International J.Math. Combin. Vol.1(2019), 88-102

Z_k-Magic Labeling of Cycle of Graphs

P.Jeyanthi

(Department of Mathematics, Govindammal Aditanar College for Women, Tiruchendur 628215, Tamilnadu, India)

K.Jeya Daisy

(Department of Mathematics, Holy Cross College, Nagercoil, Tamilnadu, India)

E-mail: jeyajeyanthi@rediffmail.com, jeyadaisy@yahoo.com

Abstract: For any non-trivial Abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An *A-magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k and these graphs are referred as *k-magic* graphs. In this paper we prove that the graphs such as cycle of generalized peterson, shell, wheel, closed helm, double wheel, triangular ladder, flower and lotus inside a circle are Z_k-magic graphs and also prove that if G is Z_k-magic graph and n is even then $C(n.G)$ is Z_k-magic.

Key Words: A-magic labeling, Z_k-magic labeling, Z_k-magic graph, cycle of graphs, Smarandachely A-magic.

AMS(2010): 05C78.

§1. Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to vertices or edges or both subject to certain conditions. A detailed survey was done by Gallian in [6]. If the labels of edges are distinct positive integers and for each vertex v the sum of the labels of all edges incident with v is the same for every vertex v in the given graph then the labeling is called a magic labeling. Sedláček [8] introduced the concept of A-magic graphs. A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all vertices. Low and Lee [7] examined the A-magic property of the resulting graph obtained from the product of two A-magic graphs. Shiu, Lam and Sun [9] proved that the product and composition of A-magic graphs were also A-magic.

For any non-trivial Abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. Otherwise, it is said to be *Smarandachely A-magic*, i.e., $|\{f^+(v), v \in V(G)\}| \geq 2$. An *$A$-magic* graph G is said to

[1]Received August 8, 2018, Accepted March 2, 2019.

be Z_k-magic graph if the group A is Z_k, the group of integers modulo k. These Z_k-magic graphs are referred to as *k-magic graphs*. Shiu and Low [10] determined all positive integers k for which fans and wheels have a Z_k-magic labeling with a magic constant 0. Motivated by the concept of A-magic graph in [8] and the results in [7], [9] and [10] Jeyanthi and Jeya Daisy [1]-[5] proved that some standard graphs admit Z_k-magic labeling. Let G be a graph with n vertices $\{u_1, u_2, \ldots u_n\}$ and consider n copies of G as $G_1, G_2, \ldots G_n$ with vertex set $V(G_i) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq n\}$. The cycle of graph G is denoted by $C(n.G)$ is obtained by identifying the vertex u_1^j of G_j with u_i of G for $1 \leq i \leq n, 1 \leq j \leq n$. In this paper we study the Z_k-magic labeling of some cycle of graphs and also prove that if G is Z_k-magic graph and n is even then $C(n.G)$ is Z_k-magic. We use the following definitions in the subsequent section.

Definition 1.1 *A generalized peterson graph $P(n, m)$, $n \geq 3, 1 \leq m < \frac{n}{2}$ is a 3 regular graph with $2n$ vertices $\{u_1, u_2, \cdots u_n, v_1, v_2 \ldots v_n\}$ and edges $(u_i v_i), (u_i u_{i+1}), (v_i v_{i+m})$ for all $1 \leq i \leq n$, where the subscripts are taken modulo n.*

Definition 1.2 *A shell S_n is the graph obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the apex.*

Definition 1.3 *The wheel W_n is obtained by joining the vertices $v_1, v_2, \cdots v_n$ of a cycle C_n to an extra vertex v called the centre.*

Definition 1.4 *The closed helm CH_n is the graph obtained from a helm H_n by joining each pendent vertex to form a cycle.*

Definition 1.5 *A double wheel graph DW_n of size n can be composed of $2C_n + K_1$, that is it consists of two cycles of size n, where the vertices of the two cycles are all connected to a common hub.*

Definition 1.6 *The triangular ladder graph TL_n, $n \geq 2$ is obtained by completing the ladder $P_2 \times P_n$ by adding the edges $v_{1,j} v_{2,j+1}$ for $1 \leq j \leq n$. The vertex set of the ladder is $\{v_{1,j}, v_{2,j} : 1 \leq j \leq n\}$.*

Definition 1.7 *The flower Fl_n is the graph obtained from a helm H_n by joining each pendent vertex to the central vertex of the helm.*

Definition 1.8 *A lotus inside a circle LC_n is a graph obtained from the cycle $C_n : u_1, u_2, \ldots u_n, u_1$ and a star $K_{1,n}$ with the central vertex v_0 and the end vertices $v_1, v_2, \cdots, v_n$ by joining each u_i and $u_{i+1}(mod\ n)$.*

§2. Main Results

Theorem 2.1 *Let G be a Z_k-magic graph with magic constant b then $C(n.G)$ is Z_k-magic if n is even.*

Proof For any integer $b \in Z_k$. Let $v_1, v_2, \cdots v_n$ be the vertices of the cycle C_n. Let G be any Z_k-magic graph with magic constant b. Therefore $f^+(v) \equiv b (mod\ k)$ for all $v \in V(G)$.

