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Arctangent Finsler Spaces With Reversible Geodesics

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Abstract: A Finsler space is said to have reversible geodesics if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. In this work, we study a class of special Finsler metrics F called arctangent Finsler metric, which is a special (α, β)-metric, $F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta$ ($\epsilon \neq 0$ are constant), where α is a Riemannian metric and β is a 1-form. The conditions for an arctangent Finsler space (M, F) to be with reversible geodesic are obtained. Further, we study some geometrical properties of F and prove that the arctangent metric F induces a generalized weighted quasi-distance d_F on M.

Key Words: Reversible Geodesics, weighted quasi metric.

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§1. Introduction

An interesting topic in Finsler geometry is to study the reversible geodesics of a Finsler metric. Recall that, a Finsler space is said to have reversible geodesics if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. In the last decade many interesting and applicable results have been obtained on the theory of Finsler spaces with reversible geodesics. In [7], Crampin gives necessary and sufficient conditions for a Finsler metric (M, F) to be with reversible and strictly reversible geodesics, respectively.

Reversible geodesic of (α, β)-metric and two dimensional Finsler spaces with (α, β)-metric were studied by Masca, Sabau and Shimada ([8],[9]). In [6], Sabau and Shimada have given some important results on reversible geodesics. In [10], Shanker and Baby have exhaust reversible geodesics for generalized (α, β)-metric.

§2. Preliminaries

Let $F^n = (M, F)$ be a connected n-dimensional Finsler manifold and let $TM = \cup_{x\in M} T_x M$ denotes the tangent bundle of M with local coordinates $u = (X, Y) = (x^i, y^i) \in TM$, where $i = 1, \cdots, n$, $y = y^i \frac{\partial}{\partial x^i}$.

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If $\gamma : [0,1] \to M$ is a piecewise C^∞ curve on M, then its Finslerian length is defined as

$$L_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))dt \tag{2.1}$$

and the Finslerian distance function $d_F : M \times M \to [0, \infty)$ is defined by $d_F(p,q) = inf_\gamma L$, where infimum is taken over all piecewise C^∞ curves γ om M joining the points $p, q \in M$. In general, this is not symmetric.

A curve $\gamma : [0,1] \to M$ is called a geodesic of (M, F) if it minimizes the Finslerian length for all piecewise C^∞ curves that keep their endpoints fixed. We denote the reverse Finsler metric of F as $\tilde{F} : TM \to (0, \infty)$, given by $\tilde{F}(x,y) = F(x, -y)$. One can easily see that $\tilde{F}$ is also a Finsler metric.

Lemma 2.1 *A Finsler metric is with a reversible geodesic if and only if for any geodesic $\gamma(t)$ of F, the reverse curve $\tilde{\gamma}(t) = \gamma(1-t)$ is also a geodesic of F.*

Lemma 2.2 *Let (M, F) be a connected, complete Finsler manifold with associated distance function $dF : M \times M \to [0, \infty)$. Then, d_F is a symmetric distance function on $M \times M$ if and only if F is a reversible Finsler metric, i.e., $F(x,y) = F(x,-y)$.*

Lemma 2.3 *A smooth curve $\gamma : [0,1] \to M$ is a constant Finslerian speed geodesic of (M, F) if and only if it satisfies $\ddot{\gamma} + 2G^i(\gamma(t), \dot{\gamma}(t)) = 0$, $i = 1, ..., n$, where the functions $G^i : TM \to \boldsymbol{R}$ given by*

$$G^i(x,y) = \Gamma^i_{jk}(x,y) y^i y^j \tag{2.2}$$

with

$$\Gamma^i_{jk}(x,y) = \frac{g^{is}}{2}\left(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s}\right).$$

Remark 2.4 It is well known [3] that the vector field

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$$

is a vector field on TM, whose integral lines are the canonical lifts $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ of the geodesics of γ. This vector field Γ is called the canonical geodesics spray of the Finsler space (M, F) and G^i are called the coefficients of the geodesics spray Γ.

