(x^5y^2,a), \\ (x^6y^2,1),(x^7y^2,a^2),(x^8y^2,a) \end{array} \right\},$$

$$M_{10} = \left\{ \begin{array}{l} (1,1),(y,a),(y^2,a^2),(x,a^2),(x^2,a),(x^3,1),(x^4,a^2),(x^5,a),(x^6,1), \\ (x^7,a^2),(x^8,a),(xy,1),(x^2y,a^2),(x^3y,a),(x^4y,1),(x^5y,a^2),(x^6y,a), \\ (x^7y,1),(x^8y,a^2),(xy^2,a),(x^2y^2,1),(x^3y^2,a^2),(x^4y^2,a),(x^5y^2,1), \\ (x^6y^2,a^2),(x^7y^2,a),(x^8y^2,1) \end{array} \right\},$$

$$M_{11} = \left\{ \begin{array}{l} (1,1),(y,a^2),(y^2,a),(x,1),(x^2,1),(x^3,1),(x^4,1),(x^5,1),(x^6,1),(x^7,1) \\ (x^8,1),(xy,a^2),(x^2y,a^2),(x^3y,a^2),(x^4y,a^2),(x^5y,a^2),(x^6y,a^2), \\ (x^7y,a^2),(x^8y,a^2),(xy^2,a),(x^2y^2,a),(x^3y^2,a),(x^4y^2,a),(x^5y^2,a), \\ (x^6y^2,a),(x^7y^2,a),(x^8y^2,a) \end{array} \right\},$$

$$M_{12} = \left\{ \begin{array}{l} (1,1),(y,a^2),(y^2,a),(x,a),(x^2,a^2),(x^3,1),(x^4,1),(x^5,a^2), \\ (x^6,1),(x^7,a),(x^8,a^2),(xy,1),(x^2y,a),(x^3y,a^2),(x^4y,1),(x^5y,a), \\ (x^6y,a^2),(x^7y,1),(x^8y,a),(xy^2,a^2),(x^2y^2,1),(x^3y^2,a),(x^4y^2,a^2), \\ (x^5y^2,1),(x^6y^2,a),(x^7y^2,a^2),(x^8y^2,1) \end{array} \right\},$$

$$M_{13} = \left\{ \begin{array}{l} (1,1),(y,a^2),(y^2,a),(x,a^2),(x^2,a),(x^3,1),(x^4,a^2),(x^5,a),(x^6,1), \\ (x^7,a^2),(x^8,a),(xy,a),(x^2y,1),(x^3y,a^2),(x^4y,a),(x^5y,1),(x^6y,a^2), \\ (x^7y,a),(x^8y,1),(xy^2,1),(x^2y^2,a^2),(x^3y^2,a),(x^4y^2,1),(x^5y^2,a^2), \\ (x^6y^2,a),(x^7y^2,1),(x^8y^2,a^2) \end{array} \right\}.$$

By the equality (c), we have

$$\begin{aligned} \frac{1}{2}h(M_{3^3} \times C_3) &= (3h(\mathbb{Z}_3 \times \mathbb{Z}_{3^2}) + 9h(M_{3^3}) + h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) \\ &\quad -[24h(\mathbb{Z}_3 \times \mathbb{Z}_3) + 54h(\mathbb{Z}_{3^2})] + [234h(\mathbb{Z}_3) + 36h(\mathbb{Z}_{3^2}) \\ &\quad +16h(\mathbb{Z}_3 \times \mathbb{Z}_3)] - [702h(\mathbb{Z}_3) + 4h(\mathbb{Z}_3 \times \mathbb{Z}_3) + 9h(\mathbb{Z}_{3^2}) \\ &\quad +1287h(\mathbb{Z}_3) - 1716h(\mathbb{Z}_3) + 1716h(\mathbb{Z}_3) \\ &\quad -1287h(\mathbb{Z}_3) + 715h(\mathbb{Z}_3) - 286h(\mathbb{Z}_3) \\ &\quad +78h(\mathbb{Z}_3) - 13h(\mathbb{Z}_3) + h(\mathbb{Z}_3) \\ &= 3h(\mathbb{Z}_3 \times \mathbb{Z}_{3^2}) + 9h(M_{3^3}) + h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \\ &\quad -27h(\mathbb{Z}_{3^2}) - 12h(\mathbb{Z}_3 \times \mathbb{Z}_3) + 27h(\mathbb{Z}_3). \end{aligned}$$

Therefore,

$$h(M_{3^3} \times C_3) = 420 + 2h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) = 420 + 2(158) = 736.$$

3.2 Determination of $h(M_{5^3} \times C_5)$

Following a careful analysis and subsequent operations on the maximal subgroups, we have an estimate given by:

$$\begin{aligned} \frac{1}{2}h(M_{5^3} \times C_5) &= \left[ph(\mathbb{Z}_p \times \mathbb{Z}_{p^2}) + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p) + p^2 h(M_{p^3})\right] \\ &\quad - \left[(p+1)h(\mathbb{Z}_p \times \mathbb{Z}_p)\binom{p+1}{2} + \binom{p+1}{2} \cdot p^2 h(\mathbb{Z}_{p^2})\right] \\ &\quad + \left[(p+1)\binom{p+1}{3} h(\mathbb{Z}_p \times \mathbb{Z}_p) + p^2 \binom{p+1}{3} h^2(\mathbb{Z}_{p^2})\right. \\ &\quad + \left[\left[\binom{1+p+p^2}{3} - (1+p+p^2)\binom{p+1}{3}\right] h(\mathbb{Z}_p)\right] \\ &\quad - \left[(p+1)\binom{p+1}{4} h(\mathbb{Z}_p \times \mathbb{Z}_p) + \binom{p+1}{4} \cdot p^2 h(\mathbb{Z}_{p^2})\right. \end{aligned}$$

$$+\left[\left(\begin{array}{c}1+p+p^2\\4\end{array}\right)-(1+p+p^2)\left(\begin{array}{c}p+1\\4\end{array}\right)\right]h(\mathbb{Z}_p)\right]$$

$$+\left[(p+1)\left(\begin{array}{c}p+1\\5\end{array}\right)h(\mathbb{Z}_p\times\mathbb{Z}_p)+p^2\cdot\left(\begin{array}{c}p+1\\5\end{array}\right)h(\mathbb{Z}_{p^2})\right.$$

$$+\left[\left(\begin{array}{c}1+p+p^2\\5\end{array}\right)-(1+p+p^2)\left(\begin{array}{c}p+1\\5\end{array}\right)\right]h(\mathbb{Z}_p)\right]$$

$$-\left[(p+1)h(\mathbb{Z}_p\times\mathbb{Z}_p)+p^2h(\mathbb{Z}_{p^2})-\left(31-\left(\begin{array}{c}31\\6\end{array}\right)\right)h(\mathbb{Z}_p)\right]$$

$$+\left[\left(\begin{array}{c}31\\7\end{array}\right)-\left(\begin{array}{c}31\\8\end{array}\right)+\cdots+\left(\begin{array}{c}31\\29\end{array}\right)-\left(\begin{array}{c}31\\30\end{array}\right)+1\right]h(\mathbb{Z}_p)$$

3.3 Determination of $h(M_{p^n}\times C_p)$

In general,

$$\begin{aligned}\frac{1}{2}h(M_{p^n}\times C_p) &= ph(\mathbb{Z}_p\times\mathbb{Z}_{p^{n-1}})+p^2h(M_{p^n})\\ &\quad +h(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}\mathbb{Z}_{p^{n-2}})-p(p+1)h(\mathbb{Z}_p\times\mathbb{Z}_{p^{n-2}})\\ &\quad -p^3h(\mathbb{Z}_{p^{n-1}})+p^3h(\mathbb{Z}_{p^{n-2}})\\ &= p(p+1)(2^{n-1}))np-p+2)-p(p+1)(2^{n-2})(np-2p+2)\\ &\quad +p^3(2^{n-2}-2^{n-1}]+h(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^{n-2}})\\ &= 2^{n-2}[p(p+1)(np+2)-p^3]+h(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^{n-2}})\end{aligned}$$

Therefore,

$$h(M_{p^n}\times C_p)=2^{n-1}[p(p+1)(np+2)-p^3]+2h(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^{n-2}}).$$

$$\begin{aligned}h(M_{p^n}\times C_p) &= 2^{n-1}[p(p+1)(np+2)-p^3]\\ &\quad +2^{n-1}[(3n-5)p+(n^2-5)p^2+(n^2-5n+8)p^3+4]-4p^3\\ &= 2^{n-1}[(p^2+p^3)n^2+(3p+p^2-4p^3)n+(7p^3-3p^2-3p+4)]-4p^3\\ &= 2^{n-1}[p^2(1+p)n^2+p(3+p-4p^2)n+(7p^3-3p^2-3p+4)]-4p^3.\end{aligned}$$

Therefore, for modular finite p-groups

$$h(M_{p^n}\times C_p)=2^{n-1}[p^2(1+p)n^2+p(3+p-4p^2)n+(7p^3-3p^2-3p+4)]-4p^3$$

for $p > 2$.

Theorem 3.2 *Let $G = M_{p^n} \times C_p$, the modular nilpotent group formed by taking the cartesian product of the modular p-group of order p^n and a cyclic group of order p, where p is a prime. Then, the number of distinct fuzzy subgroups of G for $n > 4$ is given by*

$$h(G) = 2^{n-1} \times \left[p^2(1+p)n^2 + p(3+p-4p^2)n + (7p^3 - 3p^2 - 3p + 4)\right] - 4p^3$$

for $p > 2$.

Proof For all values of p, there exist only one maximal subgroup which is isomorphic to the Abelian type $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$. p of the maximal subgroups are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$, while p^2 of them are isomorphic to M_{p^n}.

If we put these values into equation(c), we have as follows:

$$\begin{aligned}
\frac{1}{2}h(M_{p^n} \times C_p) &= ph(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) + p^2 h(M_{p^n}) \\
&\quad + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) - p(p+1)h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) \\
&\quad - p^3 h(\mathbb{Z}_{p^{n-1}}) + p^3 h(\mathbb{Z}_{p^{n-2}}) \\
&= p(p+1)(2^{n-1})(np - p + 2) - p(p+1)(2^{n-2})(np - 2p + 2) \\
&\quad + p^3(2^{n-2} - 2^{n-1}] + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) \text{ (by Theorem 2.1((1) and (5))} \\
&= 2^{n-2}[p(p+1)(np+2) - p^3] + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}).
\end{aligned}$$

Therefore,

$$h(M_{p^n} \times C_p) = 2^{n-1}[p(p+1)(np+2) - p^3] + 2h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}).$$

Here, recurrence relation was used for the purpose of $h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}})$. We now have

$$\begin{aligned}
h(M_{p^n} \times C_p) &= 2^{n-1}[p(p+1)(np+2) - p^3] \\
&\quad + 2^{n-1}[(3n-5)p + (n^2-5)p^2 + (n^2 - 5n + 8)p^3 + 4] - 4p^3 \\
&= 2^{n-1}[p^2(1+p)n^2 + p(3+p-4p^2)n + (7p^3 - 3p^2 - 3p + 4)] - 4p^3
\end{aligned}$$

for $p > 2$ □

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On Isomorphism Theorems of Torian Algebras

Ilojide Emmanuel

(Department of Mathematics, Federal University of Agriculture, Abeokuta 110101, Nigeria)

E-mail: emmailojide@yahoo.com, ilojidee@funaab.edu.ng

Abstract: In this paper, the notions of ideals and congruences in torian algebras are used to construct quotient torian algebras. The Fundamental Theorem of homomorphisms of torian algebras is established. Moreover, the three isomorphism Theorems of torian algebras are also presented.

Key Words: Torian algebras, Smarandachely torian algebras, congruences, ideals.

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§1. Introduction

Obic algebras were introduced in 2019 by Ilojide in [6]. Homomorphisms and krib maps as well as monics of obic algebras were studied. Properties of implicative obic algeras were also investigated. In [7], torian algebras were introduced. The class of torian algebras is a wider class than the class of obic algebras. It was shown that with a suitably defined binary relation, torian algebras are partially ordered sets. The partial ordering was used to investigate some of their properties. In [8], ideals of torian algebras were studied. Their properties were investigated. Moreover, the dual and nuclei of ideals as well as congruences developed on ideals of torian algebras were also studied. Right distributive torian algebras were studied in [9]. It was shown that every right distributive torian algebra fixes its zero element. Moreover, necessary and sufficient conditions for a torian algebra to be right distributive were also established. In this paper, the study of torian algebras is continued. The notions of ideals and congruences in torian algebras are used to construct quotient torian algebras. The Fundamental Theorem of homomorphisms of torian algebras is established. Moreover, the three isomorphism Theorems of torian algebras are also presented.

§2. Preliminaries

Definition 2.1(**[6]**) *A triple $(X;*,0)$; where X is a non-empty set, $*$ a binary operation on X, and 0 a constant element of X is called an obic algebra if the following axioms hold for all $x,y,z \in X$:*

(1) $x*0 = x$;

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(2) $[x*(y*z)]*x = x*[y*(z*x)]$;
(3) $x*x = 0$.

Example 2.1([6]) Consider the multiplicative group $G = \{1, -1, i, -i\}$. Define a binary operation $*$ on G by $a*b = ab^{-1}$. Then $(G;*,1)$ is an obic algebra.

Lemma 2.1([6]) *Let X be an obic algebra. Then for all $x, y \in X$, the following holds*

$$x*y = [x*(y*x)]*x.$$

Definition 2.2([6]) *A non-empty subset S of an obic algebra X is called a subalgebra if S is an obic algebra with respect to the binary operation in X.*

Definition 2.3([6]) *An obic algebra X is said to have the weak property (WP) if $x*y = 0$ and $y*x = 0$ imply that $x = y$.*

Definition 2.4([6]) *An equivalence relation $\sim^*$ on an obic algebra X is called a congruence if $(x \sim^* y)$ and $(u \sim^* v) \Rightarrow (x*u) \sim^* (y*v)$.*

Definition 2.5([6]) *Let $(X;*,0)$ and $(Y;\circ,0')$ be obic algebras. A function $f: X \to Y$ is called an obic homomorphism if $f(a*b) = f(a)\circ f(b)$ for all $a, b \in X$.*

Let $(X;*,0)$ and $(Y;\circ,0')$ be obic algebras, and let $f: X \to Y$ be a homomorphism. The set $\{x \in X : f(x) = 0'\}$ is called the kernel of f. It is denoted by $Ker(f)$. The set $\{y \in Y : y = f(x); x \in X\}$ is called the image of f. It is denoted by $Im(f)$. If f is injective, then f is called a monomorphism. If f is surjective, then f is called an epimorphism. If f is both injective and surjective, then f is called an isomorphism. If f is an isomorphism, then X is said to be isomorphic to Y.

Definition 2.6([7]) *An obic algebra X is called torian if $[(x*y)*(x*z)]*(z*y) = 0$ for all $x, y, z \in X$. Otherwise, if there are $x, y, z \in X$, such that $[(x*y)*(x*z)]*(z*y) \neq 0$, such an obic algebra X is called Smarandachely torian.*

Definition 2.7([7]) *A torian algebra which has the weak property is called a weak property torian algebra (WPTA).*

Definition 2.8([8]) *Let X be a torian algebra. A non-empty set S of X is called a left ideal of X if the following holds:*

(1) $0 \in S$;
(2) *If $x, y \in X$ such that $x, [[y*(x*y)]*y] \in S$, then $y \in S$.*

Definition 2.9([8]) *If a left ideal S of X is such that $[[x*(y*x)]*x] \in S$ for all $x, y \in X$, then S is said to be a complete left ideal of X or that S is complete in X.*

Proposition 2.10([8]) *Let $(X;*,0)$ and $(Y;\odot,0')$ be torian algebras. Let $f: X \to Y$ be a homomorphism. Then $Ker(f)$ is a complete left ideal of X.*

§3. Main Results

In this section, we establish the first, second and third isomorphism Theorems of torian algebras. But before that, some preliminary results are presented. The following Theorem is taken from [8]. It is needed in proving Theorem 3.2, which is our first preliminary result.

Theorem 3.1 *Let S be a left ideal of a torian algebra X. Let $\sim^1$ be a relation on X defined by $x \sim^1 y \Leftrightarrow [[x*(y*x)]*x]$ and $[[y*(x*y)]*y] \in S$ for all $x, y \in X$. Then $\sim^1$ is a congruence on X.*

Remark 3.1 Let X^S denote the collection of equivalence classes in the equivalence relation in Theorem 3.1. An equivalence class of $x \in X$ is denoted by $[x]$.

Definition 3.1 *Let S be a left ideal of a torian algebra X equipped with a congruence $\sim^1$. Let $[x], [y] \in X^S$. Define a binary operation $\odot$ on X^S by $[x] \odot [y] = [x*y]$ for all $x \in X$.*

Theorem 3.2 *Let S be a left ideal of a torian algebra X equipped with a congruence $\sim^1$. Let $[0]$ be the zero equivalence class. Then $(X^S; \odot, [0])$ is a torian algebra.*

Proof Let $[x], [y], [z] \in X^S$. Clearly, $[0] \in X^S$. Notice that $[x] \odot [0] = [x*0] = [x]$. Also, $[x] \odot [x] = [x*x] = [0]$. Now, $(([x] \odot [y]) \odot ([x] \odot [z])) \odot ([z] \odot [y]) = ([x*y] \odot [x*z]) \odot [z*y] = ([(x*y)*(x*z)]) \odot [z*y] = [(((x*y)*(x*z))*(z*y))] = [0]$.

