International J.Math. Combin. Vol.1(2019), 73-80

The Number of Rooted Nearly 2-Regular Loopless Planar Maps

Shude Long

(Department of Mathematics, Chongqing University of Arts and Sciences, Chongqing 402160, P.R.China)

Junliang Cai

(School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P.R.China)

E-mail: longshude@163.com, caijunliang@bnu.edu.cn

Abstract: This paper investigates the enumeration of rooted nearly 2-regular loopless planar maps and presents some formulae for such maps with the valency of the root-face, the numbers of nonrooted vertices and inner faces as three parameters.

Key Words: Loopless map, nearly 2-regular map, enumerating function, functional equation, Lagrangian inversion.

AMS(2010): 05C45, 05C30.

§1. Introduction

Since Tutte's papers on enumerating planar maps in [21–23] were published in the early 1960's, the enumerative theory has been developed greatly up to now. Eulerian maps have played a crucial role in enumerative map theory. In particular, Tutte's sum-free formula [22] for the number of eulerian planar maps, all of whose vertices are labelled and contain a distinguished edge-end, with a given sequence of (even) vertex valencies was an essential stepin obtaining his ground-breaking formula for counting rooted planar maps by number of edges [23]. Several new results on the subject have been published since then (see, e.g. [1–20, 24]).

Here we deal with the enumeration of rooted nearly 2-regular loopless planar maps with the valency of the root-face, the numbers of nonrooted vertices and inner faces as three parameters. Several explicit expressions of its enumerating functions are obtained and one of them is summation-free.

A map is a connected graph cellularly embedded on a surface. A map is rooted if an edge and a direction along that edge are distinguished. If the root is the oriented edge from u to v, then u is the root-vertex while the face on the right side of the edge as seen by an observer on the edge facing away from u is defined as the root-face. A map is called Eulerian if all the valencies of its vertices are even. A nearly 2-regular map is a rooted map such that all vertices probably except the root-vertex are of valency 2. It is clear that a nearly 2-regular map is also an Eulerian map. In this paper, maps are always rooted and planar.

[1]Supported by NNSFC under Grant No. 10371033, No. 11571044 and the Natural Science Foundation Project of CQ under Grant No. cstc2014jcyjA00041.

[2]Received May 17, 2018, Accepted February 28, 2019.

For a set of some maps $\mathscr{M}$, the enumerating function discussed in this paper is defined as

$$f_{\mathscr{M}}(x,y,z)=\sum_{M\in\mathscr{M}}x^{l(M)}y^{p(M)}z^{q(M)}, \tag{1}$$

where $l(M), p(M)$ and $q(M)$ are the root-face valency, the number of nonrooted vertices and the number of inner faces of M, respectively.

Furthermore, we introduce some other enumerating functions for $\mathscr{M}$ as follows:

$$\begin{aligned}g_{\mathscr{M}}(x,y)&=\sum_{M\in\mathscr{M}}x^{l(M)}y^{n(M)},\\ h_{\mathscr{M}}(y,z)&=\sum_{M\in\mathscr{M}}y^{p(M)}z^{q(M)},\\ H_{\mathscr{M}}(y)&=\sum_{M\in\mathscr{M}}y^{n(M)},\end{aligned} \tag{2}$$

where $l(M), p(M)$ and $q(M)$ are the same in (1) and $n(M)$ is the number of edges of M, that is

$$\begin{aligned}g_{\mathscr{M}}(x,y)=&f_{\mathscr{U}}(x,y,y),\quad h_{\mathscr{M}}(y,z)=f_{\mathscr{M}}(1,y,z),\\ H_{\mathscr{M}}(y)=&g_{\mathscr{M}}(1,y)=h_{\mathscr{M}}(y,y)=f_{\mathscr{M}}(1,y,y).\end{aligned} \tag{3}$$

For the power series $f(x),\ f(x,y)$ and $f(x,y,z)$, we employ the following notations:

$$\partial_x^l f(x),\quad \partial_{(x,y)}^{(l,p)}f(x,y)\quad \textit{and}\quad \partial_{(x,y,z)}^{(l,p,q)}f(x,y,z)$$

to represent the coefficients of x^l in $f(x)$, x^ly^p in $f(x,y)$ and $x^ly^pz^q$ in $f(x,y,z)$, respectively. Terminologies and notations not explained here can be found in [10].

