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On Helix in Minkowski 3-Space and its Retractions

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Abstract: Our aim in the present article is to introduce and study new types of retractions of helix $\mathbb{H}$ in Minkowski 3-space. The isometric and topological folding of $\mathbb{H}$ are achieved. The Frenet equations of the helix before and after retractions and foldings are achieved.

Key Words: Retraction, folding, Frenet equations, helix $\mathbb{H}$ in Minkowski 3-space.

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§1. Introduction

It is not uncommon that, the theory of retraction has always been one of the interesting topics in Euclidian and Non-Euclidian space and it has been investigated from the various viewpoints by many branches of topology and differential geometry [1-6].

Minkowski space is originally derived from the relativity in physics. In fact, a time like curve corresponds to the path of an observer moving at less than the speed of the light, a light-like curves correspond to moving at the speed of the light and a space like curves moving faster than light. The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by $g = dx_1^2 + dx_2^2 - dx_3^2$, where (x_1, x_2, x_3) is a rectangular coordinate system in E_1^3. Since g is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters, it can be space like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and light-like if $g(v, v) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can be locally space-like, time like or light-like, if all of its velocity vectors $\alpha'(s)$ are respectively, space-like, time like or light-like respectively. A curve in Lorentzian space L^n is a smooth map $\alpha : I \to L^n$ where I is the open interval in the real line $\mathbb{R}$. The interval I has a coordinate system consisting of the identity map u of I. Space-like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. The velocity of α at $t \in I$ is $\alpha' = \left.\frac{d\alpha(u)}{du}\right|_t$. Next, v, w in E_1^3 are said to be orthogonal if $g(v, w) = 0$. Vectors A curve α is said to be regular if $\alpha'(t)$ does not vanish for all in t in I, $\alpha \in L^n$ is space like if its velocity vectors α' are space like for all $t \in I$, similarly for time like and null. If α is a null curve, we can re-parameterize it such that $\langle \alpha'(t), \alpha'(t) \rangle = 0$ and $\alpha'(t) \neq 0$ [3-8].

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§2. **Preliminary Notes**

Let $\alpha(t)$ be a curve in the space-time in parameterized by arc length function s Lopez [9]. Then for the unit speed curve $\alpha(t)$ with non-null frame vectors, we distinguish three cases depending on the causal character of $T(t)$ and its Frenet equations are as follows

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ \mu_1\upsilon\kappa & 0 & \mu_2\upsilon\tau \\ 0 & \mu_3\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We write the following subcases.

Case 1. If $\alpha(s)$ is time-like curve in E_1^3, then T is time-like vector and T' is space-like vector. Then $\mu_i,\ 1 \leq i \leq 3$ read $\mu_1 = \mu_2 = 1, \mu_3 = -1$ and T, B and N are mutually orthogonal vectors satisfying the equations $g(N,N) = g(B,B) = 1,\ g(T,T) = -1$.

Case 2. If $\alpha(s)$ is space-like curve in E_1^3, then T is space-like vector, since $T'(s)$ is orthogonal to the space-like vector $T(s)$, $T'(s)$ is space-like, time-like or light-like. Thus we distinguish three cases according to $T'(s)$.

Subcase 2.1. If the vector $T'(s)$ is space-like, N is space-like vector and B is time-like vector. Then, $\mu_i,\ 1 \leq i \leq 3$ read $\mu_1 = -1,\ \mu_2 = \mu_3 = 1$, where T, N and B are mutually orthogonal vectors satisfying equations $g(T,T) = g(N,N) = 1$ and $g(B,B) = -1$.

Subcase 2.2 If the vector $T'(s)$ is time-like, N is time-like vector and B is space-like vector. Then, $\mu_i,\ 1 \leq i \leq 3$ read $\mu_1 = \mu_2 = \mu_3 = 1$, where T, N and B are mutually orthogonal vectors satisfying equations $g(T,T) = g(B,B) = 1$ and $g(N,N) = -1$.

Subcase 2.3 If the vector $T'(t)$ is light-like for all t, $N(t) = T'(t)$ is light-like vector and $B(s)$ is unique light-like vector with $g(N,B) = -1$ and it is orthogonal to T. The Frenet equations have

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \upsilon\tau & 0 \\ 1 & 0 & -\upsilon\tau \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Case 3. If $\alpha(s)$ is light-like curve in E_1^3 and $B(s)$ is unique light-like vector such that $g(T,B) = -1$ and it is orthogonal to N, the pseudo-torsion is $\tau = -\langle N', B\rangle$. Then the Frenet equations

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \upsilon\tau & 0 & 1 \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Let $\alpha = \alpha(t)$ be an arbitrary space-like curve in Minkowski space E_1^3. We have $\acute{\alpha} = \upsilon T$, $T = \frac{\acute{\alpha}}{\upsilon}$. From Frenet equations we have $\acute{T} = \upsilon\kappa N$. Then $\kappa = \frac{\|\acute{T}\|}{\upsilon}$ and $N = \frac{\acute{T}}{\upsilon\kappa} = \frac{\acute{T}}{\|\acute{T}\|}$. The vector product of $N \times T$, gives us $B = N \times T$. Using Frenet equations $\acute{N} = \upsilon\kappa T - \upsilon\tau B$. Thus

$\upsilon = \|\acute{\alpha}(t)\|$ to have the second curvature, and by inner product we obtain $\tau = g\left(-\frac{\acute{N}-\upsilon\kappa T}{\upsilon}, B\right)$.

A subset A of a topological space X is called retract of X if there exists a continuous map $r : X \to A$ called a retraction such that $r(a) = a$ for any $a \in A$ [2, 3, 6].

Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : I \to N$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ, if f does not preserve the length it is called topological folding [4, 5, 6].

Definition 2.1([9-13]) *We name that a helix is a curve where the tangent lines make a constant angle with a fixed direction. Helices are characterized by the fact that the ratio τ/κ is constant along the curve, where τ and κ denote the curvature and the torsion, respectively.*

Definition 2.2 *Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be vectors in E_1^3, the vector product in Minkowski space-time E_1^3 is defined by the determinant*

$$u \wedge v = \begin{vmatrix} e_1 & e_2 & -e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where, e_1, e_2 and e_3 are mutually orthogonal vectors (coordinate direction vectors), υ denotes speed of the curve.

