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Total Domination Stable Graphs

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Abstract: In this paper, we study the total domination number and total domination polynomials of some graph and its square. We discuss nonzero real total domination roots of these graphs. We also investigate whether all the total domination roots of some graphs lying left half plane or not.

Key Words: Total dominating set, Smarandachely total k-dominating set, total domination number, total domination polynomial, total domination root, stable.

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§1. Introduction

Let $G(V, E)$ be a simple finite graph. The order of G is the number of vertices of G. A set $S \subseteq V$ is a total dominating set if every vertex $v \in V$ is adjacent to at least one vertex in S. Generally, a set $D \subseteq V$ of G is said to be a *Smarandachely total k-dominating set* if each vertex of G is dominated by at least k vertices of S with $k \geq 1$. Clearly, a total dominating set is a Smarandachely total 1-dominating set. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of the total dominating sets in G. Let $\mathcal{D}_t(G, i)$ be the family of total dominating sets of G with cardinality i and let $d_t(G, i) = |\mathcal{D}_t(G, i)|$. The polynomial

$$D_t(G, x) = \sum_{i=\gamma_t(G)}^{|V(G)|} d_t(G, i)x^i$$

is defined as total domination polynomial of G. For more information on this polynomial the reader may refer to [8]. A root of $D_t(G, x)$ is called a total domination root of G. It is easy to see that the total domination polynomial is monic with no constant term. Consequently, 0 is a root of every total domination polynomial (in fact, 0 is a root whose multiplicity is the total domination number of the graph).

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§2. Main Results

2.1 $\mathtt{d}_t$-Number

In this section we find the number of real roots of the total domination polynomial of some graphs. We are already find out total domination polynomials of complete partite graphs [3] and square of some graphs (The square of a graph G is the graph with the same set of vertices as G and an edge between two vertices if and only if there is a path of length at most two between them, and that graph is denoted by G^2.). We are interested to find the number of real total domination roots of graphs. We define $\mathtt{d}_t$-number of a graph G as follows:

Definition 2.1 *Let G be a graph. The number of distinct real total domination roots of the graph G is called $\mathtt{d}_t$-number of G and is denoted by $\mathtt{d}_t(G)$.*

Theorem 2.1 *For any graph G, $\mathtt{d}_t(G) \geq 1$.*

Proof It follows from the fact that 0 is a total domination root of any graph. □

Theorem 2.2 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$\mathtt{d}_t(G) \leq \sum_{i=1}^{m} \mathtt{d}_t(G_i) - m + 1.$$

Proof It follows from the fact that $D_t(G, x) = \prod_{i=1}^{m} D_t(G_i, x)$. □

Theorem 2.3 *If G and H are isomorphic, then $\mathtt{d}_t(G) = \mathtt{d}_t(H)$.*

Proof It follows from the fact that if G, H are isomorphic then $D_t(G, x) = D_t(H, x)$. □

Theorem 2.4 *For $n \geq 2$ the $\mathtt{d}_t$-number of the complete graph K_n is 1 for even n and 2 for odd n.*

Proof We have the total domination polynomial of K_n is

$$D_t(K_n, x) = (1 + x)^n - nx - 1.$$

From the above equation it follows that $D_t(K_n, y - 1) = y^n - ny + n - 1.$. Clearly, $y = 1$ is a double root of $D_t(K_n, y - 1)$. By De Gua's rule for imaginary roots, there are at least $n - 2$ complex roots if n is even and there are at least $n - 3$ complex roots if n is odd. This give the result. □

Theorem 2.5 *For all m, n the $\mathtt{d}_t$-number of the complete bipartite graph $K_{m,n}$ is*

$$\mathtt{d}_t(K_{m,n}) = \begin{cases} 1 & \text{if both } m \text{ and } n \text{ are odd,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof We have the total domination polynomial of $K_{m,n}$ is

$$D_t(K_{m,n},x) = [(1+x)^m - 1][(1+x)^n - 1]. \quad (1)$$

The result follows from the transformation $y = 1 + x$ in equation (1). □

Theorem 2.6 *For $m, n \geq 2$ the $\mathtt{d}_t$-number of the complete partite graph $K_{n[m]}$ is*

$$\mathtt{d}_t(K_{n[m]}) = \begin{cases} 2 & if\ n\ is\ even, \\ 1 & if\ m\ is\ even\ and\ n\ is\ odd, \\ 2 & if\ both\ m\ and\ n\ are\ odd. \end{cases}$$