§2. Functional Equations

In this section we will set up the functional equations satisfied by the enumerating functions for rooted nearly 2-regular loopless planar maps.

Let $\mathscr{E}$ be the set of all rooted nearly 2-regular loopless planar maps with convention that the vertex map ϑ is in $\mathscr{E}$ for convenience. For any $M\in\mathscr{E}-\vartheta$, it is obvious that each edge of M is contained in only one circuit. The circuit containing the root-edge is called the root circuit of M, and denoted by $C(M)$.

It is clear that the length of the root circuit is no more than the root-face valency, and

$$\mathscr{E}=\mathscr{E}_0+\bigcup_{i\geq 2}\mathscr{E}_i, \tag{4}$$

where

$$\mathscr{E}_i=\{M\mid M\in\mathscr{E},\text{the length of }C(M)\text{ is }i\} \tag{5}$$

and $\mathscr{E}_0$ is only consist of the vertex map ϑ.

It is easy to see that the enumerating function of $\mathscr{E}_0$ is

$$f_{\mathscr{E}_0}(x, y, z) = 1. \tag{6}$$

For any $M \in \mathscr{E}_i$ $(i \geq 2)$, the root circuit divides $M - C(M)$ into two domains, the inner domain and outer domain. The submap of M in the outer domain is a general map in $\mathscr{E}$, while the submap of M in the inner domain does not contribute the valency of the root-face of M. Thus, the enumerating function of $\mathscr{E}_i$ is

$$f_{\mathscr{E}_i}(x, y, z) = x^i y^{i-1} zhf, \tag{7}$$

where $h = h_{\mathscr{E}}(y, z) = f_{\mathscr{E}}(1, y, z)$.

Theorem 2.1 *The enumerating function $f = f_{\mathscr{E}}(x, y, z)$ satisfies the following equation:*

$$f = \left(1 - \frac{x^2yzh}{1 - xy}\right)^{-1}, \tag{8}$$

where $h = h_{\mathscr{E}}(y, z) = f_{\mathscr{E}}(1, y, z)$.

Proof By (4), (6) and (7), we have

$$\begin{aligned} f =& 1 + \sum_{i \geq 2} x^i y^{i-1} zhf \\ =& 1 + \frac{x^2yzhf}{1 - xy}, \end{aligned}$$

which is equivalent to the theorem. □

Let $y = z$ in (8). Then we have

Corollary 2.1 *The enumerating function $g = g_{\mathscr{E}}(x, y)$ satisfies the following equation:*

$$g = \left(1 - \frac{x^2y^2H}{1 - xy}\right)^{-1}, \tag{9}$$

where $H = H_{\mathscr{E}}(y) = g_{\mathscr{E}}(1, y)$.

Let $x = 1$ in (8). Then we obtain

Corollary 2.2 *The enumerating function $h = h_{\mathscr{E}}(y, z)$ satisfies the following equation:*

$$yzh^2 + (y - 1)h - y + 1 = 0. \tag{10}$$

Further, let $y = z$ in (10). Then we have

Corollary 2.3 *The enumerating function $H = H_{\mathscr{E}}(y)$ satisfies the following equation:*

$$y^2H^2 + (y-1)H - y + 1 = 0. \tag{11}$$

§3. Enumeration

In this section we will find the explicit formulae for enumerating functions $f = f_{\mathscr{E}}(x, y, z), g = g_{\mathscr{E}}(x, y), h = h_{\mathscr{E}}(y, z)$ and $H = H_{\mathscr{E}}(y)$ by using Lagrangian inversion.