§3. Main Results

Lemma 3.1 *Let $H = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Since $\left\langle \dot{H}, \dot{H} \right\rangle = a^2 - b^2$,*

$$\upsilon = |\dot{H}(t)| = \sqrt{|a^2 - b^2|}, \quad T = \left(\frac{-a\sin t}{\sqrt{|a^2-b^2|}}, \frac{a\cos t}{\sqrt{|a^2-b^2|}}, \frac{b}{\sqrt{|a^2-b^2|}}\right),$$

$$|a^2 - b^2| = \begin{cases} a^2 - b^2 & if\ a^2 > b^2, \\ b^2 - a^2 & if\ a^2 < b^2 \end{cases},$$

and $\langle T, T\rangle = 1$ if $a^2 > b^2$, then the helix H is space-like, and $\langle T, T\rangle = -1$ if $a^2 < b^2$, then the helix H is time-like and a null curve if $a^2 = b^2$.

Proof Consider three cases following.

Case 1. Let $H = \{(a\cos t, a\sin t, bt)\}$ be helix in E_1^3. If $a^2 > b^2$, then $H(t)$ is a space like curve. Since $T = \frac{\dot{H}}{\upsilon}$ then

$$T = \left(\frac{-a\sin t}{\sqrt{|a^2-b^2|}}, \frac{a\cos t}{\sqrt{|a^2-b^2|}}, \frac{b}{\sqrt{|a^2-b^2|}}\right),$$

and so $\langle T,T\rangle = 1$, $B = T\times N$,

$$B = \left\{\frac{1}{\sqrt{a^2-b^2}}(b\sin t, -b\cos t, -a)\right\}, \quad N = \{(-\cos t, -\sin t, 0)\}$$

with curvature $\kappa = \frac{a}{a^2-b^2}$ and torsion $\tau = \frac{-b}{a^2-b^2}$. Also,

$$\left\langle \dot{T}, \dot{T}\right\rangle = \frac{a^2}{a^2-b^2} > 0, \quad \left|\dot{T}\right| = \frac{a}{\sqrt{a^2-b^2}}$$

and $\dot{T}$ is a space-like vector and the Frenet equations in matrix notions are

$$\begin{aligned}\begin{pmatrix}\dot{T}\\ \dot{N}\\ \dot{B}\end{pmatrix} &= \begin{pmatrix}0 & \upsilon\kappa & 0\\ -\upsilon\kappa & 0 & \upsilon\tau\\ 0 & \upsilon\tau & 0\end{pmatrix}\begin{pmatrix}T\\ N\\ B\end{pmatrix}\\ &= \begin{pmatrix}0 & \frac{a}{\sqrt{a^2-b^2}} & 0\\ \frac{-a}{\sqrt{a^2-b^2}} & 0 & \frac{-b}{\sqrt{a^2-b^2}}\\ 0 & \frac{-b}{\sqrt{a^2-b^2}} & 0\end{pmatrix}\begin{pmatrix}\left(\frac{-a\sin t}{\sqrt{a^2-b^2}}, \frac{a\cos t}{\sqrt{a^2-b^2}}, \frac{b}{\sqrt{a^2-b^2}}\right)\\ (-\cos t, -\sin t, 0)\\ \frac{1}{\sqrt{a^2-b^2}}(b\sin t, -b\cos t, -a)\end{pmatrix}.\end{aligned}$$

Case 2. If $a^2 < b^2$, then the helix $H(t)$ is a time-like curve with $\langle T,T\rangle = -1$. Since

$$T = \left(\frac{-a\sin t}{\sqrt{|a^2-b^2|}}, \frac{a\cos t}{\sqrt{|a^2-b^2|}}, \frac{b}{\sqrt{|a^2-b^2|}}\right),$$

and so $N = \{(-\cos t, -\sin t, 0)\}$, $B = T\times N$, then $B = \frac{1}{\sqrt{a^2-b^2}}(b\sin t, -b\cos t, -a)$ with curvature $\kappa = \frac{a}{b^2-a^2}$ and torsion $\tau = \frac{b}{b^2-a^2}$. Also, $\left\langle \dot{T}, \dot{T}\right\rangle = \frac{a^2}{b^2-a^2} > 0$, and $\dot{T}$ is a space-like vector and the Frenet equations in matrix notions are

$$\begin{aligned}\begin{pmatrix}\dot{T}\\ \dot{N}\\ \dot{B}\end{pmatrix} &= \begin{pmatrix}0 & \upsilon\kappa & 0\\ \upsilon\kappa & 0 & \upsilon\tau\\ 0 & -\upsilon\tau & 0\end{pmatrix}\begin{pmatrix}T\\ N\\ B\end{pmatrix}\\ &= \begin{pmatrix}0 & \frac{a}{\sqrt{b^2-a^2}} & 0\\ \frac{a}{\sqrt{b^2-a^2}} & 0 & \frac{b}{\sqrt{b^2-a^2}}\\ 0 & \frac{-b}{\sqrt{b^2-a^2}} & 0\end{pmatrix}\begin{pmatrix}\left(\frac{-a\sin t}{\sqrt{b^2-a^2}}, \frac{a\cos t}{\sqrt{b^2-a^2}}, \frac{b}{\sqrt{b^2-a^2}}\right)\\ (-\cos t, -\sin t, 0)\\ \frac{1}{\sqrt{b^2-a^2}}(b\sin t, -b\cos t, -a)\end{pmatrix}.\end{aligned}$$

Case 3. The curve is light-like if $a^2 = b^2$ and we have $\upsilon = \sqrt{a^2-b^2} = 0$, then the tangent vector T undefined, and so to calculate the Frenet equation of $H(t)$ re-parameterization by the pseudo arc length s. Let $a = b = \frac{1}{c^2}$. Thus, the equation of the light-like helix is $H(s) = \left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$, where the curvature $\kappa = 1$, the tangent vector $T(s) = \left\{\frac{1}{c}(-\sin(cs), \cos(cs), 1)\right\}$, the normal vector $N(s) = T'(s) = \{(-\cos(cs), -\sin(cs), 0)\}$. The bi-normal vector $B(s)$ is define as follows, since $B(s)$ is a unique light-like vector then (1)

$\langle B, B\rangle = 0$. Also, (2) $g(T, B) = -1$, and since B is orthogonal to N then (3) $\langle B, N\rangle = 0$. From (1), (2) and (3), then $B(s) = \frac{c}{2}(\sin(cs), -\cos(cs), 1)$, $\tau = -\langle N', B\rangle = \frac{c^2}{2}$. Then, the Frenet equations of $H(s)$ in matrix notation are