Finally, notice that $([x] \odot ([y] \odot [z])) \odot [x] = ([x] \odot [y*z]) \odot [x] = [(x*(y*z))] \odot [x] = [(x*(y*(z*x))] = [x] \odot [y*(z*x)] = [x] \odot [[y] \odot ([z] \odot [x])]$. So, $(X^S; \odot, [0])$ satisfies all the axioms of a torian algebra as required. □

The following corollary follows from Theorem 3.2.

Corollary 3.1 *Let S be a left ideal of a WPTA X equipped with a congruence $\sim^1$. Let $[0]$ be the zero equivalence class. Then $(X^S; \odot, [0])$ is a WPTA.*

Definition 3.2 *Let S be a left ideal of a torian algebra X equipped with a congruence $\sim^1$, and let $[0]$ be the zero equivalence class. The torian algebra $(X^S; \odot, [0])$ is called the quotient torian algebra induced by the left ideal S.*

Remark 3.2 The torian algebra X^S is also denoted by X/S.

Theorem 3.3 *Let S be a left ideal of a torian algebra X equipped with a congruence $\sim^1$. Then $[0]$ is a left ideal of X.*

Proof Clearly, $0 \in [0]$. So, $[0]$ is not empty. Let $x, [[y*(x*y)]*y] \in [0]$. Then $x \sim^1 0$ and $y*x = [[y*(x*y)]*y] \sim^1 0$. So, $y*x \sim^1 0$. Now, by the reflexivity of $\sim^1$, we have $y \sim^1 y$. Since $\sim^1$ is a congruence, combining $y \sim^1 y$ and $x \sim^1 0$, we have $y*x \sim^1 y$. Since $\sim^1$ is symmetric, we have $y \sim^1 y*x$. By the transitivity of $\sim^1$, combining $y \sim^1 y*x$ and $y*x \sim^1 0$, we have $y \sim^1 0$. Hence, $y \in [0]$ as required. □

Theorem 3.4 *Let S be a left ideal of a torian algebra X equipped with a congruence $\sim^1$. Then* *the map $\alpha : X \to X^S$ such that $\alpha(x) = [x]$ for all $x \in X$ is a an epimorphism.*

Proof By Theorem 3.2, $(X^S; \odot, [0])$ is a torian algebra. Let $x, y \in X$. Then $\alpha(x * y) = [x] \odot [y] = \alpha(x) \odot \alpha(y)$. Hence, α is a homomorphism. Clearly, α is surjective. Therefore, α is an epimorphism. □

Definition 3.3 *Let S be a left ideal of a torian algebra X equipped with a congruence $\sim^1$. Then the epimorphism $\alpha : X \to X^S$ such that $\alpha(x) = [x]$ for all $x \in X$ is called the natural epimorphism induced by the left ideal S.*

The following corollary is immediate from Proposition 2.1 and Theorem 3.4.

Corollary 3.2 *Let X be a torian algebra equipped with a congruence $\sim^1$. Let $f : X \to X$ be a homomorphism with $Ker(f) = K$. Then $\alpha : X \to X^K$ is a homomorphism.*

Theorem 3.5 *Let $(X; *, 0)$ and $(Y; \circ, 0')$ be torian algebras, and let $f : X \to Y$ be a homomorphism with $Ker(f) = K$. Then there exists a homomorphism $\phi : X^K \to Y$ such that the following holds:*

(1) *$f = \phi\alpha$; where α is the natural homomorphism induced by K;*
(2) *ϕ is unique;*
(3) *ϕ is a monomorphism.*

Proof By Proposition 2.1 and Theorem 3.2, X^K is a torian algebra. Now, define $\phi : X^K \to Y$ by $\phi([x]) = f(x)$ for all $[x] \in X^K$. Let $[x], [y] \in X^K$. Then $\phi([x] \odot [y]) = \phi([x*y]) = f(x*y) = f(x) \circ f(y) = \phi([x]) \circ \phi([y])$. So, ϕ is a homomorphism. Notice that $\phi\alpha(x) = \phi(\alpha(x)) = \phi([x]) = f(x)$ for all $x \in X$. Hence, $f = \phi\alpha$.

We now show that ϕ is unique. Let $\phi' : X^K \to Y$ such that $f = \phi'\alpha$, and let $[x] \in X^K$. Then $\phi'([x]) = \phi'(\alpha(x)) = f(x) = \phi\alpha(x) = \phi(\alpha(x)) = \phi([x])$. So, $\phi' = \phi$. Hence, ϕ is unique.

We now show that ϕ is a monomorphism. Let $[x], [y] \in X^K$ such that $\phi([x]) = \phi([y])$. Then $f(x) = f(y)$. Therefore, $f(((x * (y * x)) * x)) = f(x * y) = f(x) \circ f(y) = f(x) \circ f(x) = 0'$. Thus, $((x * (y * x)) * x) \in K$. Also notice that $f(((y * (x * y)) * y)) = f(y * x) = f(y) \circ f(y) = 0'$. Thus, $((y * (x * y)) * y) \in K$. Since $((x * (y * x)) * x) \in K$ and $((y * (x * y)) * y) \in K$, then $x \sim^1 y$. Hence, $[x] = [y]$. Therefore, ϕ is a monomorphism. □

Remark 3.3 Theorem 3.5 is the fundamental theorem of homomorphism of torian algebras.

Theorem 3.6 *Let X and Y be torian algebras, and let $f : X \to Y$ be a homomorphism with $Ker(f) = K$. Then X^K is isomorphic to $Im(f)$.*

Proof By Theorem 3.5, the map $\phi : X^K \to Y$ defined by $\phi([x]) = f(x)$ for all $[x] \in X^K$ is a monomorphism. Now, let $y = f(x) \in Im(f)$. Then there exists $x \in X$ such that $\phi([x]) = y$. So, for each $f(x) \in Im(f)$, there exists $[x] \in X^K$ such that $\phi([x]) = f(x)$. So, ϕ is an epimorphism, and hence an isomorphism. Therefore, X^K is isomorphic to $Im(f)$ as required. □

Remark 3.4 Theorem 3.6 is the first isomorphism theorem of torian algebras.

The following corollary is immediate from Theorem 3.6.

Corollary 3.3 *Let X and Y be torian algebras, and let $f : X \to Y$ be an epimorphism with $Ker(f) = K$. Then X^K is isomorphic to Y.*

Theorem 3.7 *Let P be a subalgebra of a torian algebra X. Let S be a complete left ideal of X. Then $P/(P \cap S)$ is isomorphic to PS/S.*

Proof Let the map $f : P \to PS/S$ be defined by $f(p) = [p]$ for all $p \in P$. Then f is a homomorphism. We claim that f is an epimorphism such that $Ker(f) = P \cap S$. Now, let $[x] \in PS/S$. Then $x \in PS$. Now, there exists $p \in P$ such that $x \in [p]$, and so $[x] = [p]$. Therefore, $f(p) = [p] = [x]$. So, f is an epimorphism. Now, let $p \in P$ such that $p \in Ker(f)$. Then $f(p) = [0]$. Since by definition of f, $f(p) = [p]$, we then have $[p] = [0]$. Hence, $p = p{*}0 \in S$. Therefore, $p \in P \cap S$. Thus, $Ker(f)$ is contained in $P \cap S$. Now, let $p \in P \cap S$. Then $p \in P$ and $p \in S$. Since S is complete, then $[p] = [0]$. Thus, $f(p) = [p] = [0]$. So, $p \in Ker(f)$. Hence, $P \cap S$ is contained in $Ker(f)$. Therefore, $Ker(f) = P \cap S$. By Corollary 3.3, $P/(P \cap S)$ is isomorphic to PS/S as required. □

Remark 3.5 Theorem 3.7 is the second isomorphism theorem of torian algebras.

Theorem 3.8 *Let X be a torian algebra. Let K be a complete left ideal of X and let A be a left ideal of K. Then X^K is isomorphic to X^A/K^A.*

Proof Since A is a left ideal of K and K is a left ideal of X, then A is a left ideal of X. By Theorem 3.2, X^K and X^A are torian algebras. Let $\odot$ and $\odot'$ be the binary operations of X^A and X^K respectively. Let the map $\phi : X^A \to X^K$ be defined by $\phi([x]_A) = [x]_K$ for all $[x]_A \in X^A$. We show that ϕ is an isomorphism with $Ker(\phi) = K^A$ and $Im(\phi) = X^K$. Now, let $[x]_A, [y]_A \in X^A$. Then $\phi([x]_A \odot [y]_A) = \phi([x{*}y]_A) = [x{*}y]_K = [x]_K \odot' [y]_K = \phi([x]_A) \odot \phi([y]_A)$. So ϕ is a homomorphism. Clearly, ϕ is a bijection. Now, notice that $Ker(\phi) = \{[x]_A \in X^A : \phi([x]_A) = [0]_K\} = \{[x]_A \in X^A : [x]_K = [0]_K\} = \{[x]^A : x \in K\} = K^A$. Notice also that $Im(\phi) = \{[x]_K \in X^K : x \in X\} = X^K$. By Theorem 3.6, therefore, X^K is isomorphic to X^A/K^A as required. □

Remark 3.6 Theorem 3.8 is the third isomorphism theorem of torian algebras.

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A Study on
Neighbourly Pseudo Irregular Neutrosophic Bipolar Fuzzy Graph

N.R.Santhi Maheswari and D.Subburaj

PG and Research Department of Mathematics, G.Venkataswamy Naidu College,
Manomaniam Sundarnar University, Kovilpatti-628 502, Tamil Nadu, India

E-mail: nrsmaths@yahoo.com, subburajphd2019@gmail.com, subburajdharmaraj@gmail.com

Abstract: In this paper, the concepts of neighbourly pseudo irregular neutrosophic bipolar fuzzy graphs, neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graphs are introduced. Some basic theorems related to the stated graphs have also been presented.

Key Words: Neutrosophic bipolar fuzzy graphs, pseudo degree in neutrosophic bipolar fuzzy graphs, total pseudo degree in neutrosophic bipolar fuzzy graphs, neighbourly pseudo irregular neutrosophic bipolar fuzzy graphs, neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graphs.

AMS(2010): 05C12, 03E72, 05C72.

§1. Introduction

F.Smarandache [4] introduced notion of neutrosophic set which is useful for dealing real life problems having imprecise, indeterminacy and inconsistent data. They are generalization of the theory of fuzzy sets, intuitionistics fuzzy set, interval valued fuzzy set and interval valued intuitionistic fuzzy sets. N. Shah and Hussain[2, 6] introduced the notion of soft neutrosophic graphs. N. Shah introduces the notion of neutrosophic graphs and different operations like union, intersection and complement in his work. A neutrosophic set is characterized by a truth membership degree (t), an indeterminacy membership degree(i), falsity membership degree(f) independently, which are with in the real standard or non standard unit interval $]^{-}0, 1^{+}[$. See [4] for details.

Divya and Dr. J. Malarvizhi[7] introduced the notion of neutrosophic fuzzy graph and few fundamental operation on neutrosophic fuzzy graph. Zhang initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. Bipolar fuzzy sets whose range of membership degree is [-1,1]. In bipolar fuzzy sets, membership degree of an element means that the element is irrelevant to the corresponding property, the membership degree within (0, 1] of an element indicates that the element somewhat satisfies the property, and the membership degree within [-1,0) of an element indicates the element somewhat satisfies the implicit counter property.

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It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible [9]. M.Akram and Wieslaw A.Dudek introduced regular and totally regular bipolar fuzzy graphs. Also, they introduced the notion of bipolar fuzzy line graphs and presents some of their properties [9].

In 2011, Akram introduced the concept of bipolar fuzzy graphs and defined different operations on it. Bipolar fuzzy graph theory is now growing and expanding its applications. The theoretical developments in this area is discussed. In 2012,Sovan Samanta and Madhumangal Pal introduced the concept of irregular bipolar fuzzy graph and defined different operations on it. N.R.Santhi Maheswari and C.Sekar introduced Pseudo regular bipolar fuzzy graph and pseudo irregular bipolar fuzzy graph and discussed its properties [16].

N.R.Santhi Maheswari and V.Jeyapratha introduced Neighbourly Pseudo irregular fuzzy graph and discussed its properties [15]. N.R.Santhi Maheswari and C.Sekar introduced Neighbourly pseudo and Strongly Pseudo irregular bipolar fuzzy graph and discussed its properties [14]. These idea motivates us to introduce Neighbourly pseudo irregular neutrosophic bipolar fuzzy graphs.

§2. Preliminaries

We present some known definition and results for ready references to go through the work presented in the paper.

Definition 2.1([4]) *Let X be a space of points with generic elements in X denoted by x. A neutrosophic set A (NSA) is an object having the form*

$$A = < x : T_A(x), I_A(x), F_A(x) >, x \in X,$$

where the functions $T, I, F \to]0^-, 1^+[$ define respectively a truth membership function, an indeterminacy membership function and a falsity membership function of the element $x \in X$ to the set A with the condition

$$0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$

The functions $T_A(x), I_A(x), F_A(x)$ are real standard or non standard subsets of $]0^-, 1^+[$.

Definition 2.2([5]) *Let V be a non empty finite set and $\sigma : V \to [0, 1]$. Again, let $\mu : V \times V \to [0, 1]$ such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $(x, y) \in V \times V$. Then the pair $G : (\sigma, \mu)$ is called a fuzzy graph over the set V. Here σ and μ are respectively called fuzzy vertex set and fuzzy edge set of the fuzzy graph $G : (\sigma, \mu)$.*

Definition 2.3([7]) *Let X be a space of points with generic elements in X denoted by x. A neutrosophic fuzzy set A(NFSA) is characterized by truth membership function $T_A(x)$, an indeterminacy membership fuctions $I_A(x)$ and a falsity membership fuction $F_A(x)$. For each point $x \in X$, $T_A(x), I_A(x), F_A(x) \in [0, 1]$. A NFS A can be written as*

$$A = < x : T_A(x), I_A(x), F_A(x) >, \quad x \in X.$$

Definition 2.4([7]) *Let* $A = (T_A(x), I_A(x), F_A(x))$ *and* $B = (T_A(x), I_A(x), F_A(x))$ *be neutrosophic fuzzy sets on a set X. If* $A = (T_A(x), I_A(x), F_A(x))$ *is a neutrosophic fuzzy relation on a set X, then* $A = (T_A(x), I_A(x), F_A(x))$ *is called a neutrosophic fuzzy relation on* $B = (T_A(x), I_A(x), F_A(x))$ *if*

$$T_B(x, y) \leq T_A(x).T_A(y),\ I_B(x, y) \leq I_A(x).I_A(y) \text{ and } F_B(x, y) \leq F_A(x).F_A(y)$$

for all $x, y \in X$*, Where the notation "." means the ordinary multiplication.*

Definition 2.5([7]) *A neutrosophic fuzzy graph (NF graph) with underlying set V is defined to be a pair* $N_G = (A, B)$, where

(*i*) *The functions* $(T_A, I_A, F_A) : V \to [0, 1]$ *denote the degree of truth membership,degree of indeterminacy membership and the degree of falsity membership of the element* $v_i \in V$ *respectively and* $0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$*;*

(*ii*) $E \subseteq V \times V$ *where the functions* $(T_B, I_B, F_B) : V \times V \to [0, 1]$ *are defined by*

$$T_B(v_i, v_j).T_A(v_i) \leq T_A(v_j), I_B(v_i, v_j).I_A(v_i) \leq I_A(v_j)$$

and

$$F_B(v_i, v_j).F_A(v_i) \leq F_A(v_j)$$

for all $v_i, v_j \in V$*, where the notation "." means ordinary multiplication denotes the degrees of truth membership, indeterminacy membership and falsity membership of the edge* $v_i, v_j \in E$ *respectively, where*

$$0 \leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$$

for all $v_i, v_j \in E$ $(j = 1, 2, \cdots, n)$*.*

Definition 2.6([17]) *A bipolar neutrosophic set A in X is defined as an object of the form*

$$A = \left\{< x, T_A^+(x), I_A^+(x), F_A^+(x), T_A^-(x), I_A^-(x), F_A^-(x) >: x \in X\right\},$$

where $(T_A^+, I_A^+, F_A^+) : X \to [0, 1]$ *and* $(T_A^-, I_A^-, F_A^-) : X \to [-1, 0]$*. The positive membership degree* $T_A^+(x), I_A^+(x), F_A^+(x)$ *denotes the truth membership, indeterminate membership and false membership of an element* $\in X$ *corresponding to a bipolar neutrosophic set A and the negative membership degree* $T_A^-(x), I_A^-(x), F_A^-(x)$ *denotes the truth membership, indeterminate membership and false membership of an element* $\in X$ *to some implicit counter-property corresponding to a bipolar neutrosophic set A.*

Definition 2.7 *A bipolar neutrosophic set B in V is defined as an object of the form*

$$B = \left\{< v, T_B^+(v), I_B^+(v), F_B^+(v), T_B^-(v), I_B^-(v), F_B^-(v) >: v \in V\right\},$$

where $(T_B^+, I_B^+, F_B^+) : V \to [0, 1]$ *and* $(T_B^-, I_B^-, F_B^-) : V \to [-1, 0]$*. The positive membership* *degree $T_B^+(v), I_B^+(v), F_B^+(v)$ denotes the truth membership, indeterminate membership and false membership of an element $\in V$ corresponding to a bipolar neutrosophic set A and the negative membership degree $T_B^-(v), I_B^-(v), F_B^-(v)$ denotes the truth membership, indeterminate membership and false membership of an element $\in V$ to some implicit counter-property corresponding to a bipolar neutrosophic set B.*

Definition 2.8 *A bipolar fuzzy graph with an underlying set V is defined to be a pair (A,B), where $A = (m_1^+, m_1^-)$ is a bipolar fuzzy set on V and $B = (m_2^+, m_2^-)$ is a bipolar fuzzy set on E such that*

$$m_2^+(x,y) \leq min\left\{m_1^+(x), m_1^+(y)\right\} \text{ and } m_2^+(x,y) \geq max\left\{m_1^-(x), m_1^-(y)\right\}$$

for all (x, y) in E. Here, A is called bipolar fuzzy vertex set on V and B is called bipolar fuzzy edge set on E.