Definition 2.5 *If F and $\tilde{F}$ are two different fundamental Finsler functions on the same manifold M, then they are said to be projectively equivalent if their geodesics coincide as set points.*

Lemma 2.6 *A Finsler structure (M, F) is with a reversible geodesic if and only if F and its reverse function $\tilde{F}$ are projectively equivalent.*

The main purpose of the current paper is to determine the reversible geodesics for special

Finsler metrics F called arctangent Finsler metric, which is a special (α, β)-metric,

$$F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta \ (\epsilon \neq 0 \text{are constant}),$$

where α is a Riemannian metric and β is a 1-form. The paper is organized as follows:

Starting with preliminary definitions on reversible geodesics in section two, in section three, we obtain the conditions for an arctangent Finsler space to be with reversible geodesics (see Theorem 3.1). in section four, we prove that if F is projectively flat then it is with reversible geodesics (see Theorem 4.2). In section five, we study metric structures associated to F and prove that the arctangent metric F induces a generalized weighted quasi-distance function d_F on the manifold M (see Theorem 5.2).

§3. Reversible Geodesics of Arctangent Finsler Metric

Consider a Finsler space (M, F) with a special (α, β)-metric

$$F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta. \ (\epsilon \neq 0 \text{ are constant}),$$

where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form.

The necessary and sufficient condition for F to have a reversible geodesic is [6]

$$\tilde{\Gamma}\frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i} = 0, \tag{3.1}$$

where $\tilde{\Gamma}$ is the reverse of Γ, the geodesic spray of F, moreover $\tilde{\Gamma}$ is geodesic spray for F. The necessary and sufficient condition for F to have strictly reversible geodesics is $\tilde{\Gamma}F = 0$. Now $F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta$. Then $\tilde{F} = \alpha + \beta \arctan(\frac{\beta}{\alpha}) - \epsilon\beta$, so $F = \tilde{F} + 2\epsilon\beta$.

We have

$$\begin{aligned}\tilde{\Gamma}F_{y^i} - F_{x^i} =& \arctan(\frac{\beta}{\alpha})(\tilde{\Gamma}\beta_{y^i} - \beta_{x^i}) + \frac{\beta}{\alpha}\Big(\frac{1}{1+(\frac{\beta}{\alpha})^2}\Big)(\tilde{\Gamma}\beta_{y^i} - \beta_{x^i}) + \epsilon(\tilde{\Gamma}\beta_{y^i} - \beta_{x^i})\\ =&\Big(\arctan(\frac{\beta}{\alpha}) + \frac{\beta}{\alpha}\Big(\frac{1}{1+(\frac{\beta}{\alpha})^2}\Big) + \epsilon\Big)(\tilde{\Gamma}\beta_{y^i} - \beta_{x^i}),\end{aligned}$$

Notice that for the spray $\tilde{\Gamma}$, we have

$$\tilde{\Gamma}\beta_{y^i} - \beta_{x^i} = (\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i})y^j,$$

So we get

$$\tilde{\Gamma}F_{y^i} - F_{x^i} = \Big(\arctan(\frac{\beta}{\alpha}) + \frac{\beta}{\alpha}\Big(\frac{1}{1+(\frac{\beta}{\alpha})^2}\Big) + \epsilon\Big)(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i})y^j. \tag{3.2}$$

Now,

$$\left\{ \arctan(\frac{\beta}{\alpha}) + \frac{\beta}{\alpha}\Big(\frac{1}{1+(\frac{\beta}{\alpha})^2}\Big) + \epsilon \right\}$$

can not be zero. Therefore, from equation (3.1) and (3.2) we conclude that F is with reversible geodesics if and only if

$$\left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right) y^j = 0,$$

i.e., $\tilde{F}$ is with reversible geodesic if and only if β is closed 1-form. Hence, we have the following theorem

Theorem 3.1 *Let (M, F) be an arctangent Finsler space with F defined by the Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and 1-form $\beta = b_iy^i$. Then F is with a reversible geodesic if and only if β is a closed 1-form on M.*