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \tau & 0 & 1 \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ \frac{-c^2}{2} & 0 & 1 \\ 0 & \frac{-c^2}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{c}(-\sin(cs), \cos(cs), 1) \\ (-\cos(cs), -\sin(cs), 0) \\ \frac{c}{2}(\sin(cs), -\cos(cs), 1) \end{pmatrix}.$$

This completes the proof. □

Definition 3.2 *Assume that* $H(t) = \{(x_1(t), x_2(t), x_3(t), \cdots, x_n(t))\}$ $t \in \mathrm{domain}H(t)$ *is any non-null curve in Minkowski n-space* E_1^n. *Then the retraction map of* $H(t)$ *called projection retraction with* $n-1$ *dimension defined as* $r_i(H) : \{(x_1(t), x_2(t), x_3(t), \cdots, x_n(t))\} \to (x_1(t), x_2(t), x_3(t), \cdots, x_n(t)) - \{(x_i)\}$ *with* $\{(x_i)\} = \{(l_1x_1(t), l_2x_2(t), \cdots, l_nx_n(t))\}$, *where* $l_n = 1$ *when* $n = i$ *and* $l_n = 0$ *when* $n \neq i$, $i, n \in \mathbb{N}$, $t \in \mathrm{Domain}H$.

Theorem 3.3 *Let* $P(t)$ *be a non-pseudo null space-like curve in* E_1^3 *with curvature* $\kappa = 1$ *and constant torsion* τ, *the projection retraction* $r_i(P) = (x_1(t), x_2(t), x_3(t), \cdots, x_n(t)) - \{(x_i)\}$, $i \in \{1, 2, 3\}$, $s \in \mathrm{Domain}r_i(P)$, *then the Frenet apparatus of* $r_i(P)$ *can be formed by Frenet apparatus of* $P(t)$.

Proof Let $P(t) = \{(a\cos t, a\sin t, bt)\}$ be a non-pseudo null space-like curve P in E_1^3 with constant curvature. For the curve $r_1(P(t)) = (0, a\sin t, bt)$, $\upsilon_r = \sqrt{a^2\cos^2(t) - b^2}$. So

$$r_1(P(t)) = (a\cos t, a\sin t, bt).$$

Differentiating this equation with respect to t, the tangent vector

$$T_r(t) = \frac{\upsilon}{\upsilon_r}T(t) - \left(\frac{-a\sin t}{\upsilon_r}, 0, 0\right),$$

where, $T = \frac{\dot{r}}{\upsilon}, T_r = \frac{\dot{r}_1}{\upsilon_r}$, $\dot{T}_r = \dot{T}(t) - (-a\cos t, 0, 0)$. Then, $\kappa = \frac{\|\dot{T}\|}{\upsilon}$, $N = \frac{\dot{T}}{\upsilon\kappa} = \frac{\dot{T}}{\|\dot{T}\|}$, the normal vector of $r_1(P)$,

$$N_r(t) = \left(\frac{\upsilon}{\upsilon_r}\right)^2 \frac{\kappa}{\kappa_r}N(t) - \frac{1}{\kappa_r\upsilon_r^2}(-a\cos t, 0, 0) - \dot{\upsilon}_r T(t).$$

And the bi-normal vector of $r_1(P)$ is

$$B_r(s) = \frac{\tau}{\kappa}T_r(s) - \frac{1}{\kappa}N_r'(s) = \frac{\tau}{\kappa}P'(s) - \frac{1}{\kappa}P'''(s).$$

If $r_1(P)$ is a null curve and $B_r(t) = T_r \wedge N_r$, if $r_1(P)$ is a non-null curve. Similarly, we have the same proof for $r_2(P(t))$ and $r_3(P(t))$. $\square$

Theorem 3.4 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Then the Frenet equations of the y-z retraction projection $r_1(H) = \{(0, a\sin t, bt)\}$ and the x-z retraction projection $r_2(H) = \{(a\cos t, 0, bt)\}$ can be formed by the Frenet equations of $H(t)$.*

Proof Let $H = \{(a\cos t, a\sin t, bt)\}$ and $r_1(H) = \{(0, a\sin t, bt)\}$ be y-z retraction projection of $H(t)$ with constant curvature $\kappa \neq 0$ and $\tau = 0$ then the ratio $\tau/\kappa = 0$ for any projection plan of $r_i(H)$. Hence, $r_1(H)$ is not a helix, $\dot{r}_1(H) = (0, a\cos t, b)$ and $\langle \dot{r}_1, \dot{r}_1\rangle = a^2\cos^2(t) - b^2$ for all $t \in$ domain$r_1(H)$, and we have three cases following.

Case 1. If $a^2\cos^2(t) - b^2 > 0$, the retraction $r_1(H)$ is a space-like curve with time like vector N_r,

$$\upsilon_r = \sqrt{a^2\cos^2 t - b^2}, \quad \kappa_r = \frac{ab\sin t}{(a^2\cos^2(t) - b^2)^{\frac{3}{2}}}.$$

From Lemma 3.1 and Theorem 3.3 we get

$$\begin{aligned} T_r &= \left(0, \frac{a\cos t}{\sqrt{a^2\cos^2(t) - b^2}}, \frac{b}{\sqrt{a^2\cos^2(t) - b^2}}\right), \\ N_r &= \left(0, \frac{b}{\sqrt{a^2\cos^2(t) - b^2}}, \frac{a\cos t}{\sqrt{a^2\cos^2(t) - b^2}}\right) \end{aligned}$$

and the torsion is $\tau_r = 0$, $\kappa_r > 0$, $B_r = (1, 0, 0)$. Then, the Frenet equations of the retraction $r_1(H)$ in matrix notation are

$$\begin{aligned} \begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} &= \begin{pmatrix} 0 & \upsilon\kappa_r & 0 \\ \upsilon\kappa_r & 0 & \upsilon\tau \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{ab\sin t}{a^2\cos^2(t)-b^2} & 0 \\ \frac{ab\sin t}{a^2\cos^2(t)-b^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \left(0, \frac{a\cos t}{\sqrt{a^2\cos^2(t)-b^2}}, \frac{b}{\sqrt{a^2\cos^2(t)-b^2}}\right) \\ (0, \frac{b}{\sqrt{a^2\cos^2(t)-b^2}}, \frac{a\cos t}{\sqrt{a^2\cos^2(t)-b^2}}) \\ (1, 0, 0) \end{pmatrix}. \end{aligned}$$