Definition 2.9 *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^* = (V, E)$. The positive degree and the negative degree of a vertex u in G is defined respectively as*

$$d^+(u) = \sum m_2^+(u,v)$$

and

$$d^-(u) = \sum m_2^-(u,v)$$

for uv in E, and the degree of a vertex u is defined as

$$d(u) = (d^+(u), d^-(u)).$$

Definition 2.10 *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^* = (V, E)$. The positive total degree and the negative total degree of a vertex u in G is defined respectively as*

$$td^+(u) = \sum m_2^+(u,v) + m_1^+(u)$$

and

$$td^-(u) = \sum m_2^-(u,v) + m_1^-(u)$$

for uv in E.

§3. Neutrosophic Bipolar Fuzzy Graph

Definition 3.1 *A neutrosophic fuzzy graph (NF graph) with underlying set V is defined to be a pair $N_G = (A, B)$, where,*

(*i*) *The functions $(T_A^+, I_A^+, F_A^+) : V \to [0, 1]$ denote the degree of truth membership,degree of indeterminacy membership and the degree of falsity membership of the element $v_i \in V$ re-*

spectively and

$$0 \leq T_A^+(v_i) + I_A^+(v_i) + F_A^+(v_i) \leq 3;$$

(*ii*) $E \subseteq V \times V$ *where the functions* $(T_B^+, I_B^+, F_B^+) : V \times V \to [0, 1]$ *are defined by*

$$T_B^+(v_i, v_j).T_A^+(v_i) \leq T_A^+(v_j), I_B^+(v_i, v_j).I_A^+(v_i) \leq I_A^+(v_j)$$

and

$$F_B^+(v_i, v_j).F_A^+(v_i) \leq F_A^+(v_j)$$

for all $v_i, v_j \in V$, *where, the notation "." means ordinary multiplication denotes the degrees of truth membership, indeterminacy membership and falsity membership of the edge* $v_i, v_j \in E$ *respectively, where*

$$0 \leq T_B^+(v_i, v_j) + I_B^+(v_i, v_j) + F_B^+(v_i, v_j) \leq 3$$

for all $v_i, v_j \in E$ $(j = 1, 2, \cdots, n)$.

Definition 3.2 *A neutrosophic fuzzy graph (NF graph) with underlying set* V *is defined to be a pair* $N_G = (A, B)$ *, where,*

(*i*) *The functions* $(T_A^-, I_A^-, F_A^-) : V \to [-1, 0]$ *denote the degree of truth membership,degree of indeterminacy membership and the degree of falsity membership of the element* $v_i \in V$ *respectively and*

$$0 \geq T_A^-(v_i) + I_A^-(v_i) + F_A^-(v_i) \geq -3;$$

(*ii*) $E \subseteq V \times V$ *where the functions* $(T_B^-, I_B^-, F_B^-) : V \times V \to [-1, 0]$ *are defined by*

$$T_B^-(v_i, v_j).T_A^-(v_i) \geq T_A^-(v_j), I_B^-(v_i, v_j).I_A^-(v_i) \geq I_A^-(v_j)$$

and

$$F_B^-(v_i, v_j).F_A^-(v_i) \geq F_A^-(v_j)$$

for all $v_i, v_j \in V$ *, where "." means ordinary multiplication denotes the degrees of truth membership, indeterminacy membership and falsity membership of the edge* $v_i, v_j \in E$ *respectively, where*

$$0 \geq T_B^-(v_i, v_j) + I_B^-(v_i, v_j) + F_B^-(v_i, v_j) \geq -3$$

for all $v_i, v_j \in E$ $(j = 1, 2, \cdots, n)$.

Definition 3.3 *Let* $BN_G = (A, B)$, *where* $A = (m_1^+, m_1^-)$ *and* $A = (m_2^+, m_2^-)$ *be a neutrosophic bipolar fuzzy graph. The neighborhood positive degree of a vertex* x *in* BN_G *defined by*

$$deg(x)^+ = (deg_T(x)^+, deg_I(x)^+,\ deg_F(x)^+),$$

where,

$$deg_T(x)^+ = \sum_{xy \in E} T_B(xy)^+,\ deg_I(x)^+ = \sum_{xy \in E} I_B(xy)^+,\ deg_F(x)^+ = \sum_{xy \in E} F_B(xy)^+.$$

The neighborhood negative degree of a vertex x in BN_G defined by

$$deg(x)^- = (deg_T(x)^-, deg_I(x)^-, deg_F(x)^-),$$

where

$$deg_T(x)^- = \sum_{xy\in E} T_B(xy)^-,\ deg_I(x)^- = \sum_{xy\in E} I_B(xy)^-,\ deg_F(x)^- = \sum_{xy\in E} F_B(xy)^-.$$

Therefore, the degree of a vertex x in BN_G is $(deg(x)^+, deg(x)^-)$.

Example 3.4 Let BN_G be the neutrosophic bipolar fuzzy graph shown in Figure 1.

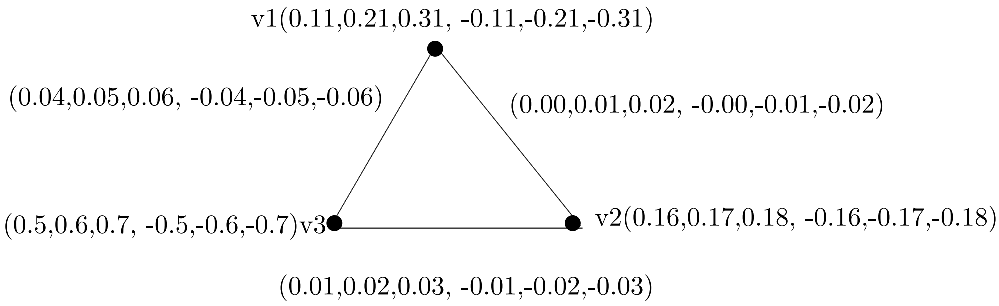

Figure 1

Then,

$$\begin{aligned} d(v_1)^+ &= (0.04, 0.06, 0.08),\ d(v_1)^- = (-0.04, -0.06, -0.08),\\ d(v_2)^+ &= (0.01, 0.03, 0.05),\ d(v_2)^- = (-0.01, -0.03, -0.05),\\ d(v_3)^+ &= (0.05, 0.07, 0.09),\ d(v_3)^- = (-0.05, -0.07, -0.09). \end{aligned}$$

Definition 3.5 *Let $BN_G = (A, B)$, where $A = (m_1^+, m_1^-)$ and $B = (m_2^+, m_2^-)$ be a neutrosophic bipolar fuzzy graph. The closed positive neighborhood degree of a vertex x in BN_G defined by*

$$deg[x]^+ = (deg_T[x]^+, deg_I[x]^+, deg_F[x]^+)$$

where,

$$\begin{aligned} deg_T(x)^+ &= \sum_{xy\in E} T_B(xy)^+ + T_A(x)^+,\\ deg_I(x)^+ &= \sum_{xy\in E} I_B(xy)^+ + I_A(x)^+,\\ deg_F(x)^+ &= \sum_{xy\in E} F_B(xy)^+ + F_A(x)^+. \end{aligned}$$

The closed negative neighborhood degree of a vertex BN_G defined by

$$deg[x]^- = (deg_T[x]^-, deg_I[x]^-, deg_F[x]^-)$$

where,

$$\begin{aligned} deg_T(x)^- &= \sum_{xy\in E} T_B(xy)^- + T_A(x)^-, \\ deg_I(x)^- &= \sum_{xy\in E} I_B(xy)^- + I_A(x)^-, \\ deg_F(x)^- &= \sum_{xy\in E} F_B(xy)^- + F_A(x)^-. \end{aligned}$$

Therefore, the closed degree of a vertex x in BN_G is $(deg[x]^+, deg[x]^-)$.

Example 3.6 Let BN_G be a neutrosophic bipolar fuzzy graph shown in Figure 1. We calculate closed degree of a vertices in the above neutrosophic bipolar fuzzy graphs as follows:

$$\begin{aligned} d[v_1]^+ &= (0.15, 0.27, 0.39), d[v_1]^- = (-0.15, -0.27, -0.39), \\ d[v_2]^+ &= (0.17, 0.20, 0.23), d[v_2]^- = (-0.17, -0.20, -0.23), \\ d[v_3]^+ &= (0.55, 0.67, 0.79), d[v_3]^- = (-0.55, -0.67, -0.79). \end{aligned}$$

§4. Pseudo Degree and Total Pseudo Degree in Neutrosophic Bipolar Fuzzy Graph

Definition 4.1 *Let BN_G be a neutrosophic bipolar fuzzy graph on $G^*(V, E)$. The 2-degree of a vertex v in BN_G is defined as the sum of the degrees of the vertices adjacent to v and is denoted by*

$$td_{BN_G}(v) = (td^+_{BN_G}(v), td^-_{BN_G}(v)).$$

That is, the positive 2-degree of v is

$$td^+_{BN_G}(v) = \sum d^+_{BN_G}(u),$$

where $d^+_{BN_G}(u)$ is the positive degree of the vertex u which is adjacent with the vertex v and the negative 2-degree of v is

$$td^-_{BN_G}(v) = \sum d^-_{BN_G}(u),$$

where $d^-_{BN_G}(u)$ is the negative degree of the vertex u which is adjacent with the vertex v.

Definition 4.2 *Let BN_G be a neutrosophic bipolar fuzzy graph on $G^*(V, E)$. A positive pseudo (average) degree of a vertex v in BN_G is denoted by $pd^+_{BN_G}(v)$ and is defined by*

$$pd^+_{BN_G}(v) = \frac{td^+_{BN_G}(v)}{d^*_{BN_G}(v)},$$

where $d^*_{BN_G}(v)$ *is the number of edges incident at v. The negative pseudo (average) degree of a vertex v in* BN_G *is denoted by* $pd^-_{BN_G}(v)$ *and is defined by*

$$pd^-_{BN_G}(v) = \frac{td^-_{BN_G}(v)}{d^*_{BN_G}(v)},$$

where $d^*_{BN_G}(v)$ *is the number of edges incident at v.*

The pseudo degree of a vertex v in neutrosophic bipolar fuzzy graph BN_G *is defined as* $pd_{BN_G}(v) = (pd^+_{BN_G}(v), pd^-_{BN_G}(v))$.

Example 4.3 Let BN_G be the neutrosophic bipolar fuzzy graph shown in Figure 2.

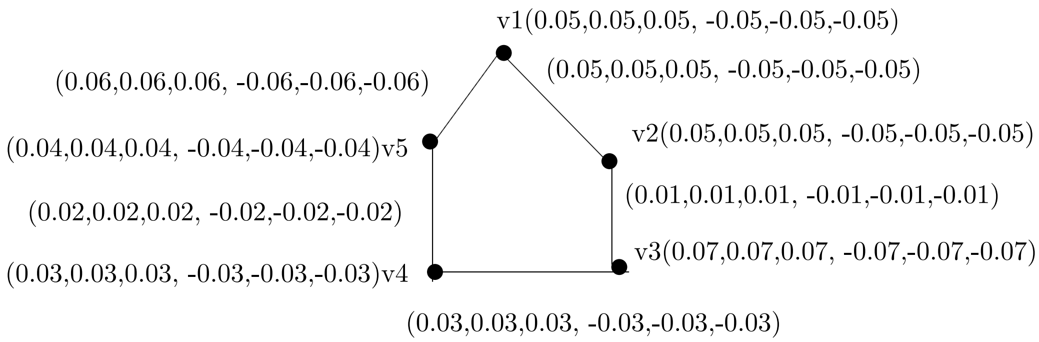

Figure 2

We calculate pseudo degree of vertex of the neutrosophic bipolar fuzzy graph in Figure 2 as follows:

$$\begin{aligned}
d(v_1)^+ &= (0.11, 0.11, 0.11), d(v_1)^- = (-0.11, -0.11, -0.11),\\
d(v_2)^+ &= (0.06, 0.06, 0.06),\ d(v_2)^- = (-0.06, -0.06, -0.06),\\
d(v_3)^+ &= (0.04, 0.04, 0.04),\ d(v_3)^- = (-0.04, -0.04, -0.04),\\
d(v_4)^+ &= (0.05, 0.05, 0.05),\ d(v_4)^- = (-0.05, -0.05, -0.05),\\
d(v_5)^+ &= (0.08, 0.08, 0.08),\ d(v_5)^- = (-0.08, -0.08, -0.08),
\end{aligned}$$

$$pd_{BN_G}(v_1)^+ = \frac{td^+_{BN_G}(v_1)}{d^*_{BN_G}(v_1)} = (-0.07, -0.07, -0.07),$$

$$pd_{BN_G}(v_1)^- = \frac{td^-_{BN_G}(v_1)}{d^*_{BN_G}(v_1)} = (-0.07, -0.07, -0.07),$$

$$\begin{aligned}
pd_{BN_G}(v_2)^+ &= (0.075, 0.075, 0.075),\\
pd_{BN_G}(v_2)^- &= (-0.075, -0.075, -0.075),\\
pd_{BN_G}(v_3)^+ &= (0.055, 0.055, 0.055),\\
pd_{BN_G}(v_3)^- &= (-0.055, -0.055, -0.055),
\end{aligned}$$

$$
\begin{aligned}
pd_{BN_G}(v_4)^+ &= (0.06, 0.06, 0.06),\\
pd_{BN_G}(v_4)^- &= (-0.06, -0.06, -0.06),\\
pd_{BN_G}(v_5)^+ &= (0.08, 0.08, 0.08),\\
pd_{BN_G}(v_5)^- &= (-0.08, -0.08, -0.08).
\end{aligned}
$$

Definition 4.4 *Let BN_G be a neutrosophic bipolar fuzzy graph on $G^*(V,E)$. A positive total pseudo degree of a vertex v in BN_G is denoted by $tpd^+_{BN_G}(v)$ and is defined by*

$$tpd^+_{BN_G}(v) = pd^+_{BN_G}(v) + (T_A, I_A, F_A)^+(v).$$

The negative total pseudo degree of a vertex v in BN_G is denoted by $tpd^-_{BN_G}(v)$ and is defined by

$$tpd^-_{BN_G}(v) = pd^-_{BN_G}(v) + (T_A, I_A, F_A)^-(v).$$

The total pseudo degree of a vertex v in BN_G is denoted by

$$tpd_{BN_G}(v) = (tpd^+_{BN_G}(v), tpd^-_{BN_G}(v))$$

for all $v \in V$.

Example 4.5 Let BN_G be the a neutrosophic bipolar fuzzy graph in Figure 2. We calculate the total pseudo degree of vertex of this neutrosophic bipolar fuzzy graphs as follows:

$$
\begin{aligned}
tpd_{BN_G}(v_1)^+ &= pd^+_{BN_G}(v_1) + (T_A, I_A, F_A)^+(v_1) = (0.12, 0.12, 0.12),\\
tpd_{BN_G}(v_1)^- &= pd^-_{BN_G}(v_1) + (T_A, I_A, F_A)^-(v_1) = (-0.12, -0.12, -0.12),\\
tpd_{BN_G}(v_2)^+ &= (0.125, 0.125, 0.125),\\
tpd_{BN_G}(v_2)^- &= (-0.125, -0.125, -0.125),\\
tpd_{BN_G}(v_3)^+ &= (0.125, 0.125, 0.125),\\
tpd_{BN_G}(v_3)^- &= (-0.125, -0.125, -0.125),\\
tpd_{BN_G}(v_4)^+ &= (0.09, 0.09, 0.09),\\
tpd_{BN_G}(v_4)^- & \quad = (-0.09, -0.09, -0.09),\\
tpd_{BN_G}(v_5)^+ &= (00.12, 0.12, 0.12),\\
tpd_{BN_G}(v_5)^- &= (-0.12, -0.12, -0.12).
\end{aligned}
$$

§5. Neighbourly Pseudo and Pseudo Totally Irregular Neutrosophic Bipolar Fuzzy Graphs

Definition 5.1 *Let $BN_G = (A, B)$ be a neutrosophic bipolar fuzzy graph. Then BN_G is said to be neighbourly pseudo irregular neutrosophic bipolar fuzzy graph if every pair of adjacent vertices have distinct pseudo degree.*

Definition 5.2 *Let $BN_G = (A, B)$ be a neutrosophic bipolar fuzzy graph. Then BN_G is said to be neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph if every pair of adjacent vertices have distinct total pseudo degree.*

Remark 5.3 A graph which is both neighbourly pseudo irregular neutrosophic bipolar fuzzy graph and neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph.

