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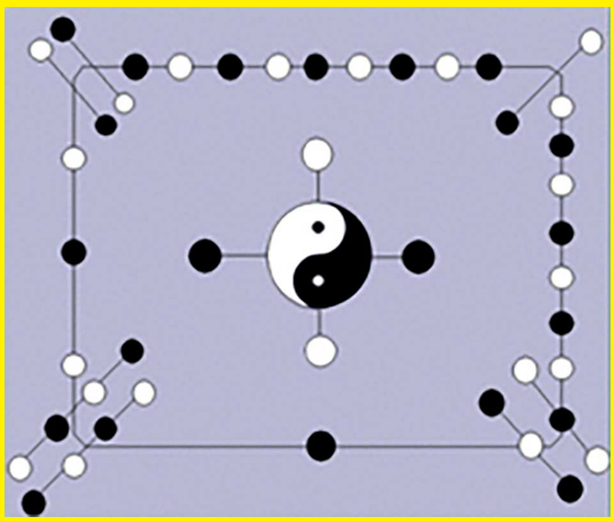

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Aims and Scope: The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly comprising 110-160 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

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Famous Words:

We must accept finite disappointment, but we must never lose infinite hope.

By Matin Luther King, a social activist and Baptist minister.

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Harmonic Flow's Dynamics on Animals in Microscopic Level With Balance Recovery

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Abstract: Actually, different models characterize things in the world, particularly, animals dependent on the microscopic level. However, there are no a mathematical subfield characterizing animals or human beings ourselves in such a level globally unless local elements such as points or spaces in classical sciences. *Could we establish a mathematics describing animal's microscopic behaviors globally?* The answer is affirmative. In fact, an animal or a human is nothing else but a skeleton or a topological graph under the electron microscope and generally, there always exist a universal connection between things, no matter which is an organic or inorganic matter in philosophy. We have found a new kind of mathematical elements, i.e., *continuity flows* or topological graphs $\overrightarrow{G}$ with each edge labeled by a vector and 2 end-operators of Banach space $\mathscr{B}$ holding with the continuity equation at vertices which globally characterizes the dynamic behavior of self-adaptive systems. However, the 12 meridians with treatment theory in Chinese medicine indicates that there is also a *harmonic flow* model, i.e., $\overrightarrow{G}^{L^2}$ with $L^2:(v,u)\in E\left(\overrightarrow{G}\right)\rightarrow(L(v,u),-L(v,u))$, $L(v,u)\in\mathscr{B}$ on human body which alludes that the Euler-Lagrange dynamic equation is more rightful for characterizing the dynamic behavior of animals in the microscopic level. In this paper, we establish such a mathematical theory on harmonic flows with dynamics, including Banach harmonic flow space closed under action of differential, integral operators. A few well-known results such as those of Banach theorem, closed graph theorem and Hahn-Banach theorem are generalized with extended Euler-Lagrange equation and balance recovery on harmonic flows. All of these results form elementary dynamics on harmonic flows for characterizing the behavior of self-adaptive systems, particularly, the animals or human beings.

Key Words: Harmonic flow, mathematical element, Banach space, harmonic flow dynamics, Smarandache multispace, mathematical combinatorics, Chinese medicine.

AMS(2010): 05C21,05C78,15A03,34B45,34K06,37N25,46A22,46B25,92B05.

§1. Introduction

Today, as the time passed into 21st century, a fundamental question on the function of mathematics is in front of scientists, i.e., *what is the nature of mathematics on reality of things*? And *what is its the original intension, is it just the minority's intellectual game on notations*?

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Certainly not because its original intension or nature is revealing the reality of things in the world. However, this aim is forgotten along with the development of mathematics in depth for many years ([24]).

As is well known, all mathematical elements came from the understanding of things by human's 5 sensory organs such as these of hearing, sight, smell, taste or touch, and also dependent on the observing is from macroscopic to microscopic or microscopic to macroscopic. The macroscopic recognizing is elementary but basic with an essential cognition in the microscopic. For example, an animal anatomy P is shown in Fig.1 in which we know that an animal is consisting of systems. For example, let μ_1 =nervous, μ_2 =circulatory, μ_3 =immune, μ_4 =endocrine, μ_5 =digestive, μ_6 =respiratory, μ_7 =urinary and μ_8 =reproductive systems with μ_9=epithelial tissue.

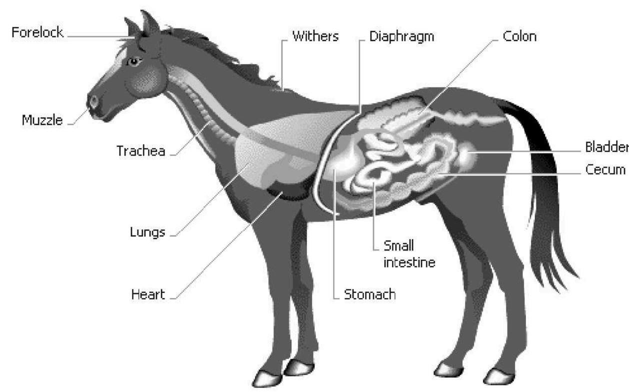


Fig.1

Whence, an animal P is understand by

$$P = \mu_1 \bigcup \mu_2 \bigcup \cdots \bigcup \mu_9 \tag{1.1}$$

in the macroscopic, which is nothing else but a *Smarandache multispace* ([8]-[10]) or *parallel universes* ([25]). But if we hold on P in the microscopic level, we know that all of its organic systems are consisted of cells, the smallest unit of life ([30]) and a cell is consisting of cytoplasm enclosed within a membrane that envelops the cell, regulates what moves in and out, maintains the electric potential of this cell and furthermore, inheres in a cytoskeleton, i.e., a stable and

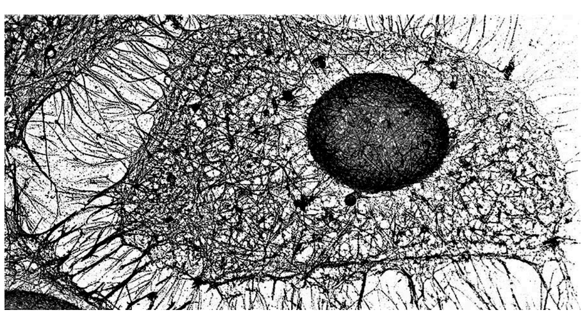


Fig.2

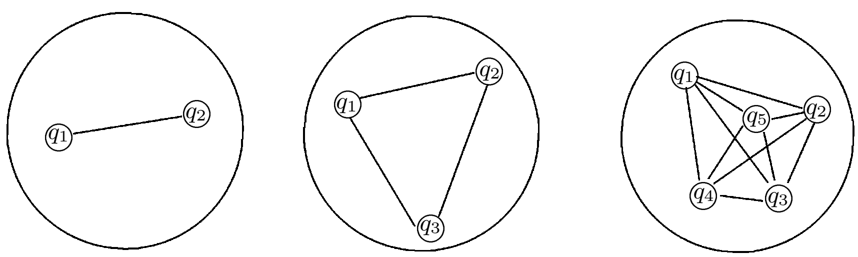
dynamic network of interlinking protein filaments that extend from the cell nucleus to the cell membrane, gives the cell's shape and structure such as those shown in Fig.2.

Let $\mathscr{N}(\mu_i)$ be the dynamic network of μ_i in cells at time t. Then, an animal P is underlying a complex network

$$P = \mathscr{N}(\mu_1) \bigcup \mathscr{N}(\mu_2) \bigcup \cdots \bigcup \mathscr{N}(\mu_9) \tag{1.2}$$

in the microscopic at time t, which is a complex network ([3], [4]).

Similarly, the divisibility of matter initiates human beings to search elementary constituting cells of matter, i.e., elementary particles such as those of quarks, leptons with interaction quanta including photons and other particles of mediated interactions, also with those of their antiparticles ([26]), and unmatters between a matter and its antimatter which is partially consisted of matter but others antimatter in the microscopic. Even though a free quark was never found in experiments, we can also get similar equalities (1.1) and (1.2) in theory such as those shown in Fig.3, where (a) a meson composed of a quark with an antiquark, (b) a baryon consisted of 3 quarks and (c) a particle composed of 5 quarks, respectively.



(a) (b) (c)

Fig.3

Notice that all these known characters on a thing P can not exist in isolation no matter which is organic or not, and the equality (1.2) is a complex network, or abstractly, a labeled graph G^L in space because they are indeed consisting of P. This fact also implies that we should find typical labeled graphs, called *continuity flows* and reviews them to be mathematical elements for revealing the reality of things ([19]) which can globally characterizes the dynamic behavior of things in the world.

Definition 1.1([22-23]) *A continuity flow* $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ *is an oriented embedded graph* $\overrightarrow{G}$ *in a topological space* $\mathscr{S}$ *associated with a mapping* $L : v \to L(v),\ (v,u) \to L(v,u)$*, 2 end-operators* $A^+_{vu} : L(v,u) \to L^{A^+_{vu}}(v,u)$ *and* $A^+_{uv} : L(u,v) \to L^{A^+_{uv}}(u,v)$ *on a Banach space* $\mathscr{B}$ *over a field* $\mathscr{F}$ *such as those shown in Fig.*4 *following*

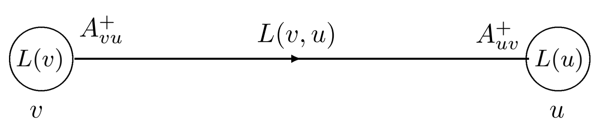


Fig.4

with $L(v,u) = -L(u,v)$, $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *holding with* *continuity equation*

$$\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u)=L(v) \quad for \quad \forall v\in V\left(\overrightarrow{G}\right)$$

and all such continuity flows are denoted by $\mathscr{G}_{\mathscr{B}}$.

Certainly, the continuity flows is such a mathematical element that its vertex equations maybe non-solvable ([11]-[14]). However, it indeed characterizes the reality of things, no matter what is organic or not. In fact, an independent energy system, including automobile, aircraft and animals is nothing else but a continuity flow, and there are monographs and papers published on continuity flows $\left(\overrightarrow{G};L,\mathscr{A}\right)$ with constraint conditions. For examples, the dynamic behavior of *complex network*, i.e., $A=\mathbf{1}_{\mathscr{V}}$ for $A\in\mathscr{A}$ with a number field $\mathbb{Z}$ or $\mathbb{R}$ is discussed in monographs [5] and [6]; an elementary $\overrightarrow{G}$*-flow theory*, i.e., $A=\mathbf{1}_{\mathscr{V}}$ for $A\in\mathscr{A}$ is established in [15]-[17] with applying to elementary particles; the *action flows*, i.e., x_v is a constant $\mathbf{v}_v$ dependent on v with applying to n-biological systems in [20]-[23] and an elementary theory on continuity flows is established in [22]-[23] with synchronization.

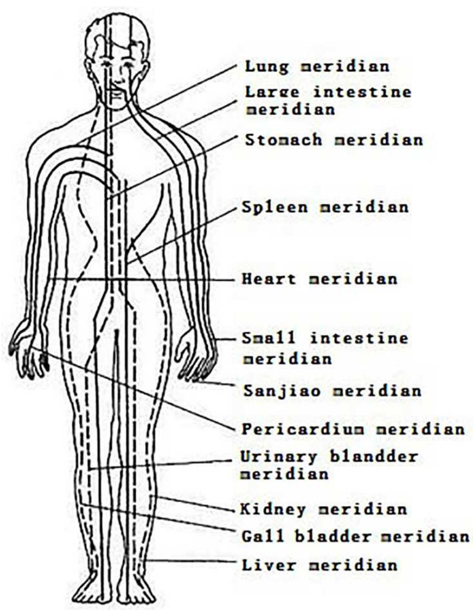


Fig.5

However, all of these results can not immediately characterize the regulatory or recovery mechanism of animals, particularly, the human body which means that we should furthermore find typical continuity flows for animals. It should be noted that preserving the balance Yin (Y^-) with Yang (Y^+) of a human body is the fundamental ruler, and there are 12 meridians in a human body which which completely reflects the physical condition in traditional Chinese medicine, i.e., the lung meridian of hand-TaiYin (LU), the large intestine meridian of hand YangMing (LI), the stomach meridian of foot-YangMing (ST), the spleen meridian of foot-TaiYin (SP), the heart meridian of hand-ShaoYin (HT), the small intestine meridian of hand-TaiYang (SI), the urinary blandder meridian of foot-TaiYang (BL), the kidney meridian of foot-ShaoYin (KI), the pericardium meridian of hand-JueYin (PC), the sanjiao meridian of hand-ShaoYang (SJ), the gall bladder meridian of foot-ShaoYang (GB), the liver meridian of foot-JueYin (LR) in Standard China National Standard (GB 12346-90), i.e., the *Body Model for Both Meridian and Extraordinary Points of China*, such as those in Fig.5, and similarly, the 12 meridians on animals such as the gall bladder meridian on a horse is shown in Fig.6.

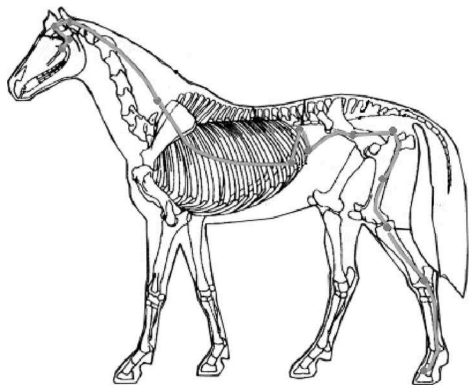


Fig.6

By the treatment theory in the traditional Chinese medicine ([29]), if there is a point on one of the 12 meridians in which $\{Y^-, Y^+\}$ are imbalance, this person must be in illness, and in turn, there must be points on the 12 meridians in which $\{Y^-, Y^+\}$ are imbalance for a patient, and the main duties of a doctor is to find out which points on which meridians are imbalance with Y^- more than Y^+ or Y^+ more than Y^-, and then by the natural ruler of the universe in traditional Chinese culture, i.e., *reducing the excess with supply the insufficient*, the doctor regulates these meridians by acupuncture or drugs so that points balance in $\{Y^-, Y^+\}$ again. This treatment theory naturally induced a subclass of continuity flows, called *harmonic flow* labeling each edge of a topological graph $\overrightarrow{G}$ by a 2-tuple vectors $(\mathbf{v}, -\mathbf{v})$ following.

Definition 1.2 *A harmonic flow* $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ *is an oriented embedded graph* $\overrightarrow{G}$ *in a topological space* $\mathscr{S}$ *associated with a mapping* $L : v \to (L(v), -L(v))$ *for* $v \in E\left(\overrightarrow{G}\right)$ *and* $L : (v,u) \to (L(v,u), -L(v,u))$*, 2 end-operators* A^+_{vu}*,* A^+_{uv} *with*

$$A^+_{vu} : (L(v,u), -L(v,u)) \to \left(L^{A^+_{vu}}(v,u), -L^{A^+_{vu}}(v,u)\right),$$
$$A^+_{uv} : (L(v,u), -L(v,u)) \to \left(L^{A^+_{uv}}(v,u), -L^{A^+_{uv}}(v,u)\right),$$

$L(v,u) = -L(u,v)$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *on a Banach space* $\mathscr{B}$ *holding with continuity equation*

$$\sum_{u \in N_G(v)} L^{A^+_{vu}}(v,u) = L(v)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$*, and all such harmonic flows are denoted by* $\mathscr{G}^{\pm}_{\mathscr{B}}$*.*

Clearly, a harmonic flow is naturally a continuity flow because of

$$\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u) + \sum_{u\in N_G(v)}\left(-L^{A^+_{vu}}(v,u)\right) = L(v) - L(v) = \mathbf{0}$$

for $\forall v \in \left(\overrightarrow{G}\right)$ and in fact, it is balanced at every where on $\overrightarrow{G}$ such as those shown in Fig.7, where $a, b, c \in \mathbb{R}$ hold with $a = b + c$.

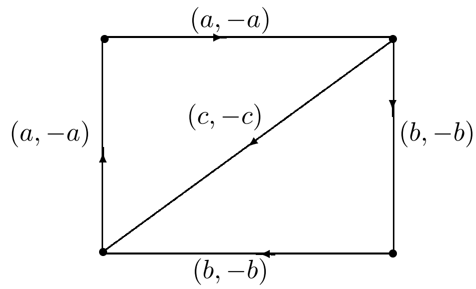


Fig.7

In this paper, we always assume that all end-operators in $\mathscr{A}$ are both *linear* and *continuous*. In this case, the result following on linear operators of Banach space is well-known.

Theorem 1.3([3]) *Let $\mathscr{B}_1, \mathscr{B}_2$ be Banach spaces over a field $\mathbb{F}$ with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then, a linear operator $\mathbf{T}:\mathscr{B}_1\to\mathscr{B}_2$ is continuous if and only if it is bounded, or equivalently,*

$$\|\mathbf{T}\| := \sup_{\mathbf{0}\neq\mathbf{v}\in\mathscr{B}_1}\frac{\|\mathbf{T}(\mathbf{v})\|_2}{\|\mathbf{v}\|_1} < +\infty.$$

The main purpose of this paper is to establish the dynamic theory on harmonic flows globally, an open problem for establishing graph dynamics in [7] including Banach harmonic flow space closed under action of differential, integral operators. A few well-known results such as those of Banach theorem, closed graph theorem and Hahn-Banach theorem are generalized with extended Euler-Lagrange equation and balance recovery on harmonic flows which is motivated by traditional Chinese medicine. We denote a continuity flow $\overrightarrow{G}^L$ with $L : (v,u) \to (L_1(v,u), L_2(v,u))$ by $\overrightarrow{G}^{L^2}$ for emphasizing L^2 mapping edges to $\mathscr{B}\times\mathscr{B}$, where $L_1(v,u), L_2(v,u) \in \mathscr{B}$ and all 2-tuple flows $\overrightarrow{G}^{L^2}$ with $L^2 : E\left(\overrightarrow{G}\right) \to \mathscr{B}\times\mathscr{B}$ by $\mathscr{G}_{\mathscr{B}^2}$.