Proof We have

$$D_t(K_{n[m]},x) = (1+x)^{mn} - m(1+x)^n + m - 1. \quad (2)$$

From the equation (2), it follows that $D_t(K_{n[m]}, y-1) = y^{mn} - my^n + m - 1$. To find the real roots of $y^{mn} - my^n + m - 1 = 0$, it is enough to find the real roots of $f_m(z) = z^m - mz + m - 1 = 0$. Clearly, $z = 1$ is a double root of $f_m(z)$. If m is even, then by De Gua's rule for imaginary roots, there are at least $m - 2$ complex roots. Therefore $z = 1$ is the only real root of $f_m(z)$. But $y^n = 1$ has exactly two real solutions, namely $y = \pm 1$ for even n and has exactly one solution, namely $y = 1$ for odd n. If m is odd, then by De Gua's rule for imaginary roots, there are at least $m - 3$ complex roots. By the intermediate value theorem, $f_m(z)$ has at least one real root in $(-3, -1)$. So the roots of $f_m(z)$ are 1 and $c \in (-3, -1)$. But $y^n = c$ has a real solution only for odd n and that solution is unique. Therefore $K_{n[m]}$ has only one nonzero real total domination root for even n and if m is even and n is odd, then $K_{n[m]}$ has no nonzero real total domination root. Finally, if both m and n are odd $K_{n[m]}$ has exactly one nonzero total domination root. □

Theorem 2.7 *For all n the $\mathtt{d}_t$-number of the star graph S_n is 1 if n is odd and 2 if n is even.*

Proof We have the total domination polynomial of S_n is

$$D_t(S_n, x) = x((1+x)^n - 1). \quad (3)$$

The result follows from the transformation $y = 1 + x$ in equation (3). □

The corona $H \circ G$ of two graphs H and G is the graph formed from one copy of H and $|V(H)|$ copies of G, where the i^{th} vertex of H is adjacent to every vertex in the i^{th} copy of G.

Theorem 2.8 *Let G be a graph of order n without isolated vertices and let H be any graph. Then the total dominating number $\gamma_t(G \circ H) = n$.*

Theorem 2.9 *Let G be a graph of order n without isolated vertices. Then the total domination* *polynomial of* $G \circ \overline{K_m}$ *is*

$$D_t(G \circ \overline{K_m}, x) = x^n (1+x)^{mn}.$$

Proof By above theorem we have $\gamma_t(G \circ \overline{K_m}) = n$. If S is a total dominating set of $G \circ \overline{K_m}$, then $V(G) \subset S$. Therefore $d_t(G \circ \overline{K_m}, n) = 1$ and for $n+1 \le i \le n(m+1)$,

$$d_t(G \circ \overline{K_m}, i) = \binom{mn}{i-n}. \qquad \square$$

Theorem 2.10 *Let* G *be a graph of order* n. *Then the total domination polynomial of* $K_1 \circ G$ *is*

$$D_t(K_1 \circ G, x) = D_t(G, x) + x\left((1+x)^n - 1\right).$$

Proof It follows from the facts that total dominating sets of G is a total dominating sets of $K_1 \circ G$ and any set of vertices of $K_1 \circ G$ containing the vertex of K_1 is also a total dominating set. $\square$

The Dutch-windmill graph G_3^n is the graph obtained by selecting one vertex in each of n triangles and identifying them.

Corollary 2.1 *The total domination polynomial of the Dutch-windmill graph* G_3^n *is*

$$D_t(G_3^n, x) = x^{2n} + x\left((1+x)^{2n} - 1\right).$$

Proof It follows from the fact that G_3^n and $K_1 \circ nK_2$ are isomorphic. $\square$

Theorem 2.11 *For all* n *the* $\mathtt{d}_t$*-number of the Dutch windmill graph* G_3^n *is greater than or equal to* 2.

Proof We have the total domination polynomial of the Dutch windmill graph G_3^n is

$$D_t(G_3^n, x) = x^{2n} + x((1+x)^{2n} - 1).$$

Consider,

$$\begin{aligned} D_t(G_3^n, -\ln n) &= (-\ln n)^{2n} + (-\ln n)((1-\ln n)^{2n} - 1) \\ &= (\ln n)^{2n}\left(1 - \ln n (\tfrac{1-\ln n}{\ln n})^{2n} + \ln n \tfrac{1}{(\ln n)^{2n}}\right). \end{aligned}$$

From Theorem **??**, we have $D_t(G_3^n, -\ln n) > 0$ for large n. Next we show that $D_t(G_3^n, -n) < 0$. Consider $f(x) = x^{2n-1} + (2n+1)x^{2n-2} + \binom{2n}{2}x^{2n-3} + \ldots + 2n$. Then

$$\begin{aligned} f(-n) &= (-1)^{2n-1}n^{2n-1} + (2n+1)n^{2n-2} + (-1)^{2n-3}\tbinom{2n}{2}n^{2n-3} + \ldots + 2n \\ &= (-1)^{2n-1}n^{2n-1}\left(1 - \tfrac{2n+1}{n} + \tfrac{\binom{2n}{2}}{n^2} - \ldots - \tfrac{2n}{n^{2n-1}}\right). \end{aligned}$$