Example 5.4 Let BN_G be the neutrosophic bipolar fuzzy graph shown in Figure 3.

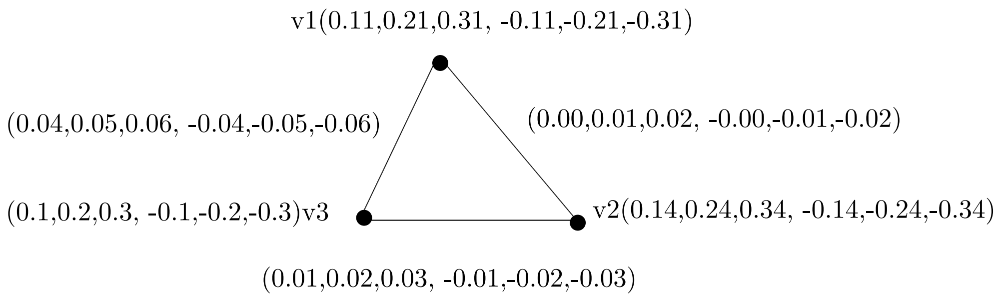

Figure 3

Then,

$$\begin{aligned} d(v_1)^+ &= (0.04, 0.06, 0.08),\ d(v_1)^- = (-0.04, -0.06, -0.08), \\ d(v_2)^+ &= (0.01, 0.03, 0.05),\ d(v_2)^- = (-0.01, -0.03, -0.05), \\ d(v_3)^+ &= (0.05, 0.07, 0.09),\ d(v_3)^- = (-0.05, -0.07, -0.09) \end{aligned}$$

and

$$\begin{aligned} pd_{BN_G}(v_1)^+ &= (0.03, 0.05, 0.07), \\ pd_{BN_G}(v_1)^- &= (-00.03, -0.05, -0.07), \\ pd_{BN_G}(v_2)^+ &= (0.045, 0.065, 0.085), \\ pd_{BN_G}(v_2)^- &= (-0.045, -0.065, -0.085), \\ pd_{BN_G}(v_3)^+ &= (0.025, 0.045, 0.065), \\ pd_{BN_G}(v_3)^- &= (-0.025, -0.045, -0.065), \\ tpd_{BN_G}(v_1)^+ &= (0.14, 0.26, 0.38), \\ tpd_{BN_G}(v_1)^- &= (-0.14, -0.26, -0.38), \\ tpd_{BN_G}(v_2)^+ &= (0.185, 0.305, 0.435), \\ tpd_{BN_G}(v_2)^- &= (-0.185, -0.305, -0.435), \\ tpd_{BN_G}(v_3)^+ &= (0.135, 0.245, 0.365), \\ tpd_{BN_G}(v_3)^- &= (-0.135, -0.245, -0.365). \end{aligned}$$

Here every pair of adjacent vertices have distinct pseudo degree and every pair of adjacent vertices have distinct total pseudo degree. Hence the graph given in Figure 3, is a neighbourly pseudo irregular neutrospphic bipolar fuzzy graph and neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph.

Remark 5.5 Every neighbourly pseudo irregular neutrosophic bipolar fuzzy graph need not be a neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph.

Example 5.6 Let BN_G be the neutrosophic bipolar fuzzy graph shown in Figure 4.

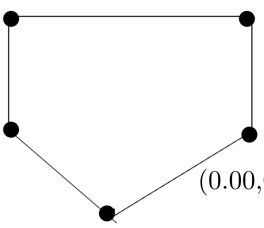

Figure 4

Then,

$$\begin{aligned}
d(v_1)^+ &= (0.04, 0.06, 0.08),\ d(v_1)^- = (-0.04, -0.06, -0.08),\\
d(v_2)^+ &= (0.05, 0.07, 0.09),\ d(v_2)^- = (-0.05, -0.07, -0.09),\\
d(v_3)^+ &= (0.01, 0.03, 0.05),\ d(v_3)^- = (-0.01, -0.03, -0.05),\\
d(v_4)^+ &= (0.01, 0.03, 0.05),\ d(v_4)^- = (-0.01, -0.03, -0.05),\\
d(v_5)^+ &= (0.01, 0.03, 0.05),\ d(v_5)^- = (-0.01, -0.03, -0.05)
\end{aligned}$$

and

$$\begin{aligned}
pd_{BN_G}(v_1)^+ &= (0.03, 0.05, 0.07),\\
pd_{BN_G}(v_1)^- &= (-0.03, -0.05, -0.07),\\
pd_{BN_G}(v_2)^+ &= (0.025, 0.065, 0.085),\\
pd_{BN_G}(v_2)^- &= (-0.025, -0.065, -0.085),\\
pd_{BN_G}(v_3)^+ &= (0.03, 0.05, 0.07),\\
pd_{BN_G}(v_3)^- &= (-0.03, -0.05, -0.07),\\
pd_{BN_G}(v_4)^+ &= (0.01, 0.03, 0.05),\\
pd_{BN_G}(v_4)^- &= (-0.01, -0.03, -0.05),\\
pd_{BN_G}(v_5)^+ &= (0.025, 0.065, 0.085),\\
pd_{BN_G}(v_5)^- &= (-0.025, -0.065, -0.085),
\end{aligned}$$

$$\begin{aligned}
tpd_{BN_G}(v_1)^+ &= (0.43, 0.55, 0.67),\\
tpd_{BN_G}(v_1)^- &= (-0.43, -0.55, -0.67),\\
tpd_{BN_G}(v_2)^+ &= (0.625, 0.765, 0.885),\\
tpd_{BN_G}(v_2)^- &= (-0.625, -0.765, -0.885),\\
tpd_{BN_G}(v_3)^+ &= (0.53, 0.65, 0.77),\\
tpd_{BN_G}(v_3)^- &= (-0.53, -0.65, -0.77),\\
tpd_{BN_G}(v_4)^+ &= (0.625, 0.765, 0.885),\\
tpd_{BN_G}(v_4)^- &= (-0.625, -0.765, -0.885),\\
tpd_{BN_G}(v_5)^+ &= (0.625, 0.765, 0.885),\\
tpd_{BN_G}(v_5)^- &= (-0.625, -0.765, -0.885).
\end{aligned}$$

Here every pair of adjacent vertices have distinct pseudo degree. But the pair of adjacent vertices v_4 and v_5 have same total pseudo degree. Hence the graph is neighbourly pseudo irregular neutrosophic bipolar fuzzy graph. But not neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph.

Remark 5.7 Every neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph need not be a neighbourly pseudo irregular neutrosophic bipolar fuzzy graph.

Example 5.8 Let BN_G be the neutrosophic bipolar fuzzy graph shown in Figure 5.

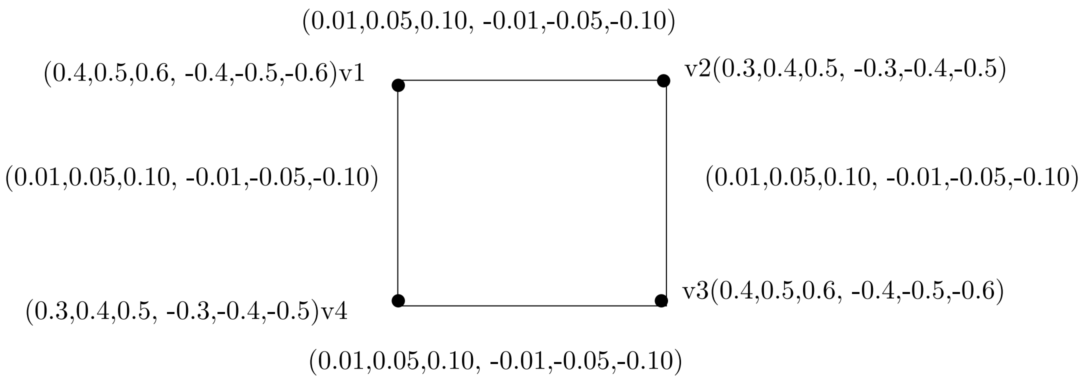

Figure 5

$$\begin{aligned}
d(v_1)^+ &= (0.02, 0.10, 0.20),\ d(v_1)^- = (-0.02, -0.10, -0.20),\\
d(v_2)^+ &= (0.02, 0.10, 0.20),\ d(v_2)^- = (-0.02, -0.10, -0.20),\\
d(v_3)^+ &= (0.02, 0.10, 0.20),\ d(v_3)^- = (-0.02, -0.10, -0.20),\\
d(v_4)^+ &= (0.02, 0.10, 0.20),\ d(v_4)^- = (-0.02, -0.10, -0.20)
\end{aligned}$$

and

$$\begin{aligned}
pd_{BN_G}(v_1)^+ &= (0.02, 0.10, 0.20),\\
pd_{BN_G}(v_1)^- &= (-0.02, -0.10, -0.20),\\
pd_{BN_G}(v_2)^+ &= (0.02, 0.10, 0.20),\\
pd_{BN_G}(v_2)^- &= (-0.02, -0.10, -0.20),\\
pd_{BN_G}(v_3)^+ &= (0.02, 0.10, 0.20),\\
pd_{BN_G}(v_3)^- &= (-0.02, -0.10, -0.20),\\
pd_{BN_G}(v_4)^+ &= (0.02, 0.10, 0.20),\\
pd_{BN_G}(v_4)^- &= (-0.02, -0.10, -0.20),\\
tpd_{BN_G}(v_1)^+ &= (0.42, 0.60, 0.80),\\
tpd_{BN_G}(v_1)^- &= (-0.42, -0.60, -0.80),\\
tpd_{BN_G}(v_2)^+ &= (0.32, 0.50, 0.70),\\
tpd_{BN_G}(v_2)^- &= (-0.32, -0.50, -0.70),\\
tpd_{BN_G}(v_3)^+ &= (0.42, 0.60, 0.80),\\
tpd_{BN_G}(v_3)^- &= (-0.42, -0.60, -0.80),\\
tpd_{BN_G}(v_4)^+ &= (0.32, 0.50, 0.70),\\
tpd_{BN_G}(v_4)^- &= (-0.32, -0.50, -0.70).
\end{aligned}$$

Here every pair of adjacent vertices have same pseudo degree. But every pair of adjacent vertices have distinct total pseudo degree. Hence the graph is neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph. But not neighbourly pseudo irregular neutrosophic bipolar fuzzy graph.

Theorem 5.9 *Let BN_G be a neutrosophic bipolar fuzzy graph and let*

$$(T_A, I_A, F_A)(u) = ((T_A, I_A, F_A)^+)(u), (T_A, I_A, F_A)^-)(u))$$

for all $u \in V$ be a constant function, then the following are equivalent:

(1) *BN_G is a neighbourly pseudo irregular neutrosophic bipolar fuzzy graph;*

(2) *BN_G is a neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph.*

Proof Assume that

$$(T_A, I_A, F_A)(u) = ((T_A, I_A, F_A)^+(u),\ (T_A, I_A, F_A)^-(u)) = ((c_T^+, c_I^+, c_F^+), (c_T^-, c_I^-, c_F^-))$$

for all $u \in V$ is a constant function. Suppose BN_G is neighbourly pseudo irregular neutrosophic bipolar fuzzy graph. Then, every pair of adjacent vertices having distinct pseudo degree.

Let v_i and v_j be two adjacent vertices having distinct pseudo degree (x_i, y_i, z_i) and (x_j, y_j, z_j)

respectively. Then $(x_i, y_i, z_i) \neq (x_j, y_j, z_j)$i.e,

$$\begin{aligned}
&pd_{BN_G}(v_i) = (pd_{BN_G}(v_i)^+, pd_{BN_G}(v_i)^-),\ pd_{BN_G}(v_j) = (pd_{BN_G}(v_j)^+, pd_{BN_G}(v_j)^-) \text{ and} \\
&pd_{BN_G}(v_i)^+ \neq pd_{BN_G}(v_j)^+ \\
&\qquad \Rightarrow (x_i, y_i, z_i)^+ \neq (x_j, y_j, z_j)^+ \\
&\qquad \Rightarrow (x_i, y_i, z_i)^+ + (c_T^+, c_I^+, c_F^+) \neq (x_j, y_j, z_j)^+ + (c_T^+, c_I^+, c_F^+) \\
&\qquad \Rightarrow (x_i, y_i, z_i)^+ + (T_A, I_A, F_A)^+(v_i) \neq (x_j, y_j, z_j)^+ + (T_A, I_A, F_A)^+(v_j) \\
&\qquad \Rightarrow tpd_{BN_G}(v_i)^+ \neq tpd_{BN_G}(v_j)^+.
\end{aligned}$$

Similarly, we prove that

$$\begin{aligned}
pd_{BN_G}(v_i)^- &\neq pd_{BN_G}(v_j)^- \\
&\Rightarrow tpd_{BN_G}(v_i)^- \neq tpd_{BN_G}(v_j)^-.
\end{aligned}$$

In general

$$pd_{BN_G}(v_i) \neq pd_{BN_G}(v_j) \Rightarrow tpd_{BN_G}(v_i) \neq tpd_{BN_G}(v_j).$$

Therefore, every pair of adjacent vertices having distinct total pseudo degree, i.e, (1) $\Rightarrow$ (2) is proved.

Now, suppose BN_G is neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph. Then every pair of adjacent vertices have distinct total pseudo degree. Let v_i and v_j be two adjacent vertices having distinct total pseudo degree (tx_i, ty_i, tz_i) and (tx_j, ty_j, tz_j) respectively. Then

$$(tx_i, ty_i, tz_i) \neq (tx_j, ty_j, tz_j),$$

i.e,

$$\begin{aligned}
tpd_{BN_G}(v_i) &= (tpd_{BN_G}(v_i)^+, tpd_{BN_G}(v_i)^-),\ tpd_{BN_G}(v_i)^+ \neq tpd_{BN_G}(v_j)^+ \text{ and} \\
tpd_{BN_G}(v_j) &= (tpd_{BN_G}(v_j)^+, tpd_{BN_G}(v_j)^-) \\
&\Rightarrow (tx_i, ty_i, tz_i)^+ \neq (tx_j, ty_j, tz_j)^+ \\
&\Rightarrow (x_i, y_i, z_i)^+ + (T_A, I_A, F_A)^+(v_i) \neq (x_j, y_j, z_j)^+ + (T_A, I_A, F_A)^+(v_j) \\
&\Rightarrow (x_i, y_i, z_i)^+ + (c_T^+, c_I^+, c_F^+) \neq (x_j, y_j, z_j)^+ + (c_T^+, c_I^+, c_F^+) \\
&\Rightarrow (x_i, y_i, z_i)^+ \neq (x_j, y_j, z_j)^+ \\
&\Rightarrow pd_{BN_G}(v_i)^+ \neq pd_{BN_G}(v_j)^+.
\end{aligned}$$

Similarly, we prove that

$$tpd_{BN_G}(v_i)^- \neq tpd_{BN_G}(v_j)^- \Rightarrow pd_{BN_G}(v_i)^- \neq pd_{BN_G}(v_j)^-.$$

In general

$$tpd_{BN_G}(v_i) \neq tpd_{BN_G}(v_j) \Rightarrow pd_{BN_G}(v_i) \neq pd_{BN_G}(v_j).$$

Therefore, every pair of adjacent vertices having distinct pseudo degree and (2) $\Rightarrow$ (1) is proved.

Hence (1) and (2) are equivalent. □

Remark 5.10 The converse of Theorem 5.9 needs not be true.

Example 5.11 Let BN_G be the neutrosophic bipolar fuzzy graph shown in Figure 6.