For terminologies and notations not mentioned here, we follow references [1] for mechanics, [3] for functional analysis, [4] for biological mathematics, [5]-[6] for complex network, [10] for combinatorial geometry, and [9], [27] for Smarandache systems and multispaces.

§2. Banach Harmonic Flow Spaces

2.1 Commutative Rings over Graphs

Let $\mathscr{G}$ be a closed family of graphs $\overrightarrow{G}$ under the union operation and let $\mathscr{B}$ be a linear space $(\mathscr{B};+,\cdot)$, or furthermore, a commutative ring $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$. For $\forall\overrightarrow{G}^{L^2}, \overrightarrow{G}'^{L'^2} \in \mathscr{G}_{\mathscr{B}^2}$, define

$$\overrightarrow{G}^{L^2} + \overrightarrow{G'}^{L'^2} = \left(\overrightarrow{G} \setminus \overrightarrow{G'}\right)^{L^2} \bigcup \left(\overrightarrow{G} \bigcap \overrightarrow{G'}\right)^{L^2+L'^2} \bigcup \left(\overrightarrow{G'} \setminus \overrightarrow{G}\right)^{L'^2}, \tag{2.1}$$

$$\overrightarrow{G}^{L^2} \cdot \overrightarrow{G'}^{L'^2} = \left(\overrightarrow{G} \setminus \overrightarrow{G'}\right)^{L^2} \bigcup \left(\overrightarrow{G} \bigcap \overrightarrow{G'}\right)^{L^2 \cdot L'^2} \bigcup \left(\overrightarrow{G'} \setminus \overrightarrow{G}\right)^{L'^2} \tag{2.2}$$

and

$$\lambda \cdot \overrightarrow{G}^{L^2} = \overrightarrow{G}^{\lambda \cdot L^2}, \tag{2.3}$$

where $\lambda \in \mathscr{F}$ and

$$\begin{aligned}
&L^2 : (v,u) \to (L_1(v,u), L_2(v,u)), \quad {L'}^2 : (v,u) \to ({L'}_1(v,u), {L'}_2(v,u)), \\
&L^2 + {L'}^2 : (v,u) \to (L_1(v,u) + L_1'(v,u),\ L_2(v,u) + L_2'(v,u)), \\
&L^2 \ \cdot \ {L'}^2 : (v,u) \to (L_1(v,u) \cdot L_1'(v,u), L_2(v,u) \cdot L_2'(v,u)), \\
&\lambda \cdot L^2(v,u) = (\lambda \cdot L_1(v,u), \lambda \cdot L_2(v,u))
\end{aligned}$$

with substituting end-operator $A : (v,u) \to A^+_{vu}(v,u) + (A')^+_{vu}(v,u)$ or $A : (v,u) \to A^+_{vu}(v,u) \cdot (A')^+_{vu}(v,u)$ for $(v,u) \in E\left(\overrightarrow{G} \bigcap \overrightarrow{G'}\right)$ in $\overrightarrow{G}^{L^2} + \overrightarrow{G'}^{L'^2}$ or $\overrightarrow{G}^{L^2} \cdot \overrightarrow{G'}^{L'^2}$ and $L_1(v,u), L_2(v,u)$, $L_1'(v,u)$, $L_2'(v,u) \in \mathscr{B}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ or $E\left(\overrightarrow{G'}\right)$.

Define

$$L^\circ_{kl}(e) = \begin{cases} L^2_k(e), & \text{if } e \in E\left(\overrightarrow{G}_k \setminus \overrightarrow{G}_l\right) \\ L^2_l(e), & \text{if } e \in E\left(\overrightarrow{G}_l \setminus \overrightarrow{G}_k\right) \\ L^2_k(e) \circ L^2_l(e) & \text{if } e \in E\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right) \end{cases}, \tag{2.4}$$

and

$$L^\circ_{kls}(e) = \begin{cases} L^2_k(e), & \text{if } e \in E\left(\overrightarrow{G}_k \setminus \left(\overrightarrow{G}_l \bigcup \overrightarrow{G}_s\right)\right) \\ L^2_l(e), & \text{if } e \in E\left(\overrightarrow{G}_l \setminus \left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_s\right)\right) \\ L^2_s(e), & \text{if } e \in E\left(\overrightarrow{G}_s \setminus \left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)\right) \\ L^\circ_{kl}(e), & \text{if } e \in E\left(\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right) \setminus \overrightarrow{G}_s\right) \\ L^\circ_{ks}(e), & \text{if } e \in E\left(\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_s\right) \setminus \overrightarrow{G}_l\right) \\ L^\circ_{ls}(e), & \text{if } e \in E\left(\left(\overrightarrow{G}_l \bigcap \overrightarrow{G}_s\right) \setminus \overrightarrow{G}_k\right) \\ L^2_k(e) {}^\circ L^2_l(e) {}^\circ L^2_s(e) & \text{if } e \in E\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l \bigcap \overrightarrow{G}_s\right) \end{cases}, \tag{2.5}$$

where $\circ$ is the operation $+$, $-$ or $\cdot$ and $\overrightarrow{G}_k, \overrightarrow{G}_l, \overrightarrow{G}_s \in \mathscr{G}$.

Clearly, if $\overrightarrow{G}^{L^2}, \overrightarrow{G'}^{L'^2} \in \mathscr{G}_{\mathscr{B}^2}$ with linear end-operators A^+_{vu}, A^+_{uv}, then $\overrightarrow{G}^{L^2} + \overrightarrow{G'}^{L'^2}$, $\overrightarrow{G}^{L^2} \cdot \overrightarrow{G'}^{L'^2}$ and $\lambda \cdot \overrightarrow{G}^{L^2} \in \mathscr{G}^2_{\mathscr{B}}$, i.e., $\mathscr{G}_{\mathscr{B}^2}$ is closed under operations (2.1)-(2.3). Furthermore, for $\forall \overrightarrow{G}_k, \overrightarrow{G}_l, \overrightarrow{G}_s \in \mathscr{G}$ calculation shows the operations "+" and "·" satisfy

(1) commutative, i.e., $\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}=\overrightarrow{G}_l^{L_l^2}+\overrightarrow{G}_k^{L_k^2}$ and $\overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_l^{L_l^2}=\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_k^{L_k^2}$ because of

$$\begin{aligned}
\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2} &= \left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_k^2}\bigcup\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{L_k^2+L_l^2}\bigcup\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2}\\
&= \left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2}\bigcup\left(\overrightarrow{G}_l\bigcap\overrightarrow{G}_k\right)^{L_l^2+L_k^2}\bigcup\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_l^2}\\
&= \overrightarrow{G}_l^{L_l^2}+\overrightarrow{G}_k^{L_k^2},\\
\overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_l^{L_l^2} &= \left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_k^2}\bigcup\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{L_k^2\cdot L_l^2}\bigcup\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2}\\
&= \left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2}\bigcup\left(\overrightarrow{G}_l\bigcap\overrightarrow{G}_k\right)^{L_l^2\cdot L_k^2}\bigcup\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_k^2}\\
&= \overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_k^{L_k^2}.
\end{aligned}$$

(2) associative, i.e., $\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)+\overrightarrow{G}_s^{L_s^2}=\overrightarrow{G}_k^{L_k^2}+\left(\overrightarrow{G}_l^{L_l^2}+\overrightarrow{G}_s^{L_s^2}\right)$ and $\left(\overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_l^{L_l^2}\right)\cdot\overrightarrow{G}_s^{L_s^2}=\overrightarrow{G}_k^{L_k^2}\cdot\left(\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_s^{L_s^2}\right)$, and distributive, i.e., $\overrightarrow{G}_s^{L_s^2}\cdot\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)=\overrightarrow{G}_s^{L_s^2}\cdot\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_s^{L_s^2}\cdot\overrightarrow{G}_l^{L_l^2}$ and $\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)\cdot\overrightarrow{G}_s^{L_s^2}=\overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_s^{L_s^2}+\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_s^{L_s^2}$ if $\mathscr{B}$ is furthermore a commutative ring $(\mathscr{B};+,\cdot)$ because of

$$\begin{aligned}
\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)+\overrightarrow{G}_s^{L_s^2} &= \left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)^{L_{kl}^+}+\overrightarrow{G}_s^{L_s^2}=\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\bigcup\overrightarrow{G}_s\right)^{L_{kls}^+}\\
&= \overrightarrow{G}_k^{L_k^2}+\left(\overrightarrow{G}_l\bigcup\overrightarrow{G}_s\right)^{L_{ls}^+}=\overrightarrow{G}_k^{L_k^2}+\left(\overrightarrow{G}_l^{L_l^2}+\overrightarrow{G}_s^{L_s^2}\right),\\
\left(\overrightarrow{G}_k^{L_k^2}\ \cdot\ \overrightarrow{G}_l^{L_l^2}\right)\cdot\overrightarrow{G}_s^{L_s^2} &= \left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)^{L^{\cdot}_{kl}}\cdot\overrightarrow{G}_s^{L_s^2}=\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\bigcup\overrightarrow{G}_s\right)^{L^{\cdot}_{kls}}\\
&= \overrightarrow{G}_k^{L_k^2}\cdot\left(\overrightarrow{G}_l\bigcup\overrightarrow{G}_s\right)^{L^{\cdot}_{ls}}=\overrightarrow{G}_k^{L_k^2}\cdot\left(\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_s^{L_s^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\overrightarrow{G}_s^{L_s^2}\cdot\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right) &= \overrightarrow{G}_s^{L_s^2}\cdot\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)^{L^{\cdot}_{kl}}=\left(\overrightarrow{G}_s\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)\right)^{L^{\cdot}_{s(kl)}}\\
&= \left(\overrightarrow{G}_s\bigcup\overrightarrow{G}_k\right)^{L^{\cdot}_{sk}}\bigcup\left(\overrightarrow{G}_s\bigcup\overrightarrow{G}_l\right)^{L^{\cdot}_{sl}}=\overrightarrow{G}_s^{L_s^2}\cdot\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_s^{L_s^2}\cdot\overrightarrow{G}_l^{L_l^2}.
\end{aligned}$$

Similarly, we can check that $\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)\cdot\overrightarrow{G}_s^{L_s^2}=\overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_s^{L_s^2}+\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_s^{L_s^2}$.

(3) There are a unique zero flow $\mathbf{O}$, i.e., $\mathbf{O}(v,u)=\{\mathbf{0},\mathbf{0}\}$ in $(\mathscr{G}_{\mathscr{B}^2};+)$ and a unique unit zero $\mathbf{1}$, i.e., $\mathbf{1}(v,u)=\{\mathbf{1},\mathbf{1}\}$ for $\forall(v,u)\in E\left(\overrightarrow{\mathscr{G}}\right)$ in $(\mathscr{G}_{\mathscr{B}^2};\cdot)$ such that $\mathbf{O}+\overrightarrow{G}^{L^2}=\overrightarrow{G}^{L^2}+\mathbf{O}=\overrightarrow{G}^{L^2}$ and $\mathbf{1}\cdot\overrightarrow{G}^{L^2}=\overrightarrow{G}^{L^2}\cdot\mathbf{1}=\overrightarrow{G}^{L^2}$;

(4) For $\forall\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}^2}$ there is a unique flow $\overrightarrow{G}^{-L^2}$ such that $\overrightarrow{G}^{L^2}+\overrightarrow{G}^{-L^2}=\mathbf{O}$;

(5) A scalar multiplication "$\cdot$ defined by (2.3) associating a flow $\overrightarrow{G}^{L^2}$ in $\mathscr{G}_{\mathscr{B}^2}$ and a scalar $\alpha\in\mathscr{F}$ with a flow $\alpha\cdot\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}^2}$ in such a way that

(*a*) $1\cdot\overrightarrow{G}^{L^2}=\overrightarrow{G}^{L^2}$;

(*b*) $(\alpha_1\alpha_2)\cdot\overrightarrow{G}^{L^2}=\alpha_1(\alpha_2\cdot\overrightarrow{G}^{L^2})$ for $\alpha_1,\alpha_2\in\mathscr{F}$;

(*c*) $\alpha\cdot(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2})=\alpha\cdot\overrightarrow{G}_k^{L_k^2}+\alpha\cdot\overrightarrow{G}_l^{L_l^2}$ for $\alpha\in\mathscr{F}$;

(*d*) $(\alpha_1+\alpha_2)\cdot\overrightarrow{G}^{L^2}=\alpha_1\cdot\overrightarrow{G}^{L^2}+\alpha_2\cdot\overrightarrow{G}^{L^2}$ for $k_1,k_2\in\mathscr{F}$.

In conclusion, we know that $(\mathscr{G}_{\mathscr{B}^2},+)$ and $(\mathscr{G}_{\mathscr{B}^2},\cdot)$ are respectively a commutative group, a commutative semigroup with unit if $\mathscr{B}$ is a commutative ring, and $(\mathscr{G}_{\mathscr{B}^2},+,\cdot)$ is a linear space if $\mathscr{B}$ is so. We therefore get the following result.

Theorem 2.1 *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a linear space $(\mathscr{B};+,\cdot)$, then, all pair flows $(\mathscr{G}_{\mathscr{B}^2};+,\cdot)$ is a linear space, and furthermore, a commutative ring if $\mathscr{B}$ is a commutative ring $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$.*

2.2 Banach Harmonic Flow Space

For $\forall\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}^2}$ with $L^2(e)=(L_1(e),L_2(e))$, $e\in E\left(\overrightarrow{G}\right)$ define

$$\left\|\overrightarrow{G}^{L^2}\right\|=\sum_{e\in E\left(\overrightarrow{G}\right)}\left(\|L_1(e)\|+\|L_2(e)\|\right),\tag{2.6}$$

where $\mathscr{B}$ is a Banach space $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$ with a norm $\|\cdot\|$. Then, for $\forall\overrightarrow{G}^{L^2},\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}_{\mathscr{B}^2}$ we are easily know that

(1) $\left\|\overrightarrow{G}^{L^2}\right\|\geq 0$ and $\left\|\overrightarrow{G}^{L^2}\right\|=0$ if and only if $L_1(e)=\mathbf{0}$ and $L_2(e)=\mathbf{0}$, i.e., $\overrightarrow{G}^{L^2}=\mathbf{O}$;

(2) $\left\|\overrightarrow{G}^{\xi L^2}\right\|=|\xi|\left\|\overrightarrow{G}^{L^2}\right\|$ for any scalar $\xi\in\mathscr{F}$;

(3) $\left\|\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right\|\leq\left\|\overrightarrow{G}_k^{L_k^2}\right\|+\left\|\overrightarrow{G}_l^{L_l^2}\right\|$ because of

$$\begin{aligned}\left\|\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right\| &= \sum_{e\in E\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)}\left(\|L_{k1}(e)\|+\|L_{k2}(e)\|\right)+\sum_{e\in E\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)}\left(\|L_{l1}(e)\|+\|L_{l2}(e)\|\right)\\ &\quad+\sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\|L_{k1}(e)+L_{l1}(e)\|+\|L_{k2}(e)+L_{l2}(e)\|\right)\\ &\leq \sum_{e\in E\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)}\left(\|L_{k1}(e)\|+\|L_{k2}(e)\|\right)+\sum_{e\in E\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)}\left(\|L_{l1}(e)\|+\|L_{l2}(e)\|\right)\\ &\quad+\sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\|L_{k1}(e)\|+\|L_{k2}(e)\|+\|L_{l1}(e)\|+\|L_{l2}(e)\|\right)\\ &= \left\|\overrightarrow{G}_k^{L_k}\right\|+\left\|\overrightarrow{G}_l^{L_l}\right\|\end{aligned}$$

by $\|\mathbf{v}_1+\mathbf{v}_2\|\leq\|\mathbf{v}_1\|+\|\mathbf{v}_2\|$ for $\forall\mathbf{v}_1,\mathbf{v}_2\in\mathscr{B}$. Therefore, $\|\cdot\|$ is also a norm on $\mathscr{G}_{\mathscr{B}^2}$.

Furthermore, if $\mathscr{B}$ is a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, for $\forall\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}_{\mathscr{B}^2}$, define

$$\left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right\rangle=\sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\langle L_{k1}(e),L_{l1}(e)\rangle+\langle L_{k2}(e),L_{l2}(e)\rangle\right)\tag{2.7}$$

Clearly, $\left\langle\overrightarrow{G}_k^{L_k^2}\bigcup\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle=\left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}^{L^2}\right\rangle+\left\langle\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle$ if $E\left(\overrightarrow{G}_k\right)\bigcap E\left(\overrightarrow{G}_l\right)=\emptyset$ and

$\left\langle \overrightarrow{G}_k^{L_{k1}^2+L_{k2}^2}, \overrightarrow{G}^{L^2} \right\rangle = \left\langle \overrightarrow{G}_k^{L_{k1}^2}, \overrightarrow{G}^{L^2} \right\rangle + \left\langle \overrightarrow{G}_k^{L_{k2}^2}, \overrightarrow{G}^{L^2} \right\rangle$ by definition (2.9), and we are easily know also that

(1) For $\forall \overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}^2}$,

$$\left\langle \overrightarrow{G}^{L^2}, \overrightarrow{G}^{L^2} \right\rangle = \sum_{e \in E\left(\overrightarrow{G}\right)} \left(\langle L_1(e), L_1(e)\rangle + \langle L_2(e), L_2(e)\rangle\right) \geq 0$$

and $\left\langle \overrightarrow{G}^{L^2}, \overrightarrow{G}^{L^2} \right\rangle = 0$ if and only if $L_1(e) = \mathbf{0}$, $L_2(e) = \mathbf{0}$, i.e., $\overrightarrow{G}^{L^2} = \mathbf{O}$.