For any integer $a \in Z_k - \{0\}$, define the edge labeling $g : E(C(n.G)) \to Z_k - \{0\}$ as follows:

$$g(v_i v_{i+1}) = \begin{cases} a & \text{for } i \text{ is odd}, \\ k-a & \text{for } i \text{ is even}, \end{cases}$$

and $g(e) = f(e)$ for other $e \in E(C(n.G))$. Then the induced vertex labeling $g^+ : V(C(n.G)) \to Z_k$ is $g^+(v) \equiv b(mod\ k)$ for all $v \in V(C(n.G))$. Hence g^+ is constant and it is equal to $b(mod\ k)$. Notice that $C(n.G)$ admits Z_k-magic labeling when n is even, then it is therefore a Z_k-magic graph. □

Theorem 2.2 *Let G be a Z_k-magic graph with magic constant b then $C(n.G)$ is Z_k-magic if k is even.*

Proof For any integer $b \in Z_k$. Let $v_1, v_2, \cdots v_n$ be the vertices of the cycle C_n. Let G be any Z_k-magic graph with magic constant b. Therefore $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(G)$.

For any integer $a \in Z_k - \{0\}$, define the edge labeling $g : E(C(n.G)) \to Z_k - \{0\}$ to be $g(v_i v_{i+1}) = \frac{k}{2}$ for $1 \le i \le n-1$, $g(v_n v_1) = \frac{k}{2}$, $g(e) = f(e)$ for other $e \in E(C(n.G))$.

Then the induced vertex labeling $g^+ : V(C(n.G)) \to Z_k$ is $g^+(v) \equiv b(mod\ k)$ for all $v \in V(C(n.G))$. Hence g^+ is constant and it is equal to $b(mod\ k)$. Since $C(n.G)$ admits Z_k-magic labeling when k is even, then it is a Z_k-magic graph. □

Theorem 2.3 *The graph $C(n.C_r)$ is Z_k-magic except r is even, n is odd and k is odd.*

Proof Let the vertex set and the edge set of $C(n.C_r)$ be $V(C(n.C_r)) = \{v_i^j : 1 \le i \le r, 1 \le j \le n\}$ and $E(C(n.C_r)) = \{v_i^j v_{i+1}^j : 1 \le i \le r-1, i \le j \le n\} \bigcup \{v_r^j v_1^j : 1 \le j \le n\} \bigcup \{v_1^j v_1^{j+1} : 1 \le j \le n-1\} \bigcup \{v_1^n v_1^1\}$.

Case 1. r is odd.

For any integer $a \in Z_k - \{0\}$, define the edge labeling $f : E(C(n.C_r)) \to Z_k - \{0\}$ as follows:

$$f(v_i^j v_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd}, 1 \le j \le n, \\ a & \text{for } i \text{ is even}, 1 \le j \le n, \end{cases}$$

$f(v_1^j v_1^{j+1}) = a$ for $1 \le j \le n-1$,

$f(v_1^n v_1^1) = a$.

Then the induced vertex labeling $f^+ : V(C(n.C_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.C_r))$.

Case 2. r is even.

Subcase 2.1 n is even.

The cycle C_r is Z_k-magic with magic constant zero when r is even. Therefore by theorem 2.1 it is Z_k-magic.

Subcase 2.2 n is odd and k is even.

By Theorem 2.2 it is Z_k-magic.

Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.C_r)$ admits Z_k-magic labeling, then it is a Z_k-magic graph. □

The example for Z_{15}-magic labeling of $C(5.C_7)$ is shown in Figure 1.

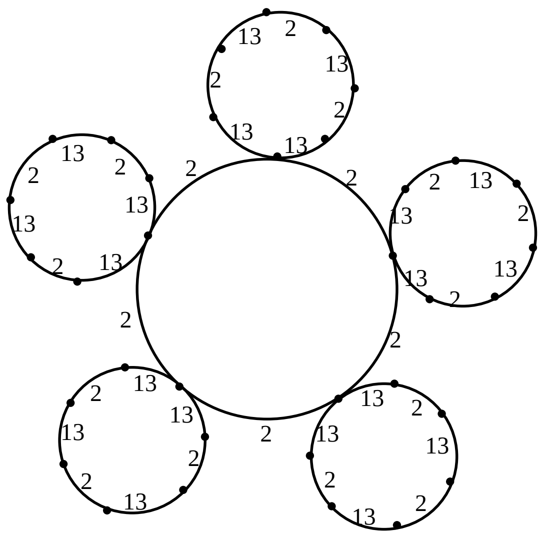

Figure 1 Z_{15}-magic labeling of $C(5.C_7)$

Conjecture 2.4 *The graph $C(n.C_r)$ is not Z_k-magic when r is even, n is odd and k is odd.*

Observation 2.1 The graph $C(n.C_{n_1}, C_{n_2} \cdots, C_{n_l})$ is Z_k-magic when $n_1, n_2, \cdots n_l$ are odd.

Theorem 2.5 *The cycle of generalized peterson graph $C(n.P(r,m))$ is Z_k-magic except r is even, n is odd and k is odd.*

Proof Let the vertex set and the edge set of $C(n.P(r,m))$ be respectively $V(C(n.P(r,m))) = \{u_i^j, v_i^j\ :\ 1 \le i \le r,\ 1 \le j \le n\}$ and $E(C(n.P(r,m))) = \{u_i^j v_i^j\ :\ 1 \le i \le r,\ 1 \le j \le n\} \bigcup \{u_i^j u_{i+1}^j :\ 1 \le i \le r-1,\ 1 \le j \le n\} \bigcup \{u_r^j u_1^j :\ 1 \le j \le n\} \bigcup \{v_i^j v_{i+m}^j :\ 1 \le i \le r,\ 1 \le j \le n\}$ where the subscripts are taken modulo r.

Case 1. r is odd.