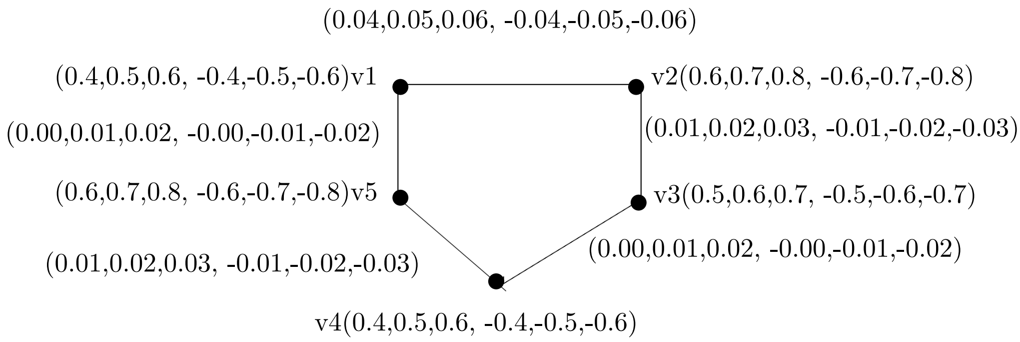

Figure 6

Then,

$$\begin{aligned}
d(v_1)^+ &= (0.04, 0.06, 0.08),\ d(v_1)^- = (-0.04, -0.06, -0.08),\\
d(v_2)^+ &= (0.05, 0.07, 0.09),\ d(v_2)^- = (-0.05, -0.07, -0.09),\\
d(v_3)^+ &= (0.01, 0.03, 0.05),\ d(v_3)^- = (-0.01, -0.03, -0.05),\\
d(v_4)^+ &= (0.01, 0.03, 0.05),\ d(v_4)^- = (-0.01, -0.03, -0.05),\\
d(v_5)^+ &= (0.01, 0.03, 0.05),\ d(v_5)^- = (-0.01, -0.03, -0.05)
\end{aligned}$$

and

$$\begin{aligned}
pd_{BN_G}(v_1)^+ &= (0.03, 0.05, 0.07),\\
pd_{BN_G}(v_1)^- &= (-0.03, -0.05, -0.07),\\
pd_{BN_G}(v_2)^+ &= (0.025, 0.065, 0.085),\\
pd_{BN_G}(v_2)^- &= (-0.025, -0.065, -0.085),\\
pd_{BN_G}(v_3)^+ &= (0.03, 0.05, 0.07),\\
pd_{BN_G}(v_3)^- &= (-0.03, -0.05, -0.07),\\
pd_{BN_G}(v_4)^+ &= (0.01, 0.03, 0.05),\\
pd_{BN_G}(v_4)^- &= (-0.01, -0.03, -0.05),\\
pd_{BN_G}(v_5)^+ &= (0.025, 0.065, 0.085),\\
pd_{BN_G}(v_5)^- &= (-0.025, -0.065, -0.085),
\end{aligned}$$

$$\begin{aligned}
tpd_{BN_G}(v_1)^+ &= (0.43, 0.55, 0.67),\\
tpd_{BN_G}(v_1)^- &= (-0.43, -0.55, -0.67),\\
tpd_{BN_G}(v_2)^+ &= (0.625, 0.765, 0.885),\\
tpd_{BN_G}(v_2)^- &= (-0.625, -0.765, -0.885),\\
tpd_{BN_G}(v_3)^+ &= (0.53, 0.65, 0.77),\\
tpd_{BN_G}(v_3)^- &= (-0.53, -0.65, -0.77),\\
tpd_{BN_G}(v_4)^+ &= (0.41, 0.53, 0.65),\\
tpd_{BN_G}(v_4)^- &= (-0.41, -0.53, -0.65),\\
tpd_{BN_G}(v_5)^+ &= (0.625, 0.765, 0.885),\\
tpd_{BN_G}(v_5)^- &= (-0.625, -0.765, -0.885).
\end{aligned}$$

Here the graph is both neighbourly pseudo irregular neutrosophic bipolar fuzzy graph and neighbourly pseudo totally irregular neutrosophic bipolar fuzzy graph. But here

$$(T_A, I_A, F_A)(u) = ((T_A, I_A, F_A)^+(u), (T_A, I_A, F_A)^-(u))$$

for all u in V is not constant.

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The Doubly Connected Hub Number of Graph

Ali Mohammed Sahal

Department of Studies in Mathematics, Faculty of Education-Zabid

University of Hodeidah, Republic of Yemen

E-mail: alisahl1980@gmail.com

Abstract: For a given connected graph $G = (V, E)$, a hub set S of G is a set of vertices with the property that for any pair of vertices outside of S, there is a path between them with all intermediate vertices in S. The hub number $h(G)$ is then defined to be the size of a smallest hub set of G. A set S is a doubly connected hub set if both $\langle S\rangle$ and $\langle V(G) - S\rangle$ are connected. The cardinality of the minimum doubly connected hub set in G is the doubly connected hub number and is denoted by $h_{cc}(G)$. In this paper, the doubly connected hub number for several classes of graphs is computed, bounds in terms of other graph parameters are also determined.

Key Words: Hub number, connected hub number, domination number, doubly connected domination number, doubly connected hub number.

AMS(2010): 05C40, 05C69.

§1. Introduction

In this paper we are concerned with simple graphs, that have no loops and no multiple or directed edges. Let G be such a graph, and let p and q be the number of its vertices and edges, respectively. Then we say that G is an (p, q)-graph. For graph theoretic terminology, we refer to [2].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \cup_{v\in S} N(v)$ and $N[S] = N(S) \cup S$.

Consider the graphs that represent transportation networks, that is the vertices can be taken to be locations or destinations, and an edge exists between two vertices precisely when there is an "easy passage" between the corresponding locations. For example, a city's network of streets, with vertices representing intersections or other points of intersect, and edges road segments. We are connected with a certain kind of connectivity, specifically we want a set S such that any traffic between disparate points in our network passes solely through vertices in this set.

Suppose that $S \subseteq V(G)$ and let $u, v \in V(G)$. An S-path between u and v is a path where all intermediate vertices are from S. (This includes the degenerate cases where the path consists

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of the single edge uv or a single vertex u if $u = v$, call such an S-path trivial.) A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $u, v \in V(G) - S$, there is an S-path in G between u and v. The smallest size of a hub set in G is called a hub number of G, and is denoted by $h(G)$. A hub set S of a connected graph G is called a connected hub set if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected hub set of G is called the connected hub number of G and is denoted by $h_c(G)$ [9].

A subset S of a graph G is called a dominating set if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set S of a connected graph G is called a doubly connected dominating set if the induced subgraphs $\langle S \rangle$ and $\langle V - S \rangle$ are connected. The minimum cardinality of a doubly connected dominating set of G is called the doubly connected domination number of G and is denoted by $\gamma_{cc}(G)$ [3].

A graph is acyclic if it has no cycles. A tree is a connected acyclic graph. A (p, q) graph is called unicyclic if it is connected and $p = q$.

A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers.

For disjoint graphs G_1 and G_2, the join $G = G_1 + G_2$ is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. Let us denote by $G - v$ the graph obtained from G by removing the vertex $v \in V(G)$ and all edges incident to v.

A connected subgraph B of G is called a block if B has no cut-vertex and every subgraph $B' \subseteq G$ with $B \subset B'$ has at least one cut-vertex. A connected graph G is called a block graph if every block in G is complete. A vertex v of a graph G is called a simplicial vertex if every two vertices of $N_G(v)$ are adjacent in G.

We need the following to prove main results.

Theorem 1.1([1]) *For any connected graph G, $\gamma_{cc}(G) = p - t$, $p \geq 3$, where t is the maximal number of simplicial vertices in a block with largest number of vertices.*

§2. The Doubly Connected Hub Number of Graph

Definition 2.1 *Let G be a connected graph. A doubly connected hub set S of G is a subset of $V(G)$ such that any pair of vertices of $V - S$ are connected by a path, whose all intermediate vertices are in S and both $\langle S \rangle$ and $\langle V(G) - S \rangle$ are connected. The cardinality of the minimum doubly connected hub set in G is the doubly connected hub number and is denoted by $h_{cc}(G)$.*

It is clear that $h_{cc}(G)$ is well-defined for any connected graph G, since $V(G) - v$ is a doubly connected hub set. In all situations of interest, we will assume G to be connected.

It is obvious that any doubly connected hub set in a graph G is also a connected hub set and any connected hub set is also a hub set, and thus we obtain the obvious bound $h(G) \leq h_c(G) \leq h_{cc}(G)$.

It is obvious that the difference $h_{cc}(G) - h_c(G)$ can be arbitrarily large in a graph G. It can be easily checked that $h_c(K_{1,n}) = 1$, while $h_{cc}(K_{1,n}) = n$, $n \geq 3$.

We now proceed to compute $h_{cc}(G)$ for some standard graphs.

Remark 2.2 Notice that

(1) For any complete graph K_p, $h_{cc}(K_p) = h_c(K_p) = 0$;
(2) For any double star $S_{n,m}$, $h_{cc}(S_{n,m}) = n + m + 1$;
(3) For any cycle C_p, $h_{cc}(C_p) = h_c(C_p) = p - 3$;
(4) For any complete bipartite graph $K_{n,m}$,

$$h_{cc}(K_{n,m}) = \begin{cases} 0, & \text{if } m = n = 1 \text{ ;} \\ 1, & \text{if } m = 1 \text{ } and \text{ } n = 2 \text{ } or \text{ } m = 2 \text{ } and \text{ } n \geq 2 \text{ ;} \\ 2, & \text{if } n, m \geq 3. \end{cases}$$

(5) For the wheel $W_{1,n}$, $n \geq 3$, $h_{cc}(W_{1,n}) = 1$.

Observation 2.3 Let $G \cong K_{m_1,m_2,\cdots,m_k}$ be the complete k-partite graph, $k \geq 3$ with $m_1 \leq m_2 \leq \cdots \leq m_k$. Then,

$$h_{cc}(G) = \begin{cases} 0, & \text{if } m_i = 1, 1 \leq i \leq k \text{ ;} \\ 1, & \text{if } m_i \leq 2 \text{ } and \text{ } m_j \geq 2, 1 \leq i \leq j \leq k \text{ ;} \\ 2, & \text{if } m_1 \geq 3. \end{cases}$$

Observation 2.4 If G_1 and G_2 are disjoint connected graphs, then

$$h_{cc}(G_1 + G_2) = \begin{cases} 0, & \text{if } h_{cc}(G_1) = 0 \text{ } and \text{ } h_{cc}(G_2) = 0 \text{ ;} \\ 1, & \text{if } h_{cc}(G_1) = 1 \text{ } or \text{ } h_{cc}(G_2) = 1 \text{ ;} \\ 2, & \text{otherwise.} \end{cases}$$

Proposition 2.5 *Let S be a minimum doubly connected hub set of a graph G. Then,*

(1) *there is at most one support vertex in $V(G) - S$;*
(2) *there is at most one cut- vertex in $V(G) - S$;*
(3) *there is at most one pendant vertex in $V(G) - S$.*

Theorem 2.6 *For any connected graph G, $h_{cc}(G) \leq \gamma_{cc}(G)$. The bound sharp for $G \cong K_{1,n}$, $n \geq 3$.*

Proof Let S be a minimum doubly connected dominating set of G. Then both $\langle S \rangle$ and $\langle V(G) - S \rangle$ are connected and for any $v \in V - S$, there exists $u \in S$ such that $v \in N(u)$. Since $\langle S \rangle$ is connected, it follows that for any $v, w \in V - S$, there is a path between them with all intermediate vertices in S. Thus, S is doubly connected hub set. Thus, $h_{cc}(G) \leq \gamma_{cc}(G)$. □

We observe that $h_{cc}(G) = \gamma_{cc}(G) = p - 1$, $p \geq 4$ only for a tree with all supports adjacent to at least two pendant vertices, otherwise $\gamma_{cc}(G) - h_{cc}(G) = 1$.

Theorem 2.7 *If G is a block graph, then $h_{cc}(G) \leq p-t$, $p \geq 3$, where t is the maximal number of simplicial vertices in a block with largest number of vertices.*

Proof The proof follows from Theorem 1.1 and Theorem 2.6. □

Proposition 2.8 *If $h_{cc}(G) = h_c(G)$, then $h_{cc}(G) \leq p-2$.*

Proof For any connected graph G with $p \geq 3$ we have $h_c(G) \leq p-2$. Since $h_c(G) \leq h_{cc}(G)$ and by hypothesis, $h_c(G) = h_{cc}(G)$, we have $h_{cc}(G) \leq p-2$. □

Theorem 2.9 *For any unicyclic graph G, $h_{cc}(G) - h_c(G) = k-1$, $k \geq 1$, where k is the number of pendant vertices of G.*

Proof Let G be a unicyclic graph of order p and let u be the only common vertex between a cycle C_n and a tree T of G. Let D be the set of all pendant vertices of G such that $|D| = k$ and let S be a minimum connected hub set of G. Then $V - S$ is the set of all pendant vertices and any two adjacent vertices v and w of a cycle C_n of G, $v \neq u$ and $w \neq u$. That is, $|S| = p-(k+2)$. Let S' be a minimum doubly connected hub set of G. Then $S' = V(G) - \{v, w, w'\}$, where $\langle vww' \rangle = P_3$, $v \neq u$, $w \neq u$ and $w' \neq u$. That is, $|S'| = p-3$. Thus,

$$h_{cc}(G) - h_c(G) = p - 3 - [p - (k+2)] = k - 1.$$ □

Corollary 2.10 *For any unicyclic graph G, $h_c(G) = h_{cc}(G)$ if and only if G is contain at most one pendant vertex.*

Proof The Proof follows from Remark 2.2 and Theorem 2.9. □

Theorem 2.11 *The difference $h_{cc}(G) - h_{cc}(G-v)$ can be arbitrarily large.*

Proof Let $H \cong K_{1,k} + K_1$ and let G be a graph obtained by adding two pendant edges and two pendant vertices u and v to the vertices of degree $k+1$ of the graph H (see Figure 1). Then $V(G) - \{u, w'\}$ is minimum doubly connected hub set of G. Therefore, $h_{cc}(G) = k+2$. Also, the set $\{v, w\}$ is minimum doubly connected hub set of $G - u$. Hence, $h_{cc}(G-u) = 2$. Thus, $h_{cc}(G) - h_{cc}(G-v) = k$. □

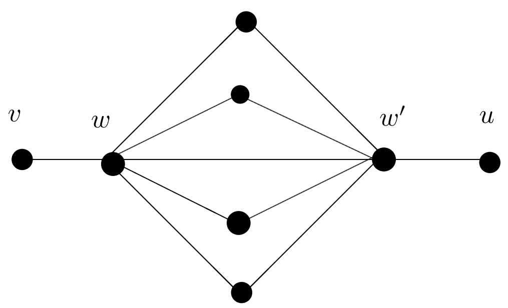

Figure 1

Theorem 2.12 *The difference $h_{cc}(G-v)-h_{cc}(G)$ can be arbitrarily large.*

Proof Let $G\cong P_n+K_1$ and let v be a vertex of K_1. Then clearly, $h_{cc}(G)=1$. Since $h_{cc}(G-v)=n-2$, it follows that $h_{cc}(G-v)-h_{cc}(G)=n-3$. □

§3. Bounds

Theorem 3.1 *For any connected (p,q) graph G with $p\geq 3$, $0\leq h_{cc}(G)\leq p-1$, with equality of the lower bound for $G\cong K_p$ and equality of the upper bound for $G\cong K_{1,n}$, $n\geq 3$.*

Proof The inequality of the lower bound is obvious. Now we prove the inequality of the upper bound. Let S be a minimum doubly connected hub set of a connected graph G. We consider the following cases.

Case 1. G is a tree.

We consider the following subcases.

Subcase 1.1 Each support vertex of G is adjacent to at least two pendant vertices, $h_{cc}(G)=p-1$.

Subcase 1.2 There exists at least one support vertex of G is adjacent to one pendant vertex, $h_{cc}(G)=p-2$.

Case 2. G is not a tree, $h_{cc}(G)\leq p-2$.

Thus, from all the above cases, $h_{cc}(G)\leq p-1$. □

Theorem 3.2 *For any connected graph G, $h_{cc}(G)\leq p-\delta(G)$. The bound sharp for $G\cong K_{1,n}$, $n\geq 3$.*

Proof Suppose that $P: v_1v_2...v_k$ be the longest path in a graph G. Assume that $H=G-\{v_1,v_2,\cdots,v_\delta\}$. We claim that H is connected. Suppose to the contrary that H is disconnected. Let C be the component of H such that $v_{\delta(G)+1}\notin C$ and let $u_1u_2\cdots u_n$ be the longest path of C. Suppose $S_1=v_1v_2\cdots v_\delta$ and $S_2=u_1u_2\cdots u_m$ such that $l=d(S_1,S_2)$. Then we may assume v_ru_i if $l=1$ and $v_rx_1x_2\cdots x_{l-1}u_i$ when $l\geq 2$ is the shortest path between S_1 and S_2, where $1\leq r\leq\delta$ and $1\leq i\leq m$. Assume that j is the largest positive integer such that $u_ou_j\in E(G)$. We consider the following cases.

Case 1. $i\geq j$. Then $u_0\cdots u_iv_rv_{r+1}\cdots v_k$ if $l=1$ and $u_0\cdots u_ix_{l-1}\cdots x_1v_rv_{r+1}\cdots v_k$ when $l\geq 2$ is a path of G longer than P which is a contradiction.

Case 2. $i\leq j$. Then $u_{i+1}\cdots u_ju_0\cdots u_iv_rv_{r+1}\cdots v_k$ if $l=1$ and $u_{i+1}\cdots u_jx_{l-1}\cdots x_1v_rv_{r+1}\cdots v_k$ when $l\geq 2$ is a path of G longer than P which is a contradiction.