But for sufficiently large n,

$$1 - \frac{2n+1}{n} + \frac{\binom{2n}{2}}{n^2} - \ldots - \frac{2n}{n^{2n-1}} < 0.$$

That is, $D_t(G_3^n, -n) < 0$ for sufficiently large n. By the intermediate value theorem, for sufficiently large n, $D_t(G_3^n, x)$ has a real root in the interval $(-n, -\ln n)$. Therefore the Dutch windmill graph G_3^n has at least two real total domination root and hence $\mathtt{d}_t(G_3^n) \geq 2$. □

Theorem 2.12 *For all n, $\mathtt{d}_t((K_n \circ K_1)^2) = 1$.*

Proof We have $D_t((K_n \circ K_1)^2, y-1) = y^{2n} - y^n - ny + n$. Let $f(y) = y^{2n} - y^n - ny + n$. Since the number of variations of the signs of the coefficients of $f(y)$ is 2, by Descartes rule, it has at most two positive real roots. Clearly, $y = 1$ is a double root of $f(y)$. Now consider, $f(-y)$.

Case 1. n is odd.

$f(-y) = y^{2n} + y^n + ny + n$. There is no sign changes, $f(y)$ has no negative real roots. Therefore the only possible real root of $D_t((K_n \circ K_1)^2, x)$ is zero.

Case 2. n is even.

$f(-y) = y^{2n} - y^n + ny + n$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 2, by Descartes rule, it has at most two negative real roots. We claim that $f(-y)$ has no positive real roots. Let $z > 0$ be a real root of $f(-y)$. Then $z^{2n} - z^n + nz + n = 0$. That is, $z^{2n} - z^n = -n(z+1)$. If $z \geq 1$, $z^{2n} - z^n \geq 0$, but right side is negative. Therefore $z \geq 1$ is not possible. If $0 < z < 1$, then $-1 \leq z^{2n} - z^n \leq 0$, but right side is greater than -1. Therefore $0 < z < 1$ is also not possible.

In both cases the only possible real roots of $D_t((K_n \circ K_1)^2, x)$ is zero. Hence we get the result. □

A spider graph Sp_{2n+1} is the graph obtained by subdividing each edges once in the star graph $K_{1,n}$.

Theorem 2.13 *The total domination polynomial of the spider graph Sp_{2n+1} is*

$$D_t(Sp_{2n+1}, x) = x^n \left((1+x)^{n+1} - 1\right).$$

Proof Let v, $V = \{v_1, v_2, \cdots, v_n\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of Sp_{2n+1} such that v is adjacent to v_i for every $i = 1, 2, \cdots, n$ and v_i and u_i are adjacent for every $i = 1, 2, \cdots, n$. It is clear that the total dominating sets of Sp_{2n+1} are exactly the sets of vertices of Sp_{2n+1} properly containing V. Hence $\gamma(Sp_{2n+1}) = n + 1$ and $d_t(Sp_{2n+1}, n+i) = \binom{n+1}{i}$ for $i = 1, 2, \cdots, n+1$. □

Theorem 2.14 *For $n \geq 2$, the $\mathtt{d}_t$-number of the spider graph Sp_{2n+1} is 1 for even n and 2 for odd n.*

Proof By Theorem 2.13 we have the total domination polynomial of the spider graph Sp_{2n+1} is

$$D_t(Sp_{2n+1}, x) = x^n \left((1+x)^{n+1} - 1\right). \tag{4}$$

The result follows from the transformation $y = 1 + x$ in (4). □

The lollipop graph $L_{n,1}$ is the graph obtained by joining a complete graph K_n to a path P_1, with a bridge.

Theorem 2.15 *The total domination polynomial of the lollipop graph $L_{n,1}$ is*

$$D_t(L_{n,1}, x) = x\left((1+x)^n - 1\right).$$

Proof Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of the complete graph K_n and v be the path P_1 and let v is adjacent to v_1. Clearly the total dominating sets of $L_{n,1}$ are exactly the set of vertices of $L_{n,1}$ properly containing v_1. Therefore , $\gamma_t(L_{n,1}) = 2$ and $d_t(L_{n,1}, i) = \binom{n}{i-1}$ for $2 \leq i \leq n+1$. □

Theorem 2.16 *The $\mathtt{d}_t$-number of the lollipop graph $L_{n,1}$ is 1 for odd n and 2 for even n.*

Proof By Theorem 2.15 we have the total domination polynomial of the lollipop graph $L_{n,1}$ is

$$D_t(L_{n,1}, x) = x\left((1+x)^n - 1\right). \quad (5)$$

The result follows from the transformation $y = 1 + x$ in equation (5). □

The bipartite cocktail party graph B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$.