For any integer a such that $k > 3a$, define the edge labeling $f : E(C(n.P(r,m))) \to Z_k - \{0\}$ as follows:

$$f(v_i^j v_{i+m}^j) = a \text{ for } 1 \le i \le r,\ 1 \le j \le n,$$
$$f(u_i^j v_i^j) = k - 2a \text{ for } 1 \le i \le r,\ 1 \le j \le n,$$
$$f(u_i^j u_{i+1}^j) = \begin{cases} 3a & \text{for } i \text{ is odd, } 1 \le j \le n, \\ k-a & \text{for } i \text{ is even, } 1 \le j \le n, \end{cases}$$

$f(u_1^j u_1^{j+1}) = k - 2a$ for $1 \leq j \leq n-1$ and $f(u_1^n u_1^1) = k - 2a$.

Then the induced vertex labeling $f^+ : V(C(n.P(r,m))) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.P(r,m)))$.

Case 2. r is even.

Subcase 2.1 n is even.

The graph $P(r,m)$ is Z_k-magic with magic constant zero. Therefore by theorem 2.1 it is Z_k-magic.

Subcase 2.2 n is odd and k is even.

By theorem 2.2 it is Z_k-magic in this case.

Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.P(r,m))$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example for Z_5-magic labeling of $C(5.P(5,2))$ is shown in Figure 2.

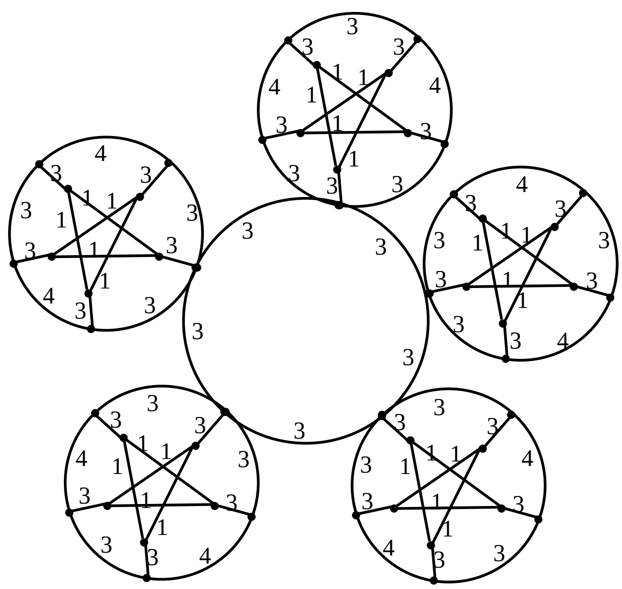

Figure 2 Z_5-magic labeling of $C(5.P(5,2))$

Conjecture 2.6 *The cycle of generalized peterson graph $C(n.P(r,m))$ is not Z_k-magic when r is even, n is odd and k is odd.*

Theorem 2.7 *The cycle of shell graph $C(n.S_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.S_r)$ be respectively $V(C(n.S_r)) = \{v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(C(n.S_r)) = \{v_i^j v_{i+1}^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \bigcup \{v_r^j v_1^j : 1 \leq j \leq n\} \bigcup \{v_1^j v_{i+2}^j : 1 \leq i \leq r-3, 1 \leq j \leq n\} \bigcup \{v_1^j v_1^{j+1} : 1 \leq j \leq n-1\} \bigcup \{v_1^n v_1^1\}$. For any integer a such that $k > (r-2)a$, define the edge labeling

$f : E(C(n.S_r)) \to Z_k - \{0\}$ as follows:

$f(v_1^j v_{i+2}^j) = 2a$ for $1 \leq i \leq r-3,\ 1 \leq j \leq n$,
$f(v_1^j v_2^j) = f(v_r^j v_1^j) = a$ for $1 \leq j \leq n$,
$f(v_i^j v_{i+1}^j) = k - a$ for $2 \leq i \leq r-1, 1 \leq j \leq n$,
$f(u_1^j u_1^{j+1}) = k - (r-2)a$ for $1 \leq j \leq n-1$,
$f(u_1^n u_1^1) = k - (r-2)a$.

Then the induced vertex labeling $f^+ : V(C(n.S_r)) \to Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(C(n.S_r))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.S_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example for Z_7-magic labeling of $C(5.S_5)$ is shown in Figure 3.