Thus, from all the above cases, H is connected. Since $\delta(G)>(G[v_1,\cdots,v_\delta])$, it follows that each $v_i,(1\leq i\leq\delta)$ has at least one neighbor $V(G)-\{v_1,\cdots,v_\delta\}$, and hence $V(G)-\{v_1,\cdots,v_\delta\}$ is a hub set of G. Since $G[v_1,\cdots,v_\delta]$ is connected, $V(G)-\{v_1,\cdots,v_\delta\}$ is a doubly connected hub set of G. □

Theorem 3.3 *For any connected graph G, $h_{cc}(G) \geq p - \Delta(G) - 1$. The bound sharp for $G \cong C_p$, $p \geq 4$.*

Theorem 3.4 *For any (p, q) graph G with both G and $\overline{G}$ are connected, $h_{cc}(G)+h_{cc}(\overline{G}) \leq 2p-1$.*

Form Theorem 3.2, $h_{cc}(G) \leq p - \delta(G)$ and $h_{cc}(\overline{G}) \leq p - \delta(\overline{G})$. We have,

$$\begin{aligned} h_{cc}(G) + h_{cc}(\overline{G}) &\leq 2p - (\delta(G) + \delta(\overline{G})) \\ &= 2p - (p - 1 - \Delta(\overline{G}) + \delta(\overline{G})) \\ &= p + 1 + (\Delta(\overline{G}) - \delta(\overline{G})). \end{aligned}$$

Since $\Delta(\overline{G}) - \delta(\overline{G})) \leq p - 2$, we have $h_{cc}(G) + h_{cc}(\overline{G}) \leq 2p - 1$.

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Graphs with Large Semitotal Domination Number

C.Sivagnanam

(Department of General Requirements, University of Technology and Applied Sciences-Sur, Sultanate of Oman)

E-mail: choshi71@gmail.com

Abstract: Let $G = (V, E)$ be a graph without isolated vertex. A set $D \subseteq V$ is a semitotal dominating set of G if it is a dominating set and every vertex in D is within distance 2 of another vertex in D. The minimaum cardinality of a semitotal dominating set is called the semitotal domination number of G and is denoted by $\gamma_{t2}(G)$. In this paper we obtain an upper bound of this parameter and characterize the corresponding extremal graphs.

Key Words: Domination number, total domination number and semitotal domination number.

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§1. Introduction

The graph $G = (V, E)$ we mean a finite, undirected, graph with neither loops nor multiple edges and without isolated vertex. The order and size of G are denoted by n and m respectively. The degree of a vertex u in G is the number of edges incident with u and is denoted by $d(u)$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [3].

Let $v \in V$. The open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$. If $S \subseteq V$ then $N(S) = \bigcup_{v \in S} N(v)$ for all $v \in S$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbor set of u with respect to S is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. For any set $S \subseteq V$, the subgraph induced by S is the maximal subgraph of G with vertex set S and is denoted by $\langle S \rangle$. The vertex has degree one is called a pendant vertex. A support is a vertex which is adjacent to a pendant vertex. A weak support is a vertex which is adjacent to exactly one pendant vertex. A strong support is a vertex which is adjacent to at least two pendant vertices. An unicyclic graph is a graph with exactly one cycle. A graph without cycle is called acyclic graph and a connected acyclic graph is called a tree. $H(m_1, m_2, \cdots, m_n)$ denotes the graph obtained from the regular graph H by attaching m_i pendant edges to the vertex $v_i \in V(H), 1 \leq i \leq n$. The graph $K_2(m_1, m_2)$ is called bistar and it is also denoted by $B(m_1, m_2)$.

A subset D of V is called a dominating set of G if every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D of a graph G is called a total

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dominating set of G if the graph induced by D has no isolated vertex.The minimum cardinality of a total dominating set is called the total domination number of G and is denoted by $\gamma_t(G)$. W.Goddard.et.al [2] introduced the concept of semitotal domination in graphs. A set D of vertices in a graph G with no isolated vertices to be a semitotal dominating set(STD-set) of G if it is a dominating set of G and every vertex in D is within distance 2 of another vertex of D. The minimum cardinality of a semitotal dominating set is called the semitotal domination number of G and is denoted by $\gamma_{t2}(G)$

Theorem 1.1 *For a cycle C_n, $\gamma_{t2}(C_n) = \left\lceil \frac{2n}{5} \right\rceil$.*

Observation 1.1 *Since any STD-set of a spanning subgraph H of a graph G is a STD-set of G, we have $\gamma_{t2}(G) \leq \gamma_{t2}(H)$.*

Observation 1.2 *If G is a disconnected graph with k components $G_1, G_2, \cdots, G_k$ then*

$$\gamma_{t2}(G) = \gamma_{t2}(G_1) + \gamma_{t2}(G_2) + \cdots + \gamma_{t2}(G_k).$$

§2. Main Results

Theorem 2.1 *For any graph G, $\gamma_{t2}(G) \leq n-\Delta+1$. Further, $\gamma_{t2}(G) = n-\Delta+1$ if and only if G is isomorphic to H or $sK_2 \cup H$ where H is any graph having a vertex v with $d(v) = |V(H)|-1$.*

Proof Let $v \in V(H)$ and $d(v) = \Delta$. Let $S = N(v) - \{u\}$ where $u \in N(v)$. Then $V - S$ is a STD-set of G and hence $\gamma_{t2}(G) \leq n - \Delta + 1$. Now, let G be a graph with $\gamma_{t2} = n - \Delta + 1$.

Case 1. G is connected.

Let $v \in V(G)$ such that $d(v) = \Delta$. If $\Delta < n-1$ then $V - N(v)$ is a STD-set of G with $|V - N(v)| = n - \Delta$, which is a contradiction. Hence $\Delta = n-1$ and $d(v) = n-1$. Thus $G = H$.

Case 2. G is disconnected.

Let $G_1, G_2, \cdots, G_k$ be the components of G and let $|V(G_i)| = n_i, 1 \leq i \leq k$. If $\Delta = 1$, then $\gamma_{t2}(G) = n$ and each G_i is isomorphic to K_2. Suppose $\Delta \geq 2$. Let $v \in V(G_1)$ be such that $d(v) = \Delta$. Since $\gamma_{t2}(G) = n-\Delta+1$, we have $\gamma_{t2}(G_1) = n_1-\Delta+1$ and $\gamma_{t2}(G_i) = n_i, i \geq 2$. Hence by case 1, G_1 is isomorphic to H where H is any graph contains a vertex v with $d(v) = |V(H)|-1$ and G_i is isomorphic to $K_2, i \geq 2$. The converse is obvious. □

Theorem 2.2 *Let G be a connected graph with $\Delta < n-1$. Then*

$$\gamma_{t2}(G) \leq n - \Delta.$$

Proof Let $v \in V(G), d(v) = \Delta$. Clearly, $V - N(v)$ is a STD-set of G, which implies that

$$\gamma_{t2}(G) \leq n - \Delta.$$

□

Theorem 2.3 *Let T be a tree with $n \geq 3$. Then $\gamma_{t2}(T) = n - \Delta$ if and only if T can be obtained from a star by subdividing k of its edges, $1 \leq k \leq \Delta - 1$.*

Proof Let T be a tree with $\gamma_{t2}(T) = n - \Delta$. Let $u \in V(T)$ and $d(u) = \Delta$. It is clear that T is not a star graph and hence $\Delta < n - 1$. Let $N(u) = \{u_1, u_2, \cdots, u_\Delta\}, X = V(T) - N[v] = \{x_1, x_2, \cdots, x_k\}$ and let $T_1 = \langle X \rangle$.

Suppose $E(T_1) \neq \phi$. Let G_1 be a nontrivial component of T_1 and we assume that $u_1 \in N(x_1)$, where $x_1 \in V(G_1)$. Since G_1 is nontrivial, there exists a vertex $x_2 \in V(G_1)$ such that $x_1x_2 \in E(G_1)$. Then $D = V - (N(u) \cup \{x_2\})$ is a STD-set of cardinality $n - \Delta - 1$, which is a contradiction. Hence G_1 is trivial and hence $E(T_1) = \phi$.

If $d(u_i) \geq 3$ for some $u_i \in N(u)$, then $D = \{u, u_i\} \cup [X - (N(u_i) \cap X)]$ is a STD-set of T and $|D| \leq n - \Delta - 1$, which is a contradiction. Hence $d(u_i) \leq 2$. Suppose $d(u_i) = 2$ for all $i, 1 \leq i \leq \Delta$. Then $D = N(u)$ is a STD-set of G and $|D| = |N(u)| = \Delta = n - (n - \Delta) = n - (2\Delta + 1 - \Delta) = n - (\Delta + 1) = n - \Delta - 1$, which is a contradiction. Hence $d(u_i) = 1$ for some i. Thus T is obtained from $K_{1.\Delta}$ by subdividing k of its edges, $1 \leq k \leq \Delta - 1$. The converse is obvious. □

Notation 2.1 We define the graphs $G_i, 1 \leq i \leq 7$ as follows:

(1) $G_1 = C_3(m_1, 1, 0), m_1 \geq 1$;

(2) $G_2 = C_3(m_1, 1, 1), m_1 \geq 1$;

(3) G_3 is a graph obtained from $C_3(m_1, 0, 0), m_1 \geq 1$, by subdividing at least one pendant edge once;

(4) G_4 is a graph obtained from $C_3(m_1, 1, 0), m_1 \geq 2$, by subdividing t pendant edges which are incident with a vertex of degree $\Delta, 1 \leq t \leq m_1 - 1$;

(5) G_5 is a graph obtained from $C_3(m_1, 1, 1), m_1 \geq 2$, by subdividing t pendant edges which are incident with a vertex of degree $\Delta, 1 \leq t \leq m_1 - 1$;

(6) $G_6 = C_4(m_1, 0, 0, 0), m_1 \geq 1$;

(7) G_7 is a graph obtained from $C_4(m_1, 0, 0, 0), m_1 \geq 1$, by subdividing at least one pendant edge once.

Theorem 2.4 *Let G be a connected unicyclic graph with cycle $C = (v_1, v_2, \cdots, v_r, v_1)$. Then $\gamma_{t2}(G) = n - \Delta$ if and only if G is isomorphic to either C_4 or $G_i, 1 \leq i \leq 7$.*

Proof Let G be an unicyclic graph with cycle C and $\gamma_{t2} = n - \Delta$. If $G = C$ then it follows from Theorem 1.1 that $n \leq 4$ and hence G is isomorphic to C_4.

Suppose $G \neq C$. Let X denote the set of all pendant vertices in G and let $|X| = k$. Clearly,

$$\Delta - 2 \leq k \leq \Delta. \tag{1}$$

Claim 1. If $v \in V(G)$ and $d(v) = \Delta$ then v lies on C.

Suppose v is not on C. Then $k = \Delta - 1$ or Δ. Let $v_1 \in V(C)$ such that $d(v, v_1) = d(v, C)$. Then $D = V - (X \cup \{v_2, v_3\})$ is a STD-set with $|D| \leq n - \Delta - 1$, which is a contradiction. Hence v lies on C. Let $C = (v_1, v_2, \cdots, v_r, v_1)$ and let $d(v_1) = \Delta$.

Claim 2. $d(x) = 1$ or 2 for all $x \in V(G) - V(C)$.

Suppose there exists a vertex $x \in V(G) - V(C)$ with $d(x) \geq 3$. Then $k = \Delta - 1$ or Δ. If $k = \Delta - 1$ then all the vertices of $V(C) - \{v_1\}$ have degree 2 and hence $D = V(G) - [X \cup \{v_2, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$.

If $k = \Delta$ then at least one vertex v_i on C has degree 2. Then $D = V(G) - [X \cup \{v_i\}]$ is a STD-set of G with $|D| < n - \Delta$. Hence $d(x) = 1$ or 2 for all $x \in V(G) - V(C)$.

Claim 3. Every vertex of $V(C) - \{v_1\}$ has degree 2 or 3.

Inequality (1) gives that $d(v_i) \leq 4$ for all $i \neq 1$. Suppose that $v_i \in V(C)$ with $d(v_i) = 4$ for some i. Then $k = \Delta$ and $d(v_j) = 2$ for all $j \neq 1, i$. Hence $D = V(G) - [X \cup \{v_i\}]$ is a STD-set of G with $|D| < n - \Delta$. This proves claim 3.

Claim 4. $r \leq 4$.

Suppose $r \geq 5$. If $k = \Delta$, then there exists a vertex v_i such that $d(v_i) = 2$ and $D = V(G) - [X \cup \{v_i\}]$ is a STD-set of G with $|D| = n - \Delta - 1$. If $k = \Delta - 1$, then there exist two adjacent vertices v_i and v_j such that $d(v_i) = d(v_j) = 2$. Hence $D = V(G) - [X \cup \{v_i, v_j\}]$ is a STD-set of G with $|D| = n - \Delta - 2$.. If $k = \Delta - 2$, then every vertex of $V(C) - \{v_1\}$ has degree 2 and hence $D = V(G) - [X \cup \{v_2, v_3, v_5\}]$ is a STD-set of G with $|D| < n - \Delta$. Thus $r \leq 4$.

Now, we only need to consider two cases following.

Case 1. $r = 3$

Suppose, there exists a vertex $x_1 \in X$ such that $d(x_1, C) \geq 3$. Let $(x_1, x_2, \cdots, x_s, v_i), s \geq 3$ be the unique $x_1 - v_i$ path. If $k = \Delta - 2$, then $D = V(G) - [X \cup \{x_s, v_2, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$. Let $k = \Delta - 1$. We assume $d(v_2) = 3$. If $i = 1$ then $D = V(G) - [X \cup \{x_s, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$. If $i = 2$ then $D = V(G) - [X \cup \{v_2, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$, which is a contradiction. If $k = \Delta$ then $D = V(G) - [X \cup \{x_s\}]$ is a STD-set of G with $|D| < n - \Delta$, which is a contradiction. Hence every $x \in X, d(x, C) \leq 2$.

Suppose $d(x, C) = 1$ for all $x \in X$. If $k = \Delta - 2$ then $d(v_1) = n - 1$ and hence $\gamma_{t2}(G) = n - \Delta + 1$,which is a contradiction. If $k = \Delta - 1$ then G is isomorphic to G_1. If $k = \Delta$ then G is isomorphic to G_2.

Suppose $d(x, C) = 2$ for some $x \in X$. If $k = \Delta - 2$ then G is isomorphic to G_3 If $k = \Delta - 1$ then G is isomorphic to G_4. If $k = \Delta$ then G is isomorphic to G_5.

Case 2. $r = 4$.

Suppose, there exists a vertex $x_1 \in X$ such that $d(x_1, C) \geq 3$. Let $(x_1, x_2, \cdots, x_s, v_i), s \geq 3$ be the unique $x_1 - v_i$ path. If $k = \Delta - 2$, then $D = V(G) - [X \cup \{x_s, v_2, v_3\}]$ is a STD-set of G with $|D| < n - \Delta$. If $k = \Delta - 1$ or Δ, then there exists a vertex in C, say v_2 with $d(v_2) = 2$. Then $D = V(G) - [X \cup \{x_s, v_2\}]$ is a STD-set of G with $|D| < n - \Delta$. Hence $d(x, C) \leq 2$ for all $x \in X$. Now, if $k = \Delta - 2$, then G is isomorphic to $G_i, 6 \leq i \leq 7$. If $k = \Delta - 1$ or Δ, then there is no graphs satisfy $\gamma_{t2}(G) = n - \Delta$. The converse is obvious. □

Theorem 2.5 *Let G be a connected graph with $\gamma_{t2}(G) = n - \Delta$ and let v be a vertex of G with $d(v) = \Delta$. Then, each vertex $u \in N(v), |N(u) \cap V(G - N[v])| \leq 1$.*

Proof Suppose there exists a vertex $u \in N(v)$ such that u is adjacent to k vertices in $G - N[v], k \geq 2$. Let $X = \{x_1, x_2, \cdots, x_k\}$ be the set of vertices in $G - N[v]$ such that $ux_i \in E(G), 1 \leq i \leq k$. Then $D = [V - (N(v) \cup X)] \cup \{u\}$ is a STD-set of G and $|D| = n - (\Delta + k) + 1 \leq n - \Delta - 1$, which is a contradiction. Hence the result follows. $\square$

Theorem 2.6 *Let G be a graph with $\Delta(G) = 2$. Then $\gamma_{t2}(G) = n - \Delta$ if and only if G is isomorphic to one of the following graphs:*

(1) $C_3 \cup P_3 \cup sK_2, 6 + 2s = n$;

(2) $C_4 \cup sK_2, 4 + 2s = n$;

(3) $2C_3 \cup sK_2, 6 + 2s = n$;

(4) $P_4 \cup sK_2, 4 + 2s = n$;

(5) $2P_3 \cup sK_2, 6 + 2s = n$.

Proof Let G be a graph with $\Delta = 2$ and $\gamma_{t2} = n - \Delta$. It is clear that every component of G is either a path or a cycle. If there exists a component G_1 of G with $|V(G_1)| = n_1 \geq 5$, then $\gamma_{t2}(G_1) \leq n_1 - 3$ and hence $\gamma_{t2}(G) \leq n - 3$, which is a contradiction. Thus the order of each component of G is at most 4.