Theorem 2.17 *The total domination polynomial of the bipartite Cocktail party graph B_n is*

$$D_t(B_n, x) = \left((1+x)^n - nx - 1\right)^2.$$

Proof Let $V = \{v_1, v_2, \cdots, v_n\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of B_n such that every vertex v_i in V and every vertex u_i in U are adjacent if $i \neq j$. The total dominating set S of B_n are exactly the set of vertices of B_n such that S contains at least two v_i and at least two u_i. Note that sets of this form are of size greater than or equal to 4. Therefore $\gamma_t(B_n) = 4$. Also for $4 \leq i \leq n$, $d_t(B_n, i) = \binom{2n}{i} - 2\binom{n}{i} - 2\binom{n}{i-1}$, $d_t(B_n, n+1) = \binom{2n}{n+1} - 2n$ and for $n+2 \leq i \leq 2n$, $d_t(B_n, i) = \binom{2n}{i}$. □

Theorem 2.18 *For $n \geq 2$ the $\mathtt{d}_t$-number of the bipartite cocktail party graph B_n is 1 for even n and 2 for odd n.*

Proof The proof is similar to the proof of Theorem 2.4. □

Theorem 2.19 *For $n \geq 3$, the total domination polynomial of square of the bipartite cocktail party graph B_n is*

$$D_t(B_n^2, x) = (1+x)^{2n} - n(1+x)^2 + (n-1).$$

Proof Let $V = \{v_1, v_2, \cdots, v_n\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of B_n such that every vertex v_i in V and every vertex u_i in U are adjacent if $i \neq j$. Clearly, any subset of vertices of B_n of cardinality 2 forms a total dominating set of B_n^2 excluding $\{v_i, u_i\}$ for all $i = 1, 2, \cdots, n$.

Therefore $\gamma_t(B_n^2) = 2$, $d_t(B_n^2, 2) = \binom{2n}{2} - n$ and $d_t(B_n^2, i) = \binom{2n}{i}$ for all $3 \le i \le 2n$. □

Theorem 2.20 *The* $\mathtt{d}_t$*-number of the square of the bipartite cocktail party graph* B_n *is* 2 *for* $n \ge 3$.

Proof We have $D_t(B_n^2, y-1) = y^{2n} - ny^2 + n - 1$. Then by De Gua's rule for imaginary roots, there are at least $2n-4$ complex roots. Clearly, $y = 1$ and $y = -1$ are double roots of $D_t(B_n^2, y-1)$. Therefore $x = 0$ and $x = -2$ are the only real roots. □

The generalized barbell graph $B_{m,n,1}$ is the simple graph obtained by connecting two complete graphs K_m and K_n by a path P_1.

Theorem 2.21 *For* $m \le n$*, the total domination polynomial of generalized barbell graph* $B_{m,n,1}$ *is*

$$D_t(B_{m,n,1}, x) = \left[(1+x)^m - (m-1)x - 1\right]\left[(1+x)^n - (n-1)x - 1\right].$$

Proof Let $V = \{v_1, v_2, \cdots, v_m\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i \ne j$ every vertices V are adjacent, every vertices U are adjacent and v_m and u_n is adjacent. The only two element total dominating set of $B_{m,n,1}$ is $\{v_m, u_n\}$. Therefore $\gamma_t(B_{m,n,1}) = 2$ and $d(B_{m,n,1}, 2) = 1$. Also observe that for $2 \le i \le 2n$, a subset S of vertices $B_{m,n,1}$ of cardinality i is not a total domination set if and only if (i) $S \subset V$ or (ii) $S \subset U$ or (iii) S contains one element from $V - \{v_n\}$ and $i-1$ elements from U or (iv)S contains one element from $U - \{u_n\}$ and $i-1$ elements from V. Therefore

$$d_t(B_{m,n,1}, i) = \begin{cases} 1 & \text{if } i = 2 \\ \binom{m+n}{i} - \binom{n}{i} - \binom{m}{i} - (n-1)\binom{m}{i-1} - (m-1)\binom{n}{i-1} & \text{if } 3 \le i \le m \\ \binom{m+n}{m+1} - \binom{n}{m+1} - (n-1) - (m-1)\binom{n}{m} & \text{if } i = m+1 \\ \binom{m+n}{i} - \binom{n}{i} - (m-1)\binom{n}{i-1} & \text{if } m+2 \le i \le n \\ \binom{m+n}{n+1} - (m-1) & \text{if } i = n+1 \\ \binom{m+n}{i} & \text{if } n+2 \le i \le m+n \end{cases}.$$