§4. Projective Flatness of Arctangent Finsler metric

A Finsler space (M, F) is called (locally) projectively flat if all its geodesics are straight lines [4]. An equivalent condition is that the spray coefficients G^i of F can be expressed as $G^i = P(x, y)y^i$, where $P(x, y) = \frac{1}{2F}\frac{\partial F}{\partial x^k}y^k$. An equivalent characterization of projective flatness is the Hamels relation [5].

$$\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 0.$$

Proposition 4.1 *Let (M, F) be an arctangent space with F defined by the Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and 1-form $\beta = b_iy^i$. Then F is projectively flat if and only if $\tilde{F}$ is projectively flat.*

Outline of the Proof Recall (see Theorem 3.1 of [6]) that if $F = F_0 + \epsilon\beta$ is a Finsler metric, where F_0 is an absolute homogeneous (α, β)-metric, then any two of the following properties imply the third one:

(a) F is projectively flat;

(b) F_0 is projectively flat;

(c) β is closed.

In our case

$$F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta = \tilde{F} + 2\epsilon\beta,$$

where

$$\tilde{F} = \alpha + \beta \arctan(\frac{\beta}{\alpha}) - \epsilon\beta$$

which is absolute homogeneous.

Proof of Proposition 4.1 Let (M, F) be projectively flat, then by Hamels relation for projective flatness, we have

$$\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 0.$$

Thus we have

$$\begin{aligned}
&\frac{\partial^2 (\tilde{F} + 2\epsilon\beta)}{\partial x^m \partial y^k} y^m - \frac{\partial (\tilde{F} + 2\epsilon\beta)}{\partial x^k} = 0, \\
&\frac{\partial^2 \tilde{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \tilde{F}}{\partial x^k} + 2\epsilon\Big(\frac{\partial^2 \beta}{\partial x^m \partial y^k} y^m - \frac{\partial \beta}{\partial x^k}\Big) = 0, \\
&\frac{\partial^2 \tilde{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \tilde{F}}{\partial x^k} = -2\epsilon\Big(\frac{\partial^2 \beta}{\partial x^m \partial y^k} y^m - \frac{\partial \beta}{\partial x^k}\Big).
\end{aligned}$$

Since β is closed, Then we have

$$\frac{\partial^2 \tilde{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \tilde{F}}{\partial x^k} = 0$$

Therefore $\tilde{F}$ is projectively flat.

Conversely, suppose that $\tilde{F}$ is projectively flat. Since $\tilde{F}$ is projectively flat, therefore $\tilde{F}$ will satisfy Hamels equation

$$\frac{\partial^2 \tilde{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \tilde{F}}{\partial x^k} = 0,$$

So we have

$$\begin{aligned}
&\frac{\partial^2 (F - 2\epsilon\beta)}{\partial x^m \partial y^k} y^m - \frac{\partial (F - 2\epsilon\beta)}{\partial x^k} = 0, \\
&\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} - 2\epsilon\Big(\frac{\partial^2 \beta}{\partial x^m \partial y^k} y^m - \frac{\partial \beta}{\partial x^k}\Big) = 0, \\
&\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 2\epsilon\Big(\frac{\partial^2 \beta}{\partial x^m \partial y^k} y^m - \frac{\partial \beta}{\partial x^k}\Big).
\end{aligned}$$

Since β is closed, Then we get:

$$\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 0$$

Therefore F is projectively flat. □

Theorem 4.2 *Let (M, F) be be an arctangent space with F defined by the Riemannian metric $\alpha = \sqrt{a_{ij} y^i y^j}$ and 1-form $\beta = b_i y^i$. If F is projectively flat, then it is with a reversible geodesic.*

Proof By Hamels relation, we can see that F is projectively flat if and only if $\tilde{F}$ is projectively flat. This implies that F and $\tilde{F}$ both are projectively equivalent to the standard Euclidean metric and therefore F must be projective to $\tilde{F}$. Thus F must be with a reversible geodesic. □