If $a^2\cos^2(t) - b^2 < 0$, the retraction $r_1(H)$ is space-like curve with time like vector N_r, $\upsilon_r = \sqrt{b^2 - a^2\cos^2 t}$, $\kappa_r = \frac{ab\sin t}{(b^2 - a^2\cos^2(t))^{\frac{3}{2}}}$. From Lemma 3.1 and Theorem 3.3 we get

$$\begin{aligned} T_r &= \left(0, \frac{a\cos t}{\sqrt{b^2 - a^2\cos^2(t)}}, \frac{b}{\sqrt{b^2 - a^2\cos^2(t)}}\right), \\ N_r &= \left(0, \frac{-b}{\sqrt{b^2 - a^2\cos^2(t)}}, \frac{-a\cos t}{\sqrt{b^2 - a^2\cos^2(t)}}\right) \end{aligned}$$

and the torsion is $\tau_r = 0,\ \kappa_r > 0,\ B_r = (1,0,0)$. Then, the Frenet equations of the retraction $r_1(H)$ in matrix notation are

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa_r & 0 \\ \upsilon\kappa_r & 0 & \upsilon\tau \\ 0 & -\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{ab\sin t}{b^2-a^2\cos^2(t)} & 0 \\ \frac{ab\sin t}{b^2-a^2\cos^2(t)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \left(0, \frac{a\cos t}{\sqrt{b^2-a^2\cos^2(t)}}, \frac{b}{\sqrt{b^2-a^2\cos^2(t)}}\right) \\ (0, \frac{-b}{\sqrt{b^2-a^2\cos^2(t)}}, \frac{-a\cos t}{\sqrt{b^2-a^2\cos^2(t)}}) \\ (1,0,0) \end{pmatrix}.$$

If $a^2\cos^2(t) - b^2 = 0$, then the retraction $r_1(H) = \{(0, a\sin t, bt)\}$ is a light like curve and the Frenet equations of the retraction $r_1(H)$ cannot appoints.

Case 2. The helix $H = \{(a\cos t, a\sin t, bt)\}$ and $r_2(H) = \{(a\cos t, 0, bt)\}$ is x-z retraction projection of the curve $H(t)$, and $\langle \dot{r}, \dot{r}\rangle = a^2\sin^2(t) - b^2$ for all $t \in \text{domain} r_2(H)$. WE have three cases should be discussed.

If $a^2\sin^2(t) - b^2 > 0$, then the retraction $r_2(H)$ is space-like, $\upsilon = |\dot{r}| = \sqrt{a^2\sin^2(t) - b^2}$,

$$T = \frac{\dot{r}_2}{\upsilon} = \left(\frac{-a\sin t}{\sqrt{a^2\sin^2(t) - b^2}}, 0, \frac{b}{\sqrt{a^2\sin^2(t) - b^2}}\right)$$

and

$$\left\langle \dot{T}, \dot{T} \right\rangle = \frac{-a^2b^2\cos^2(t)}{\left(a^2\sin^2(t) - b^2\right)^2} < 0, \quad \left|\dot{T}\right| = \frac{ab\cos t}{a^2\sin^2(t) - b^2},$$

then $\dot{T}$ is time-like, $\langle T, T\rangle = 1$. So, T is space-like with curvature

$$\kappa = \frac{ab\cos t}{\left(a^2\sin^2(t) - b^2\right)^{\frac{3}{2}}}, \quad N = \left(\frac{b}{\sqrt{a^2\sin^2(t) - b^2}}, 0, \frac{-a\sin t}{\sqrt{a^2\sin^2(t) - b^2}}\right)$$

and $B = (0,-1,0)$, $\langle B, B\rangle = 1 > 0$, which is positively oriented, and the torsion $\tau = 0$. Then, the Frenet equations of $r_2(H)$ in matrix notation are

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa_r & 0 \\ \upsilon\kappa & 0 & \upsilon\tau \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{ab\cos t}{a^2\sin^2(t)-b^2} & 0 \\ \frac{ab\cos t}{a^2\sin^2(t)-b^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{a\sin t}{\sqrt{a^2\sin^2(t)-b^2}}, 0, \frac{b}{\sqrt{a^2\sin^2(t)-b^2}}\right) \\ (\frac{b}{\sqrt{a^2\sin^2(t)-b^2}}, 0, \frac{-a\sin t}{\sqrt{a^2\sin^2(t)-b^2}}) \\ (0,-1,0) \end{pmatrix}.$$

If $a^2\sin^2(t)-b^2<0$, then the retraction $r_2(H)$ is a time-like curve. Since $T=\frac{\dot{r}_2}{\upsilon}$, so

$$T=\left(\frac{-a\sin t}{\sqrt{b^2-a^2\sin^2(t)}},0,\frac{b}{\sqrt{b^2-a^2\sin^2(t)}}\right),\langle T,T\rangle=-1,\upsilon=|\dot{r}_2|=\sqrt{b^2-a^2\sin^2(t)}$$

and

$$\left\langle\dot{T},\dot{T}\right\rangle=\frac{a^2b^2\cos^2(t)}{\left(b^2-a^2\sin^2(t)\right)^2}>0,\quad\left|\dot{T}\right|=\frac{ab\cos t}{b^2-a^2\sin^2(t)}.$$

