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Null Quaternionic Slant Helices in Minkowski Spaces

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Abstract: In this study, we give definition of null quaternionic slant helices by definition of slant helices in Euclidean 3-space. Besides, relationships between curvatures of null quaternionic curves are given for null quaternionic slant helices in Minkowski 3-space E_1^3. In other section, we give definition of null quaternionic W-slant helix in Minkowski 4-space E_1^4. We obtain that in which case curve is null quaternionic W-slant helix. Moreover, we have relationships between the curvatures of null quaternionic W-slant helix.

Key Words: Null quaternionic curves, slant helices, W-slant helices, Serret-Frenet formulae.

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§1. Introduction

In differential geometry of curves, a curve of constant slope or general helix in Euclidean 3-space E^3 is defined by the property that the tangent makes a constant angle with a fixed straight line. As other definition, a necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant [8,10,11]. In 2004, Izumiya and Takeuchi defined slant helices and conical geodesic curve. In the light of definition, slant helices are adapted to more spaces and surfaces constructed on slant helices by the researchers [1,6,9,12,13,15].

The quaternions were firstly introduced by Hamilton. Later, quaternions were very popular for researcher and many books were written about them. These papers are very important for examining differential geometry of quaternionic curves [7,14]. In 1987, the Serret-Frenet formulas for a quaternionic curve in E^3 and E^4 was defined by Bharathi and Nagaraj [2] and then in 2004, Serret-Frenet formulas for quaternionic curves and quaternionic inclined curves have been defined in Semi-Euclidean space by Çöken and Tuna [3]. In 2015, they defined Serret-Frenet formulas for null quaternionic curves in semi-Euclidean spaces [4,5].

In this paper, we give the definition of null quaternionic slant helices in Minkowski 3-space E_1^3 and null quaternionic W-slant helices in Minkowski 4-space E_1^4. We examine that in which case curve is null quaternionic W-slant helix or null quaternionic slant helix. Besides, we get relationships between the curvatures of these slant helices.

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§2. Preliminaries

In this section, we give basic concepts related to the semi-real quaternions. For more detailed information, we refer ref. [5,6].

The set of semi-real quaternions is given by

$$Q = \{ q| \quad q = ae_1 + be_2 + ce_3 + d\,; \quad a, b, c, d \in IR\}$$

where $e_1, e_2, e_3 \in E_1^3$, $h(e_i, e_i) = \varepsilon(e_i), \quad 1 \leqslant i \leqslant 3$ and

$$e_i \times e_i = -\varepsilon(e_i),$$

$$e_i \times e_j = \varepsilon(e_i)\varepsilon(e_i)e_k \ \in E_1^3.$$

The multiplication of two semi real quaternions p and q are defined by

$$p \times q = S_p S_q + S_p V_q + S_q V_p + h\left(V_p, V_q\right) + V_p \wedge V_q$$

where S_p and V_p is show scaler and vectoral parts of quaternion p.

Herein, we have inner and cross products in semi-Euclidean space E_1^3. $q = ae_1 + be_2 + ce_3 + d$ and$\alpha q = -ae_1 - be_2 - ce_3 + d$ are semi real quaternion and its conjugate, respectively and inner product h are defined by ([5])

$$h(p, q) = \frac{1}{2}\left[\varepsilon(p)\varepsilon(\alpha q)\left(p \times \alpha q\right) + \varepsilon(q)\varepsilon(\alpha p)\left(q \times \alpha p\right)\right]$$

The three-dimensional semi-Euclidean space E_1^3 is identified with the space of null spatial quaternionic curves $\left\{\gamma \in Q_{E_1^3}\middle| \quad \gamma + \alpha\gamma = 0\right\}$ in an obvious manner,

$$\gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i, \quad 1 \leqslant i \leqslant 3.$$

where $\{l, n, u\}$ are Frenet frames of the null quaternionic curves in E_1^3 and e_2 be timelike vector. Then, the Frenet formulae are

$$\begin{bmatrix} l' \\ n' \\ u' \end{bmatrix} = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & \tau \\ -\tau & -k & 0 \end{bmatrix} \begin{bmatrix} l \\ n \\ u \end{bmatrix} \tag{2.1}$$

where k and τ are curvatures of null spatial quaternionic curve and

$$h\left(l, l\right) = h\left(n, n\right) = h\left(l, u\right) = h\left(n, u\right) = 0\ ,\ h\left(l, n\right) = h\left(u, u\right) = 1. \tag{2.2}$$