§5. Weighted Quasi Metric Associated with Arctangent Finsler Metric

It is well known that the Riemannian spaces can be represented as metric spaces. Indeed, for a Riemannian space (M,α), one can define the induced metric space (M,d_α) with the metric

$$d_\alpha : M\times M\to[0,\infty)\ \ d_\alpha(x,y)=\inf_{\gamma\in\Gamma_{xy}}\int_a^b \alpha(\gamma(t),\dot\gamma(t))dt, \tag{5.1}$$

where $\Gamma_{xy}=\{\gamma:[a,b]\to M|\gamma$ is piecewise, $\gamma(a)=x,\ \gamma(b)=y\}$ is the set of curves joining x and y, $\dot\gamma(t)$ is the tangent vector to γ at $\gamma(t)$. Then d_α is a metric on M satisfying the following conditions:

(1) Positiveness: $d_\alpha(x,y)>0$ if $x\neq y,\ d_\alpha(x,x)=0,\ x,y\in X$;

(2) Symmetry: $d_\alpha(x,y)=d_\alpha(y,x),\ \ \forall x,y\in M$;

(3) Triangle inequality: $d_\alpha(x,y)\leq d_\alpha(x,z)+d_\alpha(z,y),\ \ \forall x,y,z\in M$.

Similar to the Riemannian space, one can induce the metric d_F to a Finsler space (M,F), given by

$$d_F : M\times M\to[0,\infty)\ \ d_F(x,y)=\inf_{\gamma\in\Gamma_{xy}}\int_a^b F(\gamma(t),\dot\gamma(t))dt, \tag{5.2}$$

but unlike the Riemannian case, here d_F lacks the symmetric condition. In fact, d_F is a special case of quasi metric defined below.

Definition 5.1([1]) *A quasi metric d on a set X is a function $d : X\times X\to[0,\infty)$ that satisfies the following axioms:*

(1) *Positiveness:* $d(x,y)>0$ *if* $x\neq y$, $d(x,x)=0,\ x,y\in X$*;*

(2) *Triangle inequality:* $d(x,y)\leq d(x,z)+d(z,y),\ \forall x,y,z\in X$*;*

(3) *Separation axiom:* $d(x,y)=d(y,x)=0\ \Rightarrow\ \ x=y,\forall x,y\in X$.

One special class of quasi metric spaces are the so called weighted quasi metric spaces (M,d,w), where d is a quasi-metric on M and for each d, there exists a function $w : M\to[0,\infty)$, called the weight of d, that satisfies

(4) *Weightability:* $d(x,y)+w(x)=d(y,x)+w(y),\ \forall x,y\in M$.

In this case, the weight function w is R-valued, and is called generalized weight.

Theorem 5.2 *Let M be an n-dimensional simply connected smooth manifold. Arctangent Finsler metric $F=\alpha+\beta\arctan(\frac{\beta}{\alpha})+\epsilon\beta$ ($\epsilon\neq0$ are constant) induces a generalized weighted quasi-distance d_F on M.*

Proof Consider an arctangent Finsler space (M,F) with F defined by the Riemannian metric $\alpha=\sqrt{a_{ij}y^iy^j}$ and 1-form $\beta=b_iy^i$. ($\epsilon\neq0$ are constant) Let $\gamma_{xy}\in\Gamma_{xy}$ be a Finslerian geodesic, then from (5.2), we have

$$\begin{aligned} d_F(x,y) &= \int_a^b F(\gamma(t),\dot\gamma(t))dt \\ &= \int_a^b (\alpha+\beta\arctan(\frac{\beta}{\alpha})+\epsilon\beta)dt \\ &= \int_a^b (\alpha+\beta\arctan(\frac{\beta}{\alpha}))dt+\epsilon\int_{\gamma_{xy}}\beta \\ &= \int_{\gamma_{xy}} \alpha+\beta\arctan(\frac{\beta}{\alpha})+\epsilon\int_{\gamma_{xy}}\beta. \end{aligned} \tag{5.3}$$