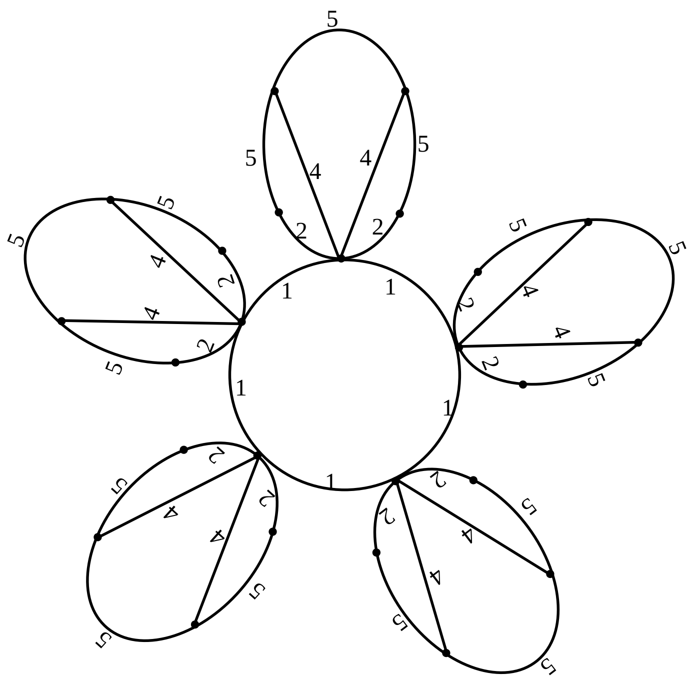

Figure 3 Z_7-magic labeling of $C(5.S_5)$

Theorem 2.8 *The cycle of wheel graph $C(n.W_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.W_r)$ be respectively $V(C(n.W_r)) = \{w_j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(C(n.W_r)) = \{u_i^j u_{i+1}^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \bigcup \{u_r^j u_1^j : 1 \leq j \leq n\} \bigcup \{w_j u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \bigcup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\} \bigcup \{u_1^n u_1^1\}$. For any integer a such that $k > 2(r-1)a$, define the edge labeling $f : E(C(n.W_r)) \to Z_k - \{0\}$ as follows:

$f(w_j u_i^j) = 2a$ for $2 \leq i \leq r,\ 1 \leq j \leq n$,
$f(w_j u_1^j) = k - 2(r-1)a$ for $1 \leq j \leq n$,
$f(u_i^j u_{i+1}^j) = k - a$ for $1 \leq i \leq r-1, 1 \leq j \leq n$,

$f(u_r^j u_1^j) = k - a$ for $1 \leq j \leq n$,
$f(u_1^j u_1^{j+1}) = ra$ for $1 \leq j \leq n-1$,
$f(u_1^n u_1^1) = ra$.

Then the induced vertex labeling $f^+ : V(C(n.W_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.W_r))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.W_r)$ admits Z_k-magic labeling , the cycle of wheel graph $C(n.W_r)$ is a Z_k-magic graph. □

The example of Z_{13}-magic labeling of $C(5.W_7)$ is shown in Figure 4.

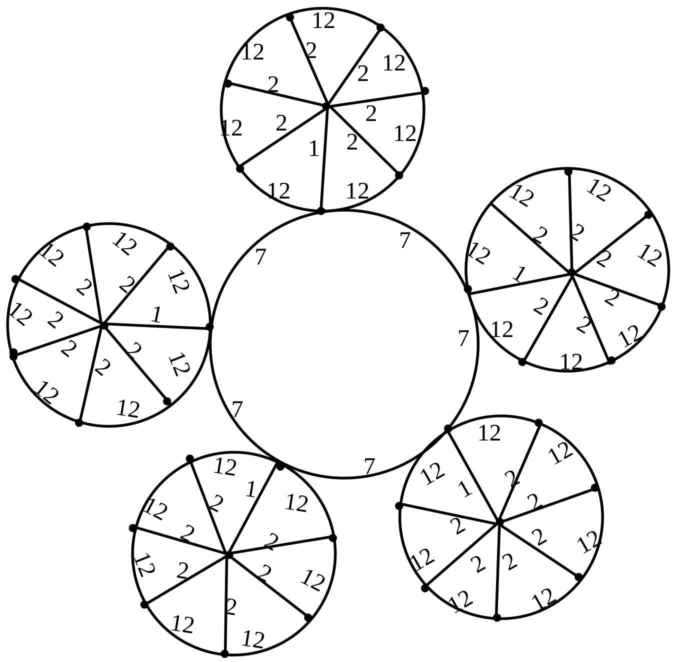

Figure 4 Z_{13}-magic labeling of $C(5.W_7)$

Theorem 2.9 *The cycle of closed helm graph $C(n.CH_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.CH_r)$ be respectively $V(C(n.CH_r)) = \{w_j, u_i^j, x_i^j : 1 \leq i \leq r,\ 1 \leq j \leq n\}$ and $E(C(n.CH_r)) = \{u_i^j u_{i+1}^j : 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{u_r^j u_1^j : 1 \leq j \leq n\} \bigcup \{x_i^j x_{i+1}^j : 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{x_r^j x_1^j : 1 \leq j \leq n\} \bigcup \{w_j x_i^j : 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{x_i^j u_i^j : 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\} \bigcup \{u_1^n u_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > (r+1)a$, define the edge labeling $f : E(C(n.CH_r)) \to Z_k - \{0\}$ as follows:

$$f(u_i^j u_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd},\ 1 \leq j \leq n, \\ 2a & \text{for } i \text{ is even},\ 1 \leq j \leq n, \end{cases}$$

$$f(x_i^j x_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd}, \ 1 \leq j \leq n, \\ a & \text{for } i \text{ is even}, \ 1 \leq j \leq n, \end{cases}$$

$f(w_j x_1^j) = k - (r-1)a$ for $1 \leq j \leq n$,
$f(w_j x_i^j) = a$ for $2 \leq i \leq r, 1 \leq j \leq n$,
$f(x_1^j u_1^j) = (r+1)a$ for $1 \leq j \leq n$,
$f(x_i^j u_i^j) = k - a$ for $2 \leq i \leq r, 1 \leq j \leq n$,
$f(u_1^j u_1^{j+1}) = k - \frac{(r-1)a}{2}$ for $1 \leq j \leq n-1$,
$f(u_1^n u_1^1) = k - \frac{(r-1)a}{2}$.

Then the induced vertex labeling $f^+ : V(C(n.CH_r)) \to Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(C(n.CH_r))$.

Case 2. r is even.