Further, if G has three components which are cycles of order 3 or 4 then $\gamma_{t2} \leq n-3 < n-\Delta$. Hence at most two components of G are a cycle of order 3 or 4. Suppose two components of G be cycles. If G contains cycles C_3 and C_4 then $\gamma_{t2} \leq n - 3 < n - \Delta$, which is a contradiction. If G contains $2C_4$ then $\gamma_{t2} \leq n - 4 < n - \Delta$, which is a contradiction. Thus G contains $2C_3$ and hence $G = 2C_3 \cup sK_2$ where $6 + 2s = n$.

Suppose exactly one component of G be a cycle. Let it be C. Suppose $C = C_4$. If G contains a path of order 3 or 4, then $\gamma_{t2} \leq n-3$, which is a contradiction. Hence $G = C_4 \cup sK_2$ where $4 + 2s = n$. Let $C = C_3$. If G contains a path of order 4, then $\gamma_{t2} \leq n - 3$, which is a contradiction. If G contains $2P_3$, then $\gamma_{t2} \leq n-3$, which is a contradiction. If G contains no P_3 then $\gamma_{t2} = n-1$ which is a contradiction. Thus G has exactly one P_3 Hence $G = C_3 \cup P_3 \cup sK_2$ where $6 + 2s = n$.

Suppose no components of G is a cycle. Then all the components of G are paths. If G has three components which are paths of order 3 or 4, then $\gamma_{t2} \leq n - 3 < n - \Delta$. Hence at most two components of G are paths of order 3 or 4. If G contains a P_3 and a P_4, then $\gamma_{t2} \leq n - 3 < n - \Delta$, which is a contradiction. If G has $2P_4$ then $\gamma_{t2} \leq n - 4 < n - \Delta$, which is a contradiction.Hence G contains $2P_3$ or one P_4. Hence G is isomorphic to $2P_3 \cup sK_2$ where $6 + 2s = n$ or $P_4 \cup sK_2$ where $4 + 2s = n$. The converse is obvious. $\square$

Theorem 2.7 *Let G be a connected graph and let v be a vertex of degree Δ . If $V - N[v]$ is an independent set and every vertex in $N(v)$ is adjacent to at most one vertex in $V - N[v]$, then $\gamma_{t2}(G) = n - \Delta$ or $n - \Delta - 1$*

Proof Let D be a γ_{t2}−set. Since every vertex of $N(v)$ is adjacent to at most one vertex in $V - N[v]$, it follows that $|D| \geq |V - N[v]|$. Hence $\gamma_{t2} \geq n - (\Delta + 1)$. Also $V - (N(v))$ is a STD-set and hence $\gamma_{t2}(G) \leq n - \Delta$. Thus $\gamma_{t2} = n - \Delta$ or $n - \Delta - 1$. $\square$

Theorem 2.8 *Let G be a connected graph and let v be a vertex of degree Δ. If*

(1) $V - N[v]$ is an independent set;

(2) Every vertex in $N(v)$ is adjacent to at most one vertex in $V - N[v]$;

(3) $N(v)$ Contains a vertex of degree one,

then, $\gamma_{t2}(G) = n - \Delta$.

Proof Let D be a γ_{t2}-set of G. Let $u \in N(v)$ be a vertex of degree 1. It follows from Theorem 2.7 that $|D| = n - \Delta$ or $n - \Delta - 1$. Since u is a pendent vertex of $G, v \in D$. Also it follows from $i)$ and $ii)$ that D contains $n - \Delta - 1$ vertices for dominating the vertices of $V - N[v]$. Hence $\gamma_{t2} = |D| = n - \Delta$. □

Theorem 2.9 *Let G be a connected graph with bipartition $\{V_1, V_2\}$ and let $v \in V_1$ with $d(v) = \Delta$. Suppose $\gamma_{t2} = n - \Delta$. Then the following conditions are satisfied. $i)|V_2| = \Delta(G)$. $ii)$ Every pair of vertices $u, w \in V_1, u \neq v, w \neq v$ such that $N(u) \cap N(w) = \phi$. $iii)$ Each vertex in V_2 has degree at most 2 and at least one vertex of V_2 has degree 2.*

Proof Let $N(v) = \{v_1, v_2, \cdots, v_\Delta\}$. Since $N(v) \subseteq V_2, \Delta(G) \leq |V_2|$. If there exists a vertex $x \in V_2 - N(v)$, then since G is connected, $D = [V - (N(v) \cup \{x\})]$ is a STD-set of G with $|D| = n - \Delta - 1$ which is a contradiction. Hence $|V_2| = \Delta$.

Suppose $N(u) \cap N(w) \neq \phi$. Let $y \in N(u) \cap N(w)$. It is clear that $y \in N(v)$. Then $D = [V - (N(v) \cup \{u, w\})] \cup \{y\}$ is a STD-set of G with $|D| = n - \Delta - 1$ which is a contradiction. Hence $N(u) \cap N(w) \neq \phi$.

Suppose there exists a vertex $z \in V_2$ with $d(z) \geq 3$. Let $u, w \in V_1$ be such that $uz, wz \in E$. Then $N(u) \cap N(w) \neq \phi$ which is a contradiction. Hence each vertex in V_2 has degree at most 2. Also if all the vertices of V_2 have degree 1, then G is a star and $\gamma_{t2} \neq n - \Delta$. Hence at least one vertex of V_2 has degree 2. □

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Degree Affinity Number of Graphs

J. Kok

(City of Tshwane, South Africa & Visiting Faculty at CHRIST, Bangalore, India)

E-mail: jacotype@gmail.com; johan.kok@christuniversity.in

Abstract: This paper initiates a study on a new graph parameter called the *degree affinity number* of a graph. The degree affinity number of a graph G is obtained by iteratively constructing graphs, $G_1, G_2, \cdots, G_k$ of increased size by adding a maximal number of edges between absolute distinct pairs of distinct, non-adjacent vertices of equal degree. Results for cycle and path graphs and certain general results are presented.

Key Words: Degree affinity number, degree affinity edge, partition.

AMS(2010): 05C15, 05C38, 05C75, 05C85.

§1. Introduction

For general notation and concepts in graphs see [1, 3, 6]. Throughout the study only finite, simple and undirected graphs will be considered. Recall that an edge $uv \in E(G)$ may also be denoted by, $\{u, v\}$. The latter notation emphasizes that an edge may be considered to be a 2-element set of vertices joined by an edge. It is indeed the properties of sets which clarify notions such as, edges $\{u, v\}, \{x, z\} \in E(G)$ are identical if and only if, $|\{u, v\} \cap \{x, z\}| = 2$, and the edges are incident if and only if, $|\{u, v\} \cap \{x, z\}| = 1$. A pair of distinct vertices which are not necessarily joined by an edge is denoted by, $\{u, v\}^{\pm}$. Two pairs of distinct vertices say, $\{u, v\}^{\pm}$ and $\{x, z\}^{\pm}$ are said to be *distinct pairs* if and only if, $|\{u, v\}^{\pm} \cap \{x, z\}^{\pm}| \leq 1$. Two pairs of distinct vertices are said to be *absolutely distinct* if and only if, $|\{u, v\}^{\pm} \cap \{x, z\}^{\pm}| = 0$. A pair of distinct vertices say, $u, v \in V(G)$ which are not joined by an edge (non-adjacent) is denoted by, $\{u, v\}^{-}$. In similar fashion, two pairs of distinct non-adjacent vertices $\{u, v\}^{-}$ and $\{x, z\}^{-}$ are said to be absolute distinct if and only if $|\{u, v\}^{-} \cap \{x, z\}^{-}| = 0$. This absolute distinct relation between two pairs of distinct vertices is denoted by, $\{u, v\}^{\pm} \nmid \{x, z\}^{\pm}$.

Various studies with regards to the addition or deletion of edges in a graph G and the effect thereof on parameters of graphs appear in the literature. To mention a few with corresponding references, see [2, 4, 5, 7]. The wide research interest in the addition or deletion of edges and the principle that, researching *mathematics for the sake of mathematics* is acceptable, motivate the study of a new graph parameter called the *degree affinity number* of a graph. In a more general sense the idea of *degree affinity* can be conceptualized to be chemical affinity between atoms or molecular affinity in molecular structures or social affinity between the members in

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social networks. Such notions could lead to real world applications with regards to the changes in graph theoretical properties of these structures. The recent *contact tracing models* utilised by countries in managing SARS-CoV-2 virus spread through communities suggest real world informatica application of this new notion. This new parameter is introduced in the next section.

§2. Degree Affinity Number of Cycle Graphs

It is known that a graph of order $n \geq 2$ has at least two vertices of equal degree. If two non-adjacent vertices $u, v \in V(G)$ with $deg_G(u) = deg_G(v)$ exist then the added edge uv to obtain G' is called a *degree affinity edge*.

Maximal Degree Affinity Convention (MDAC): For a graph G the 1^{st}-iteration is a maximal addition of degree affinity edges in respect of absolute distinct pairs of distinct non-adjacent vertices of equal degree. The graph obtained is labeled G_1. Hence, by the same convention it is possible to construct G_i from G_{i-1} provided that at least one (absolute distinct) pair of distinct non-adjacent vertices of equal degree exists in G_{i-1}. When no further edges can be added on the k^{th}-iteration the MDAC terminates.

Remark 2.1 A subtle feature of the MDAC is that all degrees are considered per iteration. Hence, if in the i^{th}-iteration say, $\{u, v\}^- \nsim \{x, z\}^-$ exist such that, $deg_{G_{i-1}}(u)[= deg_{G_{i-1}}(v)] \neq deg_{G_{i-1}}(x)[= deg_{G_{i-1}}(z)]$ then both the degree affinity edges uv and xz must be added during the i^{th}-iteration. The procedure can be viewed as a graph operation denoted by say, $\Lambda(G)$ whereby:

(*i*) $V(G)$ is partitioned into sets, $\{X_i : u, v \in X_i \Leftrightarrow deg_G(u) = deg_G(v)\}$;

(*ii*) For each X_i the maximum partition Y_i of random unordered pairs of distinct non-adjacent vertices is considered;

(*iii*) For each such pair $\{u, v\}$ the edge uv is added to G.

It is obvious that in applying the MDAC a finite number of iterations are possible. Let $\eta(k)$ be the number of degree affinity edges added when applying the MDAC has exhausted (say at k^{th}-iteration).

Definition 2.1 *The degree affinity number of a graph G is given by*

$\eta(G) = max\{\eta(k)$: *over all choices of maximal number of absolute distinct pairs of distinct non-adjacent vertices of equal degree, through exhaustive application of the MDAC*$\}$.

An upperbound on the *exhaustive iteration count* follows immediately.

Corollary 2.1 *For any graph G of order $n \geq 2$, the exhaustive iteration count for obtaining $\eta(G)$ is*

$$k \leq \frac{n(n-1)}{2} - \varepsilon(G).$$

Proof Observe that any degree affinity edge e of G is an edge in $\overline{G}$. Hence, if a graph exists which permits the addition of only one degree affinity edge per iteration and on exhaustion a complete graph is obtained. Then,

$$k = \frac{n(n-1)}{2} - \varepsilon(G).$$

Therefore, for graphs in general

$$k \leq \frac{n(n-1)}{2} - \varepsilon(G). \qquad \square$$

Corollary 2.1 implies that $0 \leq k \leq \eta(G) \leq \varepsilon(\overline{G})$. This observation can be illustrated by adding the maximum number of degree affinity edges to the cycle C_6. See Figures 1 and 2.

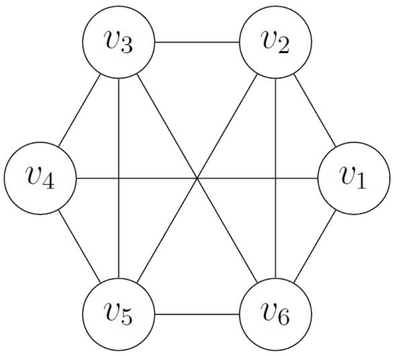

Figure 1 Graph C_6 for which a non-optimal application of the MDAC yields $k = 2$ and $\eta(k) = 5 < 9 = \varepsilon(\overline{C_6})$.

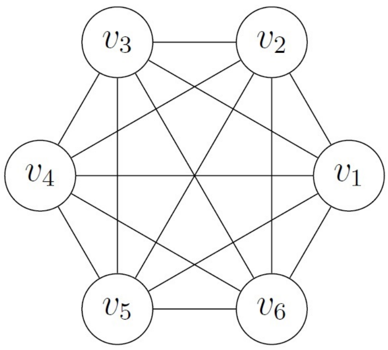

Figure 2 Graph C_6 for which an optimal application of the MDAC yields $k = 3$ and $\eta(3) = 9 = \varepsilon(\overline{C_6})$.

Figure 1 can be explained by: (i) begin with $G = C_6$ and (ii) add the degree affinity edges v_1v_4, v_2v_5, v_3v_6. Note this is a maximal addition of degree affinity edges during the 1^{st}-iteration to obtain G_1. Proceed with the 2^{nd}-iteration by considering G_1 and adding a maximal (the maximum in this case) degree affinity edges say, v_2v_6 and v_3v_5 to obtain G_2. Applying the MDAC is exhausted since no $\{u, v\}^-$, $u, v \in V(G_2)$ with $deg_{G_2}(u) = deg_{G_2}(v)$ exists. Note that $G_2 \ncong K_6$.

Figure 2 can be explained by: (i) begin with $G = C_6$, (ii) add the degree affinity edges, v_1v_4, v_2v_6, v_3v_5 to obtain G_1 and (iii) add the degree affinity edges v_1v_5, v_2v_4, v_3v_6 to obtain G_2. Proceed with the 3^{rd}-iteration by considering G_2 and add degree affinity edges v_1v_3, v_2v_5, v_4v_6 to obtain G_3. Since no $\{u,v\}^-$, $u,v \in V(G_3)$ with $deg_{G_3}(u) = deg_{G_3}(v)$ exists, applying the MDAC is exhausted. Note that $G_3 \cong K_6$. From Definition 2.1 it follows that, $\eta(C_6) = max\{5,9\} = 9$.

Theorem 2.2 *For an even cycle C_n, $n \geq 4$ the MDAC exhausts after $k = n-3$ iterations and*

$$\eta(C_n)_{n,even} = \frac{n(n-3)}{2} \quad and \quad G_{n-3} \cong K_n.$$

Proof We describe an imbedded induction procedure for even cycles (EIPC). Construct $G = C_4$ as follows; take the disjoint union of paths P and Q on the vertices v_1, v_2 and u_1, u_2, respectively and add the edges v_1u_1, v_2u_2. For the exhaustive 1^{st}-iteration of the MDAC add the edges v_1u_2 and u_1v_2 to obtain K_4. Clearly, $k = 1 = 4-3$ and $\eta(C_4) = 2 = \frac{4(4-3)}{2}$. Also, $G_1 \cong K_4$.

Similarly, construct the cycle $G = C_6$. For the 1^{st}-iteration of MDAC add the edges v_1u_3 and u_1v_2, u_2v_3. For the 2^{nd}-iteration of the MDAC add the edges u_1v_3 and v_1u_2, v_2u_3. For the exhaustive 3^{rd}-iteration of the MDAC add the edges v_1v_3, u_1u_3 and v_2u_2 to obtain K_6. Clearly, $k = 3 = 6-3$,

$$\eta(C_6) = 9 = \frac{6(6-3)}{2} \text{ and } G_3 \cong K_6.$$

For a similar construction of $G = C_8$ add the degree affinity edges v_1u_4, u_1v_2, u_2v_3, and u_3v_4 on the 1^{st}-iteration. For the 2^{nd}-iteration add u_1v_4, v_1u_2, v_2u_3, v_3u_4. For the 3^{rd}-iteration add v_1u_3, u_1v_3, u_2v_4, v_2u_4. For the 4^{th}-iteration add v_2u_2, v_3u_3, v_1v_4, u_1u_4. For the exhaustive 5^{th}-iteration add v_1v_3, v_2v_4, u_1u_3, u_2u_4. During all iterations the maximal degree affinity edges was a maximum. Therefore, $k = 5 = 2 \times 4 - 3$, $\eta(C_8) = 20 = \frac{(2\times 4)((2\times 4)-3)}{2}$ and $G_{k=5} \cong K_{2\times 4}$. Through immediate induction it follows that the EIPC can be utilized to add all degree affinity edges of the cycle $C_{2\ell}$ to obtain $K_{2\ell}$ through, $k = 2\ell - 3$ iterations. Finally,

$$\eta(C_{2\ell}) = \frac{2\ell(2\ell-1)}{2} - 2\ell = \frac{2\ell(\ell-3)}{2}.$$

This settles the result. □

An immediate corollary follows.