Hence

$$D_t(B_{m,n,1}, x) = \left[(1+x)^m - (m-1)x - 1\right]\left[(1+x)^n - (n-1)x - 1\right]. \quad \square$$

Theorem 2.22 *For* $m, n \ge 2$; $m \ne n$*, the* $\mathtt{d}_t$*-number of the generalized barbell graph* $B_{m,n,1}$ *is*

$$\mathtt{d}_t(B_{m,n,1}) = \begin{cases} 3 & \textit{if both } m \text{ and } n \text{ are even,} \\ 5 & \textit{if both } m \text{ and } n \text{ are odd,} \\ 4 & \textit{if } m \text{ and } n \text{ have opposite parity.} \end{cases}$$

Proof By Theorem 2.21 we have the total domination polynomial of generalized barbell graph $B_{m,n,1}$ is

$$D_t(B_{m,n,1}, x) = \left[(1+x)^m - (m-1)x - 1\right]\left[(1+x)^n - (n-1)x - 1\right].$$

Since there is no real number satisfying both the following equations

$$(1+x)^m - (m-1)x - 1 = 0$$
$$(1+x)^n - (n-1)x - 1 = 0$$

simultaneously. So it is enough to show that $f(x) = x^n - (n-1)x + n - 2$ has exactly one nonzero real root for even n and has exactly two nonzero real roots for odd n. Clearly $x = 1$ is a simple root of $f(x)$. For even n, by De Gua's rule for imaginary roots, there are at least $n-2$ complex roots. Therefore the remaining root is real number different from 1. For odd n by De Gua's rule for imaginary roots, there are at least $n-3$ complex roots. Observe that $f(-1) > 0$ and $f(-2) < 0$. By the intermediate value theorem, we have $f(x)$ has a root in the interval $(-2, -1)$. Therefore the remaining roots real numbers different from 1. It remains to show that $f(x)$ has no double roots. Suppose $a \in \mathbb{R}$ is a double root of $f(x)$. Then

$$a^n - (n-1)a + n - 2 = 0, \tag{6}$$

$$na^{n-1} - (n-1) = 0. \tag{7}$$

Solving these equations we get $a = \frac{n(n-2)}{(n-1)^2}$. This implies that $a \geq 0$, a contradiction, since $a < 0$. So we have the result. □

The $n-$barbell graph $B_{n,1}$ is the simple graph obtained by connecting two copies of complete graph K_n by a bridge.

Corollary 2.2 *The total domination polynomial of the n-barbell graph $B_{n,1}$ is*

$$D_t(B_{n,1}) = \left((1+x)^n - (n-1)x - 1\right)^2.$$

Proof It follows from the fact that the n-barbell graph $B_{n,1}$ and the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

Corollary 2.3 *For $n \geq 2$ the $\mathtt{d}_t$-number of the n-barbell graph $B_{n,1}$, is*

$$\mathtt{d}_t(B_{n,1}) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

2.2 Total Domination Stable Graphs

In this section we introduce $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs. We obtained some examples of $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs.

Definition 2.2 *Let $G = (V(G), E(G))$ be a graph. The graph G is said to be a total domination stable graph or simply $\mathtt{d}_t$-stable graph if all the nonzero total domination roots lie in the left open half-plane, that is, if real part of the nonzero total domination roots are negative. If G is not $\mathtt{d}_t$-stable graph, then G is said to be a total domination unstable graph or simply $\mathtt{d}_t$-unstable* *graph.*

Theorem 2.23 *If G and H are isomorphic graphs then G is $\mathtt{d}_t$-stable if and if H is $\mathtt{d}_t$-stable.*

Proof It follows from the fact that if G and H are isomorphic graphs then $D_t(G,x) = D_t(H,x)$. □

Corollary 2.4 *If G and H are isomorphic graphs then G is $\mathtt{d}_t$-unstable if and if H is $\mathtt{d}_t$-unstable.*

Theorem 2.24 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}_t$-stable if and if each G_i is $\mathtt{d}_t$-stable.*

Proof It follows from the fact that $D_t(G,x) = \prod_{i=1}^{m} D_t(G_i,x)$. □

Corollary 2.5 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}_t$-unstable if and if one of the G_i is $\mathtt{d}_t$-unstable.*

Theorem 2.25 *Let G be a graph of order n without isolated vertices. Then $G \circ \overline{K_m}$ is $\mathtt{d}_t$-stable for all m, n.*