For any integer a such that $k > ra$, define the edge labeling $f : E(C(n.CH_r)) \to Z_k - \{0\}$ as follows:

$$f(u_i^j u_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd}, \ 1 \leq j \leq n, \\ 2a & \text{for } i \text{ is even}, \ 1 \leq j \leq n, \end{cases}$$

$$f(x_i^j x_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd}, \ 1 \leq j \leq n, \\ a & \text{for } i \text{ is even}, \ 1 \leq j \leq n, \end{cases}$$

$f(w_j x_1^j) = k - (r-1)a$ for $1 \leq j \leq n$,
$f(w_j x_i^j) = a$ for $2 \leq i \leq r, 1 \leq j \leq n$,
$f(x_1^j u_1^j) = (r-1)a$ for $1 \leq j \leq n$,
$f(x_i^j u_i^j) = k - a$ for $2 \leq i \leq r, 1 \leq j \leq n$,
$f(u_1^j u_1^{j+1}) = k - \frac{ra}{2}$ for $1 \leq j \leq n-1$,
$f(u_1^n u_1^1) = k - \frac{ra}{2}$.

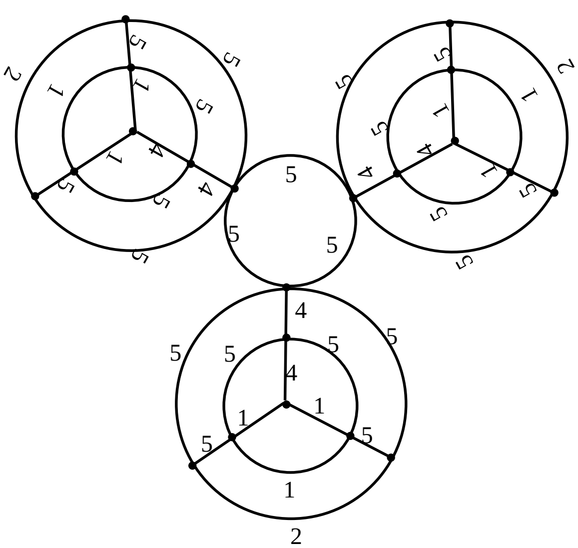

Figure 5 Z_6-magic labeling of $C(3.CH_3)$

Then the induced vertex labeling $f^+ : V(C(n.CH_r)) \rightarrow Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.CH_r))$.

Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.CH_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_6-magic labeling of $C(3.CH_3)$ is shown in Figure 5.

Theorem 2.10 *The cycle of double wheel graph $C(n.DW_r)$ is Z_k-magic except r is even, n is odd and k is odd.*

Proof Let the vertex set and the edge set of $C(n.DW_r)$ be respectively $V(C(n.DW_r)) = \{v_j, u_i^j :\ 1 \leq i \leq r,\ 1 \leq j \leq n\}$ and $E(C(n.DW_r)) = \{v_iv_i^j, v_iu_i^j :\ 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{v_i^jv_{i+1}^j :\ 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{v_r^jv_1^j :\ 1 \leq j \leq n\} \bigcup \{u_i^ju_{i+1}^j :\ 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{u_r^ju_1^j :\ 1 \leq j \leq n\} \bigcup \{u_1^ju_1^{j+1} :\ 1 \leq j \leq n-1\} \bigcup \{u_1^nu_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > 3a$, define the edge labeling $f : E(C(n.DW_r)) \rightarrow Z_k - \{0\}$ as follows:

$f(v_iv_i^j) = 2a$ for $1 \leq i \leq r,\ 1 \leq j \leq n$,

$f(v_iu_i^j) = k - 2a$ for $1 \leq i \leq r,\ 1 \leq j \leq n$,

$f(v_i^jv_{i+1}^j) = k - a$ for $1 \leq i \leq r-1, 1 \leq j \leq n$,

$$f(u_i^ju_{i+1}^j) = \begin{cases} 3a & \text{for } i \text{ is odd},\ 1 \leq j \leq n, \\ k-a & \text{for } i \text{ is even},\ 1 \leq j \leq n, \end{cases}$$

$f(v_r^jv_1^j) = k - a$ for $1 \leq j \leq n$,

$f(u_1^ju_1^{j+1}) = k - 2a$ for $1 \leq j \leq n-1$,

$f(u_1^nu_1^1) = k - 2a$.

Case 2. r is even.

Subcase 2.1 n is even.

The graph DW_r is Z_k-magic with magic constant zero. Therefore by theorem 2.1 it is Z_k-magic.

Subcase 2.2 n is odd and k is even.

By Theorem 2.2 it is Z_k-magic in this case.

Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.DW_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_9-magic labeling of $C(5.DW_3)$ is shown in Figure 6.

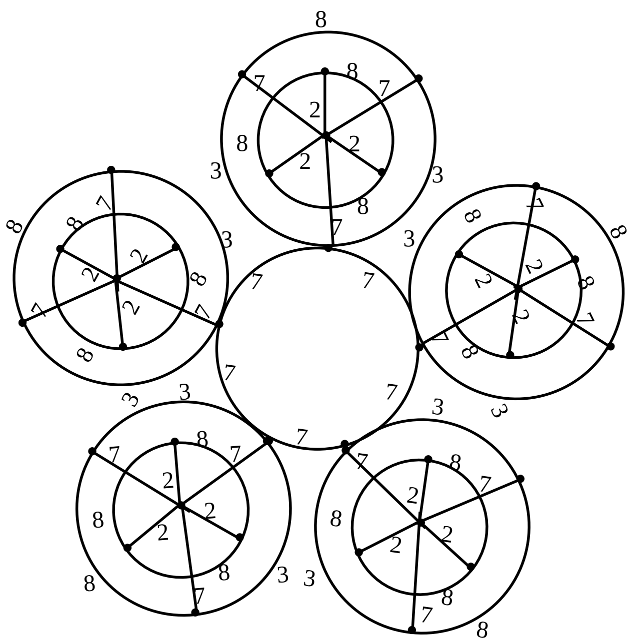

Figure 6 Z_9-magic labeling of $C(5.DW_3)$

Conjecture 2.11 *The cycle of double wheel graph $C(n.DW_r)$ is not Z_k-magic when r is even, n is odd and k is odd.*

Obsevation 2.2 The graph $C(n.DW_{n_1}, DW_{n_2} \ldots, DW_{n_l})$ is Z_k-magic when $n_1, n_2, \ldots n_l$ are odd.