(2) For $\forall \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}_{\mathscr{B}^2}$, $\left\langle \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \right\rangle = \overline{\left\langle \overrightarrow{G}_l^{L_l^2}, \overrightarrow{G}_k^{L_k^2} \right\rangle}$ because of

$$\begin{aligned}
\left\langle \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \right\rangle &= \sum_{e \in E\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)} \left(\langle L_{k1}(e), L_{l1}(e)\rangle + \langle L_{k2}(e), L_{l2}(e)\rangle\right) \\
&= \sum_{e \in E\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)} \left(\overline{\langle L_{l1}(e), L_{k1}(e)\rangle} + \overline{\langle L_{l2}(e), L_{k2}(e)\rangle}\right) \\
&= \overline{\left\langle \overrightarrow{G}_l^{L_l^2}, \overrightarrow{G}_k^{L_k^2} \right\rangle}
\end{aligned}$$

for $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \overline{\langle \mathbf{v}_2, \mathbf{v}_1 \rangle}$, $\mathbf{v}_1, \mathbf{v}_2$ in Hilbert space $\mathscr{B}$.

(3) For $\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}_{\mathscr{B}^2}$ and $\lambda \in \mathscr{F}$, $\left\langle \overrightarrow{G}_k^{L_k^2}, \lambda \overrightarrow{G}_l^{L_l^2}) \right\rangle = \lambda \left\langle \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \right\rangle$ because of

$$\begin{aligned}
\left\langle \overrightarrow{G}_k^{L_k^2}, \lambda \overrightarrow{G}_l^{L_l^2} \right\rangle &= \left\langle \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{\lambda L_l^2} \right\rangle \\
&= \sum_{e \in E\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)} \left(\langle L_{k1}(e), \lambda L_{l1}(e)\rangle + \langle L_{k2}(e), \lambda L_{l2}(e)\rangle\right) \\
&= \sum_{e \in E\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)} \lambda \left(\langle L_{k1}(e), L_{l1}(e)\rangle + \langle L_{k2}(e), L_{l2}(e)\rangle\right) \\
&= \lambda \sum_{e \in E\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)} \left(\langle L_{k1}(e), L_{l1}(e)\rangle + \langle L_{k2}(e), L_{l2}(e)\rangle\right) \\
&= \lambda \left\langle \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \right\rangle.
\end{aligned}$$

by definition (2.7).

(4) For $\overrightarrow{G}^{L^2}, \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}_{\mathscr{B}^2}$, $\left\langle \overrightarrow{G}_k^{L_k^2} + \overrightarrow{G}_l^{L_l^2}, \overrightarrow{G}^{L^2} \right\rangle = \left\langle \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}^{L^2} \right\rangle + \left\langle \overrightarrow{G}_l^{L_l^2}, \overrightarrow{G}^{L^2} \right\rangle$ because of

$$\begin{aligned}
\left\langle \overrightarrow{G}_k^{L_k^2} + \overrightarrow{G}_l^{L_l^2}, \overrightarrow{G}^{L^2} \right\rangle &= \left\langle \left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)^{L_{kl}^+}, \overrightarrow{G}^{L^2} \right\rangle \\
&= \left\langle \left(\overrightarrow{G}_k \setminus \overrightarrow{G}_l\right)^{L_k^2} \bigcup \left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)^{L_k^2+L_l^2} \bigcup \left(\overrightarrow{G}_l \setminus \overrightarrow{G}_k\right)^{L_l^2}, \overrightarrow{G}^{L^2} \right\rangle
\end{aligned}$$

$$\begin{aligned}
&= \left\langle\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_k^2},\overrightarrow{G}^{L^2}\right\rangle+\left\langle\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle\\
&\quad+\left\langle\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{L_k^2+L_l^2},\overrightarrow{G}^{L^2}\right\rangle\\
&= \left\langle\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_k^2},\overrightarrow{G}^{L^2}\right\rangle+\left\langle\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{L_k^2},\overrightarrow{G}^{L^2}\right\rangle\\
&\quad+\left\langle\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle+\left\langle\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle\\
&= \left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}^{L^2}\right\rangle+\left\langle\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle.
\end{aligned}$$

by definition (2.9). Whence, $(\mathscr{G}_{\mathscr{B}^2};+,\cdot)$ is also an inner product space and a normed space with

$$\left\|\overrightarrow{G}^{L^2}\right\|=\sqrt{\left\langle\overrightarrow{G}^{L^2},\overrightarrow{G}^{L^2}\right\rangle}$$

for $\forall\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}^2}$.

Definition 2.2 *For $\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}_{\mathscr{B}^2}$, the distance between $\overrightarrow{G}_k^{L_k^2}$, $\overrightarrow{G}_l^{L_l^2}$ is defined by*

$$d\left(\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right)=\left\|\overrightarrow{G}_k^{L_k^2}-\overrightarrow{G}_l^{L_l^2}\right\|=\left\|\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{-L_l^2}\right\|. \tag{2.8}$$

Clearly, $(\mathscr{G}_{\mathscr{B}^2};+,\cdot)$ is also a distance space by Definition 2.2 with previous properties $(1)-(3)$ or $(1)-(4)$ of Banach or Hilbert space, respectively.

Definition 2.3 *A sequence $\forall\overrightarrow{G}_1^{L_1^2},\overrightarrow{G}_2^{L_2^2},\cdots,\overrightarrow{G}_n^{L^2}$ in $\mathscr{G}_{\mathscr{B}^2}$ is called Cauchy sequence if for any number $\varepsilon>0$, there always exists an integer $N(\varepsilon)$ such that*

$$\left\|\overrightarrow{G}_n^{L_k^2}-\overrightarrow{G}_l^{L_l^2}\right\|<\varepsilon$$

for integers $k,l\geq N(\varepsilon)$.

Let $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ be a Cauchy sequence of $\mathscr{G}_{\mathscr{B}^2}$ and $\overrightarrow{\Pi}=\bigcup\limits_{\overrightarrow{G}\in\mathscr{G}}\overrightarrow{G}$. Notice that $\mathscr{G}$ is closed under operation union by assumption. We know that $\overrightarrow{\Pi}\in\mathscr{G}$ is finite and embed each $\overrightarrow{G}_n^{L_n^2}$ into a subflows $\overrightarrow{\Pi}^{\widehat{L^2}}\in\mathscr{G}_{\mathscr{B}^2}$ by defining

$$\widehat{L}_n^2(e)=\left\{\begin{array}{ll} L_n^2(e) & \text{if} \quad e\in E\left(\overrightarrow{G}_n\right),\\ \{\mathbf{0},\mathbf{0}\} & \text{if} \quad e\in E\left(\overrightarrow{\Pi}\setminus\overrightarrow{G}_n\right).\end{array}\right.$$

Clearly,

$$\begin{aligned}
\left\|\overrightarrow{G}^{L_k^2}-\overrightarrow{G}^{L_l^2}\right\| &= \left\|\overrightarrow{\Pi}^{\widehat{L}_k^2}-\overrightarrow{\Pi}^{\widehat{L}_l^2}\right\|=\left\|\overrightarrow{\Pi}^{\widehat{L}_k^2-\widehat{L}_l^2}\right\|\\
&= \sum_{e\in E\left(\overrightarrow{\Pi}\right)}\left(\left\|\widehat{L}_{k1}(e)-\widehat{L}_{l1}(e)\right\|+\left\|\widehat{L}_{k2}(e)-\widehat{L}_{l2}(e)\right\|\right).
\end{aligned}$$

Now, for $\forall\varepsilon > 0$ if $\left\|\overrightarrow{G}^{L_k^2} - \overrightarrow{G}^{L_l^2}\right\| \leq \varepsilon$ for integers $k, l \geq N(\varepsilon)$ then there must be $\left\|\widehat{L}_{k1}(e) - \widehat{L}_{l1}(e)\right\| \leq \varepsilon$ and $\left\|\widehat{L}_{k2}(e) - \widehat{L}_{l2}(e)\right\| \leq \varepsilon$ for integers $k, l \geq N(\varepsilon)$, i.e., $\left\{\widehat{L}_n^2\right\}$ is a Cauchy sequence for $\forall e \in E\left(\overrightarrow{\Pi}\right)$, which is convergent in $\mathscr{B}$ by assumption. Without loss of generality, let $\lim\limits_{n\to\infty} \widehat{L}_n^2 = (L_{01}, L_{02}) = L_0^2$. Then, $\lim\limits_{n\to\infty} \overrightarrow{G}_n^{L_n^2} = \lim\limits_{n\to\infty} \overrightarrow{\Pi}^{\widehat{L}_n^2} = \overrightarrow{\Pi}^{\lim\limits_{n\to\infty} \widehat{L}_n^2} = \overrightarrow{\Pi}^{L_0^2}$, i.e., $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ is convergent in $\mathscr{G}_{\mathscr{B}^2}$ by definition. We therefore get the result following.

Theorem 2.4 *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a Banach space $(\mathscr{B}; +, \cdot)$, then, $\mathscr{G}_{\mathscr{B}^2}$ with linear operators A_{vu}^+, A_{uv}^+ for $\forall(v,u) \in E\left(\bigcup\limits_{G\in\mathscr{G}} \overrightarrow{G}\right)$ is a Banach space, and furthermore, if $\mathscr{B}$ is a Hilbert space, $\mathscr{G}_{\mathscr{B}^2}$ is a Hilbert space too.*

We have known that all continuity flows $\overrightarrow{G}^L$ form a Banach or Hilbert space $\mathscr{G}_{\mathscr{B}}$ respect to that $\mathscr{B}$ is a Banach or Hilbert space in [24] and [25]. By definition, $\left(\overrightarrow{G}_i^L, \overrightarrow{G}_j^L\right) \in \mathscr{G}_{\mathscr{B}}^2$ for $\overrightarrow{G}_i^L, \overrightarrow{G}_j^L \in \mathscr{G}_{\mathscr{B}}$. Notice that a harmonic flow $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}^{\pm}$ is isomorphic to a continuity flow $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$ because flows $(L^{A_{vu}^+}(v,u), -L^{A_{vu}^+}(v,u))$ is isomorphic to $L^{A_{vu}^+}(v,u)$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$. Thus, all harmonic flows $\mathscr{G}_{\mathscr{B}}^{\pm}$ is in fact a Banach or Hilbert subspace of $\mathscr{G}_{\mathscr{B}}^2$ by Theorem 2.4.

Theorem 2.5 *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a Banach space $(\mathscr{B}; +, \cdot)$, then, all harmonic flows $\mathscr{G}_{\mathscr{B}}$ with linear operators A_{vu}^+, A_{uv}^+ for $\forall(v,u) \in E\left(\bigcup\limits_{G\in\mathscr{G}} \overrightarrow{G}\right)$ under operations $+$ and $\cdot$ form a Banach or Hilbert space respect to that $\mathscr{B}$ is a Banach or Hilbert space with inclusions*

$$\mathscr{G}_{\mathscr{B}}^{\pm} \subset \mathscr{G}_{\mathscr{B}}^2 \subset \mathscr{G}_{\mathscr{B}^2}.$$

2.3 Operators on Banach Harmonic Flow Space

Definition 2.7 *Let $\mathbf{T} : \mathscr{G}_{\mathscr{B}}^{\pm} \to \mathscr{B}^{\pm}$ be an operator on Banach harmonic flow space $\mathscr{G}_{\mathscr{B}}^{\pm}$ over a field $\mathscr{F}$. Then, $\mathbf{T}$ is linear if*

$$\mathbf{T}\left(\lambda\overrightarrow{G}_k^{L_k^2} + \mu\overrightarrow{G}_l^{L_l^2}\right) = \lambda\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right) + \mu\mathbf{T}\left(\overrightarrow{G}_l^{L_l^2}\right)$$

for $\forall\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ and $\lambda, \mu \in \mathscr{F}$, is continuous at $\overrightarrow{G}_0^{L_0^2}$ if there always exist a number $\delta(\varepsilon)$ for $\forall\epsilon > 0$ such that

$$\left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right) - \mathbf{T}\left(\overrightarrow{G}_0^{L_0^2}\right)\right\| < \varepsilon$$

if $\left\|\overrightarrow{G}^{L^2} - \overrightarrow{G}_0^{L_0^2}\right\| < \delta(\varepsilon)$, bounded if $\left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\| \leq \xi\left\|\overrightarrow{G}^{L^2}\right\|$ for $\forall\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ with a constant $\xi \in [0, \infty)$ and furthermore, a contractor if

$$\left\|\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right) - \mathbf{T}\left(\overrightarrow{G}_l^{L_l^2}\right)\right\| \leq \xi\left\|\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right\|$$

for $\forall\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ with $\xi \in [0, 1)$.

Theorem 2.8(Fixed Harmonic Flow Theorem) *If* $\mathbf{T}: \mathscr{G}^{\pm}_{\mathscr{B}} \to \mathscr{G}^{\pm}_{\mathscr{B}}$ *is a linear continuous contractor, then there is a uniquely harmonic flow* $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$ *such that*

$$\mathbf{T}\left(\overrightarrow{G}^{L^2}\right) = \overrightarrow{G}^{L^2}.$$

Proof Let $\overrightarrow{G}^{L_0^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$ be a harmonic flow. We define a sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ by

$$\begin{array}{l}
\overrightarrow{G}_1^{L_1^2} = \mathbf{T}\left(\overrightarrow{G}^{L_0^2}\right), \\
\overrightarrow{G}_2^{L_2^2} = \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2}\right) = \mathbf{T}^2\left(\overrightarrow{G}^{L_0^2}\right), \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
\overrightarrow{G}_n^{L_n} = \mathbf{T}\left(\overrightarrow{G}_{n-1}^{L_{n-1}^2}\right) = \mathbf{T}^n\left(\overrightarrow{G}^{L_0^2}\right), \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}$$

We prove the sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ is a Cauchy sequence in $\mathscr{G}^{\pm}_{\mathscr{B}}$. By assumption $\mathbf{T}$ is a contractor, there is a constant $\xi \in [0,1)$ such that $\left\|\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right) - \mathbf{T}\left(\overrightarrow{G}_l^{L_l^2}\right)\right\| \le \xi \left\|\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right\|$ for $\forall \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$. We therefore know that

$$\begin{aligned}
\left\|\overrightarrow{G}_{m+1}^{L_{m+1}^2} - \overrightarrow{G}_m^{L_m^2}\right\| &= \left\|\mathbf{T}\left(\overrightarrow{G}_m^{L_m^2}\right) - \mathbf{T}\left(\overrightarrow{G}_{m-1}^{L_{m-1}^2}\right)\right\| \le \xi \left\|\overrightarrow{G}_m^{L_m^2} - \overrightarrow{G}_{m-1}^{L_{m-1}^2}\right\| \\
&= \left\|\mathbf{T}\left(\overrightarrow{G}_{m-1}^{L_{m-1}^2}\right) - \mathbf{T}\left(\overrightarrow{G}_{m-2}^{L_{m-2}^2}\right)\right\| \le \xi^2 \left\|\overrightarrow{G}_{m-1}^{L_{m-1}^2} - \overrightarrow{G}_{m-2}^{L_{m-2}^2}\right\| \\
&\le \cdots \le \xi^m \left\|\overrightarrow{G}_1^{L_1^2} - \overrightarrow{G}^{L_0^2}\right\|,
\end{aligned}$$

where $m \ge 1$ is an integer. Applying the triangle inequality, we generally know that

$$\begin{aligned}
\left\|\overrightarrow{G}_m^{L_m^2} - \overrightarrow{G}_n^{L_n^2}\right\| &\le \left\|\overrightarrow{G}_m^{L_m^2} - \overrightarrow{G}_{m-1}^{L_{m-1}^2}\right\| + + \cdots + \left\|\overrightarrow{G}_{n-1}^{L_{n-1}^2} - \overrightarrow{G}_n^{L_n^2}\right\| \\
&\le \left(\xi^m + \xi^{m-1} + \cdots + \xi^{n-1}\right) \times \left\|\overrightarrow{G}_1^{L_1^2} - \overrightarrow{G}^{L_0^2}\right\| \\
&= \frac{\xi^{n-1} - \xi^m}{1-\xi} \times \left\|\overrightarrow{G}_1^{L_1^2} - \overrightarrow{G}^{L_0^2}\right\| \le \frac{\xi^{n-1}}{1-\xi} \times \left\|\overrightarrow{G}_1^{L_1^2} - \overrightarrow{G}^{L_0^2}\right\|
\end{aligned}$$

with $m \ge n$ and $0 < \xi < 1$. Consequently, $\left\|\overrightarrow{G}_m^{L_m^2} - \overrightarrow{G}_n^{L_n^2}\right\| \to 0$ if $m, n \to \infty$. Whence, $\left\{\overrightarrow{G}_n^{L_n}\right\}$ is a Cauchy sequence convergent to a harmonic flow $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$ because of

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sum_{u \in N_{\overrightarrow{G}_n}(v)} L_n^{A^+_{vu}}(v,u), -\sum_{u \in N_{\overrightarrow{G}_n}(v)} L_n^{A^+_{vu}}(v,u)\right) \\
&= \left(\lim_{n\to\infty}\sum_{u \in N_{\overrightarrow{G}_n}(v)} L_n^{A^+_{vu}}(v,u), -\lim_{n\to\infty}\sum_{u \in N_{\overrightarrow{G}_n}(v)} L_n^{A^+_{vu}}(v,u)\right) = (L(v), -L(v))
\end{aligned}$$

for $\forall v \in V\left(\overrightarrow{G}\right)$. Notice that

$$\begin{aligned}\left\|\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\| &\leq \left\|\overrightarrow{G}^{L^2} - \overrightarrow{G}_m^{L_m^2}\right\| + \left\|\overrightarrow{G}_m^{L_m^2} - \mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\| \\ &\leq \left\|\overrightarrow{G}^{L^2} - \overrightarrow{G}_m^{L_m^2}\right\| + \xi\left\|\overrightarrow{G}_{m-1}^{L_{m-1}^2} - \overrightarrow{G}^{L^2}\right\|.\end{aligned}$$

Thus, if $m \to \infty$, we get that $\left\|\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\| = 0$, i.e., $\mathbf{T}\left(\overrightarrow{G}^{L^2}\right) = \overrightarrow{G}^{L^2}$.