Hence, $\dot{T}$ is space-like, $\langle T,T\rangle=1$, T is time-like with curvature

$$\kappa=\frac{ab\cos t}{\left(b^2-a^2\sin^2(t)\right)^{\frac{3}{2}}},\quad N=\left(\frac{-b}{\sqrt{b^2-a^2\sin^2(t)}},0,\frac{a\sin t}{\sqrt{b^2-a^2\sin^2(t)}}\right)$$

and $B=(0,-1,0)$, $\langle B,B\rangle=1>0$, which is positively oriented, so the torsion $\tau=0$. Then, the Frenet equations of $r_2(H)$ in matrix notation are

$$\begin{aligned}\begin{pmatrix}\dot{T}\\ \dot{N}\\ \dot{B}\end{pmatrix}&=\begin{pmatrix}0&\upsilon\kappa_r&0\\ \upsilon\kappa&0&\upsilon\tau\\ 0&-\upsilon\tau&0\end{pmatrix}\begin{pmatrix}T\\ N\\ B\end{pmatrix}\\ &=\begin{pmatrix}0&\frac{ab\cos t}{b^2-a^2\sin^2(t)}&0\\ \frac{ab\cos t}{b^2-a^2\sin^2(t)}&0&0\\ 0&0&0\end{pmatrix}\begin{pmatrix}\left(\frac{-a\sin t}{\sqrt{b^2-a^2\sin^2(t)}},0,\frac{b}{\sqrt{b^2-a^2\sin^2(t)}}\right)\\ (\frac{-b}{\sqrt{b^2-a^2\sin^2(t)}},0,\frac{a\sin t}{\sqrt{b^2-a^2\sin^2(t)}})\\ (0,-1,0)\end{pmatrix}.\end{aligned}$$

If $a^2\sin^2(t)-b^2=0$, then the retraction $r_2(H)$ is a light-like curve and the tangent vector T is undefined, and then the Frenet equations cannot be appointed, see Figure 1 for details.

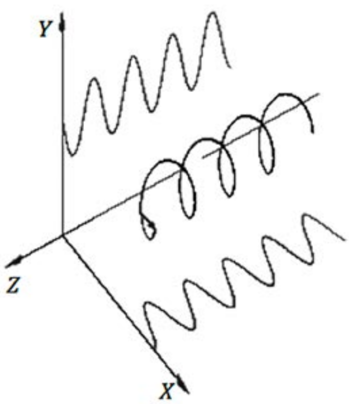

Figure 1

This completes the proof. □

Theorem 3.5 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Then the Frenet equations are variants under the x-y retraction projection $r_3(H) = \{(a\cos t, a\sin t, 0)\}$,*

Proof Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ and let $r_3(H) = \{(a\cos t, a\sin t, 0)\}$ be x-y retraction projection of the curve $H(t)$. Clearly, $\langle \dot{r}_3, \dot{r}_3\rangle = a^2 > 0$ for all $t \in \text{domain} r_3(H)$, and $|\dot{r}_3| = a$,

$$T = \frac{\dot{r}_3}{\upsilon} = (-\sin t, \cos t, 0),$$

where $\upsilon = a$, $\langle T, T\rangle = 1$. Then, $r_3(H)$ is a space-like curve. And $\left\langle \dot{T}, \dot{T}\right\rangle = 1$, i.e., $|\dot{T}| = 1$. Thus $\dot{T}$ is space-like with curvature $\kappa = \frac{\|\dot{T}\|}{\upsilon} = \frac{1}{a} > 0$ and $\tau = 0$, $N = (-\cos t, -\sin t, 0)$, $B = (0, 0, -1)$. Then, the Frenet equations in matrix notation are

$$\begin{aligned}\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} &= \begin{pmatrix} 0 & \upsilon\kappa_r & 0 \\ -\upsilon\kappa & 0 & \upsilon\tau \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (-\sin t, \cos t, 0) \\ (-\cos t, -\sin t, 0) \\ (0, 0, -1) \end{pmatrix}.\end{aligned}$$

This completes the proof. □

Corollary 3.6 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Then, the Frenet equations can not be appointed under the retraction projection $r_{1,2}(H) = \{((l_1)a\cos t, (l_2)a\sin t, (l_3)bt)\} = \{(0, 0, bt)\}$, $i_1 = l_2 = 0$, $l_3 = 1$, and $r_{1,2}(H)$ is a straight line $z = bt$.*

Proof Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3 and let $r_{1,2}(H) = \{(0, 0, bt)\}$ be a retraction of the curve $H(t)$ and $\langle \dot{r}_{1,2}, \dot{r}_{1,2}\rangle = -b^2 < 0$ for all $t \in \text{domain} H$, then $r_{1,2}(H)$ is a time-like curve and $|\dot{r}_{1,2}| = b$, $T = \frac{\dot{r}_{1,2}}{\upsilon} = (0, 0, 1)$, where $\upsilon = a$ and $\left\langle \dot{T}, \dot{T}\right\rangle = 0, |\dot{T}| = 0$. Thus, $\dot{T}$ is light-like and T is time-like, the normal and bi-normal vectors N, B are undefined. And the Frenet equations cannot be appointed. The ratio τ/κ is undefined. Since $\kappa = \tau = 0$ then $r_{1,2}(H)$ is a straight line. □

Definition 3.7 *Let $P(t), P(t) \subset E_1^3$ be any curve in Minkowski 3-space $E_1^3, t \in \mathbb{R}$, we have an arbitrary point, $t_0 \in \text{Domain} P \ni I = (\delta - t_0, \delta + t_0)$ with $\delta > 0$, $I \subset \mathbb{R}$ and $\delta \in \mathbb{R}$, then there exists related retractions for every interval I define as $r_I(P(t)) \subset P(t) \subset E_1^3$.*

Theorem 3.8 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a helix in E_1^3. Then, the Frenet equations of the retraction $r_i(H) = H(f(t))$ are different from the Frenet equations of $H(t)$.*

Proof Notice that $r_i(H) = \{(a\cos f(t), a\sin f(t), bf(t))\}$. Since $\langle \dot{r}, \dot{r}\rangle = \dot{f}^2\left(a^2 - b^2\right)$, $\dot{f}^2 > 0$ for all t, the retraction $r_i(H)$ of the helix $H(t)$ is a space-like curve when $a^2 - b^2 > 0$, a time-like curve if $a^2 - b^2 < 0$ and a null-like curve if $a^2 = b^2$. And since the ratio $\tau/\kappa = c$ is a constant for all t, then the retraction $r_i(H)$ is a helix. We have three cases should be discussed.