Theorem 2.12 *The cycle of triangular ladder graph $C(n.TL_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.TL_r)$ be respectively $V(C(n.TL_r)) = \{u_i^j, v_i^j : 1 \le i \le r, 1 \le j \le n\}$ and $E(C(n.TL_r)) = \{u_i^j u_{i+1}^j, v_i^j v_{i+1}^j : 1 \le i \le r-1, 1 \le j \le n\} \bigcup \{u_i^j v_{i+1}^j : 1 \le i \le r-1, 1 \le j \le n\} \bigcup \{u_i^j v_i^j : 1 \le i \le r, 1 \le j \le n\} \bigcup \{v_1^j v_1^{j+1} : 1 \le j \le n-1\} \cup \{v_1^n v_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > 2a$, define the edge labeling $f : E(C(n.TL_r)) \to Z_k - \{0\}$ as follows:

$$f(u_i^j u_{i+1}^j) = \begin{cases} 2a & \text{for } i \text{ is odd}, \ 1 \le j \le n, \\ k-a & \text{for } i \text{ is even}, \ 1 \le j \le n, \end{cases}$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd}, \ 1 \le j \le n, \\ a & \text{for } i \text{ is even}, \ 1 \le j \le n, \end{cases}$$

$f(u_1^j v_1^j) = k - a$ for $1 \le j \le n$,

$f(u_i^j v_i^j) = a$ for $2 \le i \le r, 1 \le j \le n$,

$f(u_1^j v_2^j) = k - a$ for $1 \leq j \leq n$,

$f(u_i^j v_{i+1}^j) = k - 2a$ for $2 \leq i \leq r - 1, 1 \leq j \leq n$,

$f(v_1^j v_1^{j+1}) = a$ for $1 \leq j \leq n - 1$,

$f(v_1^n v_1^1) = a$.

Then the induced vertex labeling $f^+ : V(C(n.TL_r)) \to Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(C(n.TL_r))$.

Case 2. r is even.

For any integer a such that $k > 2a$, define the edge labeling $f : E(C(n.TL_r)) \to Z_k - \{0\}$ as follows:

$$f(u_i^j u_{i+1}^j) = \begin{cases} 2a & \text{for } i \text{ is odd, } 1 \leq j \leq n, \\ k - 2a, & \text{for } i \text{ is even, } 1 \leq j \leq n, \end{cases}$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} k - a & \text{for } i \text{ is odd}, i \neq (r-1)\ 1 \leq j \leq n, \\ a & \text{for } i \text{ is even, } 1 \leq j \leq n, \end{cases}$$

$f(v_{r-1}^j v_r^j) = a$ for $1 \leq j \leq n$,

$f(u_1^j v_1^j) = k - a$ for $1 \leq j \leq n$,

$f(u_i^j v_i^j) = a$ for $2 \leq i \leq r - 2, 1 \leq j \leq n$,

$f(u_{r-1}^j v_{r-1}^j) = k - a$ for $1 \leq j \leq n$,

$f(u_r^j v_r^j) = a$ for $1 \leq j \leq n$,

$f(u_i^j v_{i+1}^j) = k - a$ for $1 \leq i \leq r - 2, 1 \leq j \leq n$,

$f(u_{r-1}^j v_r^j) = a$ for $1 \leq j \leq n$,

$f(v_1^j v_1^{j+1}) = a$ for $1 \leq j \leq n - 1$,

$f(v_1^n v_1^1) = a$.

Then the induced vertex labeling $f^+ : V(C(n.TL_r)) \to Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(C(n.TL_r))$. Hence f^+ is constant and it is equal to $0 (mod\ k)$. Since $C(n.TL_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_{10}-magic labeling of $C(5.TL_5)$ is shown in Figure 7.