If there is another harmonic flow $\overrightarrow{G'}^{L'^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ holding with $\mathbf{T}\left(\overrightarrow{G'}^{L'^2}\right) = \overrightarrow{G}^{L^2}$, by

$$\left\|\overrightarrow{G}^{L^2} - \overrightarrow{G'}^{L'^2}\right\| = \left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right) - \mathbf{T}\left(\overrightarrow{G'}^{L'^2}\right)\right\| \leq \xi\left\|\overrightarrow{G}^{L^2} - \overrightarrow{G'}^{L'^2}\right\|,$$

it is true only in the case of $\overrightarrow{G}^{L} = \overrightarrow{G'}^{L'}$, i.e., $\overrightarrow{G}^{L^2}$ is unique. □

Theorem 2.9 *A linear operator* $\mathbf{T} : \overrightarrow{\mathscr{G}}^{\pm} \to \mathscr{G}_{\mathscr{B}}^{\pm}$ *is continuous if and only if it is bounded.*

Proof If $\mathbf{T}$ is bounded, then

$$\left\|\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right) - \mathbf{T}\left(\overrightarrow{G}_l^{L_l^2}\right)\right\| = \left\|\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right)\right\| \leq \xi\left\|\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right\|$$

for a constant $\xi \in [0, \infty)$ and $\forall \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ by definition. Let $\delta(\varepsilon) = \frac{\varepsilon}{\xi}$ with $\xi \neq 0$. Clearly, $\left\|\mathbf{T}\left(\overrightarrow{G}^{L} - \overrightarrow{G}^{L_0}\right)\right\| < \varepsilon$ if $\left\|\overrightarrow{G}^{L} - \overrightarrow{G}^{L_0}\right\| < \delta(\varepsilon)$, i.e., $\mathbf{T}$ is continuous. If $\xi = 0$ then it is obvious that $\mathbf{T}$ is bounded.

Now, if $\mathbf{T}$ is continuous but unbounded, there must be a sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ in $\mathscr{G}_{\mathscr{B}}^{\pm}$ such that $\left\|\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right)\right\| \geq n\left\|\overrightarrow{G}_n^{L_n^2}\right\|$. Let $\overrightarrow{G^*}_n^{L_n^2} = \frac{\overrightarrow{G}_n^{L_n^2}}{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|}$. Then $\left\|\overrightarrow{G^*}_n^{L_n^2}\right\| = \frac{1}{n}$, which implies that $\left\|\mathbf{T}\left(\overrightarrow{G^*}_n^{L_n^2}\right)\right\| = \frac{1}{n} \to 0$ if $n \to \infty$. However, by definition

$$\left\|\mathbf{T}\left(\overrightarrow{G^*}_n^{L_n^2}\right)\right\| = \left\|\mathbf{T}\left(\frac{\overrightarrow{G}_n^{L_n^2}}{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|}\right)\right\| = \frac{\left\|\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right)\right\|}{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|} \geq \frac{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|}{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|} = 1,$$

a contradiction. Thus, such a sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ can not exists in $\mathscr{G}_{\mathscr{B}}^{\pm}$ and $\mathbf{T}$ is bounded. □

The following results generalize the Banach inverse mapping theorem, closed graph theorem in classical Banach space to Banach harmonic flow space.

Theorem 2.10(Banach) *Let* $\mathbf{T} : \mathscr{G}_{\mathscr{B}_1}^{\pm} \to \mathscr{G}_{\mathscr{B}_2}^{\pm}$ *be a linear continuous operator with Banach spaces* $\mathscr{B}_1$ *and* $\mathscr{B}_2$*. If* $\mathbf{T}$ *is bijective then its inverse operator* $\mathbf{T}^{-1}$ *is continuous.*

Proof Clearly, the inverse operator $\mathbf{T}^{-1}$ exists by the assumption that $\mathbf{T}$ is bijective. For integers $n \in \mathbb{Z}^+$, let $O_n = \left\{\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}_1}^{\pm} \middle| \left\|\overrightarrow{G}^{L^2}\right\| \leq n\right\}$ and $M_n = \mathbf{T}(O_n)$. Notice that $\bigcup\limits_{n=1}^{\infty} O_n = \mathscr{G}_{\mathscr{B}_1}^{\pm}$. Whence, $\mathscr{G}_{\mathscr{B}_2}^{\pm} = \bigcup\limits_{n=1}^{\infty} \mathbf{T}(O_n)$. We prove that $\mathbf{T}^{-1}$ is continuous which follows by

3 claims following.

Claim 1. there is an integer n_0 such that the closure of M_{n_0} is closed, i.e., $Cl\left(M_{n_0}\right) = M_{n_0}$ in sphere $B\left(\overrightarrow{G}_0^{L_0^2}, r_0\right) = \left\{\overrightarrow{G}^{L^2} \middle| \left\|\overrightarrow{G}^{L^2} - \overrightarrow{G}_0^{L_0^2}\right\| \leq r_0\right\} \subset \mathscr{G}_{\mathscr{B}_2}^{\pm}$.

If Claim 1 is not true, there always exists a closed sphere $\mathbb{B}'_n$ in the interior of closed sphere $\mathbb{B}_n$ of $\mathscr{G}_{\mathscr{B}_2}^{\pm}$ such that $B_n \bigcap M_n = \emptyset$ for integers $n \geq 1$. Now let $\mathbb{B}_0$ be a closed sphere of $\mathscr{G}_{\mathscr{B}_2}^{\pm}$. Then there is a closed sphere $\mathbb{B}_1 \subset \mathbb{B}_0$ with $\mathbb{B}_1 \bigcap M_1 = \emptyset$. Similarly, there is a closed sphere $\mathbb{B}_2 \subset \mathbb{B}_1$ with $\mathbb{B}_2 \bigcap M_2 = \emptyset$. Continuing this process, we get a sequence $\{\mathbb{B}_n\}$ of closed spheres with $\mathbb{B}_n \supset \mathbb{B}_{n+1}$.

Without loss of generality, assume the diameter $\mathrm{Diam}\left(\mathbb{B}_n\right) \to 0$ with $\mathbb{B}_n \neq \mathbb{B}_{n+1}$ as $n \to \infty$. We can always choose harmonic flow $\overrightarrow{G}_n^{L_n^2} \in \mathbb{B}_n - \mathbb{B}_{n-1}$ and get a harmonic flow sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ of $\mathscr{G}_{\mathscr{B}_2}^{\pm}$. Clearly, $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ is a Cauchy sequence for $d\left(\overrightarrow{G}_n^{L_n^2}, \overrightarrow{G}_m^{L_m^2}\right) \leq \mathrm{Diam}\left(\mathbb{B}_m\right) \to 0$ as $m \to \infty$ if $n \geq m$. Thus, there is a harmonic flow $\overrightarrow{G}_\infty^{L_\infty^2} \in \bigcap\limits_{n\geq 1} \mathbb{B}_n$ but $\overrightarrow{G}_\infty^{L_\infty^2} \notin \bigcup\limits_{n\geq 1} M_n$, a contradiction to $\mathscr{G}_{\mathscr{B}_2}^{\pm} = \bigcup\limits_{n=1}^{\infty} \mathbf{T}\left(O_n\right)$.

Define $\lambda_0 = \frac{r_0}{n_0}$ and $\mathbb{B}_{\lambda_0} = \left\{\overrightarrow{G}^{L^2} \middle| \left\|\overrightarrow{G}^{L^2}\right\| \leq \lambda_0\right\}$. By Claim 1, $M_1 \subset \mathbb{B}_{\lambda_0}$.

Claim 2. $Cl\left(M_1\right) = M_1$.

Clearly, if $\overrightarrow{G}^{L^2} \in \mathbb{B}_{\lambda_0}$ then $\overrightarrow{G}_0^{L_0} + n_0\overrightarrow{G}^{L^2}, \overrightarrow{G}_0^{L_0} - n_0\overrightarrow{G}^{L^2} \in B\left(\overrightarrow{G}_0^{L_0^2}, r_0\right)$. Whence, there are sequence $\left\{\overrightarrow{G}_k^{L_k^2}\right\}$ and $\left\{\overrightarrow{G'}_k^{L_k^2}\right\}$ in $\mathbb{B}_{n_0}$ such that

$$\lim_{k\to\infty} \mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right) = \overrightarrow{G}_0^{L_0} + n_0\overrightarrow{G}^{L^2} \quad \text{and} \quad \lim_{k\to\infty} \mathbf{T}\left(\overrightarrow{G'}_k^{L_k^2}\right) = \overrightarrow{G}_0^{L_0} + n_0\overrightarrow{G}^{L^2}.$$

Whence, we get that

$$\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G'}_k^{L_k^2}\right) = 2n_0\overrightarrow{G}^{L^2}, i.e., \quad \mathbf{T}\left(\frac{\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G'}_k^{L_k^2}}{2n_0}\right) = \overrightarrow{G}^{L^2}.$$

Clearly, $\frac{\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G'}_k^{L_k^2}}{2n_0} \in O_1$. We know that $Cl\left(M_1\right) = M_1$.

Let $O_{\frac{1}{2^n}} = \left\{\overrightarrow{G}^{L^2} \middle| \left\|\overrightarrow{G}^{L^2}\right\| \leq \frac{1}{2^n}\right\}$ and $\mathbb{B}_{\frac{\lambda_0}{2^n}} = \left\{\overrightarrow{G}^{L^2} \middle| \left\|\overrightarrow{G}^{L^2}\right\| \leq \frac{1}{2^n}\right\}$ for integers $n \geq 1$. We are easily know that $Cl\left(\mathbf{T}\left(O_{\frac{1}{2^n}}\right)\right) = \mathbf{T}\left(O_{\frac{1}{2^n}}\right)$ in closed sphere $\mathbb{B}_{\frac{\lambda_0}{2^n}}$ by Claim 2.

Claim 3. $\mathbf{T}\left(O_1\right) \supset \mathbb{B}_{\frac{\lambda_0}{2}}$.

In fact, let $\overrightarrow{G}^{L^2} \in \mathbb{B}_{\frac{\lambda_0}{2}}$. Notice that $Cl\left(\mathbf{T}\left(O_{\frac{1}{2}}\right)\right) = \mathbf{T}\left(O_{\frac{1}{2}}\right)$ in $\mathbb{B}_{\frac{\lambda_0}{2}}$. There is $\overrightarrow{G}_1^{L_1^2} \in O_{\frac{1}{2}}$ such that $\left\|\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2}\right)\right\| \leq \frac{\lambda_0}{2^2}$, i.e., $\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2}\right) \in \mathbb{B}_{\frac{\lambda_0}{2^2}}$. Similarly, by $Cl\left(\mathbf{T}\left(O_{\frac{1}{2^2}}\right)\right) = \mathbf{T}\left(O_{\frac{1}{2^2}}\right)$ in $\mathbb{B}_{\frac{\lambda_0}{2^2}}$ we know that there is $\overrightarrow{G}_2^{L_2^2} \in O_{\frac{1}{2^2}}$ such that $\left\|\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2}\right) - \mathbf{T}\left(\overrightarrow{G}_2^{L_1^2}\right)\right\| \leq \frac{\lambda_0}{2^3}$, i.e., $\left\|\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2} + \overrightarrow{G}_2^{L_1^2}\right)\right\| \leq \frac{\lambda_0}{2^3}$. Continuing this process, we generally know that there is a harmonic flow sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ with $\overrightarrow{G}_n^{L_n^2} \in O_{\frac{1}{2}}$ for integers $n \geq 1$ such that

$$\left\|\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2} + \overrightarrow{G}_2^{L_2^2} + \cdots + \overrightarrow{G}_n^{L_n^2}\right)\right\| \leq \frac{\lambda_0}{2^{n+1}} \tag{2.9}$$

by mathematical induction. Notice that

$$\left\|\sum_{i=1}^{n}\overrightarrow{G}_i^{L_i^2}\right\| \leq \sum_{i=1}^{n}\left\|\overrightarrow{G}_i^{L_i^2}\right\| \leq \sum_{i=1}^{n}\frac{1}{2^n} = 1.$$

We therefore know that $\sum\limits_{i=1}^{n}\overrightarrow{G}_i^{L_i^2}$ is convergent in O_1. Denoted by $\overrightarrow{G}_{\Sigma}^{L^2} = \sum\limits_{i=1}^{n}\overrightarrow{G}_i^{L_i^2}$, i.e., $\overrightarrow{G}_{\Sigma}^{L^2} \in O_1$. By the continuous assumption of $\mathbf{T}$, we get immediately that $\overrightarrow{G}^{L^2} = \mathbf{T}\left(\overrightarrow{G}_{\Sigma}^{L^2}\right)$ by letting $n\to\infty$ in (2.9), which implies that $\mathbf{T}(O_1) \supset \mathbb{B}_{\frac{\lambda_0}{2}}$, i.e., $O_1 \supset \mathbf{T}^{-1}\left(\mathbb{B}_{\frac{\lambda_0}{2}}\right)$.

Now we prove $\mathbf{T}^{-1}$ is bounded. Let $\mathbf{O} \neq \overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$. Clearly, $\frac{\lambda_0\overrightarrow{G}^{L^2}}{2\left\|\overrightarrow{G}^{L^2}\right\|} \in \mathbb{B}_{\frac{\lambda_0}{2}}$, we know that

$$\mathbf{T}^{-1}\left(\frac{\lambda_0\overrightarrow{G}^{L^2}}{2\left\|\overrightarrow{G}^{L^2}\right\|}\right) \in O_1, \quad \text{i.e.,} \quad \left\|\mathbf{T}^{-1}\left(\frac{\lambda_0\overrightarrow{G}^{L^2}}{2\left\|\overrightarrow{G}^{L^2}\right\|}\right)\right\| \leq 1.$$

Whence we get that

$$\left\|\mathbf{T}^{-1}\left(\overrightarrow{G}^{L^2}\right)\right\| \leq \frac{2}{\lambda_0}\left\|\overrightarrow{G}^{L^2}\right\|,$$

i.e., $\mathbf{T}^{-1}$ is bounded. Applying Theorem 2.9 we know that $\mathbf{T}^{-1}$ is continuous. □

Definition 2.11 *Let $\mathbf{T}: \mathscr{G}_{\mathscr{B}_1}^{\pm} \to \mathscr{G}_{\mathscr{B}_2}^{\pm}$ be a linear continuous operator with Banach spaces $\mathscr{B}_1$, $\mathscr{B}_2$. The graph of $\mathbf{T}$ in $\mathscr{G}_{\mathscr{B}_2}^{\pm}$ is defined by*

$$\mathrm{Grap}\mathbf{T} = \left\{\left(\overrightarrow{G}^{L^2}, \mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right)\middle|\, \overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}_1}^{\pm}\right\}$$

and $\mathbf{T}$ is closed if $Cl\,(\mathrm{Grap}\mathbf{T}) = \mathrm{Grap}\mathbf{T}$, i.e., a closed subspace.

Theorem 2.12 *Let $\mathbf{T}: \mathscr{G}_{\mathscr{B}_1}^{\pm} \to \mathscr{G}_{\mathscr{B}_2}^{\pm}$ be a linear operator with Banach spaces $\mathscr{B}_1$, $\mathscr{B}_2$. Then $\mathbf{T}$ is closed if and only if for any harmonic flow sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\} \in \mathscr{G}_{\mathscr{B}_1}^{\pm}$ with $\lim\limits_{n\to\infty}\overrightarrow{G}_n^{L_n^2} = \overrightarrow{G}_0^{L_0^2} \in \mathscr{G}_{\mathscr{B}_1}^{\pm}$, $\lim\limits_{n\to\infty}\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right) = \overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}_2}^{\pm}$ and $\mathbf{T}\left(\overrightarrow{G}_0^{L_0^2}\right) = \overrightarrow{G}^{L^2}$.*

Proof For $\left(\overrightarrow{G}_0^{L_0^2}, \overrightarrow{G}^{L^2}\right) \in Cl\,(\mathrm{Grap}\mathbf{T})$, there is a harmonic flow sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ such that $\left(\overrightarrow{G}_n^{L_n^2}, \mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right)\right) \to \left(\overrightarrow{G}_0^{L_0^2}, \overrightarrow{G}^{L^2}\right)$ as $n\to\infty$ by definition. We therefore get that $\overrightarrow{G}_n^{L_n^2} \to \overrightarrow{G}_0^{L_0^2}$ and $\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right) \to \overrightarrow{G}^{L^2}$ as $n\to\infty$. If $\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right) = \overrightarrow{G}^{L^2}$, then $\left(\overrightarrow{G}_0^{L_0^2}, \overrightarrow{G}^{L^2}\right) \in \mathrm{Grap}\mathbf{T}$. We know that $\mathrm{Grap}\mathbf{T}$ is a closed subspace, i.e., $\mathbf{T}$ is a closed operator.