Note that l and n are null vectors and u is a spacelike vector. Herein, the quaternion product is given by ([5])

$$\begin{aligned} &\times n = -1 - u, \quad n \times l = -1 + u, \quad n \times u = -n, \quad u \times n = n \\ &u \times l = -l, \quad l \times u = l, \quad u \times u = -1\ , \quad l \times l = n \times n = 0 \end{aligned} \tag{2.3}$$

Let $\gamma(s) = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3$ be a quaternionic curve in E_1^3. An orthonormal basis of E_1^4 is $\{e_1, e_2, e_3, e_4 = 1\}$ and let e_2 be timelike vector and $\beta = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4$ be a null quaternionic curve in E_1^4 defined over the interval I and $\{L, N, U, W\}$ be the Frenet components of β in E_1^4. Then, Frenet formulae are given by

$$\begin{bmatrix} L' \\ N' \\ U' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K \\ 0 & 0 & \tau + p & p \\ \tau + p & 0 & 0 & 0 \\ p & K & 0 & 0 \end{bmatrix} \begin{bmatrix} L \\ N \\ U \\ W \end{bmatrix} \tag{2.4}$$

where K is the first curvature of β in E_1^4. Here,

$$\begin{aligned} &h\left(L, L\right) = h\left(N, N\right) = h\left(L, U\right) = h\left(N, U\right) = h\left(W, U\right) = 0 \\ &h\left(U, U\right) = h\left(W, W\right) = 1, \quad h\left(L, N\right) = -1, \quad h\left(N, W\right) = h\left(L, W\right) = 0 \end{aligned} \tag{2.5}$$

where L and N are null vectors, U and W are spacelike vectors for which the quaternion product is given by ([6])

$$\begin{aligned} &L \times N = 1 - U\ ,\ N \times L = 1 + U\ ,\ N \times U = N\ ,\ U \times N = -N \\ &U \times L = L\ ,\ L \times U = -L\ ,\ U \times U = -1\ ,\ L \times L = N \times N = 0 \end{aligned} \tag{2.6}$$

§3. Null Quaternionic Slant Helices in E_1^3

In this section, we give definition of null quaternionic slant helices by using definition of the slant helices. Besides, relationships between curvatures of null quaternionic curves are given for null quaternionic slant helices.

Definition 3.1 *Let $\gamma(s)$ be a null quaternionic curve in Minkowski 3-space. If principal normal vector u of γ makes a constant angle θ by a constant direction v , then, the curve γ is called null quaternionic slant helix.*

Theorem 3.1 *Let $\gamma(s)$ be a null quaternionic curve in E_1^3. If γ is null quaternionic slant helix in E_1^3. Then, following results are obtained:*

(1) *If $v = a\,l + b\,n + c\,u$, $a, b, c \in IR$ is spacelike and plane spanned by u and v is timelike,*

$$v = a\,l + b\,n + (\cos\theta)\ u,\ ab = \frac{1}{2}\sin^2\theta$$

(2) *If* $v = a\,l + b\,n + c\,u,\ a, b, c \in IR$ *is spacelike and plane spanned by* u *and* v *is spacelike,*

$$v = a\,l + b\,n + (\cosh\theta)\ u, \quad ab = \frac{-\sinh^2\theta}{2} = \frac{1-\cosh^2\theta}{2}$$

(3) *If* $v = a\,l + b\,n + c\,u,\ a, b, c \in IR$ *is timelike,*

$$v = a\,l + b\,n + (\sinh\theta)\ u, \quad ab = \frac{-\cosh^2\theta}{2}$$

where u *is principal normal vector of* γ *and* θ *is constant angle between principal normal vector* u *and constant direction* v*.*

Proof Let γ be null quaternionic slant helix in E_1^3. Then, principal normal vector u of γ make a constant angle θ by a constant direction v. Suppose that constant direction v is

$$v = a\,l + b\,n + c\,u, \quad a, b, c \in IR. \tag{3.1}$$

Therefore, by using the quaternionic inner product, we have

$$h(u, v) = h(u, al + bn + cu) = a\,h(u, l) + b\,h(u, n) + c\,h(u, u),$$

$$h(u, v) = c. \tag{3.2}$$

(1) If $v = a\,l + b\,n + c\,u,\ a, b, c \in IR$ is spacelike and spanned plane by u and v is timelike, since principal normal vector u is spacelike, we write that

$$h(u, v) = c = \cos\theta. \tag{3.3}$$

Since constant direction v is spacelike, we have

$$\begin{aligned} h(v, v) &= ab\ h(l, n) + ba\ h(n, l) + c^2\,h(u, u) \\ &= 2ab + c^2 \end{aligned}. \tag{3.4}$$