Case 1. The curve $r_i(H)$ is a space-like retraction helix if $a^2 > b^2$, and we have

$$\langle \dot{r}, \dot{r} \rangle = \dot{f}^2(a^2 - b^2),\ \upsilon = \dot{f}\sqrt{a^2 - b^2},\ T_r = \left(\frac{-a\sin f}{\sqrt{a^2 - b^2}}, \frac{a\cos f}{\sqrt{a^2 - b^2}}, \frac{b}{\sqrt{a^2 - b^2}} \right)$$

and $\langle T_r, T_r \rangle = 1$. Hence, $r_i(H)$ is space-like, $N_r = \{(-\cos f, -\sin f, 0)\}$, $B = T \times N$ and

$$B_r = \left\{ \frac{1}{\sqrt{a^2 - b^2}}(b\sin f, -b\cos f, -a) \right\}$$

with curvature $\kappa_r = \frac{a}{a^2 - b^2}$ and torsion $\tau_r = \frac{-b}{a^2 - b^2}$. Also,

$$\left\langle \dot{T}_r, \dot{T}_r \right\rangle = \frac{\dot{f}^2 a^2}{a^2 - b^2} > 0,\quad |\dot{T}| = \frac{\dot{f}a}{\sqrt{a^2 - b^2}}$$

and $\dot{T}_r$ is a space-like vector, and the Frenet equations of $r_i(H)$ in matrix notation are

$$\begin{aligned}
\begin{pmatrix} \dot{T}_r \\ \dot{N}_r \\ \dot{B}_r \end{pmatrix} &= \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ -\upsilon\kappa & 0 & \upsilon\tau \\ 0 & \upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T_r \\ N_r \\ B_r \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{a\dot{f}}{\sqrt{a^2-b^2}} & 0 \\ \frac{-a\dot{f}}{\sqrt{a^2-b^2}} & 0 & \frac{-b\dot{f}}{\sqrt{a^2-b^2}} \\ 0 & \frac{-b\dot{f}}{\sqrt{a^2-b^2}} & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{-a\sin f}{\sqrt{a^2-b^2}}, \frac{a\cos f}{\sqrt{a^2-b^2}}, \frac{b}{\sqrt{a^2-b^2}} \right) \\ (-\cos f, -\sin f, 0) \\ \frac{1}{\sqrt{a^2-b^2}}(b\sin f, -b\cos f, -a) \end{pmatrix}.
\end{aligned}$$

Case 2. The curve $r_i(H)$ is a time-like retraction helix if $a^2 < b^2$, and then we have

$$|\dot{r}(H)| = \dot{f}^2(|a^2 - b^2|),\ \upsilon = \dot{f}\sqrt{b^2 - a^2},\ T_r = \left(\frac{-a\sin f}{\sqrt{b^2 - a^2}}, \frac{a\cos f}{\sqrt{b^2 - a^2}}, \frac{b}{\sqrt{b^2 - a^2}} \right)$$

and $\langle T_r, T_r \rangle = -1$.

So, $N_r = \{(-\cos f, -\sin f, 0)\}$, $B = T \times N$, $B_r = \left\{ \frac{1}{\sqrt{b^2-a^2}}(b\sin f, -b\cos f, -a) \right\}$, the basis $\{T, N, B\}$ is positive oriented because $\langle B, B \rangle = 1$, the curvature is $\kappa_r = \frac{a}{b^2-a^2}$ and the torsion is $\tau_r = \frac{b}{b^2-a^2}$. Also, $\left\langle \dot{T}_r, \dot{T}_r \right\rangle = \frac{\dot{f}^2 a^2}{b^2-a^2} > 0$, and $\dot{T}_r$ is a space-like vector. So, the Frenet equations of $r_i(H)$ in matrix notation are

$$\begin{aligned}
\begin{pmatrix} \dot{T}_r \\ \dot{N}_r \\ \dot{B}_r \end{pmatrix} &= \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ \upsilon\kappa & 0 & \upsilon\tau \\ 0 & -\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} T_r \\ N_r \\ B_r \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{a\dot{f}}{\sqrt{b^2-a^2}} & 0 \\ \frac{a\dot{f}}{\sqrt{b^2-a^2}} & 0 & \frac{b\dot{f}}{\sqrt{b^2-a^2}} \\ 0 & \frac{-b\dot{f}}{\sqrt{b^2-a^2}} & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{-a\sin f}{\sqrt{b^2-a^2}}, \frac{a\cos f}{\sqrt{b^2-a^2}}, \frac{b}{\sqrt{b^2-a^2}} \right) \\ (-\cos f, -\sin f, 0) \\ \frac{1}{\sqrt{b^2-a^2}}(b\sin f, -b\cos f, -a) \end{pmatrix}.
\end{aligned}$$

This completes the proof. $\square$

Theorem 3.9 *Let $H(t) = \{(a\cos t, a\sin t, bt)\}$ be a null helix in E_1^3. Then, the Frenet equations of the retraction $r_i(H) = H(f(t))$ are different from the Frenet equations of $H(t)$.*

Proof Let $r_i(H)$ be a light-like retraction helix of $H(t)$, re-parameterization by the pseudo arc length s. So, $a = b = \frac{1}{c^2}$, $f(t) = cf(s)$ and the equation of a light-like helix is $H(s) = \left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$, where $r_i(H) = \left\{\frac{1}{c^2}(\cos(cf(s)), \sin(cf(s)), cf(s))\right\}$ (I) with curvature $\kappa_r = 1$, the tangent vector

$$T_r(s) = \left\{\frac{1}{c}(f'(-\sin(cf(s))), \cos(cf(s)), 1)\right\}, \quad \langle T_r, T_r\rangle = 0.$$

Then $r_i(H)$ is a light-like helix and the norma vector is

$$N_r(s) = T_r'(s) = \left\{\left(-\frac{f''}{c}\sin(cf) - f'^2\cos(cf), \frac{f''}{c}\cos(cf) - f'^2, \frac{f''}{c}\right)\right\},$$

the bi-normal vector $B_r(s)$ is defined as follows:

Notice that $B(s)$ is a unique light-like vector then (1) $\langle B, B\rangle = 0$; Also, (2) $g(T, B) = 1$ and since B is orthogonal to N, then (3) $\langle B, N\rangle = 0$. From (1),(2) and (3), the bi-normal vector is

$$B_r(s) = \{(J, K, L)\}$$

$$\begin{aligned} J &= \frac{f''}{f'^3}(\tan^2(cf)\sin(cf) - \sec(cf)) + \left(\frac{c}{2f'} - \frac{f''}{2cf'^5}\right)\sin(cf), \\ K &= \frac{f''}{2cf'^5}\cos(cf) - \frac{f''}{f'^3}\tan^2(cf)\cos(cf) - \frac{c}{2f'}\cos(cf), \\ L &= \frac{f''}{2f'^3}(\tan(cf) - \tan^2(cf)) + \frac{f''}{2cf'^5} + \frac{c}{2f'} \end{aligned}$$

with torsion

$$\tau_r = -\langle N_r', B_r\rangle = \frac{f'''}{f'} + \frac{f''}{2f'} - \frac{3f''^2}{f'^2} + (\tan(cf) - \tan^2(cf))cf'' - \frac{c^2}{2}f'^2$$