Conversely, if $\mathbf{T}$ is a closed operator, let $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ be a harmonic flow sequence in $\mathscr{G}_{\mathscr{B}_1}^{\pm}$ with $\left(\overrightarrow{G}_n^{L_n^2}, \mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right)\right) \to \left(\overrightarrow{G}_0^{L_0^2}, \overrightarrow{G}^{L^2}\right) \in \mathrm{Grap}\mathbf{T}$ as $n\to\infty$ by definition. Whence, $\mathbf{T}\left(\overrightarrow{G}_0^{L_0^2}\right) = \overrightarrow{G}^{L^2}$. This completes the proof. □

Theorem 2.13(Closed Graph Theorem) *If $\mathbf{T}: \mathscr{G}_{\mathscr{B}_1}^{\pm} \to \mathscr{G}_{\mathscr{B}_2}^{\pm}$ is a closed linear operator with Banach spaces $\mathscr{B}_1$, $\mathscr{B}_2$, then $\mathbf{T}$ is continuous.*

Proof Notice that $\mathscr{G}_{\mathscr{B}_1}^{\pm}\bigoplus\mathscr{G}_{\mathscr{B}_2}^{\pm}$ with norm $\left\|\overrightarrow{G}^{L^2}\right\| + \left\|\overrightarrow{G'}^{L^2}\right\|$ for $\forall\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}_1}^{\pm}, \overrightarrow{G'}^{L^2} \in \mathscr{G}_{\mathscr{B}_2}^{\pm}$ is also a Banach space with subspace Grap$\mathbf{T}$ by definition. Define $\widehat{\mathbf{T}}:\left(\overrightarrow{G}^{L^2},\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right)\rightarrow\overrightarrow{G}^{L^2}$ for $\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}_1}$. Clearly, $\widehat{\mathbf{T}}$ is bijective from Grap$\mathbf{T}$ to $\mathscr{G}^{\pm}_{\mathscr{B}_1}$. By Theorem 4.10, we know that $\widehat{\mathbf{T}}^{-1}$ is continuous or bounded, i.e.,

$$\left\|\widehat{\mathbf{T}}^{-1}\left(\overrightarrow{G}^{L^2}\right)\right\|=\left\|\left(\overrightarrow{G}^{L^2},\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right)\right\|=\left\|\overrightarrow{G}^{L^2}\right\|+\left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\|\leq\left\|\widehat{\mathbf{T}}^{-1}\right\|\left\|\overrightarrow{G}^{L^2}\right\|.$$

We therefore get that $\left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\|\leq\left\|\widehat{\mathbf{T}}^{-1}\right\|\left\|\overrightarrow{G}^{L^2}\right\|$, i.e., $\mathbf{T}$ is bounded and continuous by Theorem 2.9. $\square$

Notice that harmonic flow spaces $\mathscr{G}^{\pm}_{\mathscr{B}_1}$ and $\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ are both labeled graph families. A harmonic flow space $\mathscr{G}^{\pm}_{\mathscr{B}_1}$ is *isomorphic* to $\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ if there is a linear continuous operator $\mathbf{T}:\mathscr{G}^{\pm}_{\mathscr{B}_1}\rightarrow\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ of bijection with $\mathbf{T}:\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}_1}\rightarrow\overrightarrow{G'}^{L'^2}\in\mathscr{G}^{\pm}_{\mathscr{B}_2}$ such that

$$\mathbf{T}\left(A^+_{vu},(L(v,u),-L(v,u)),A^+_{uv}\right)=\left(A'^+_{vu},(L'(v,u),-L'(v,u)),A'^+_{uv}\right)$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. The following result characterizes isomorphic harmonic flow spaces.

Theorem 2.14 *A harmonic flow spaces $\mathscr{G}^{\pm}_{\mathscr{B}_1}$ is isomorphic to $\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ with $\mathbf{T}:\overrightarrow{G}^{L^2}\rightarrow\overrightarrow{G'}^{L'^2}$ if and only if $\mathscr{G}=\mathscr{G}'$ and $\mathscr{B}_1$ is isomorphic to $\mathscr{B}_2$.*

Proof Clearly, if $T:\mathscr{B}_1\rightarrow\mathscr{B}_2$ is an isomorphism and $\mathscr{G}=\mathscr{G}'$, there is an identical mapping $id:G\in\mathscr{G}\rightarrow G\in\mathscr{G}'$. We are easily know that the operator $\mathbf{T}=T\circ id:\mathscr{G}^{\pm}_{\mathscr{B}_1}\rightarrow\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ with $\mathbf{T}:\overrightarrow{G}^{L^2}\rightarrow\overrightarrow{G'}^{L'^2}$ is an isomorphism between $\mathscr{G}^{\pm}_{\mathscr{B}_1}$ and $\mathscr{G}'^{\pm}_{\mathscr{B}_2}$.

Conversely, if $\mathbf{T}:\mathscr{G}^{\pm}_{\mathscr{B}_1}\rightarrow\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ is an isomorphism with $\mathbf{T}:\overrightarrow{G}^{L^2}\rightarrow\overrightarrow{G'}^{L'^2}$, we know that

$$\begin{aligned}\mathbf{T}\quad&:\left(A^+_{vu},(L(v,u),-L(v,u)),A^+_{uv}\right)\in\overrightarrow{G}^{L^2}\rightarrow\left(A'^+_{v'u'},(L'(v',u'),-L'(v',u')),A'^+_{u'v'}\right)\in\overrightarrow{G}^{L^2},\\ \mathbf{T}^{-1}&:\left(A'^+_{v'u'},(L'(v',u'),-L'(v',u')),A'^+_{u'v'}\right)\in\overrightarrow{G}^{L^2}\rightarrow\left(A^+_{vu},\{L(v,u),-L(v,u)\},A^+_{uv}\right)\in\overrightarrow{G}^{L^2}\end{aligned}$$

which naturally induces

$$\mathbf{T}_v:\left\{L(v,u),u\in N_{\overrightarrow{G}}(v)\right\}\rightarrow\left\{L'(v',u'),u'\in N_{\overrightarrow{G}}(v')\right\},$$

i.e., an isomorphism $\mathbf{T}_v:v'\in V\left(\overrightarrow{G}\right)\rightarrow v'\in V\left(\overrightarrow{G'}\right)$ preserving the adjacency of vertices. We therefore know that $\overrightarrow{G}$ and $\overrightarrow{G'}$ are isomorphic, i.e., $\mathscr{G}=\mathscr{G}'$.

Notice that an isomorphism $\mathbf{T}$ is linear continuous. By Theorem 4.10 we know that $\mathbf{T}^{-1}$ is continuous also. Thus, $\mathbf{T}$, $\mathbf{T}^{-1}$ induce operators $\mathbf{T}_{vu}:\{L(v,u)\in\mathscr{B}_1\}\rightarrow\{L'(v',u')\in\mathscr{B}_2\}$, $\mathbf{T}^{-1}_{vu}:\{L'(v',u')\in\mathscr{B}_2\}\rightarrow\{L(v,u)\in\mathscr{B}_1\}$ for edges $(v,u)\in E\left(\overrightarrow{G}\right)$, $(v',u')\in E\left(\overrightarrow{G'}\right)$ and both of them are bijective. Consequently, $\mathbf{T}_{vu}$ is also linear continuous with a continuously inverse $\mathbf{T}^{-1}_{vu}$, i.e., preserving the topology on $\mathscr{B}_1$ and $\mathscr{B}_2$. Whence, $\mathbf{T}_{vu}$ is an isomorphisms between Banach spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ for $(v,u)\in E\left(\overrightarrow{G}\right)$ by definition. $\square$

Certainly, there maybe existed more than one norm on a harmonic flow space $\mathscr{G}^{\pm}_{\mathscr{B}}$, We need to distinguish them by the equivalence following.

Definition 2.15 *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms in $\mathscr{G}^{\pm}_{\mathscr{B}}$. If there are positive numbers K_1, K_2 such that*

$$K_1\left\|\overrightarrow{G}^{L^2}\right\|_1 \le \left\|\overrightarrow{G}^{L^2}\right\|_2 \le K_2\left\|\overrightarrow{G}^{L^2}\right\|_1$$

for $\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}}$, then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be equivalent on $\mathscr{G}^{\pm}_{\mathscr{B}}$.

Theorem 2.16 *Let $\|\cdot\|_1$, $\|\cdot\|_2$ be norms defining Banach spaces on $\mathscr{G}^{\pm}_{\mathscr{B}}$. If there is a positive number K such that $\left\|\overrightarrow{G}^{L^2}\right\|_2 \le K\left\|\overrightarrow{G}^{L^2}\right\|_1$ for $\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}}$, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*

Proof Denoted by $\mathscr{G}^{\pm}_{\mathscr{B}_1}$, $\mathscr{G}^{\pm}_{\mathscr{B}_2}$ the Banach spaces with norm $\|\cdot\|_1$ or $\|\cdot\|_2$, respectively. Define an operator $\mathbf{I}:\mathscr{G}^{\pm}_{\mathscr{B}_1}\to\mathscr{G}^{\pm}_{\mathscr{B}_2}$ by $\mathbf{I}\left(\overrightarrow{G}^{L^2}\right)=\overrightarrow{G}^{L^2}$ for $\forall\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}_1}$. Clearly, $\mathbf{I}$ is linear and bijective.

Now, if $\left\|\overrightarrow{G}^{L^2}\right\|_2 \le K\left\|\overrightarrow{G}^{L^2}\right\|_1$ for $\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}_2}$, then $\mathbf{I}$ is bounded. Applying Theorem 2.10 we know that $\mathbf{I}^{-1}$ is continuous, i.e., bounded by Theorem 2.9. Whence, $\left\|\mathbf{I}^{-1}\left(\overrightarrow{G}^{L^2}\right)\right\|_1 \le \left\|\mathbf{I}^{-1}\right|\left\|\overrightarrow{G}^{L^2}\right\|_2$, i.e., $\left\|\overrightarrow{G}^{L^2}\right\|_1 \le \left\|\mathbf{I}^{-1}\right|\left\|\overrightarrow{G}^{L^2}\right\|_2$ by definition. We get that

$$\frac{1}{\left\|\mathbf{I}^{-1}\right|}\left\|\overrightarrow{G}^{L^2}\right\|_1 \le \left\|\overrightarrow{G}^{L^2}\right\|_2 \le K\left\|\overrightarrow{G}^{L^2}\right\|_1. \qquad \square$$

Notice that the far or near degree of $\overrightarrow{G}_k^{L_k^2}$ and $\overrightarrow{G}_l^{L_l^2}$ is measured by the sum of norms on edges in Definition 2.2. Sometimes, we also need it to measure by the residue norms on vertices such as the synchronization of complex networks, i.e., the conception following.

Definition 2.17 *For $\forall\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}^{\pm}_{\mathscr{B}}$, the distance $D\left(\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right)$ between vertices of $\overrightarrow{G}_k^{L_k^2}$ and $\overrightarrow{G}_l^{L_l^2}$ is defined by the sum of vertices norms of $\overrightarrow{G}_k^{L_k^2}-\overrightarrow{G}_l^{L_l^2}=\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{-L_l^2}$, i.e.,*

$$D\left(\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right)=\sum_{v\in V\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)}\left\|L^-_{kl_1}(v)\right\|, \tag{2.10}$$

where,

$$L^-_{kl}(v)=\begin{cases} L_k^2(v) \text{ or } L_l^2(v) & if\ v\in V\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right) \text{ or } V\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)\\ L_k^2(v)-L_l^2(v) & if\ v\in V\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)\end{cases}.$$

Clearly, $\left(\mathscr{G}^{\pm}_{\mathscr{B}};D\right)$ is not a distance space because we have $D\left(\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right)=0$ if the residue flows on vertices in $\overrightarrow{G}_k^{L_l^2}$, $\overrightarrow{G}_l^{L_l^2}$ are a constant. However, we can measure the near degree of $\overrightarrow{G}_k^{L_k^2}$ and $\overrightarrow{G}_l^{L_l^2}$ by norms on edges, i.e., it is stronger than that on vertex for harmonic flows.

Theorem 2.18 *For $\forall\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}^{\pm}_{\mathscr{B}}$, if all end-operators on $\overrightarrow{G}_k^{L_k^2}$ and $\overrightarrow{G}_l^{L_l^2}$ are linear continuous, then there exists a constant $c>0$ such that*

$$D\left(\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right)\le c\left(\left\|\overrightarrow{G}_k^{L_k^2}-\overrightarrow{G}_l^{L_l^2}\right\|\right).$$

Proof By Theorem 1.3 we know that there are positive constants $c_{vu}^1, c_{vu}^2 \in \mathbb{R}$ such that $\left\|L_1^{A_{vu}^+}(v,u)\right\| \leq c_1^{vu}\left\|L_1(v,u)\right\|$ and $\left\|L_2^{A_{vu}^+}(v,u)\right\| \leq c_2^{vu}\left\|L_2(v,u)\right\|$ for $\forall(v,u) \in \overrightarrow{G}$ if the end-operator A_{vu}^+ is linear continuous. Without loss of generality, let

$$c^{\max}\left(\overrightarrow{G}^{L^2}\right) = \max\left\{c_1^{vu}, c_2^{vu} | v,u \in V\left(\overrightarrow{G}\right)\right\}$$

and $\overrightarrow{H} = \overrightarrow{G}_k \bigcup \overrightarrow{G}_l$. We are easily know that

$$\begin{aligned}
d\left(\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2}\right) &= \sum_{v \in V\left(\overrightarrow{H}\right)} \left\|L_{kl1}^-(v)\right\| = \sum_{v \in V\left(\overrightarrow{H}\right)} \sum_{u \in N_{\overrightarrow{H}}(v)} \left\|L_{kl1}^{-\ A_{vu}^+}(v,u)\right\| \\
&\leq c^{\max} \sum_{v \in V\left(\overrightarrow{H}\right)} \sum_{u \in N_{\overrightarrow{H}}(v)} \left\|L_{kl1}^-(v,u)\right\| \\
&= 2c^{\max}\left(\overrightarrow{H}^{L_{kl1}^-}\right) \sum_{(v,u) \in E\left(\overrightarrow{H}\right)} \left\|L_{kl1}^-(v,u)\right\| \\
&= 2c^{\max}\left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)^{L_{kl}^-} \left\|\left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)^{L_{kl}^-}\right\| \\
&= 2c^{\max}\left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)^{L_{kl}^-} \left\|\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right\|
\end{aligned}$$

by the assumption. We therefore get that

$$d\left(\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2}\right) \leq c\left(\left\|\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right\|\right)$$

with $c = 2c^{\max}\left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)^{L_{kl}^-}$. This completes the proof. □

A linear operator $\mathbf{T} : \mathscr{G}_{\mathscr{B}_1}^{\pm} \rightarrow \mathscr{G}_{\mathscr{B}_2}^{\pm}$ is a *functional* if $\mathscr{G}_{\mathscr{B}_2}^{\pm} = \mathbb{R}$ or $\mathbb{C}$, and there is a fundamental question on functionals should be answered, i.e., *are there really linear continuous functionals on harmonic flow spaces* $\mathscr{G}_{\mathscr{B}}^{\pm}$? Certainly, its answer is affirmative by results following.

Definition 2.19 *A functional* $p : \mathscr{G}_{\mathscr{B}_1}^{\pm} \rightarrow \mathbb{R}$ *is sublinear if* $p\left(\overrightarrow{G}^{L^2} + \overrightarrow{G'}^{L'^2}\right) \leq p\left(\overrightarrow{G}^{L^2}\right) + p\left(\overrightarrow{G'}^{L'^2}\right)$ *and* $p\left(\alpha\overrightarrow{G}^{L^2}\right) = \alpha p\left(\overrightarrow{G}^{L^2}\right)$ *for* $\overrightarrow{G}^{L^2}, \overrightarrow{G'}^{L'^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ *and* $\alpha \geq 0$.

We can similarly extend the Hahn-Banach theorem, i.e., the existence of functionals in a Banach space to the harmonic flow space $\mathscr{G}_{\mathscr{B}}^{\pm}$ following.

Theorem 2.20(Hahn-Banach) *Let* $\mathscr{H}_{\mathscr{B}}^{\pm}$ *be a harmonic flow subspace of* $\mathscr{G}_{\mathscr{B}}^{\pm}$ *and let* $F : \mathscr{H}_{\mathscr{B}}^{\pm} \rightarrow \mathbb{C}$ *be a linear continuous functional on* $\mathscr{H}_{\mathscr{B}}^{\pm}$. *Then, there is a linear continuous functional* $\widetilde{F} : \mathscr{G}_{\mathscr{B}}^{\pm} \rightarrow \mathbb{C}$ *hold with*

(1) $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ *if* $\overrightarrow{G}^{L^2} \in \mathscr{H}_{\mathscr{B}}^{\pm}$;

(2) $\left\|\widetilde{F}\right\| = \|F\|$.

Proof The proof is consisting of claims following.

Claim 1. If there is a linear functional $F:\mathscr{H}_{\mathscr{B}}^{\pm}\rightarrow\mathbb{R}$ and a sublinear functional $p:\mathscr{G}_{\mathscr{B}}^{\pm}\rightarrow\mathbb{R}$ with $F\left(\overrightarrow{G}^{L^2}\right)\leq p\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}}^{\pm}$, then there exists a linear functional $\widetilde{F}:\mathscr{G}_{\mathscr{B}}^{\pm}\rightarrow\mathbb{R}$ such that $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)=F\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2}\in\mathscr{H}_{\mathscr{B}}^{\pm}$ and $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\leq p\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}}^{\pm}$.