From (**??**), we obtain the desired equality

$$v = a\,l + b\,n + (\cos\theta)\ u \quad ab = \frac{1}{2}\sin^2\theta. \tag{3.5}$$

(2) If $v = a\,l + b\,n + c\,u,\ \ a, b, c \in IR$ is spacelike and spanned plane by u and v is spacelike, since principal normal vector u is spacelike, we write that

$$h(u, v) = c = \cosh\theta. \tag{3.6}$$

Since constant direction v is spacelike and from (3.4), we have the desired equality.

(3) If $v = a\,l + b\,n + c\,u,\ \ a, b, c \in IR$ is timelike, since principal normal vector u is spacelike,

we write that

$$h\,(u,v) = c = \sinh\theta. \tag{3.7}$$

Since constant direction v is timelike and from (**??**), we find the desired result. □

Theorem 3.2 *Let $\gamma(s)$ be a null quaternionic curve in E_1^3. γ is null quaternionic slant helix in E_1^3 if and only if*

$$\tau b + \kappa a = 0. \tag{3.8}$$

Proof Let γ be null quaternionic slant helix, Then, principal normal vector u of γ make a constant angle θ by a constant direction v,

$$h\,(u,v) = \text{constant}. \tag{3.9}$$

By taking the derivation of (**??**), we get

$$\begin{aligned} h\,(u',v) &= h\,(-\tau l - \kappa n\,,\,v) \\ &= h\,(-\tau l - \kappa n\,,\;a\,l + b\,n + c\,u) \end{aligned} \tag{3.10}$$

$$h\,(u',v) = -\tau b\,h\,(l,n) - \kappa a\,h\,(n,l) = 0. \tag{3.11}$$

Therefore, we obtain the equation (**??**).

Conversely, we suppose that equation (**??**) is provided. By using the Frenet formulae, from (**??**) and (**??**), we obtain that $h\,(u',v) = -\tau b - \kappa a = 0$. Thus, it is clear that

$$h\,(u,v) = \text{constant}.$$

This means that γ is null quaternionic slant helix. □

§4. Null Quaternionic W-Slant Helices in E_1^4

In this section, we obtain that in which case curve is null quaternionic W-slant helix. Besides, we have relationships between the curvatures of null quaternionic W-slant helix.

Definition 4.1 *Let $\beta(s)$ be a null quaternionic curve in Minkowski 4-space. If binormal vector W of β makes a constant angle θ by a constant direction v . Then, the curve β is called null quaternionic W-slant helix.*

Theorem 4.1 *Let $\beta(s)$ be a null quaternionic curve in E_1^4. If β is null quaternionic W-slant helix in E_1^4. Then, following results are obtained:*

(1) *If $v^* = a\,L + b\,N + c\,U + d\,W$, $\;a,b,c,d \in IR$ is spacelike and plane spanned by W and v^* is timelike,*

$$v^* = a\,L + b\,N + c\,U + (\cos\theta)\;W, \quad 2ab - c^2 = -\sin^2\theta$$

(2) If $v^* = a\,L + b\,N + c\,U + d\,W, \ \ a,b,c,d \in IR$ is spacelike and plane spanned by W and v^* is spacelike,

$$v^* = a\,L + b\,N + c\,U + (\cosh\theta)\;W, \quad 2ab - c^2 = \sinh^2\theta$$

(3) If $v^* = a\,L + b\,N + c\,U + d\,W, \ \ a,b,c,d \in IR$ is timelike,

$$v^* = a\,L + b\,N + c\,U + (\sinh\theta)\;W, \quad 2ab - c^2 = \cosh^2\theta$$

where u is principal normal vector of γ and θ is constant angle between principal normal vector u and constant direction v.