Corollary 2.3 *For an odd cycle C_n, $n \geq 5$ the MDAC exhausts after $k = n-3$ iterations and*

$$\eta(C_n)_{n,odd} = \frac{(n-2)(n-3)}{2}.$$

Proof Let $\ell = \lfloor \frac{n}{2} \rfloor$. Construct an odd cycle C_n, $n \geq 5$ as follows; take the disjoint union of paths P and Q on the vertices $v_1, v_2, v_3, \cdots, v_\ell$ and $u_1, u_2, u_3, \cdots, u_{\ell+1}$, respectively. Add the edges v_1u_1 and $v_\ell u_{\ell+1}$. Consider the path $v_\ell u_{\ell+1}u_\ell$ equivalent to the *ghost edge* $v_\ell u_\ell$ and apply the EIPC to add degree affinity edges to $v_1, v_2, v_3, \cdots, v_\ell$ and $u_1, u_2, u_3, \cdots, u_\ell$ similar to that for, C_{n-1}. The aforesaid step minimized the number of vertices which could not be paired to permit additional degree affinity edges. The exhaustive iteration of the MDAC is the addition of the single edge $v_\ell u_\ell$. Hence, a total of $\frac{(n-1)(n-4)}{2}+1=\frac{(n-2)(n-3)}{2}$ degree affinity edges have been added.

Clearly the procedure yields $\eta(C_n)_{n,odd} = max\{\eta(k)$: over all choices of maximal number of absolute distinct pairs of distinct non-adjacent vertices of equal degree through exhaustive application of the MDAC$\} = \frac{(n-2)(n-3)}{2}$. The result $k = n-3$ follows by similar reasoning. □

Theorem 2.4 *If a non-complete r-regular graph G of even order n reaches completeness on exhaustion of the MDAC then,*

$$k = n-(r+1) \quad and \quad \eta(G) = \frac{n(n-(1+r))}{2}.$$

Proof Let G be a non-complete regular graph of even order n. Since G reaches completeness on exhaustion of the MDAC it is possible to find $\frac{n}{2}$ absolutely distinct pairs of distinct, non-adjacent vertices for the 1^{st}-iteration. If the aforesaid is not possible it implies that some vertex $v \in V(G)$ exists which is adjacent to all vertices in $V(G)\backslash v$. Hence, $deg_G(v) = n-1$ implying that G is complete. This is a contradiction.

Add the $\frac{n}{2}$ degree affinity edges to obtain G_1. Clearly, G_1 is regular. If G_1 is complete then the MDAC has exhausted. Else, since G reaches completeness on exhaustion of the MDAC, so does G_1. Hence, repeat the reasoning iteratively.

Since, the number of degree affinity edges which must be added to any vertex $v \in V(G)$ is, $(n-1)-r = n-(r+1)$ the result, $k = n-(r+1)$ is settled. The aforesaid implies that

$$\eta(G) = \frac{n(n-(1+r))}{2}. \qquad \square$$

§3. Degree Affinity Number of Path Graphs

It is assumed that the reader is familiar with a path graph (simply, a path) P_n, $n \geq 1$ on the vertices, $v_1v_2, v_3, \cdots, v_n$. Both paths P_1, P_2 are complete and path P_3 requires one iteration to exhaust the MDAC by adding one edge and is thereafter complete. Path P_4 requires two iterations to add three degree affinity edges to reach completeness. For the path $G = P_6$ consider the disjoint union on the paths $v_1v_2v_3$, $u_1u_2u_3$ and add the edge v_1u_1. During the 1^{st}-iteration of the MDAC the maximal (not maximum) degree affinity edges v_2u_2 and v_3u_3 are added. Then add the edges, v_1v_3 and u_1u_3 where-after, the application of the EIPC is followed to reach completeness. This procedure renders, $k = 4$, $\eta(P_6) = 10$ and $G_4 \cong K_6$.

Next we present a lemma for which the proof is left for the reader. Thereafter, the result for paths of even order $n \geq 8$ follows.

Lemma 3.1 *For the positive integers, n_1, m_1 and n_2, m_2 with, $n_1 \geq n_2$; $m_1 \leq m_2$ and $n_1 + m_1 = n_2 + m_2$ we have that,*

$$\frac{n_1(n_1-1)}{2} + \frac{m_1(m_1-1)}{2} \geq \frac{n_2(n_2-1)}{2} + \frac{m_2(m_2-1)}{2}.$$

From a graph theoretical perspective, Lemma 3.1 states that under the stated conditions,

$$\varepsilon(K_{n_1}) + \varepsilon(K_{m_1}) \geq \varepsilon(K_{n_2}) + \varepsilon(K_{m_2}).$$

Theorem 3.2 *For an even path P_n, $n \geq 8$ the MDAC exhausts after $k = n - 4$ iterations and*

$$\eta(P_n)_{n,even} = \frac{n^2 - 7n + 14}{2}.$$

Proof Consider the path $P_{2\ell}$, $\ell = 4, 5, \cdots$. Construct the path as follows; take the disjoint union of paths P and Q on the vertices $v_1, v_2, v_3, \cdots, v_\ell$ and $u_1, u_2, u_3, \ldots, u_\ell$, respectively and add the edge v_1u_1. From Lemma 3.1 it follows that for the 1^{st}-iteration add the degree affinity edges $v_\ell u_\ell$. Consider the path $v_{\ell-1}v_\ell u_\ell u_{\ell-1}$ equivalent to the *ghost edge* $v_{\ell-1}u_{\ell-1}$ and add those degree affinity edges to $C_{2\ell-2}$ for vertices v_i, u_i, $1 \leq i \leq (\ell - 1)$ by the EIPC in Theorem 2.3. The aforesaid iteration ensures the maximum number of vertices of degree 3. Hence, it minimizes the number of vertices i.e. two vertices, which exhaust on 1^{st}-iteration. Finally, on exhaustion of the EIPC add the degree affinity edge, $v_{\ell-1}u_{\ell-1}$. Relying on the induction reasoning of Theorem 2.3 the results, $k = (n-2) - 3 + 1 = n - 4$ and follows

$$\eta(P_n)_{n,even} = \frac{(n-2)((n-2)-3)}{2} + 2 = \frac{n^2 - 7n + 14}{2}. \qquad \square$$

Corollary 3.3 *An even path P_n, $n \geq 8$ does not reach completeness with the application of the MDAC.*

Proof The proof is a direct consequence of the proof of Theorem 3.2. $\square$

A path P_2 i.e. v_1v_2 and path Q_3 i.e. $u_1u_2u_3$ with the added edge v_1u_1 yield the odd path P_5. It is easy to see that three iterations are needed to exhaust the MDAC. For path P_5 we have, $k = 3$, $\eta(P_5) = 4$.

Theorem 3.4 *For an odd path P_n, $n \geq 7$ the MDAC exhausts after $k = n - 5$ iterations and*

$$\eta(P_n)_{n,odd} = \frac{n^2 - 9n + 24}{2}.$$

Proof Let path $P_{2\ell+1}$ be constructed from paths P and Q on the vertices $v_1, v_2, v_3, \cdots, v_\ell$ and $u_1, u_2, u_3, \cdots, u_{\ell+1}$ by adding edge v_1u_1.

Case 1. From Lemma 3.1 it follows that, for P_7 add the degree affinity edge, v_3u_4 and consider the path $v_2v_3u_4u_3u_2$ equivalent to the *ghost edge* v_2u_2 to add those degree affinity edges to C_4. In the 2^{nd} iteration which exhausts the MDAC, add degree affinity edges, v_2u_2 and v_3u_3.

Case 2. From Lemma 3.1 it follows that for P_n, $n \geq 9$ add the degree affinity edges, $v_\ell u_{\ell+1}$ and consider the path $v_{\ell-1}v_\ell u_{\ell+1}u_\ell u_{\ell-1}$ equivalent to the *ghost edge* $v_{\ell-1}u_{\ell-1}$ to add those degree affinity edges to $C_{\ell-1}$. During the 2^{nd}-iteration include the degree affinity edge $v_\ell u_\ell$. For the exhaustive iteration add the edge $v_{\ell-1}u_{\ell-1}$.

In both cases the maximum number of degree affinity edges were obtained in that, the minimum number of vertices exhausted respectively at one vertex of degree 1 and two vertices of degree 3. The parameters $k = n - 5$ and

$$\eta(P_n)_{n,odd} = \frac{n^2 - 9n + 24}{2}$$

follow easily through similar reasoning found for even paths. □

§4. Certain General Results

Denote a null graph (edgeless) of order n by, $\mathfrak{N}_n$. Consider two null graphs i.e. $\mathfrak{N}_{n_1}$, $\mathfrak{N}_{n_2}$ on the vertices v_i,$i = 1, 2, 3, \cdots, n_1$ and u_j, $j = 1, 2, 3, \ldots, n_2$, respectively. Partial MDAC is performed by considering only distinct pairs of vertices $\{v_i, u_j\}$. This partial MDAC operation between the null graphs are denoted by, $\mathfrak{N}_{n_1} \uplus \mathfrak{N}_{n_2}$. The next lemma has a trivial proof.

Lemma 4.1 *For the null graphs, $\mathfrak{N}_{n_1}$, $\mathfrak{N}_{n_2}$ with $n_1 = n_2$, graph $\mathfrak{N}_{n_1} \uplus \mathfrak{N}_{n_2}$ is a complete bipartite graph.*

A direct consequence of Lemma 4.1 is given as a corollary.

Corollary 4.2 *A null graph $\mathfrak{N}_{n_1}$, n_1 is even, reaches completeness on exhaustion of the MDAC.*

It is obvious that a null graph of odd order will have an isolated vertex on exhaustion of the MDAC. The general result implied by Lemma 4.1 is stated below.

Theorem 4.3 *Consider two graphs G and H of equal order say, n, which reach completeness on exhaustion of the MDAC, respectively. Then the disjoint union $G \cup H$ reaches completeness on exhaustion of the MDAC.*

Proof Since both G and H reach completeness independently in compliance to MDAC the first phase is to apply the MDAC independently to G and H. Thereafter, apply the MDAC to the distinct pairs $\{v, u\}$, $v \in V(G)$, $u \in V(H)$ as required. Lemma 4.1 guarantees that completeness is reached. □

Example 4.1 Let G and H be two copies of $K_1 + C_6$. Label the two K_1-vertices as v_0 and u_0, respectively. During the 1^{st}-iteration of the MDAC add the appropriate edges to the $C_6^{'s}$ (see figure 2) as well as the edge v_0u_0. Thereafter, apply the MDAC independently until both copies reach completeness and apply the MDAC to the two distinct complete subgraphs, $K_6^{'s}$, finally.

In respect of the join of two graphs a straight forward result is presented.

Theorem 4.4 *For the join of any two graphs G and H it follows that,* $\eta(G+H) = \eta(G)+\eta(H)$.

Proof Let graphs G and H be of order n and m, respectively. In the join $G+H$ each vertex $v \in V(G)$ has $deg_{G+H}(v) = deg_G(v) + m$. Similarly, $\forall\ u \in V(H)$, $deg_{G+H}(u) = deg_H(u) + n$. Furthermore, the edge vu exists for all pairs $\{v, u\}$, $v \in V(G)$, $u \in V(H)$. Note that the join operation did not change the adjacency property within G and H *per se*. Therefore, the prescriptions of the MDAC applies to G and H as before. Hence, applying the MDAC to $G+H$ is equivalent to applying the MADC independently to G and H simultaneously. Therefore the result. □

Theorem 4.4 yields a useful corollary.

Corollary 4.5 *Let G of order n have a single vertex v with $deg_G(v) = n-1$ then,* $\eta(G) = \eta(G-v)$.

Proof Since $v \in V(G)$ is the only vertex with $deg_G(v) = n - 1 = \Delta(G)$ the graph can be denoted by, $K_{1_{(=v)}} + (G-v)$. By Theorem 4.4, $\eta(G) = \eta(K_1) + \eta(G-v) = 0 + \eta(G-v)$. □

Corollary 4.5 permits immediate results for specialized classes of graphs such as wheels, fans, stars, windmill graphs and others. In the next result Nordhaus-Gaddum type bounds are presented.

Proposition 4.6 *For any graph G of order n,*

$$\max\{\eta(G), \eta(\overline{G})\} \leq \eta(G) + \eta(\overline{G}) \leq \varepsilon(G),$$

$$0 \leq \eta(G) \cdot \eta(\overline{G}) \leq \left\lceil \frac{\varepsilon(K_n)}{2} \right\rceil \left\lfloor \frac{\varepsilon(K_n)}{2} \right\rfloor.$$

Proof The proof is respectively on the two inequalities following.

(i) If both G and $\overline{G}$ reach completeness on exhaustion of the MDAC then $\eta(G) + \eta(\overline{G}) = \varepsilon(G)$. Note that equally does not always hold as can be seen by say, $G = \mathfrak{N}_5$ since $\overline{\mathfrak{N}_5} = K_5$. Since $\eta(G) + \eta(\overline{G}) > \varepsilon(G)$ is impossible for simple graphs it follows that, $\eta(G) + \eta(\overline{G}) \leq \varepsilon(G)$. The lower bound is trivial. Hence, $max\{\eta(G), \eta(\overline{G})\} \leq \eta(G) + \eta(\overline{G}) \leq \varepsilon(G)$.

(ii) It is known that for any two non-negative integers a, b the product,

$$ab \leq \left\lceil \frac{a+b}{2} \right\rceil \left\lfloor \frac{a+b}{2} \right\rfloor.$$

Hence,

$$\eta(G) \cdot \eta(\overline{G}) \leq \left\lceil \frac{\varepsilon(K_n)}{2} \right\rceil \left\lfloor \frac{\varepsilon(K_n)}{2} \right\rfloor$$

is immediate. The lower bound is trivial. Therefore,

$$0 \leq \eta(G) \cdot \eta(\overline{G}) \leq \left\lceil \frac{\varepsilon(K_n)}{2} \right\rceil \left\lfloor \frac{\varepsilon(K_n)}{2} \right\rfloor . \qquad \Box$$

The number of vertices of graph G which has degree equal to $\delta(G)$ is denoted by, $\theta(G)$.

Theorem 4.7 *If an incomplete graph G or an iterative graph G_i has odd value $\theta(G)$ or, odd value $\theta(G_i)$ then G cannot reach completeness on exhaustion of the MDAC.*

Proof It is sufficient to proof the result for G only. Since G is incomplete it has order $n \geq 2$. Let G have odd value $\theta(G) \geq 3$. Since exactly one vertex say, u, $deg_G(u) = \delta(G)$ cannot be paired it follows that $deg_{G_1}(u) = deg_G(u)$ after the first iteration. Clearly, $\delta(G_1) = \delta(G)$, $\theta(G_1) = 1$ and $\delta(G_i) = \delta(G)$, $\theta(G_i) = 1$ for all G_i through to exhaustion of the MDAC. Therefore, the results holds. $\Box$

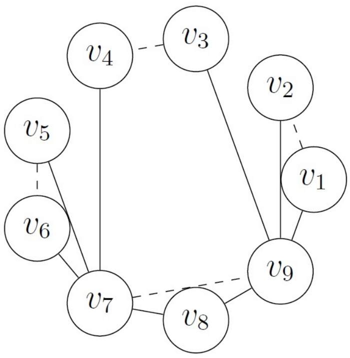

Figure 1 Graph G for which
$\delta(G) = 1$, $\theta(G) = 6$ and $\delta(G_1) = 2$, $\theta(G_1) = 7$.

Figure 3 illustrates the applicability of Theorem 4.7. Note that G is a tree on solid lines and G_1 has the dotted degree affinity edges added on 1^{st}-iteration.

§5. Conclusion

The numerous well-defined classes of graphs indicate that a wide scope for further research exists. It is conjectured that a salient dual problem has been solved without explicit mentioning or proof thereof. The claim is that the number of MDAC iterations k required to yield $\eta(G)$, is a minimum.

Problem 5.1 *Prove or disprove the claim that in applying the MDAC exhaustively, the number of iterations k to yield $\eta(G)$ is, $k = min\{k' : \eta(k') = \eta(G)\}$.*

Finding an efficient algorithm to ensure that during each iteration of the MDAC, the maximum (rather than maximal) number of degree affinity edges is added, is a worthy research problem.

Theorem 4.4 provides a result for the join of graphs. This prompts the next problems.

Problem 5.2 *Prove or disprove that, for the disjoint union* $G \cup H$ *of any two graphs* G, H *we have:* $\eta(G) + \eta(H) \leq \eta(G \cup H)$.

Problem 5.3 *Prove or disprove that, for the corona of any two graphs* G *and,* H *of order* m *we have:* $\eta(G) + m\eta(H) \leq \eta(G \circ H)$.

Theorem 4.7 as illustrated in Figure 3 suggests that it could be hard or impossible to characterize graphs which reach completeness on exhaustion of the MDAC.

Problem 5.4 *If possible, characterize graphs which reach completeness on exhaustion of the MDAC.*

The graph K_1 has the property that, $\eta(K_1) = 0 = \eta(\overline{K_1})$.

Problem 5.5 *Do graphs* $G \not\cong K_1$ *exist for which,* $\eta(G) = 0 = \eta(\overline{G})$*?*

Figures 1 and 2 suggest that there is a *flawed strategy* in application of the MDAC in that it may result in not yielding $\eta(G)$. Therefore, the problem could possibly be approached as a two person zero-sum game.

Problem 5.6 *If possible, analize the optimal application of the MDAC as a two person zero-sum game.*

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Famous Words

The world can be changed by man's endeavor, and that this endeavor can lead to something new and better. No man can sever the bonds that unite him to his society simply by averting his eyes. He must ever be receptive and sensitive to the new; and have sufficient courage and skill to face novel facts and to deal with them.

By Franklin Roosevelt, an American president

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