Let $\overrightarrow{G}_0^{L_0^2}\in\mathscr{G}_{\mathscr{B}}^{\pm}\setminus\mathscr{H}_{\mathscr{B}}^{\pm}$ and $\mathscr{H}_{1\mathscr{B}}^{\pm}=\left\{\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\middle|\,\alpha\in\mathbb{R},\overrightarrow{G}^{L^2}\in\mathscr{H}_{\mathscr{B}}^{\pm}\right\}$, a linear space spanned by $\overrightarrow{G}_0^{L_0^2}$ and $\mathscr{H}_{\mathscr{B}}^{\pm}$. For $\forall\overrightarrow{G}^{L^2},\overrightarrow{G'}^{L'^2}\in\mathscr{H}_{\mathscr{B}}^{\pm}$ calculation shows that

$$\begin{aligned}F\left(\overrightarrow{G}^{L^2}\right)-F\left(\overrightarrow{G'}^{L'^2}\right) &= F\left(\overrightarrow{G}^{L^2}-\overrightarrow{G'}^{L'^2}\right)\leq p\left(\overrightarrow{G}^{L^2}-\overrightarrow{G'}^{L'^2}\right)\\ &\leq p\left(\overrightarrow{G}^{L^2}+\overrightarrow{G}_0^{L_0^2}\right)+p\left(-\overrightarrow{G'}^{L'^2}-\overrightarrow{G}_0^{L_0^2}\right),\end{aligned}$$

i.e.,

$$-p\left(-\overrightarrow{G'}^{L'^2}-\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G'}^{L'^2}\right)\leq p\left(\overrightarrow{G}^{L^2}+\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G}^{L^2}\right).$$

Notice that $\overrightarrow{G}^{L^2}$, $\overrightarrow{G'}^{L'^2}$ are arbitrarily selected in $\mathscr{H}_{\mathscr{B}}^{\pm}$. There are must be

$$\sup_{\overrightarrow{G'}^{L'^2}\in\mathscr{H}_{\mathscr{B}}^{\pm}}\left\{-p\left(-\overrightarrow{G'}^{L'^2}-\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G'}^{L'^2}\right)\right\}\leq\inf_{\overrightarrow{G'}^{L'^2}\in\mathscr{H}_{\mathscr{B}}^{\pm}}\left\{\left(\overrightarrow{G}^{L^2}+\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G}^{L^2}\right)\right\},$$

which enables one to choose a number c hold with

$$\sup_{\overrightarrow{G'}^{L'^2}\in\mathscr{H}_{\mathscr{B}}^{\pm}}\left\{-p\left(-\overrightarrow{G'}^{L'^2}-\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G'}^{L'^2}\right)\right\}\leq c\leq\inf_{\overrightarrow{G'}^{L'^2}\in\mathscr{H}_{\mathscr{B}}^{\pm}}\left\{\left(\overrightarrow{G}^{L^2}+\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G}^{L^2}\right)\right\}$$

and define a functional F' by $F'\left(\overrightarrow{G^*}^{L^{*2}}\right)=\alpha c+F\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G^*}^{L^{*2}}\in\mathscr{H}_{1\mathscr{B}}^{\pm}$.

Clearly, F' is indeed a linear functional on $\mathscr{H}_{1\mathscr{B}}^{\pm}$ because $\mathscr{H}_{1\mathscr{B}}^{\pm}$ is linear spanned by $\overrightarrow{G}_0^{L_0^2}$ and $\mathscr{H}_{\mathscr{B}}^{\pm}$. We prove

$$F'\left(\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\right)\leq p\left(\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\right),\tag{2.11}$$

and without of loss of generality, assume $\alpha\neq 0$ because the assertion is obvious if $\alpha=0$.

Now if $\alpha>0$, by

$$c\leq p\left(\overrightarrow{G}_0^{L_0^2}+\frac{\overrightarrow{G}^{L^2}}{\alpha}\right)-F\left(\frac{\overrightarrow{G}^{L^2}}{\alpha}\right)$$

we are easily know that

$$F'\left(\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\right)\leq p\left(\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\right),$$

i.e., (2.11) is true, and if $\alpha<0$, by

$$c\geq -p\left(-\overrightarrow{G}_0^{L_0^2}-\frac{\overrightarrow{G}^{L^2}}{\alpha}\right)-F\left(\frac{\overrightarrow{G}^{L^2}}{\alpha}\right)$$

we can know that (2.11) hold also. Whence, F' is a linear extension of F by $\overrightarrow{G}_0^{L_0^2}$. All such extensions of F' are denoted by $\mathscr{H}(F)$, i.e., $F'\left(\overrightarrow{G}^{L^2}\right)=F\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2}\in\mathscr{H}_{\mathscr{B}}^{\pm}$ and all extensions F' of F further with $F'\left(\overrightarrow{G}^{L^2}\right)\leq p\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2}\in\mathscr{H}(F)$ are denoted by $\widetilde{\mathscr{H}}(F)$.

We define an order $\prec$ in $\widetilde{\mathscr{H}}(F)$ by:

For $F_1, F_2 \in \widetilde{\mathscr{H}}(F)$, if $\mathscr{H}(F_1) \subset \mathscr{H}(F_2)$ and $F_1\left(\overrightarrow{G}^{L^2}\right) = F_2\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2} \in \mathscr{H}(F_1)$, then F_1 is precedent of F_2, denoted by $F_1 \prec F_2$.

Then $\left(\widetilde{\mathscr{H}}(F); \prec\right)$ is a partial order set.

Let $\mathscr{M}(F) \subset \widetilde{\mathscr{F}}(F)$ be with $(\mathscr{M}(F); \prec)$ an order subset and let

$$\mathscr{D}(F) = \bigcup_{F \in \mathscr{M}(F)} \mathscr{H}(F).$$

Notice that for $\overrightarrow{G}^{L^2} \in \mathscr{D}(F)$ there must be a $\mathscr{H}(F)$ such that $\overrightarrow{G}^{L^2} \in \mathscr{H}(F)$. By this fact, we can define a linear functional $\widehat{F}$ on $\mathscr{D}(F)$ by $\widehat{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2} \in \mathscr{H}(F)$. Since $\mathscr{M}(F)$ is an order set we know such a $\widetilde{F}$ is a uniquely linear functional with $\widehat{F}\left(\overrightarrow{G}^{L^2}\right) \leq p\left(\overrightarrow{G}^{L^2}\right)$ on $\mathscr{H}(F)$. Thus, $\widehat{F} \in \mathscr{D}(F)$ and it is an upper bound of $\mathscr{M}(F)$.

By Zorn's Lemma, there is a maximal element $\widetilde{F}$ in $\widetilde{\mathscr{H}}(F)$ with $\mathscr{H}\left(\widetilde{F}\right) = \mathscr{G}_{\mathscr{B}}^{\pm}$. Otherwise, let $\overrightarrow{G}_0^{L_0^2} \in \mathscr{G}_{\mathscr{B}}^{\pm} \setminus \mathscr{H}\left(\widetilde{F}\right)$, then we can extend $\widetilde{F}$ to a linear space spanned by $\mathscr{H}\left(\widetilde{F}\right)$ with $\overrightarrow{G}_0^{L_0^2}$, contradicts to the maximality of $\widetilde{F}$. We therefore know that $\mathscr{H}\left(\widetilde{F}\right) = \mathscr{G}_{\mathscr{B}}^{\pm}$.

Claim 2. If $F : \mathscr{H}_{\mathscr{B}}^{\pm} \to \mathbb{C}$ is a linear continuous functional on $\mathscr{H}_{\mathscr{B}}^{\pm}$, then there is a linear continuous functional $\widetilde{F} : \mathscr{G}_{\mathscr{B}}^{\pm} \to \mathbb{C}$ hold with $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2} \in \mathscr{H}_{\mathscr{B}}^{\pm}$ and $\left\|\widetilde{F}\right\| = \|F\|$.

Let $F\left(\overrightarrow{G}^{L^2}\right) = F_1\left(\overrightarrow{G}^{L^2}\right) + iF_2\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2} \in \mathscr{H}_{\mathscr{B}}^{\pm}$, where $F_1, F_2 : \mathscr{H}_{\mathscr{B}}^{\pm} \to \mathbb{R}$ and $i^2 = 1$. Notice that

$$i\left(F_1\left(\overrightarrow{G}^{L^2}\right) + iF_2\left(\overrightarrow{G}^{L^2}\right)\right) = iF\left(\overrightarrow{G}^{L^2}\right) = F\left(i\overrightarrow{G}^{L^2}\right) = F_1\left(i\overrightarrow{G}^{L^2}\right) + iF_2\left(i\overrightarrow{G}^{L^2}\right).$$

We know that $F_1\left(i\overrightarrow{G}^{L^2}\right) = -F_2\left(\overrightarrow{G}^{L^2}\right)$. Let $p\left(\overrightarrow{G}^{L^2}\right) = \|F\| \left\|\overrightarrow{G}^{L^2}\right\|$. Then $p\left(\overrightarrow{G}^{L^2}\right)$ is a linear functional with

$$F_1\left(\overrightarrow{G}^{L^2}\right) \leq \left\|F\left(\overrightarrow{G}^{L^2}\right)\right\| \leq \|F\| \left\|\overrightarrow{G}^{L^2}\right\| = p\left(\overrightarrow{G}^{L^2}\right)$$

on $\mathscr{H}_{\mathscr{B}}^{\pm}$, i.e., F_1 is holding with conditions of Claim 1. We know that F_1 can be extended to a linear functional F_{10} on $\mathscr{G}_{\mathscr{B}}^{\pm}$ with $F_{10}\left(\overrightarrow{G}^{L^2}\right) \leq p\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$.

Define

$$\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F_{10}\left(\overrightarrow{G}^{L^2}\right) + iF_{20}\left(\overrightarrow{G}^{L^2}\right) = F_{10}\left(\overrightarrow{G}^{L^2}\right) - iF_{10}\left(i\overrightarrow{G}^{L^2}\right). \tag{2.12}$$

We prove $\widetilde{F}$ is a linear continuous functional satisfying conditions of Claim 2. For $\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$, calculation shows that

$$\begin{aligned}\widetilde{F}\left(i\overrightarrow{G}^{L^2}\right) &= F_{10}\left(i\overrightarrow{G}^{L^2}\right) - iF_{10}\left(-\overrightarrow{G}^{L^2}\right) = F_{10}\left(i\overrightarrow{G}^{L^2}\right) + iF_{10}\left(i\overrightarrow{G}^{L^2}\right)\\ &= i\left(F_{10}\left(\overrightarrow{G}^{L^2}\right) - iF_{10}\left(\overrightarrow{G}^{L^2}\right)\right) = i\widetilde{F}\left(\overrightarrow{G}^{L^2}\right).\end{aligned}$$

Whence, for $\forall\alpha_1 + i\alpha_2 \in \mathbb{C}$ we have

$$\begin{aligned}\widetilde{F}\left(\alpha\overrightarrow{G}^{L^2}\right) &= \widetilde{F}\left((\alpha_1 + i\alpha_2)\overrightarrow{G}^{L^2}\right) = \alpha_1 F_{10}\left(\overrightarrow{G}^{L^2}\right) + \alpha_2 F_{10}\left(i\overrightarrow{G}^{L^2}\right)\\ &= \alpha_1 F_{10}\left(\overrightarrow{G}^{L^2}\right) + i\alpha_2 F_{10}\left(\overrightarrow{G}^{L^2}\right) = \alpha\widetilde{F}\left(\overrightarrow{G}^{L^2}\right),\end{aligned}$$

i.e., $\widetilde{F}$ is a linear functional on $\mathscr{G}^{\pm}_{\mathscr{B}}$. By Claim 1 we know that $F_{10}\left(\overrightarrow{G}^{L^2}\right) = \widetilde{F}_1\left(\overrightarrow{G}^{L^2}\right)$ and $F_{20}\left(\overrightarrow{G}^{L^2}\right) = \widetilde{F}_2\left(\overrightarrow{G}^{L^2}\right)$, i.e., $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2} \in \mathscr{H}^{\pm}_{\mathscr{B}}$.

Clearly, $\widetilde{F}$ is continuous by definition. We show that $\left\|\widetilde{F}\right\| = \|F\|$. Let $\theta = \arg\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)$. By definition, $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = \left\|\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\right\| e^{i\theta}$. Therefore,

$$\left\|\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\right\| = e^{-i\theta}\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = \widetilde{F}\left(e^{-i\theta}\overrightarrow{G}^{L^2}\right) = F_{10}\left(e^{-i\theta}\overrightarrow{G}^{L^2}\right) - iF_{10}\left(ie^{-i\theta}\overrightarrow{G}^{L^2}\right).$$

Notice that $\left\|\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\right\| \geq 0$ is a real number, we know that

$$\left\|\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\right\| = F_{10}\left(e^{-i\theta}\overrightarrow{G}^{L^2}\right) \leq p\left(e^{-i\theta}\overrightarrow{G}^{L^2}\right) = \|F\|\left\|\overrightarrow{G}^{L^2}\right\|.$$

Whence, $\left\|\widetilde{F}\right\| \leq \|F\|$. However, $\left\|\widetilde{F}\right\| \geq \|F\|$ for $\mathscr{G}^{\pm}_{\mathscr{B}} \supset \mathscr{H}^{\pm}_{\mathscr{B}}$. We get $\left\|\widetilde{F}\right\| = \|F\|$. □

Corollary 2.21 *Let $\mathscr{G}^{\pm}_{\mathscr{B}}$ be harmonic flows space with $\mathbf{O} \neq \overrightarrow{G}_0^{L_0^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$. Then, there always exists a linear continuous functional F with $\|F\| = 1$ and $\left\|F\left(\overrightarrow{G}_0^{L_0^2}\right)\right\| = \left\|\overrightarrow{G}_0^{L_0^2}\right\|$ on $\mathscr{G}^{\pm}_{\mathscr{B}}$.*

Proof Define $\mathscr{H}^{\pm}_{\mathscr{B}} = \left\{\alpha\overrightarrow{G}_0^{L_0^2}\middle|\alpha \in \mathbb{C}\right\}$ and a liner functional $F\left(\alpha\overrightarrow{G}_0^{L_0^2}\right) = \alpha\left\|\overrightarrow{G}_0^{L_0^2}\right\|$ on $\mathscr{H}^{\pm}_{\mathscr{B}}$. Clearly, $\left\|F\left(\overrightarrow{G}^{L^2}\right)\right\| = |\alpha|\left\|\overrightarrow{G}_0^{L_0^2}\right\| = \left\|\alpha\overrightarrow{G}_0^{L_0^2}\right\| = \left\|\overrightarrow{G}^{L^2}\right\|$ if $\overrightarrow{G}^{L^2} = \alpha\overrightarrow{G}_0^{L_0^2}$. We know that $\|F\| = 1$ on $\mathscr{H}^{\pm}_{\mathscr{B}}$ with $F\left(\overrightarrow{G}_0^{L_0^2}\right) = \overrightarrow{G}_0^{L_0^2}$. By Theorem 2.20, F can be extended to $\mathscr{G}^{\pm}_{\mathscr{B}}$. □

Corollary 2.22 *For $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$, if $F\left(\overrightarrow{G}^{L^2}\right) = 0$ hold with all linear functionals F on $\mathscr{G}^{\pm}_{\mathscr{B}}$ then $\overrightarrow{G}^{L^2} = \mathbf{O}$.*

§3. Harmonic Flow Dynamics

3.1 Harmonic Flow Calculus

Let $\overrightarrow{G}^{L^2} \in \mathscr{G}^2_{\mathscr{B}}$ with $L^2 : (v,u) \rightarrow (L_1(v,u), L_2(v,u))$ for $(v,u) \in E\left(\overrightarrow{G}^{L^2}\right)$. We transform L^2 to $L^2 : (v,u) \rightarrow L_1(v,u) + iL_2(v,u)$, i.e., a complex vector and particularly, a complex number if $\mathscr{B}=\mathbb{C}$, where $i=\sqrt{-1}$, which enables one to establish calculus on harmonic flows.

Definition 3.1 *Let D be a boundary subset of $\mathbb{C}^n=\{(x_1,x_2,\cdots,x_n)|x_i\in\mathbb{C},1\le i\le n\}$, $\mathscr{B}=\mathbb{C}(D)$ of differentiable functions on D and all end-operators in $\mathscr{A}$ satisfying $[A,\frac{\partial}{\partial x_i}]=\mathbf{0}$ for $\forall A\in\mathscr{A}$. Define n differential operators $\partial_i:\mathscr{G}^2_{\mathscr{B}}\to\mathscr{G}^2_{\mathscr{B}}$, $1\le i\le n$ by*

$$\partial_i\overrightarrow{G}^{L^2}=\overrightarrow{G}^{\frac{\partial L^2}{\partial x_i}},$$

and denoted by $\frac{d\overrightarrow{G}^{L^2}}{dz}$ if $D\subset\mathbb{C}$ for $\overrightarrow{G}^{L^2}\in\mathscr{G}^2_{\mathscr{B}}$ in which the integral flow of $\overrightarrow{G}^{L^2}\in\mathscr{G}^2_{\mathscr{B}}$ along a curve $C=\{\ z(t)\mid\alpha\le t\le\beta\ \}$ of length$<+\infty$ is defined by

$$\int_C\overrightarrow{G}^{L^2}dz=\overrightarrow{G}^{\int_C L^2dz}.$$

For $\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}^2_{\mathscr{B}}$ and $\lambda,\mu\in\mathbb{C}$, calculation shows that

$$\begin{aligned}\partial_i\left(\lambda\overrightarrow{G}_k^{L_k^2}+\mu\overrightarrow{G}_l^{L_l^2}\right)&=\partial_i\left(\overrightarrow{G}_k^{\lambda L_k^2}+\overrightarrow{G}_l^{\mu L_l^2}\right)\\&=\partial_i\left(\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{\lambda L_k^2}\bigcup\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{\mu L_l^2}\bigcup\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{\lambda L_k^2+\mu L_l^2}\right)\\&=\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{\lambda\frac{\partial L_k^2}{\partial x_i}}\bigcup\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{\mu\frac{\partial L_l^2}{\partial x_i}}\bigcup\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{\lambda\frac{\partial L_k^2}{\partial x_i}+\mu\frac{\partial L_l^2}{\partial x_i}}\\&=\lambda\partial_i\overrightarrow{G}_k^{L_k^2}+\mu\partial_i\overrightarrow{G}_l^{L_l^2}\end{aligned}$$

and $\partial_i\overrightarrow{G}^{L^2}\to\partial\overrightarrow{G}_0^{L_0^2}$ if $\overrightarrow{G}^{L^2}\to\overrightarrow{G}_0^{L_0^2}$, i.e., linear continuous on the boundary domain D for integers $1\le i\le n$. Similarly, we can also show that the integral operator $\int_C$ is linear continuous on the boundary domain D. We get the following result.