Proof Let β be null quaternionic W-slant helix in E_1^4. Then, binormal vector W of β make a constant angle θ by a constant direction v^*. Suppose that constant direction v^* is

$$v^* = a\,L + b\,N + c\,U + d\,W, \quad a,b,c,d \in IR. \tag{4.1}$$

Thus, by using the quaternionic inner product, we get

$$h\,(W, v^*) = d. \tag{4.2}$$

(1) If $v^* = a\,L + b\,N + c\,U + d\,W, \ \ a,b,c,d \in IR$ is spacelike and plane spanned by W and v^* is timelike, since binormal vector W is spacelike, we write that

$$h\,(W, v^*) = \cos\theta. \tag{4.3}$$

Since constant direction v is spacelike, we have

$$\begin{aligned} h\,(v^*, v^*) &= ab\;h\,(L,N) + ba\;h\,(N,L) + c^2\,h\,(U,U) + d^2\,h\,(W,W) \\ &= -2ab + c^2 + d^2 \\ &= 1 \end{aligned} \quad . \tag{4.4}$$

From (**??**), (**??**) and (**??**), the proof is completed.

(2) If $v^* = a\,L + b\,N + c\,U + d\,W, \ \ a,b,c,d \in IR$ is spacelike and plane spanned by W and v^* is spacelike, since binormal vector W is spacelike, we have

$$h\,(W, v^*) = \cosh\theta. \tag{4.5}$$

Since constant direction v^* is spacelike, we have equality (**??**). From (**??**), (**??**) and (**??**), the desired result is found.

(3) If $v^* = a\,L + b\,N + c\,U + d\,W, \ \ a,b,c,d \in IR$ is timelike, since binormal vector W is spacelike, we write that

$$h\,(W, v^*) = \sinh\theta. \tag{4.6}$$

Since constant direction v^* is timelike, we obtain that

$$\begin{aligned} h\,(v^*, v^*) &= ab\; h\,(L, N) + ba\; h\,(N, L) + c^2\, h\,(U, U) + d^2\, h\,(W, W) \\ &= -2ab + c^2 + d^2 = -1 \end{aligned} \quad . \tag{4.7}$$

From (4.2), (4.6) and (4.7), the proof is completed. □

Theorem 4.2 *Let $\beta(s) = \gamma_1\, e_1 + \gamma_2\, e_2 + \gamma_3\, e_3 + \gamma_4\, e_4$ be null quaternionic curve in E_1^4, Then, β is null quaternionic W-slant helix if and only if*

$$p\, b = -K\, a. \tag{4.8}$$

Proof Suppose that β be null quaternionic W-slant helix in E_1^4. Then, binormal vector W of β make a constant angle θ by a constant direction v^*. Constant direction v^* is

$$v^* = a\, L + b\, N + c\, U + d\, W, \quad a, b, c, d \in IR.$$

Thus, we get equation (**??**). By differentiating equation (**??**) according to arclength parameter s and using the Frenet formulae of null quaternionic curves, we have

$$h(W', v^*) + h(W, {v^*}') = 0,$$

$$h(W', v^*) = h(p\, L + K\, N,\, v^*) = 0, \tag{4.9}$$

$$p\;h(L,\, v^*) + K\;h(N,\, v^*) = 0,$$

$$p\, b\;h(L,\, N) + K\, a\, h(N,\, L) = 0, \tag{4.10}$$

$$p\, b\;\; = -K\, a.$$

Conversely, we have equality (**??**). Suppose that the inner product of binormal vector W of β and a constant direction v^* is function $F(s)$. By differentiating of the inner product, we get that

$$-p\, b\; - K\, a\; = F'(s).$$

From (**??**), we obtain that $F(s)$ is a constant. That is, β is the null quaternionic W-slant helix in E_1^4. □

Theorem 4.3 *If the curve β is null quaternionic W-slant helix in E_1^4. Then, curvatures of null quaternionic W-slant helix is provided that*

$$2dpK + cK\,(\tau + p) - b\, p' - a\, K' = 0. \tag{4.11}$$

Proof If the curve β is null quaternionic W-slant helix in E_1^4. Then, we have equation

(??). By differentiating (??), we obtain that

$$p'\,h(L,v^*)+p\,h(L',v^*)+K'\,h(N,v^*)+K\,h(N',v^*)=0. \tag{4.12}$$

By substituting Frenet formulae in (4.12), we have

$$-b\,p'+p\,K\,h(W,v^*)+K'\,(-a)+K\,h\,((\tau+p)U+pW,v^*)=0 \tag{4.13}$$

and the desired equation is found. □

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