Proof We have the total domination polynomial of $G \circ \overline{K_m}$ is

$$D_t(G \circ \overline{K_m}, x) = x^n(1+x)^{mn}.$$

Therefore $\mathbb{Z}(D_t(G \circ \overline{K_m}, x)) = \{0, -1\}$, hence $G \circ \overline{K_m}$ is $\mathtt{d}_t$-stable for all m, n. □

Theorem 2.26 *The spider graph Sp_{2n+1} is $\mathtt{d}_t$-stable for all n.*

Proof We have the total domination polynomial of the spider graph Sp_{2n+1} is

$$D_t(Sp_{2n+1}, x) = x^n\left((1+x)^{n+1} - 1\right).$$

Therefore

$$\mathbb{Z}(D_t(Sp_{2n+1}, x)) = \left\{\exp\left(\frac{2k\pi i}{n+1}\right) - 1 | k = 0, 1, \ldots, n\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the spider graph Sp_{2n+1} is $\mathtt{d}_t$-stable for all n. □

Theorem 2.27 *The lollipop graph $L_{n,1}$ is $\mathtt{d}_t$-stable for all n.*

Proof We have the total domination polynomial of the lollipop graph $L_{n,1}$ is

$$D_t(L_{n,1}, x) = x\left((1+x)^n - 1\right).$$

Therefore

$$\mathbb{Z}(D_t(L_{n,1}, x)) = \left\{\exp\left(\frac{2k\pi i}{n}\right) - 1 | k = 0, 1, \ldots, n\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the lollipop graph $L_{n,1}$ is $\mathtt{d}_t$-stable for all n. □

Theorem 2.28 *The bi-star graph $B_{(m,n)}$ is $\mathtt{d}_t$-stable for all m, n.*

Proof We have the total domination polynomial of the bi-star graph $B_{(m,n)}$ is

$$D_t(B_{(m,n)}, x) = x^2(1+x)^{m+n}.$$

Therefore

$$\mathbb{Z}(D_t(B_{(m,n)}, x) = \{0, -1\},$$

hence the bi-star graph $B_{(m,n)}$ is $\mathtt{d}_t$-stable for all m, n. □

Corollary 2.6 *The corona graph $K_2 \circ \overline{K_n}$ is $\mathtt{d}_t$-stable for all n.*

Proof It follows from the fact that the corona graph $K_2 \circ \overline{K_n}$ and the bi-star graph $B_{(n,n)}$ are isomorphic. □

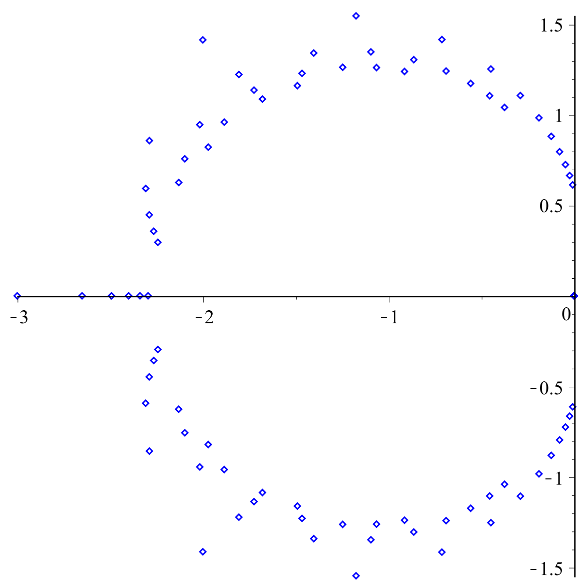

Figure 1 Total domination roots of K_n for $1 \le n \le 14$.

Remarks 2.1 Using maple, we find that the complete graph K_n is $\mathtt{d}_t$-stable for $1 \le n \le 14$ and is $\mathtt{d}_t$-unstable for $15 \le n \le 30$. We have the total domination polynomial of K_n is

$$D_t(K_n, x) = (1+x)^n - nx - 1.$$

Put $y = 1 + x$ and consider $f(y) = y^n - ny + n - 1$. Then $y = 1$ is a double root of $f(y)$.

Therefore $f(y) = (y-1)^2 g(y)$, where

$$g(y) = y^{n-2} + 2y^{n-3} + 3y^{n-4} + \ldots + (n-2)y + n - 1.$$

We have if $f(z) = a_n z^n + a_{n-1} z^n + \ldots + a_0$ is a polynomial with real coefficient satisfying $a_0 \geq a_1 \geq \ldots \geq a_n > 0$ then no roots of $f(z)$ lie in $\{z \in \mathbb{C} : |z| < 1\}$ [6]. Therefore all the roots z of $g(y)$ satisfy $|z| > 1$. This implies that all the nonzero roots of $D_t(K_n, x)$ are out side the unit circle centered at $(-1, 0)$. So we conjectured that the complete graph K_n is not $\mathtt{d}_t$-stable for all but finite values of n.