Theorem 3.2 *All partial differential operators ∂_i and the integral operator $\int_C$ are linear continuous on $\mathscr{G}^2_{\mathscr{B}}$, and furthermore, on $\mathscr{G}^{\pm}_{\mathscr{B}}$ for integers $1\le i\le n$.*

Proof We have shown that each ∂_i is linear continuous for integers $1\le i\le n$. Now, we prove that $\partial_i\overrightarrow{G}^{L^2},\int_C\overrightarrow{G}^{L^2}dz\in\mathscr{G}^2_{\mathscr{B}}$ if $\overrightarrow{G}^{L^2}\in\mathscr{G}^2_{\mathscr{B}}$, i.e., hold with the continuity equations on vertices. In fact, by assumption $\overrightarrow{G}^{L^2}\in\mathscr{G}^2_{\mathscr{B}}$ and $[A,\frac{\partial}{\partial x_i}]=0$ for $\forall A\in\mathscr{A}$ there must be

$$\sum_{u\in N_G(v)}\left(L_1^{A^+_{vu}}(v,u)+iL_2^{A^+_{vu}}(v,u)\right)=L_1(v)+iL_2(v)$$

for $\forall v\in V\left(\overrightarrow{G}\right)$ by definition. Whence,

$$\begin{aligned}&\partial_i\sum_{u\in N_G(v)}\left(L_1^{A^+_{vu}}(v,u)+iL_2^{A^+_{vu}}(v,u)\right)=\sum_{u\in N_G(v)}\left(\partial_iL_1^{A^+_{vu}}(v,u)+i\partial_iL_2^{A^+_{vu}}(v,u)\right)\\&=\sum_{u\in N_G(v)}\left((\partial_iL_1)^{A^+_{vu}}(v,u)+i\left(\partial_iL_2\right)^{A^+_{vu}}(v,u)\right)=\partial_iL_1(v)+i\partial_iL_2(v),\end{aligned}$$

i.e., $\partial_i : \mathscr{G}^2_{\mathscr{B}} \to \mathscr{G}^2_{\mathscr{B}}$ for integers $1 \le i \le n$.

Now, if $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$, then we are easily know that

$$\partial_i \left(\sum_{u \in N_G(v)} \left(L^{A^+_{vu}}(v,u) - iL^{A^+_{vu}}(v,u) \right) \right) = \partial_i L(v) - i\partial_i L(v),$$

i.e., $\partial_i : \mathscr{G}^{\pm}_{\mathscr{B}} \to \mathscr{G}^{\pm}_{\mathscr{B}}$ for integers $1 \le i \le n$.

Similarly, we can also show that the integral operators

$$\int_C : \mathscr{G}^2_{\mathscr{B}} \to \mathscr{G}^2_{\mathscr{B}} \quad \text{and} \quad \int_C : \mathscr{G}^{\pm}_{\mathscr{B}} \to \mathscr{G}^{\pm}_{\mathscr{B}}$$

are linear continuous and hold also with the continuity equation on vertices of $\overrightarrow{G}$. $\square$

Now, if $\frac{d\overrightarrow{G}^{L^2}}{dz}$ exists, then $\frac{d}{dz}L^2(v,u)$, a complex function for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ also exists. Let $L^2(v,u)(z) = M^{vu}(x,y) + iN^{vu}(x,y)$ for $(v,u) \in E\left(\overrightarrow{G}\right)$, where $z = x + iy$ and $M(x,y), N(x,y) \in \mathbb{R}^2$. Applying the Cauchy-Riemann equations in complex analysis, we are easily know that

$$\frac{dL^2}{dz} = \frac{\partial M}{\partial x} + i\frac{\partial N}{\partial x} = \frac{\partial N}{\partial y} - i\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x} - i\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} + i\frac{\partial N}{\partial x}.$$

By definition,

$$\begin{aligned} \frac{d\overrightarrow{G}^{L^2}}{dz} &= \overrightarrow{G}^{\frac{dL^2}{dz}} = \overrightarrow{G}^{\frac{\partial M}{\partial x} + i\frac{\partial N}{\partial x}} = \frac{\partial \overrightarrow{G}^M}{\partial x} + i\frac{\partial \overrightarrow{G}^N}{\partial x}, \\ \frac{d\overrightarrow{G}^{L^2}}{dz} &= \overrightarrow{G}^{\frac{dL^2}{dz}} = \overrightarrow{G}^{\frac{\partial N}{\partial y} - i\frac{\partial M}{\partial y}} = \frac{\partial \overrightarrow{G}^N}{\partial y} - i\frac{\partial \overrightarrow{G}^M}{\partial y}. \end{aligned}$$

Similarly, if $\frac{d\overrightarrow{G}^{L^2}}{dz} = \overrightarrow{G}'^{L'^2}$, i.e., $d\overrightarrow{G}^{L^2} = \overrightarrow{G}'^{L'^2}dz$, then $\overrightarrow{G}^{L^2}$ is called the *primitive flow* of $\overrightarrow{G}'^{L'^2}$ and denoted by $\int \overrightarrow{G}^{L^2}dz$. Calculation shows that

$$\int_C \overrightarrow{G}^{L^2}dz = \overrightarrow{G}^{\int_C L^2 dz} = \overrightarrow{G}^{\int L^2 dz|_\beta - \int L^2 dz|_\alpha} = \int \overrightarrow{G}^{L^2}\bigg|_\beta - \int \overrightarrow{G}^{L^2}\bigg|_\alpha$$

and particularly,

$$\int_C \overrightarrow{G}^{L^2}dz = \mathbf{O}$$

if C is the boundary curve of a simply connected domain on $\mathbb{R}^2$ and furthermore,

$$\overrightarrow{G}^{L^2}(z) = \frac{1}{2\pi i}\int_C \frac{\overrightarrow{G}^{L^2}(\zeta)}{\zeta - z}d\zeta$$

with $z \in D$ if $\overrightarrow{G}^{L^2}$ is differentiable on D and continuous on $Cl(D) = D + C$ by definition. We therefore generalize a few well-known results of complex analysis to $\mathscr{G}^2_{\mathbb{C}}$ following.

Theorem 3.3 *Let $D \subset \mathbb{C}$ be a domain with boundary curve C and $\mathscr{B} = \mathbb{C}(D)$. Then,*

(1)(C-R Equations) *A flow $\overrightarrow{G}^{L^2} \in \mathscr{G}^2_{\mathscr{B}}$ or $\mathscr{G}^{\pm}_{\mathscr{B}}$ is differentiable at $x + iy = z \in D$ if and only if*

$$\frac{\partial \overrightarrow{G}^M}{\partial x} = \frac{\partial \overrightarrow{G}^N}{\partial y} \quad \text{and} \quad \frac{\partial \overrightarrow{G}^N}{\partial x} = -\frac{\partial \overrightarrow{G}^M}{\partial y}$$

where $L^2(v,u)(z) = M^{vu}(x,y) + iN^{vu}(x,y)$, $K_1^{vu}(x,y), K_2^{vu}(x,y) \in \mathbb{R}^2$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ and both of them differentiable at (x,y);

(2)(Cauchy) *If D is simply connected on $\mathbb{R}^2$ and flow $\overrightarrow{G}^{L^2} \in \mathscr{G}^2_{\mathscr{B}}$ or $\mathscr{G}^{\pm}_{\mathscr{B}}$ is differentiable on D, then*

$$\int_{\alpha}^{\beta} \overrightarrow{G}^{L^2} dz = \int \overrightarrow{G}^{L^2}\Big|_{\beta} - \int \overrightarrow{G}^{L^2}\Big|_{\alpha},$$

where $z(\alpha)$ and $z(\beta)$ are two points on C and particularly, $\int_C \overrightarrow{G}^{L^2} dz = \mathbf{O}$;

(3)(Cauchy Integral Formula) *If $\overrightarrow{G}^{L^2} \in \mathscr{G}^2_{\mathscr{B}}$ or $\mathscr{G}^{\pm}_{\mathscr{B}}$ is differentiable on D and continuous on $Cl(D) = D + C$, then*

$$\overrightarrow{G}^{L^2}(z) = \frac{1}{2\pi i} \int_C \frac{\overrightarrow{G}^{L^2}(\zeta)}{\zeta - z} d\zeta,$$

where $z \in D$.

3.2 Harmonic Flow Dynamics

A self-adaptive system is naturally a harmonic flow over its underlying skeleton or a topological graph $\overrightarrow{G}$, particularly, an animal or a human, and all animals are in motion, internal, external or both which motives the harmonic flow dynamics, i.e., harmonic flow's status $\overrightarrow{G}^{L^2}[t]$ changes on time t for fields such as those of life or social systems. For example, the differential $\frac{d\overrightarrow{G}^{L^2}[t]}{dt}$ can be viewed as the harmonic change rate of a national economy, both in the internal and external if one models a notional economy by harmonic flow $\overrightarrow{G}^{L^2}[t]$, which is more scientific than that of the current rate of GDP, the gross domestic product of a country.

As it is well-known, the dynamic behavior of a self-adaptive system S, particularly, an animal or a human consisting of subsystems can be characterized by Lagrangians with continuity equations holds. If S is characterized by a harmonic flow $\overrightarrow{G}^{L^2}[t]$ with all subsystems by edges of $\overrightarrow{G}^{L^2}[t]$, this fact implies that Lagrangians on edges of $\overrightarrow{G}^{L^2}[t]$ hold with the continuity equation at vertices, i.e., if $L^2 : (v,u) \to L(v,u)[t] - iL(v,u)[t]$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ then $\overrightarrow{G}^{L^2}[t]$ is a harmonic flow with $L(v,u)[t] \in \mathbb{R}$ for edges $(v,u) \in E\left(\overrightarrow{G}\right)$. Whence, $\frac{d\overrightarrow{G}^{L^2}[t]}{dt}$ and $\int_{t_1}^{t_2} \overrightarrow{G}^{L^2} dt$ both are existed in $\mathscr{G}^{\pm}_{\mathscr{B}}$ by Section 3.1.

Now, if

$$\mathscr{L}\left[L^2(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\right] : (v,u) \in E\left(\overrightarrow{G}\right) \to \mathscr{L}\left[L^2(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)\right]$$

is a differentiable functional with $[\mathscr{L}, A] = \mathbf{0}$ for $A \in \mathscr{A}$, there must be $\overrightarrow{G}^{\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]} \in \mathscr{G}^{\pm}_{\mathscr{B}}$, i.e., hold with the continuity equations on vertices of $\overrightarrow{G}$, where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Consider the variational action

$$J\left[\overrightarrow{G}^{L^2}[t]\right] = \int_{t_1}^{t_2} \overrightarrow{G}^{\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]} dt. \tag{3.1}$$

on a harmonic flow $\overrightarrow{G}^{L^2}[t] \in \mathscr{G}_{\mathscr{B}}^{\pm}$. By variational calculus we know that

$$\begin{aligned}\delta J\left[\overrightarrow{G}^{L^2}[t]\right] &= \delta \int_{t_1}^{t_2} \overrightarrow{G}^{\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]} dt \\ &= \overrightarrow{G}^{\delta \int_{t_1}^{t_2} \mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right] dt} = \overrightarrow{G}^{\int_{t_1}^{t_2} \sum\limits_{i=1}^{n} \left(\frac{\partial \mathscr{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}_i}\right)\delta x_i dt}.\end{aligned}$$

According to the Hamiltonian principle there must be $\delta J\left[\overrightarrow{G}^{L^2}[t]\right] = \mathbf{O}$, i.e.,

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\left(\frac{\partial \mathscr{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}_i}\right)\Big|_{(v,u)}\right)\delta x_i dt = 0 \tag{3.2}$$

for $(v,u) \in E\left(\overrightarrow{G}\right)$. However, this can be only happened only if each coefficient of δx_i is 0 in (3.2), i.e.,

$$\left(\frac{\partial \mathscr{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}_i}\right)\Big|_{(v,u)} = 0, \quad 1 \le i \le n \tag{3.3}$$

for $(v,u) \in E\left(\overrightarrow{G}\right)$ which results in Euler-Lagrange equations on $\overrightarrow{G}^{L^2}[t]$ following.

Theorem 3.4 *If $L^2(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)$ is a Lagrangian on edge (v,u) and $\mathscr{L}\left[L^2(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\right]$: $(v,u) \to \mathscr{L}\left[L^2(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)\right]$ is a differentiable functional on a harmonic flow $\overrightarrow{G}^{L^2}[t]$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ with $[\mathscr{L}, A] = \mathbf{0}$ for $A \in \mathscr{A}$, then*

$$\frac{\partial \overrightarrow{G}^{\mathscr{L}}}{\partial x_i} - \frac{d}{dt}\frac{\partial \overrightarrow{G}^{\mathscr{L}}}{\partial \dot{x}_i} = \mathbf{O}, \quad 1 \le i \le n. \tag{3.4}$$

Let the polynomial expansion of $\mathscr{L}$ be

$$\mathscr{L}\left[L^2\right] = \mathscr{L}(0) + \frac{1}{1!}\frac{d\mathscr{L}}{d(L^2)}\Big|_{L=0} L^2 + \cdots + \frac{1}{m!}\frac{d^m\mathscr{L}}{d(L^2)^m}\Big|_{L=0} \left(L^2\right)^m + o\left(\left(L^2\right)^m\right) \tag{3.5}$$

on L^2 with an approximation

$$\mathscr{L}\left[L^2\right] = \mathscr{L}(0) + \frac{1}{1!}\frac{d\mathscr{L}}{d(L^2)}\Big|_{L=0} L^2 + \cdots + \frac{1}{m!}\frac{d^m\mathscr{L}}{d(L^2)^m}\Big|_{L=0} \left(L^2\right)^m$$

of m terms. Calculation shows that

$$\frac{\partial \mathscr{L}}{\partial x_i} = \left.\frac{d\mathscr{L}}{d(L^2)}\right|_{L=0} \frac{\partial L^2}{\partial x_i} + \cdots + \frac{1}{(m-1)!} \left.\frac{d^m \mathscr{L}}{d(L^2)^m}\right|_{L=0} \left(L^2\right)^{m-1} \frac{\partial L^2}{\partial x_i}$$

and

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}_i} = \left.\frac{d\mathscr{L}}{d(L^2)}\right|_{L=0} \frac{d}{dt}\frac{\partial L^2}{\partial \dot{x}_i} + \cdots + \frac{1}{(m-1)!} \left.\frac{d^m \mathscr{L}}{d(L^2)^m}\right|_{L=0} \frac{d}{dt}\left(\left(L^2\right)^{m-1} \frac{\partial L^2}{\partial \dot{x}_i}\right).$$

Whence,

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}_i} &= \left.\frac{d\mathscr{L}}{d(L^2)}\right|_{L=0} \left(\frac{\partial L^2}{\partial x_i} - \frac{d}{dt}\frac{\partial L^2}{\partial \dot{x}_i}\right) + \cdots \\ &\quad + \frac{1}{(m-1)!} \left.\frac{d^m \mathscr{L}}{d(L^2)^m}\right|_{L=0} \left(\left(L^2\right)^{m-1} \frac{\partial L^2}{\partial x_i} - \frac{d}{dt}\left(\left(L^2\right)^{m-1} \frac{\partial L^2}{\partial \dot{x}_i}\right)\right) = 0\end{aligned}$$

for $(v,u) \in E\left(\overrightarrow{F}\right)$ and integers $1 \leq i \leq n$. Particularly, if $\mathscr{L}$ is linear dependent on L^2, we get the following conclusion.

Corollary 3.5 *If $\mathscr{L}$ is linear dependent on L^2, then*

$$\frac{\partial \overrightarrow{G}^{L^2}}{\partial x_i} - \frac{d}{dt}\frac{\partial \overrightarrow{G}^{L^2}}{\partial \dot{x}_i} = \mathbf{O}, \quad 1 \leq i \leq n.$$

Corollary 3.5 enables one to define the *Lagrangian* of a harmonic flow $\overrightarrow{G}^{L^2}[t]$ by $\mathscr{L}\left[\overrightarrow{G}^{L^2}[t]\right]$: $(v,u) \to L(v,u) - iL(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ which is generally dependent on $L(v,u)$, $(v,u) \in E\left(\overrightarrow{G}\right)$ in Theorem 3.4. If it is independent on $L(v,u)$, $(v,u) \in E\left(\overrightarrow{G}\right)$, we get an interesting result following.

Corollary 3.6(Euler-Lagrange) *If the Lagrangian $\mathscr{L}\left[\overrightarrow{G}^{L^2}[t]\right]$ of a harmonic flow $\overrightarrow{G}^{L^2}[t]$ is independent on (v,u), i.e., all Lagrangians $L^2(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)$, $(v,u) \in E\left(\overrightarrow{G}\right)$ are synchronized, then the dynamic behavior of $\overrightarrow{G}^{L^2}[t]$ can be characterized by n equations*

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \qquad 1 \leq i \leq n, \tag{3.6}$$

which are essentially equivalent to the Euler-Lagrange equations of bouquet $\overrightarrow{B}_1^{L^2} \in \overrightarrow{B}_{1\mathscr{B}}^{\pm}$, i.e, dynamic equations on a particle P.

For example, let

$$\mathscr{L}\left[L^2(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)\right] = \sum_{i=1}^{n} c_i \dot{x}_i^2 - \sum_{1 \leq i,j \leq n} c_{ij} x_i x_j,$$

which is independent on $(v,u)\in\in E\left(\overrightarrow{G}\right)$. Then

$$\frac{\partial\mathscr{L}}{\partial x_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{x}_i}=2c_{ij}x_j-2c_i\ddot{x}_i.$$

We get a system of n differential equations

$$\begin{cases} c_1\ddot{x}_1-\sum\limits_{j\neq1}c_{1j}x_j=0,\\ c_2\ddot{x}_2-\sum\limits_{j\neq2}c_{2j}x_j=0,\\ \cdots\cdots\cdots\cdots\cdots\cdots,\\ c_n\ddot{x}_n-\sum\limits_{j\neq n}c_{nj}x_j=0\end{cases}$$

for harmonic flow $\overrightarrow{G}^{L^2}[t]$ by Corollary 3.6. The solvability of equations (3.4) is answered in results following.