Let us consider a fixed point $a \in M$ and define the function $w_a : M \to R, w_a(x) = d_F(a,x) - d_F(x,a)$. From (5.3) it follows that

$$\begin{aligned} w_a(x) &= d_F(a,x) - d_F(x,a) \\ &= \int_{\gamma_{ax}} \alpha+\beta\arctan(\frac{\beta}{\alpha})+\epsilon\int_{\gamma_{ax}}\beta \\ &\quad - \int_{\gamma_{xa}} \alpha+\beta\arctan(\frac{\beta}{\alpha})-\epsilon\int_{\gamma_{xa}}\beta \\ &= \epsilon\int_{\gamma_{ax}}\beta-\epsilon\int_{\gamma_{xa}}\beta \\ &= -2\int_{\gamma_{xa}}\beta, \end{aligned} \tag{5.4}$$

where we have used the Stokes theorem for the 1-form β on the closed domain D with boundary $\partial D = \gamma_{ax} \cup \gamma_{xa}$. One can see that w_a is the anti derivative of β. This is well defined if and only if the path integral in right hand side of (5.4) is path independent, that is, β must be exact. Then d_F is a weighted quasi-metric with generalized weight w_a. Indeed, we have

$$\begin{aligned} d_F(x,y)+w_a(x) &= \int_{\gamma_{xy}} (\alpha+\beta\arctan(\frac{\beta}{\alpha}))+\epsilon\int_{\gamma_{xy}}\beta+\epsilon\int_{\gamma_{ax}}\beta-\epsilon\int_{\gamma_{xa}}\beta \\ &= \int_{\gamma_{xy}} (\alpha+\beta\arctan(\frac{\beta}{\alpha}))-\epsilon\int_{\gamma_{xa}}\beta-\epsilon\int_{\gamma_{ya}}\beta, \end{aligned} \tag{5.5}$$

where we have again used the Stokes theorem for the 1-form β on the closed domain with boundary $\gamma_{ax} \cup \gamma_{xy} \cup \gamma_{ya}$.

Similarly,

$$d_F(y,x)+w_a(y) = \int_{\gamma_{yx}} (\alpha+\beta\arctan(\frac{\beta}{\alpha}))-\int_{\gamma_{ya}}\beta-\int_{\gamma_{xa}}\beta. \tag{5.6}$$

From equations (5.5) and (5.6) we conclude that d_F is weighted quasimetric with generalized weight w_a. This completes the proof. □

Next, recall the following result.

Lemma 5.3([1],[2]) *Let (M, d) be any quasi-metric space. Then d is weightable if and only if there exists $w : M \to [0, \infty)$ such that*

$$d(x, y) = \rho(x, y) + \frac{1}{2}[w(x) - w(y)], \quad \forall x, y \in M, \tag{5.7}$$

where ρ is the symmetrized distance function of d. Moreover, we have

$$\frac{1}{2}[w(x) - w(y)] \leq \rho(x, y), \quad \forall x, y \in M. \tag{5.8}$$

The proof is trivial from the definition of weighted quasi-metric.

Remark 5.4 If (M, F) is an arctangent Finsler space with $F = \alpha + \beta \arctan(\frac{\beta}{\alpha}) + \epsilon\beta$, ($\epsilon \neq 0$ are constant). then the induced quasi-metric d_F and the symmetrized metric ρ induce the same topology on M. This follows immediately from ([3],[4]).

Remark 5.5 From Lemma 5.3, it can be seen that the assumption of w to be smooth is not essential.

Next, we discuss an interesting geometric property concerning the geodesic triangles.

Proposition 5.6 *Let (M, F) be an arctangent space with F defined by the Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and 1-form $\beta = b_iy^i$. ($\epsilon \neq 0$ are constant). Then the perimeter length of any geodesic triangle on M does not depend on the orientation, that is,*

$$d_F(x, y) + d_F(y, z) + d_F(z, x) = d_F(x, z) + d_F(z, y) + d_F(y, x), \ \forall x, y, z \in M. \tag{5.9}$$

Proof From Theorem 3.1, it follows that the quasi-metric is weightable and therefore (5.7) holds good. By using this formula an elementary computation proves (5.9). □

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