By (10) we have

$$h = \frac{(1-y)\left(1 - \sqrt{1 - \frac{4yz}{1-y}}\right)}{2yz}. \tag{12}$$

Let

$$y = \frac{\theta}{1+\theta}, \quad z = \eta(1 - \theta\eta). \tag{13}$$

By substituting (13) into (12), one may find that

$$h = \frac{1}{1 - \theta\eta}. \tag{14}$$

By (13) and (14), we have the following parametric expression of $h = h_{\mathscr{E}}(y, z)$:

$$\begin{aligned} y =& \frac{\theta}{1+\theta}, \quad z = \eta(1-\theta\eta), \\ h =& \frac{1}{1-\theta\eta} \end{aligned} \tag{15}$$

and from which we get

$$\Delta_{(\theta,\eta)} = \begin{vmatrix} \frac{1}{1+\theta} & 0 \\ * & \frac{1-2\theta\eta}{1-\theta\eta} \end{vmatrix} = \frac{1 - 2\theta\eta}{(1+\theta)(1-\theta\eta)}. \tag{16}$$

Theorem 3.1 *The enumerating function $h = h_{\mathscr{E}}(y, z)$ has the following explicit expression:*

$$h_{\mathscr{E}}(y, z) = 1 + \sum_{p \geq 1} \sum_{q=1}^{p} \frac{(2q)!(p-1)!}{q!(q+1)!(p-q)!(q-1)!} y^p z^q. \tag{17}$$

Proof By employing Lagrangian inversion with two parameters, from (15) and (16) one may find that

$$
\begin{aligned}
h_{\mathscr{E}}(y,z) &= \sum_{p,q\geq 0} \partial_{(\theta,\eta)}^{(p,q)} \frac{(1+\theta)^{p-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+2}} y^p z^q \\
&=1+\sum_{p,q\geq 1}\left[\partial_{(\theta,\eta)}^{(p,q)} \frac{(1+\theta)^{p-1}}{(1-\theta\eta)^{q+2}} - 2\partial_{(\theta,\eta)}^{(p-1,q-1)} \frac{(1+\theta)^{p-1}}{(1-\theta\eta)^{q+2}}\right] y^p z^q \\
&=1+\sum_{p\geq 1}\sum_{q=1}^{p} \frac{(2q)!}{q!(q+1)!} \partial_{\theta}^{p-q}(1+\theta)^{p-1} y^p z^q = 1+\sum_{p\geq 1}\sum_{q=1}^{p} \frac{(2q)!(p-1)!}{q!(q+1)!(p-q)!(q-1)!} y^p z^q,
\end{aligned}
$$

which is just the theorem. □

In what follows we present a corollary of Theorem 3.1.

Corollary 3.1 *The enumerating function $H = H_{\mathscr{E}}(y)$ has the following explicit expression:*

$$
H_{\mathscr{E}}(y) = 1 + \sum_{n\geq 2}\sum_{q=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(2q)!(n-q-1)!}{q!(q+1)!(n-2q)!(q-1)!} y^n. \tag{18}
$$

Proof It follows immediately from (17) by putting $y = z$ and $n = p + q$. □

Now, let

$$
x = \frac{\xi(1+\theta)}{1+\xi\theta}. \tag{19}
$$

By substituting (15) and (19) into Equ. (8), one may find that

$$
f = \frac{1}{1 - \frac{\xi^2\theta\eta(1+\theta)}{1+\xi\theta}}. \tag{20}
$$

By (15), (19) and (20), we have the parametric expression of the function $f = f_{\mathscr{E}}(x,y,z)$ as follows:

$$
\begin{aligned}
x &= \frac{\xi(1+\theta)}{1+\xi\theta}, \quad y = \frac{\theta}{1+\theta}, \\
z &= \eta(1-\theta\eta), \quad f = \frac{1}{1 - \frac{\xi^2\theta\eta(1+\theta)}{1+\xi\theta}}.
\end{aligned} \tag{21}
$$

According to (21), we have

$$
\Delta_{(\xi,\theta,\eta)} = \begin{vmatrix} \frac{1}{1+\xi\theta} & * & 0 \\ 0 & \frac{1}{1+\theta} & 0 \\ 0 & * & \frac{1-2\theta\eta}{1-\theta\eta} \end{vmatrix} = \frac{1-2\theta\eta}{(1+\xi\theta)(1+\theta)(1-\theta\eta)}. \tag{22}
$$