This completes the proof. □

Corollary 3.10 *Let $H(s) = \left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$ be a null helix in E_1^3. Then, the Frenet equations of the retraction $r_i(H) = H(f(s))$ can be formed by the Frenet equations of $H(s)$ if $f''(s) = 0$.*

Proof Let $H(s) = \left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$ be a null helix in E_1^3, where

$$r_i(H) = \left\{\frac{1}{c^2}(\cos(cf(s)), \sin(cf(s)), cf(s))\right\}$$

with curvature $\kappa_r = 1$. If $f''(s) = 0$ then $f'(s) = c_1$ and $f(s) = c_1 s + c_2$ be a linear function, where $c_1 = 1$ and $c_2 < 0$ are constants. By substituting there conditions in the equations of

$T_r(s), N_r(s)$ and $B_r(s)$ in Theorem 3.9 and applying the value of $T(s), N(s)$ and $B(s)$ following

$$\begin{cases} T(s) = \left\{\frac{1}{c}(-\sin(cs), \cos(cs), 1)\right\} \\ N(s) = (-\cos(cs), -\sin(cs), 0) \\ B(s) = \frac{c}{2}\{(\sin(cs), -\cos(cs), 1)\} \end{cases}$$

Then ,we have $r_i(H) = \left\{\frac{1}{c^2}(\cos(cs + c_1)), \sin(cs + c_1, cs + c_1)\right\}$. The tangent vector of $r_i(H)$ is $T_r(s) = \lambda_1 T + \lambda_2 N + \lambda$, where $\lambda = (0, 0, 1 - \lambda_1)$ and λ_1, λ_2 are constants. Then the normal vector of $r_i(H)$ is $N_r(s) = \nu_1 N + \nu_2 B + \nu$, where $\nu = (0, 0, -\nu_2)$ and ν_1, ν_2 are constants, the bi-normal vector of $r_i(H)$ is $B_r(s) = \eta_1 B - \eta_2 N + \eta$, where $\eta = (0, 0, 1 - \eta_1)$ and η_1, η_2 are constants and the curve $r_i(H)$ has torsion $\tau_r = \frac{-c^2}{2} f'^2 = f'^2 \tau$, curvature $\kappa_r = \kappa = 1$. Then, the Frenet equations of the retraction $r_i(H) = H(f(s))$ can be formed by the Frenet equations of $H(s)$ if $f''(s) = 0$ and so

$$\begin{cases} T_r(s) = \lambda_1 T(s) + \lambda_2 N(s) + \lambda \\ N_r(s) = \nu_N(s) + \nu_2 B(s) + \nu \\ B_r(s) = \eta_1 B(s) - \eta_2 N(s) + \eta \end{cases}$$

This completes the proof. □

Now, we introduce types of conditional foldings of the helix $H = \{(a\cos t, a\sin t, bt)\}$ in E_1^3. Clearly, $H' = \{(a\sin t, a\cos t, b)\}$. Define

$$\Psi : \{(a\cos t, a\sin t, bt)\} \to \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}, \quad m > 1.$$

Theorem 3.11 *The Frenet equations of the non-null helix $H = \{(a\cos t, a\sin t, bt)\}$ in E_1^3 are invariant under the folding $\Psi(H) = \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}$ for integers $m > 1$.*

Proof Let $\Psi(H) = \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}$ be a folding of the helix $H = \{(a\cos t, a\sin t, bt)\}$ and $\left\langle \dot{\Psi}, \dot{\Psi} \right\rangle = \frac{1}{m^2}\left(a^2 - b^2\right)$. Since $\frac{1}{m^2} > 0$, this folding is a space-like curve if $a^2 > b^2$, a time-like curve if $a^2 < b^2$ and a null curve if $a^2 = b^2$. Since $H(t)$ is a helix then the ratio $\tau/\kappa = c$ is a constant. So the folding $\Psi(t)$ has the ratio $\tau_f/\kappa_f = \tau/\kappa = -\frac{b}{a}$, is also a constant and then $\Psi(t)$ is a helix. We have three cases should be discussed.

Case 1. The folded helix is space-like if $a^2 > b^2$. So, $|\Psi(t)| = v = \sqrt{\left(\frac{a}{m}\right)^2 - \left(\frac{b}{m}\right)^2}$ and

$$\begin{aligned} T_\Psi &= \left(\frac{-a\sin t}{\sqrt{a^2 - b^2}}, \frac{a\cos t}{\sqrt{a^2 - b^2}}, \frac{b}{\sqrt{a^2 - b^2}}\right), \quad \langle T, T \rangle = 1, \\ B_\Psi &= T_\Psi \times N_\Psi = \left\{\frac{1}{\sqrt{a^2 - b^2}}(b\sin t, -b\cos t, -a)\right\}, \\ N_\Psi &= \{(-\cos t, -\sin t, 0)\} \end{aligned}$$

with curvature $\kappa_\Psi = \frac{ma}{a^2 - b^2}$ and torsion $\tau_\Psi = \frac{-mb}{a^2 - b^2}$. Also, $\left\langle \dot{T}_\Psi, \dot{T}_\Psi \right\rangle = \frac{a^2}{a^2 - b^2} > 0$, $|\dot{T}_\Psi| =$

$\frac{a}{\sqrt{a^2-b^2}}$. Thus, $\dot{T}_\Psi$ is a space-like vector, and the Frenet equations in matrix notation are

$$\begin{aligned}\begin{pmatrix}\dot{T}\\ \dot{N}\\ \dot{B}\end{pmatrix} &= \begin{pmatrix}0 & \upsilon\kappa & 0\\ -\upsilon\kappa & 0 & \upsilon\tau\\ 0 & \upsilon\tau & 0\end{pmatrix}\begin{pmatrix}T\\ N\\ B\end{pmatrix}\\ &= \begin{pmatrix}0 & \frac{a}{\sqrt{a^2-b^2}} & 0\\ \frac{-a}{\sqrt{a^2-b^2}} & 0 & \frac{-b}{\sqrt{a^2-b^2}}\\ 0 & \frac{-b}{\sqrt{a^2-b^2}} & 0\end{pmatrix}\begin{pmatrix}\left(\frac{-a\sin t}{\sqrt{a^2-b^2}}, \frac{a\cos t}{\sqrt{a^2-b^2}}, \frac{b}{\sqrt{a^2-b^2}}\right)\\ (-\cos t, -\sin t, 0)\\ \frac{1}{\sqrt{a^2-b^2}}(b\sin t, -b\cos t, -a)\end{pmatrix}.\end{aligned}$$