The total domination roots of the complete graph K_n for $1 \leq n \leq 14$ and $1 \leq n \leq 30$ are shown in Figures 1 and 2 respectively.

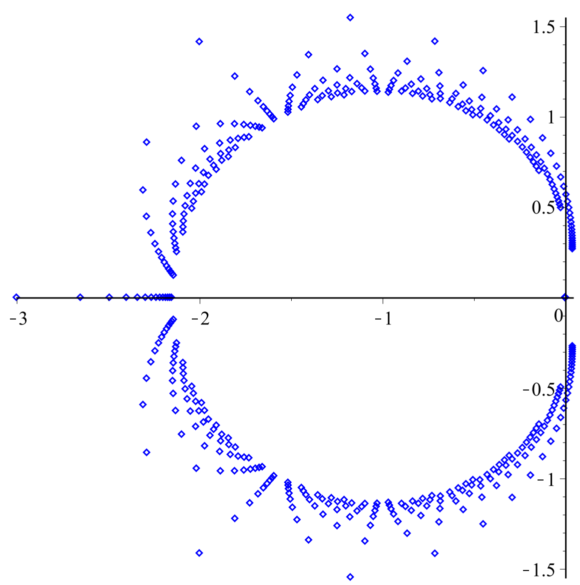

Figure 2 Total domination roots of K_n for $1 \leq n \leq 30$.

We use the following definitions and results to prove some graphs which are $\mathtt{d}_t$-unstable. These definitions and theorems are taken from [12].

Definition 2.3 *If $f_n(x)$ is a family of complex polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathbb{Z}(f_n(x))$, $\mathbb{Z}(f_n(x))$ is the set of the roots of the family $f_n(x)$.*

Now, a family $f_n(x)$ of polynomials is a recursive family of polynomials if $f_n(x)$ satisfy a homogeneous linear recurrence

$$f_n(x) = \sum_{i=1}^{k} a_i(x) f_{n-i}(x), \qquad (8)$$

where the $a_i(x)$ are fixed polynomials, with $a_k(x) \neq 0$. The number k is the order of the recurrence. We can form from equation (8) its associated characteristic equation

$$\lambda^k - a_1(x)\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_k(x) = 0 \tag{9}$$

whose roots $\lambda = \lambda(x)$ are algebraic functions, and there are exactly k of them counting multiplicity.

If these roots, say $\lambda_1(x), \lambda_2(x), \cdots, \lambda_k(x)$, are distinct, then the general solution to equation (8) is known to be

$$f_n(x) = \sum_{i=1}^{k} \alpha_i(x)\lambda_i(x)^n \tag{9}$$

with the usual variant if some of the $\lambda_i(x)$ are repeated. The functions

$$\alpha_1(x), \alpha_2(x), \cdots, \alpha_k(x)$$

are determined from the initial conditions, that is, the k linear equations in the α_i obtained by letting $n = 0, 1, \cdots, k-1$ in equation (10) or its variant. The details are available in [12]. Beraha, Kahane and Weiss [12] proved the following results on recursive families of polynomials and their roots.

Theorem 2.29 *If $f_n(x)$ is a recursive family of polynomials, then a complex number z is a limit of roots of $f_n(x)$ if and only if there is a sequence (z_n) in $\mathbb{C}$ such that $f_n(z_n) = 0$ for all n and $z_n \to z$ as $n \to \infty$.*

Theorem 2.30 *Under the non-degeneracy requirements that in equation* (10) *no $\alpha_i(x)$ is identically zero and that for no pair $i \neq j$ is it true that $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some complex number ω of unit modulus, then $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if and only if either*

(1) *two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or*

(2) *for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$.*

Corollary 2.7(see [2]) *Suppose $f_n(x)$ is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \ldots + \alpha_k(x)\lambda_k(x)^n \tag{11}$$

where the $\alpha_i(x)$ and the $\lambda_i(x)$ are fixed non-zero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then the limits of roots of $f_n(x)$ are exactly those z satisfying (i) or (ii) of Theorem 2.30.