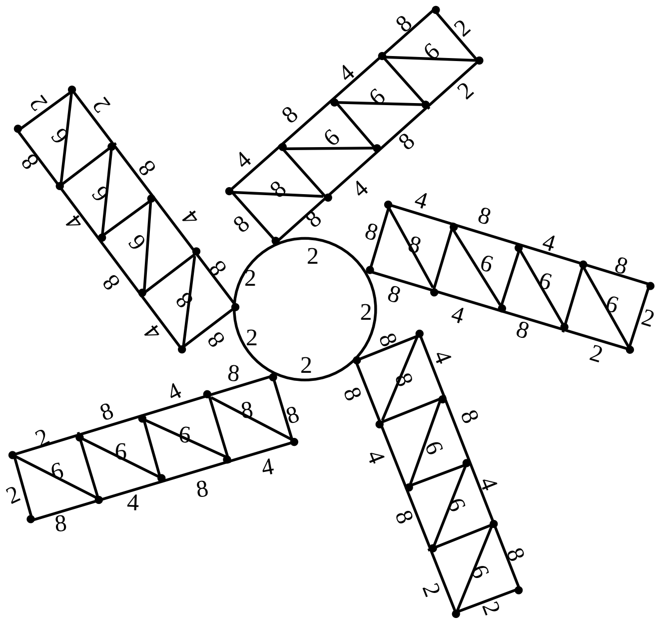

Figure 7 Z_{10}-magic labeling of $C(5.TL_5)$

Theorem 2.13 *The cycle of flower graph $C(n.Fl_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.Fl_r)$ be respectively $V(C(n.Fl_r)) = \{v_j, v_i^j, u_i^j :\ 1 \leq i \leq r,\ 1 \leq j \leq n\}$ and $E(C(n.Fl_r)) = \{v_j v_i^j :\ 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{v_i^j u_i^j :\ 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{v_j u_i^j :\ 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{v_i^j v_{i+1}^j :\ 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{v_r^j v_1^j :\ 1 \leq j \leq n\} \bigcup \{v_1^j v_1^{j+1} :\ 1 \leq j \leq n-1\} \bigcup \{v_1^n v_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > 3a$, define the edge labeling $f : E(C(n.Fl_r)) \to Z_k - \{0\}$ as follows:

$f(v_j v_i^j) = a$ for $1 \leq i \leq r,\ 1 \leq j \leq n$,
$f(v_i^j u_i^j) = a$ for $1 \leq i \leq r,\ 1 \leq j \leq n$,
$f(u_i^j v_j) = k - a$ for $1 \leq i \leq r,\ 1 \leq j \leq n$,

$$f(v_i^j v_{i+1}^j) = \begin{cases} a & \text{for } i \textit{ is odd},\ 1 \leq j \leq n, \\ k - 3a & \text{for } i \text{ is even},\ 1 \leq j \leq n, \end{cases}$$

$f(v_1^j v_1^{j+1}) = k - 2a$ for $1 \leq j \leq n-1$,
$f(v_1^n v_1^1) = k - 2a$.

Then the induced vertex labeling $f^+ : V(C(n.Fl_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.Fl_r))$.

Case 2. r is even.

For any integer a such that $k > 2a$, define the edge labeling $f : E(C(n.Fl_r)) \to Z_k - \{0\}$ as follows:

$f(v_j v_1^j) = 2a$ for $1 \leq j \leq n$,
$f(v_j v_i^j) = a$ for $2 \leq i \leq r,\ 1 \leq j \leq n$,
$f(v_1^j u_1^j) = 2a$ for $1 \leq j \leq n$,
$f(v_i^j u_i^j) = a$ for $2 \leq i \leq r,\ 1 \leq j \leq n$,
$f(u_1^j v_j) = k - 2a$ for $1 \leq j \leq n$,
$f(u_i^j v_j) = k - a$ for $2 \leq i \leq r,\ 1 \leq j \leq n$,
$f(v_i^j v_{i+1}^j) = k - a$ for $1 \leq i \leq r-1,\ 1 \leq j \leq n$,
$f(v_n^j v_1^j) = k - a$,
$f(v_1^j v_1^{j+1}) = k - a$ for $1 \leq j \leq n-1$,
$f(v_1^n v_1^1) = k - a$.

Then the induced vertex labeling $f^+ : V(C(n.Fl_r)) \to Z_k$ is $f^+(v) \equiv\ 0(mod\ k)$ for all $v \in V(C(n.Fl_r))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.Fl_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_5-magic labeling of $C(3.Fl_3)$ is shown in Figure 8.

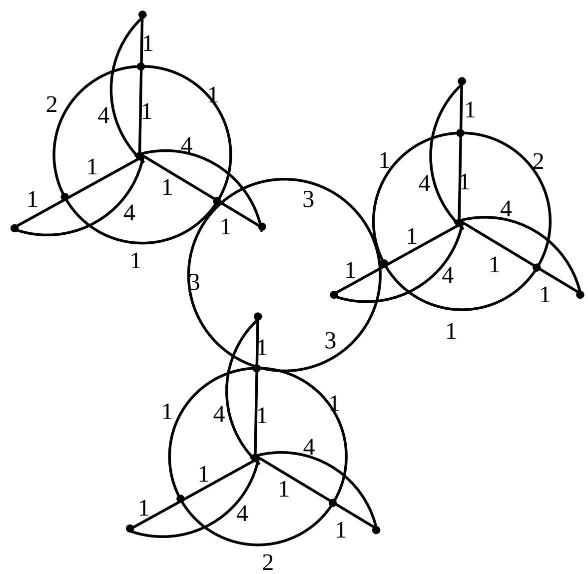

Figure 8 Z_5-magic labeling of $C(3.Fl_3)$

Theorem 2.14 *The cycle of lotus inside a circle graph $C(n.LC_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.LC_r)$ be respectively $V(C(n.LC_r)) = \{v_j, v_i^j, u_i^j\ :\ 1 \leq i \leq r,\ 1 \leq j \leq n\}$ and $E(C(n.LC_r)) = \{v_j v_i^j\ :\ 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{v_i^j u_i^j\ :\ 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{u_i^j v_{i+1}^j\ :\ 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{u_r^j v_1^j\ :\ 1 \leq j \leq n\} \bigcup \{u_i^j u_{i+1}^j\ :\ 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{u_r^j u_1^j\ :\ 1 \leq j \leq n\} \bigcup \{u_1^j u_1^{j+1}\ :\ 1 \leq j \leq n-1\} \bigcup \{u_1^n u_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > (r-1)a$, define the edge labeling $f : E(C(n.LC_r)) \to Z_k - \{0\}$ as follows:

$f(v_jv_1^j) = k-(n-1)a$ for $1 \le j \le n$,

$f(v_jv_i^j) = a$ for $2 \le i \le r,\ 1 \le j \le n$,

$f(v_1^ju_1^j) = (r-2)a$ for $1 \le j \le n$,

$f(v_i^ju_i^j) = k-2a$ for $2 \le i \le r, 1 \le j \le n$,

$f(u_i^jv_{i+1}^j) = a$ for $1 \le i \le r-1, 1 \le j \le n$,

$f(u_r^jv_1^j) = a$ for $1 \le j \le n$,

$$f(u_i^ju_{i+1}^j) = \begin{cases} 2a & \text{for } i \text{ is odd}, \ 1 \le j \le n, \\ k-a & \text{for } i \text{ is even}, \ 1 \le j \le n, \end{cases}$$

$f(u_1^ju_1^{j+1}) = k - \frac{(r+3)a}{2}$ for $1 \le j \le n-1$,

$f(u_1^nu_1^1) = k - \frac{(r+3)a}{2}$.

Then the induced vertex labeling $f^+ : V(C(n.LC_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.LC_r))$.

Case 2. r is even.

For any integer a such that $k > (r-1)a$, define the edge labeling $f : E(C(n.LC_r)) \to Z_k - \{0\}$ as follows:

$f(v_jv_1^j) = k-(n-1)a$ for $1 \le j \le n$,

$f(v_jv_i^j) = a$ for $2 \le i \le r,\ 1 \le j \le n$,

$f(v_1^ju_1^j) = (r-2)a$ for $1 \le j \le n$,

$f(v_i^ju_i^j) = k-2a$ for $2 \le i \le r, 1 \le j \le n$,

$f(u_i^jv_{i+1}^j) = a$ for $1 \le i \le r-1, 1 \le j \le n$,

$f(u_r^jv_1^j) = a$ for $1 \le j \le n$,

$$f(u_i^ju_{i+1}^j) = \begin{cases} 2a & \text{for } i \text{ is odd}, \ 1 \le j \le n, \\ k-a & \text{for } i \text{ is even}, \ 1 \le j \le n, \end{cases}$$

$f(u_1^ju_1^{j+1}) = k - \frac{ra}{2}$ for $1 \le j \le n-1$,

$f(u_1^nu_1^1) = k - \frac{ra}{2}$.

Then the induced vertex labeling $f^+ : V(C(n.LC_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.LC_r))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.LC_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_8-magic labeling of $C(5.LC_5)$ is shown in Figure 9.

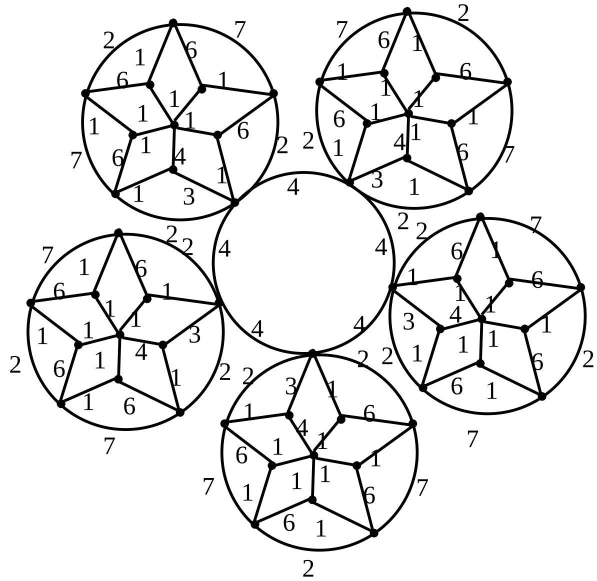

Figure 9 Z_8-magic labeling of $C(5.LC_5)$

References

[1] P.Jeyanthi and K.Jeya Daisy, Z_k-magic labeling of open star of graphs, *Bulletin of the International Mathematical Virtual Institute*, 7 (2017), 243–255.

[2] P.Jeyanthi and K.Jeya Daisy, Z_k-magic labeling of subdivision graphs, *Discrete Math. Algorithm. Appl.*, 8(3) (2016), [19 pages] DOI: 10.1142/ S1793830916500464.

[3] P.Jeyanthi and K.Jeya Daisy, Certain classes of Z_k-magic graphs, *Journal of Graph Labeling*, 4(1) (2018), 38–47.

[4] P.Jeyanthi and K.Jeya Daisy, Z_k-Magic labeling of some families of graphs, *Journal of Algorithms and Computation*, 50(2) (2018),1-12.

[5] P.Jeyanthi and K.Jeya Daisy, Some results on Z_k-magic labeling, *Palestine Journal of Mathematics*, to appear.

[6] J.A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 2017.

[7] R.M. Low and S.M Lee, On the products of group-magic graphs, *Australian J. Combin.*, 34 (2006), 41–48.

[8] J. Sedláček, On magic graphs, *Math. Slov.*, 26 (1976), 329–335.

[9] W.C. Shiu, P.C.B. Lam and P.K. Sun, Construction of magic graphs and some A-magic graphs with A of even order, *Congr. Numer.*, 167 (2004), 97–107.

[10] W.C. Shiu and R.M. Low, Z_k-magic labeling of fans and wheels with magic-value zero, *Australian. J. Combin.*, 45 (2009), 309–316.