Case 2. The folded helix is time-like if $a^2 < b^2$. Then, we have $|\dot{H}(t)| = \sqrt{|a^2-b^2|}$, where, $\upsilon = \sqrt{\left(\frac{b}{m}\right)^2 - \left(\frac{a}{m}\right)^2}$ and

$$\begin{aligned}T_\Psi &= \left(\frac{-a\sin t}{\sqrt{b^2-a^2}}, \frac{a\cos t}{\sqrt{b^2-a^2}}, \frac{b}{\sqrt{b^2-a^2}}\right),\\ N_\Psi &= \{(-\cos t, -\sin t, 0)\}.\end{aligned}$$

So, $\langle T, T\rangle = -1$ and

$$B_\Psi = T_\Psi \times N_\Psi = \left\{\frac{1}{\sqrt{b^2-a^2}}(b\sin t, -b\cos t, -a)\right\},$$

the basis $\{T, N, B\}$ is positive oriented because $\langle B, B\rangle = 1$, $\dot{T} = \left(\frac{-a\cos t}{\sqrt{b^2-a^2}}, \frac{-a\sin t}{\sqrt{b^2-a^2}}, 0\right)$. The curvature $\kappa_\Psi = \frac{ma}{b^2-a^2}$ and the torsion $\tau_\Psi = \frac{-mb}{b^2-a^2}$. Also, $\left\langle \dot{T}_\Psi, \dot{T}_\Psi\right\rangle = \frac{a^2}{b^2-a^2} > 0$. So, $\dot{T}_\Psi$ is a space-like vector, and the Frenet equations in matrix notation are

$$\begin{aligned}\begin{pmatrix}\dot{T}\\ \dot{N}\\ \dot{B}\end{pmatrix} &= \begin{pmatrix}0 & \upsilon\kappa & 0\\ \upsilon\kappa & 0 & \upsilon\tau\\ 0 & -\upsilon\tau & 0\end{pmatrix}\begin{pmatrix}T\\ N\\ B\end{pmatrix}\\ &= \begin{pmatrix}0 & \frac{a}{\sqrt{b^2-a^2}} & 0\\ \frac{a}{\sqrt{b^2-a^2}} & 0 & \frac{b}{\sqrt{b^2-a^2}}\\ 0 & \frac{-b}{\sqrt{b^2-a^2}} & 0\end{pmatrix}\begin{pmatrix}\left(\frac{-a\sin t}{\sqrt{b^2-a^2}}, \frac{a\cos t}{\sqrt{b^2-a^2}}, \frac{b}{\sqrt{b^2-a^2}}\right)\\ (-\cos t, -\sin t, 0)\\ \frac{1}{\sqrt{b^2-a^2}}(b\sin t, -b\cos t, -a)\end{pmatrix}.\end{aligned}$$

This completes the proof. □

Theorem 3.12 *Let $H = \{(a\cos t, a\sin t, bt)\}$ be a null helix in E_1^3, $\Psi(H) = \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}$ a folding of $H(t)$ for integers $m > 1$. Then, the Frenet equation of $\Psi(t)$ can be formed by the Frenet equation of $H(t)$.*

Proof Let $\Psi(t) = \left\{\frac{a}{m}\cos t, \frac{a}{m}\sin t, \frac{b}{m}t\right\}$ be a folding of the null helix $H = \{(a\cos t, a\sin t, bt)\}$, $\upsilon = 0$. Then, T is undefined, re-parameterization $H(t)$ by the pseduo arc length s. Let $a = b = \frac{1}{c^2}, t = cs$. Thus the equation of the light-like helix is $H(s) = \left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$,

$\Psi(s) = \frac{1}{m}\left\{\frac{1}{c^2}(\cos(cs), \sin(cs), cs)\right\}$ with curvature $\kappa = 1$. The tangent vector is $T_\Psi(s) = \dot{\Psi}(s) = \frac{1}{m}\left\{\frac{1}{c}(-\sin(cs), \cos(cs), 1)\right\}$, the normal vector is $N_\Psi(s) = T'_\Psi(s) = \frac{1}{m}\left\{(-\cos(cs), \sin(cs), 0)\right\}$ with the the bi-normal vector defined as follows.

Since $B_\Psi(s)$ is a unique light-like vector, we know that (1) $\langle B, b\rangle = 0$. Also, (2) $g(T_\Psi, B_\Psi) = -1$. Notice that B_Ψ is orthogonal to N_Ψ, there must be (3) $\langle B_\Psi, N_\Psi\rangle = 0$. From (1), (2) and (3),

$$B_\Psi(s) = \frac{mc}{2}(\sin(cs), -\cos(cs), 1)$$

and $\tau_\Psi = -\langle N'_\Psi, B_\Psi\rangle = \frac{c^2}{2}$. So, the Frenet equations of folded curve $\Psi(H)$ in matrix notation are

$$\begin{aligned}\begin{pmatrix} T'_\Psi \\ N'_\Psi \\ B'_\Psi \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ \tau_\Psi & 0 & 1 \\ 0 & \tau_\Psi & 0 \end{pmatrix}\begin{pmatrix} T_\Psi \\ N_\Psi \\ B_\Psi \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{-c^2}{2} & 0 & 1 \\ 0 & \frac{-c^2}{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{mc}(-\sin(cs), \cos(cs), 1) \\ \frac{1}{m}(-\cos(cs), -\sin(cs), 0) \\ \frac{mc}{2}(\sin(cs), -\cos(cs), 1) \end{pmatrix},\end{aligned}$$

where, $\kappa_\Psi = \kappa = 1$, $\tau_\Psi = \tau = \frac{-c^2}{2}$ and

$$\begin{cases} T_\Psi = \frac{1}{m}T \\ N_\Psi = \frac{1}{m}N \\ B_\Psi = mB \end{cases}$$

This completes the proof. □

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