Remark 2.2 *We have the total domination polynomial of G_3^n is*

$$D_t(G_3^n, x) = x(1+x)^{2n} - x + x^{2n}.$$

Rewrite $D_t(G_3^n, x)$ *as*

$$D_t(G_3^n, x) = f_{2n}(x) \quad = x(1+x)^{2n} + (-x)(1)^{2n} + (1)x^{2n}. \\ = \alpha_1\lambda_1^{2n} + \alpha_2\lambda_2^{2n} + \alpha_3\lambda_3^{2n},$$

where $\alpha_1 = x$, $\lambda_1 = 1 + x$, $\alpha_2 = -x$, $\lambda_2 = 1$, $\alpha_3 = 1$ *and* $\lambda_3 = x$. *Clearly* α_1,α_2 *and* α_3 *are not identically zero and* $\lambda_i \neq \omega\lambda_j$ *for* $i \neq j$ *and any complex number* ω *of modulus* 1. *Therefore the initial conditions of Theorem* 2.30 *are satisfied. Now, applying part(i) of Theorem* 2.30, we consider the following four different cases:

(1) $|\lambda_1| = |\lambda_2| = |\lambda_3|$;

(2) $|\lambda_1| = |\lambda_2| > |\lambda_3|$;

(3) $|\lambda_1| = |\lambda_3| > |\lambda_2|$;

(4) $|\lambda_2| = |\lambda_3| > |\lambda_1|$.

Case 1. Assume that $|1 + x| = |1| = |x|$. Then $|x - (-1)| = |x - 0|$ implies that x lies on the vertical line $z = -\frac{1}{2}$, $|x - (-1)| = 1$ implies that x lies on the unit circle centered at $(-1, 0)$ and $1 = |x - 0|$ implies that x lies on the unit circle centered at the origin. Therefore the two points of intersection, $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ are the limits of roots.

Case 2. Assume that $|1 + x| = |1| > |x|$. Then $|x - (-1)| = 1$ implies that x lies on the unit circle centered at $(-1, 0)$, $|x - (-1)| > |x - 0|$ implies that x lies to the right of the vertical line $z = -\frac{1}{2}$. Therefore the complex numbers x that satisfy $|x - (-1)| = 1$ and $\mathcal{R}(x) > -\frac{1}{2}$ are the limits of roots.

Case 3. Assume that $|1 + x| = |x| > |1|$. Then $|x - (-1)| = |x - 0|$ implies that x lies on the vertical line $z = -\frac{1}{2}$ and $|x - 0| > 1$ implies that x lies outside the unit circle centered at the origin. Therefore the complex numbers x that satisfy $|x| > 1$ and $\mathcal{R}(x) > -\frac{1}{2}$ are the limits of roots.

Case 4. Assume that $|1| = |x| > |1 + x|$. Then $1 = |x - 0|$ implies that x lies on the unit circle centered at the origin and $|x - 0| > |x - (-1)|$ implies that x lies to the left of the vertical line $z = -\frac{1}{2}$. Therefore the complex numbers x that satisfy $|x| = 1$ and $\mathcal{R}(x) < -\frac{1}{2}$ are the limits of roots.

The union of the curves and points above yield that, the limits of roots of the total domination polynomial of the Dutch windmill graph G_3^n consists of the part of the circle $|z| = 1$ with real part at most $-\frac{1}{2}$, the part of the circle $|z + 1| = 1$ with real part at least $-\frac{1}{2}$ and the part of the line $\mathcal{R}(z) = -\frac{1}{2}$ with modulus at least 1. So we conjectured that the Dutch windmill graph G_3^n is $\mathtt{d}_t$-stable for all n.

The total domination roots of the Dutch windmill graph G_3^n for $1 \leq n \leq 30$ are shown in Figure 3.

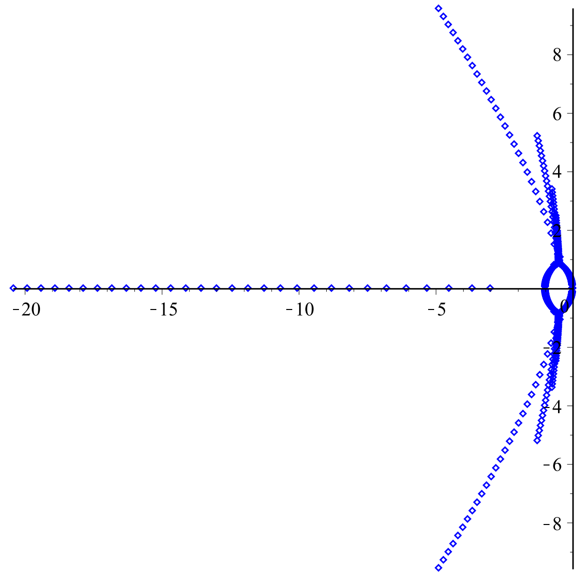

Figure 3 Total domination roots of G_3^n for $1 \leq n \leq 30$.

Remark 2.3 We have the total domination polynomial of B_n is

$$D_t(B_n, x) = ((1+x)^n - nx - 1)^2 .$$

Because of the same reason as mentioned in Remark 2.1, we conjectured that the bipartite cocktail party graph B_n is not a $\mathtt{d}_t$-stable for all but finite values of n.

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