Theorem 3.2 *The enumerating function $f = f_{\mathscr{E}}(x,y,z)$ has the following explicit expression:*

$$f_{\mathscr{E}}(x,y,z) = 1 + \sum_{p\geq 1}\sum_{q=1}^{p}\sum_{l=2}^{p+q}\sum_{k=\max\{1,\lceil\frac{l+q-p}{2}\rceil\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \frac{k}{q}\binom{2q-k-1}{q-k}\binom{l-k-1}{l-2k} \times \binom{p-l+k-1}{p-q-l+2k} x^l y^p z^q. \quad (23)$$

Proof By using Lagrangian inversion with three variables, from (21) and (22) one may find that

$$\begin{aligned}
f_{\mathscr{E}}(x,y,z) &= \sum_{l,p,q\geq 0} \partial^{(l,p,q)}_{(\xi,\theta,\eta)} \frac{(1+\xi\theta)^{l-1}(1+\theta)^{p-l-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+1}\left[1-\frac{\xi^2\theta\eta(1+\theta)}{(1+\xi\theta)}\right]} x^l y^p z^q \\
&= \sum_{l,p,q\geq 0}\sum_{k=0}^{\min\{\lfloor\frac{l}{2}\rfloor,p,q\}} \partial^{(l-2k,p-k,q-k)}_{(\xi,\theta,\eta)} \frac{(1+\xi\theta)^{l-k-1}}{(1-\theta\eta)^{q+1}} \\
&\quad \times (1+\theta)^{p-l+k-1}(1-2\theta\eta) x^l y^p z^q \\
&= 1 + \sum_{p,q\geq 1}\sum_{l=2}^{p+q}\sum_{k=\max\{1,l-p\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \binom{l-k-1}{l-2k} \\
&\quad \times \partial^{(p-l+k,q-k)}_{(\theta,\eta)} \frac{(1+\theta)^{p-l+k-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+1}} x^l y^p z^q \\
&= 1 + \sum_{p,q\geq 1}\sum_{l=2}^{p+q}\sum_{k=\max\{1,l-p\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \binom{l-k-1}{l-2k} \\
&\quad \times \left[\partial^{(p-l+k,q-k)}_{(\theta,\eta)} \frac{(1+\theta)^{p-l+k-1}}{(1-\theta\eta)^{q+1}} \right. \\
&\quad \left. - 2\partial^{(p-l+k-1,q-k-1)}_{(\theta,\eta)} \frac{(1+\theta)^{p-l+k-1}}{(1-\theta\eta)^{q+1}}\right] x^l y^p z^q \\
&= 1 + \sum_{p\geq 1}\sum_{q=1}^{p}\sum_{l=2}^{p+q}\sum_{k=\max\{1,\lceil\frac{l+q-p}{2}\rceil\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \frac{(2q-k-1)!k}{(q-k)!q!}\binom{l-k-1}{l-2k} \\
&\quad \times \partial^{p-q-l+2k}_{\theta}(1+\theta)^{p-l+k-1} x^l y^p z^q \\
&= 1 + \sum_{p\geq 1}\sum_{q=1}^{p}\sum_{l=2}^{p+q}\sum_{k=\max\{1,\lceil\frac{l+q-p}{2}\rceil\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \frac{(2q-k-1)!k}{(q-k)!q!}\binom{l-k-1}{l-2k} \\
&\quad \times \binom{p-l+k-1}{p-q-l+2k} x^l y^p z^q,
\end{aligned}$$

which is what we wanted. □

Finally, we give a corollary of Theorem 3.2.

Corollary 3.2 *The enumerating function $g = g_{\mathscr{E}}(x,y)$ has the following explicit expression:*

$$g_{\mathscr{E}}(x,y) = 1 + \sum_{n\geq 2}\sum_{l=2}^{n}\sum_{q=1}^{\lfloor \frac{n}{2}\rfloor}\sum_{k=\max\{1,\lceil\frac{l+2q-n}{2}\rceil\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \frac{k}{q}\binom{2q-k-1}{q-k}\binom{l-k-1}{l-2k} \times \binom{n-q-l+k-1}{n-2q-l+2k} x^l y^n. \quad (24)$$

Proof It follows soon from (23) by putting $y = z$ and $n = p + q$. □

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