Theorem 3.7 *Let $\overrightarrow{G}\in\mathscr{G}$ and $n\geq1$ be an integer. A Cauchy problem*

$$\begin{cases}\mathcal{F}\left(\mathbf{x},X,\frac{\partial X}{\partial x_1},\cdots,\frac{\partial X}{\partial x_n},\frac{\partial^2X}{\partial x_1\partial x_2},\cdots\right)=0,\\ X|_{\mathbf{x}_0}=\overrightarrow{G}^{L^0}\end{cases}\qquad(3.7)$$

is solvable in a boundary domain $D\subset\mathscr{G}^{\pm}_{\mathbb{C}^n}$ if and only if the Cauchy problem

$$\begin{cases}\mathcal{F}\left(\mathbf{x},X_{vu},\frac{\partial X_{vu}}{\partial x_1},\cdots,\frac{\partial X_{vu}}{\partial x_n},\frac{\partial^2X_{vu}}{\partial x_1\partial x_2},\cdots\right)=0,\ u\in N_G(v),\\ X_{vu_1}^{A^+_{vu_1}}+X_{vu_2}^{A^+_{vu_2}}+\cdots+X_{vu_{\rho(v)}}^{A^+_{vu_{\rho(v)}}}=L(v),\ u_i\in N_G(v),\ 1\leq i\leq\rho(v),\\ X_{vu_i}|_{\mathbf{x}_0}=L_0(v,u_i),\ 1\leq i\leq\rho(v),\\ L_0^{A^+_{vu_1}}(v,u_1)+L_0^{A^+_{vu_2}}(v,u_2)+\cdots+L_0^{A^+_{vu_{\rho(v)}}}(v,u_{\rho(v)}))=L_0(v)\end{cases}\qquad(3.8)$$

is solvable in $D|_v$ for $\forall v\in V\left(\overrightarrow{G}\right)$, where D_v, $\rho(v)$ denote respectively the domain of D constraint on the closure neighborhood $N_{\overrightarrow{G}}(v)\bigcup\{v\}$ and the valency of v in $\overrightarrow{G}$, and particularly, if the solution of

$$\mathcal{F}\left(\mathbf{x},X_{vu},\frac{\partial X_{vu}}{\partial x_1},\cdots,\frac{\partial X_{vu}}{\partial x_n},\frac{\partial^2X_{vu}}{\partial x_1\partial x_2},\cdots\right)=0\qquad(3.9)$$

is linearly dependent on the initial value $L_0(v,u)$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ with $\overrightarrow{G}^{L_0^2}\in\mathscr{G}^{\pm}_{\mathbb{C}^n}$ and $[\frac{\partial}{\partial x_i},A]=\mathbf{0}$ for $A\in\mathscr{A}$, then (3.7) is solvable on $D\subset\mathscr{G}^{\pm}_{\mathbb{C}^n}$.

Proof Clearly, if (3.7) is solvable in $D\subset\mathscr{G}^{\pm}_{\mathbb{C}^n}$, without loss of generality, let the solution be $\overrightarrow{G}^{L^2}$, then $\overrightarrow{G}^{L^2}$ holds with (3.8). Conversely, if (3.8) is solvable in D_v for $\forall v\in V\left(\overrightarrow{G}\right)$, then there are solutions X_{vu} on edges $(v,u)\in E\left(\overrightarrow{G}\right)$, hold with the continuity equations on vertices of $\overrightarrow{G}^{L^2}$.

Now, if the solution X_{vu} of

$$\mathcal{F}\left(\mathbf{x}, X_{vu}, \frac{\partial X_{vu}}{\partial x_1}, \cdots, \frac{\partial X_{vu}}{\partial x_n}, \frac{\partial^2 X_{vu}}{\partial x_1 \partial x_2}, \cdots\right) = 0$$

is linearly dependent on the initial value $L_0(v,u)$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$, there is a linear functional H such that $X = H\left(\mathbf{x}, L_0(v,u)\right)$ holds with (3.9).

Notice that $\overrightarrow{G}^{L_0^2} \in \mathscr{G}^{\pm}_{\mathbb{C}^n}$ and $[\frac{\partial}{\partial x_i}, A] = \mathbf{0}$ for $A \in \mathscr{A}$ by assumption. We know that

$$\sum_{u \in N_{G^0}(v)} L^{2^{A^+_{vu}}}_0(v,u) = L_0^2(v)$$

for $v \in V\left(\overrightarrow{G}\right)$ by definition. Whence,

$$H\left(L_0(v)\right) = H\left(\sum_{u \in N_{G^0}(v)} L^{2^{A^+_{vu}}}_0(v,u)\right) = \sum_{u \in N_{G^0}(v)} \left(H\left(L_0^2(v,u)\right)\right)^{A^+_{vu}}$$

i.e., hold with the continuity equation at vertex v for $v \in V\left(\overrightarrow{G}\right)$. Therefore, if we define $L^2 : v \to H\left(L_0(v)\right)$ for $v \in V\left(\overrightarrow{G}\right)$ and $L^2 : (v,u) \to H\left(\mathbf{x}, L_0(v,u)\right)$ for $(v,u) \in E\left(\overrightarrow{G}\right)$, we get a harmonic flow $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{C}^n}$ which holds with (3.7). $\square$

Theorem 3.7 enables one to extend solutions of differential equations in a domain $D \subset \mathbb{C}^n$ to $\mathscr{G}^{\pm}_{\mathbb{C}^n}$ if the solution is linearly dependent on initial values. For example, we have know the solution of the heat equation

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

is linearly dependent on the initial values $X(\mathbf{x}, t_0) = \varphi(\mathbf{x})$ in $\mathbb{R}^n \times \mathbb{R}$ if $\varphi(\mathbf{x})$ is continuous and bounded in $\mathbb{R}^n$, where c is a non-zero constant in. We get the conclusion following.

Corollary 3.8 *Let $\overrightarrow{G} \in \mathscr{G}$ with $\overrightarrow{G}^{L_0^2} \in \mathscr{G}^{\pm}_{\mathbb{R}^n \times \mathbb{R}}$ and $n \geq 1$ be an integer. If $[\frac{\partial}{\partial x_i}, A] = \mathbf{0}$ for $A \in \mathscr{A}$, then the Cauchy problem*

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

with $X|_{t=t_0} = \overrightarrow{G}^{L_0^2} \in \mathscr{G}^{\pm}_{\mathscr{V}}$ is solvable on a domain $D \subset \mathscr{G}^{\pm}_{\mathbb{R}\times\mathbb{R}}$ if $L_0(v,u)$ is continuous and bounded in $\mathbb{R}^n$ for $(v,u) \in E\left(\overrightarrow{G}\right)$.

§4. Balance Recovery

A flow $\overrightarrow{G}^{L^2}$ maybe not continuity. Even it is, it maybe not harmonic. *How to transform a non-continuity or non-harmonic flow to a continuity or harmonic flow, i.e., balance recovery?* We consider this problem in the following.

Definition 4.1 *Let $\overrightarrow{G}^{L^2}$ be a flow with $L^2(v,u) = (L_1(v,u), L_2(v,u))$, $L_1(v,u), L_2(v,u) \in \mathscr{B}$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$ and $v \in V\left(\overrightarrow{G}\right)$. Define an action operations O on v, i.e., input or output an additional flow A at vertex v with $O(v) = (A, A)$, where $A \in \mathscr{B}$ such as those shown in Fig.*8 *following.*

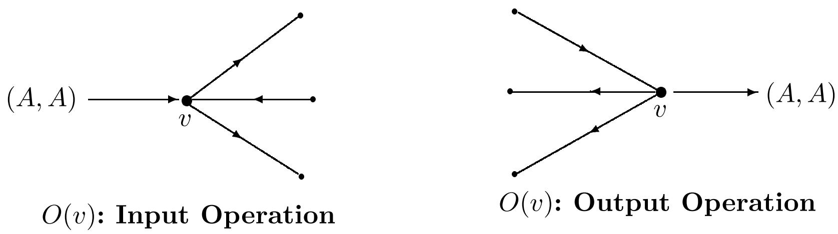


Fig.8

Observation 4.2 If a continuity flow $\overrightarrow{G}^L$ is influenced externally by $A_1, A_2, \cdots, A_s$ respectively on vertices $v_i \in V\left(\overrightarrow{G}\right)$, $1 \leq i \leq s$ which results in an imbalanced flow $\overrightarrow{G}^{L'}$, we can offset inputs $-A_1, -A_2, \cdots, -A_s$ on vertices $v_i \in V\left(\overrightarrow{G}\right)$, $1 \leq i \leq s$ and obtain the continuity flow $\overrightarrow{G}^L$ immediately.

Denoted by $\varpi\left(\overrightarrow{G}^L\right)$ the number of acted vertices by O and $o\left(\overrightarrow{G}^L\right)$ the number of conservation vertices in $\overrightarrow{G}^L$. Then Observation 4.2 implies the following result.

Proposition 4.3 $\varpi\left(\overrightarrow{G}^L\right) + o\left(\overrightarrow{G}^L\right) = \left|\overrightarrow{G}\right|$.

A continuity flow $\overrightarrow{G}^{L^2}$ maybe not a harmonic flow even for the circuit $\overrightarrow{C}$ such as those shown in Fig.9,

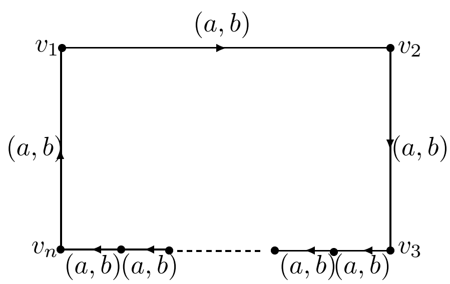


Fig.9

where $a \neq -b$, which naturally brings about the following problem.

Problem 4.4 *Can a flow $\overrightarrow{G}^{L^2}$ (continuity or not) be transformed to a harmonic flow, i.e., balanced at everywhere by input and output operations O on vertices of $\overrightarrow{G}$? And generally, if a flow $\overrightarrow{G}^{L^2}[t]$ evolves on the time t, can it be transformed to a harmonic flow by action O within interval $[t_1, t_2]$ of times?*

The answer of this problem is affirmative, which in fact consists of the foundation of traditional Chinese medicine theory.

Theorem 4.5 *For a flow $\overrightarrow{G}^{L^2}$ (continuity or not) over a Banach space $\mathscr{B}$ with $\left[A, \frac{d}{dt}\right] = \mathbf{0}$ and $\left[A, \int_t^0\right] = \mathbf{0}$ for $A \in \mathscr{A}$, there are input or output operations O on vertices of $\overrightarrow{G}$ which transforms $\overrightarrow{G}^{L^2}$ to a harmonic flow, i.e., balanced on everywhere, and generally, if $\overrightarrow{G}^{L^2}[t]$ is continuous on time t holding with a harmonic flow $\overrightarrow{G}^{L^2}[0]$, then there are input or output operations O on vertices of $\overrightarrow{G}$ which transforms $\overrightarrow{G}^{L^2}[t]$ to a harmonic flow on time t.*

Proof Notice that $L^2: (v,u) \to (L_1(v,u), L_2(v,u))$ with $L_1(v,u), L_2(v,u) \in \mathscr{B}$. First, it is clear that a non-continuity flow $\overrightarrow{G}^{L^2}$ can be transferred to a continuity flow. Without loos of generality, let $v \in V\left(\overrightarrow{G}\right)$ be such a vertex with

$$\sum_{u\in V\left(\overrightarrow{G}\right)} L_1^{A_{vu}}(v,u) \neq \sum_{w\in V\left(\overrightarrow{G}\right)} L_1^{A_{wv}}(w,v), \quad \text{or} \quad \sum_{u\in V\left(\overrightarrow{G}\right)} L_2^{A_{vu}}(v,u) \neq \sum_{w\in V\left(\overrightarrow{G}\right)} L_2^{A_{wv}}(w,v).$$

Then, we can let O act on v by input $O(v) = (S_1, S_2)$ with

$$\begin{aligned} S_1 &= \sum_{u\in V\left(\overrightarrow{G}\right)} L_1^{A_{vu}}(v,u) - \sum_{w\in V\left(\overrightarrow{G}\right)} L_1^{A_{wv}}(w,v), \\ S_2 &= \sum_{u\in V\left(\overrightarrow{G}\right)} L_2^{A_{vu}}(v,u) - \sum_{w\in V\left(\overrightarrow{G}\right)} L_2^{A_{wv}}(w,v). \end{aligned}$$

Clearly, v becomes a conservation vertex after such an action. Notice that such action can be acted on all non-conservation vertices in $\overrightarrow{G}^{L^2}$ and get a continuity flow $\overrightarrow{G}^{L^2}$ finally.

Second, there exists a labeled graph $\overrightarrow{G}^{L'}$ on $\overrightarrow{G}$ such that $\overrightarrow{G}^{L^2} + \overrightarrow{G}^{L'}$ is a harmonic flow. Define a labeling L' on $\overrightarrow{G}$ by

$$L': (v,u) \to \begin{cases} \left(-\dfrac{L_1(v,u)+L_2(v,u)}{2}, -\dfrac{L_1(v,u)+L_2(v,u)}{2}\right) & \text{if } u \in N^(v), \\ (\mathbf{0}, \mathbf{0}) & \text{otherwise.} \end{cases}$$

Then, calculation shows that

$$\begin{aligned} L^2 + L': (v,u) &\to \left(L_1(v,u) - \frac{L_1(v,u)+L_2(v,u)}{2}, \ L_2(v,u) - \frac{L_1(v,u)+L_2(v,u)}{2}\right) \\ &= \left(\frac{L_1(v,u)-L_2(v,u)}{2}, -\frac{L_1(v,u)-L_2(v,u)}{2}\right), \end{aligned}$$

i.e., the flows on the edge (v,u) are in balance which implies that $\overrightarrow{G}^{L^2} + \overrightarrow{G}^{L'}$ is harmonic.

For $\forall v \in V\left(\overrightarrow{G}\right)$ let

$$V_v = \sum_{w\in N^-(v)} \frac{L_1^{A_{wv}}(w,v) + L_2^{A_{wv}}(w,v)}{2} - \sum_{u\in N^+(v)} \frac{L_1^{A_{vu}}(v,u) + L_2^{A_{vu}}(v,u)}{2}$$

and define $O(v) = (V_v,\ V_v)$ on the vertex v with allocation

$$\left(-\frac{L_1(v,u)+L_2(v,u)}{2},\ -\frac{L_1(v,u)+L_2(v,u)}{2}\right),$$
$$\left(\frac{L_1(w,v)+L_2(w,v)}{2},\ \frac{L_1(w,v)+L_2(w,v)}{2}\right)$$

respective on edges (v,u) for $u \in N^-(v)$ and (w,v) for $w \in N^+(v)$. Clearly, O transforms $\overrightarrow{G}^{L^2}$ to a flow with

$$\left(\frac{L_1(v,u)-L_2(v,u)}{2},\ -\frac{L_1(v,u)-L_2(v,u)}{2}\right)$$

on edges (v,u) for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. Whence, we get a harmonic flow.

Now, if $\overrightarrow{G}^{L^2}[t]$ is continuous on time t holding with a harmonic flow $\overrightarrow{G}^{L^2}[0]$, we consider the differential flow

$$\frac{d}{dt}\left(\overrightarrow{G}^{L^2}[t]\right) = \overrightarrow{G}^{\frac{d}{dt}\left(L^2[t]\right)}$$

and apply the method of the previous. For $\forall v \in V\left(\overrightarrow{G}\right)$ we define

$$\begin{aligned} V'_v &= \sum_{w \in N^-(v)} \frac{\frac{d}{dt}\left(L_1^{A_{wv}}(w,v)[t]\right) + \frac{d}{dt}\left(L_2^{A_{wv}}(w,v)[t]\right)}{2} \\ &\quad - \sum_{u \in N^+(v)} \frac{\frac{d}{dt}\left(L_1^{A_{vu}}(v,u)[t]\right) + \frac{d}{dt}\left(L_2^{A_{vu}}(v,u)[t]\right)}{2}, \end{aligned}$$

i.e., V'_v in $\frac{d}{dt}\left(\overrightarrow{G}^{L^2}[t]\right)$ and let $O'(v) = (V'_v,\ V'_v)$ on vertex v with allocation

$$\left(-\frac{\frac{d}{dt}\left(L_1(v,u)[t]\right) + \frac{d}{dt}\left(L_2(v,u)[t]\right)}{2},\ -\frac{\frac{d}{dt}\left(L_1(v,u)[t]\right) + \frac{d}{dt}\left(L_2(v,u)[t]\right)}{2}\right),$$
$$\left(\frac{\frac{d}{dt}\left(L_1(w,v)[t]\right) + \frac{d}{dt}\left(L_2(w,v)[t]\right)}{2},\ \frac{\frac{d}{dt}\left(L_1(w,v)[t]\right) + \frac{d}{dt}\left(L_2(w,v)[t]\right)}{2}\right)$$

respective on edges (v,u) for $u \in N^-(v)$ and (w,v) for $w \in N^+(v)$ in this case. Clearly, $O' : v \in V\left(\overrightarrow{G}\right) \to O'(v)$ transforms $\frac{d}{dt}\left(\overrightarrow{G}^{L^2}[t]\right)$ to a flow with

$$\left(\frac{\frac{d}{dt}\left(L_1(v,u)[t]\right) - \frac{d}{dt}\left(L_2(v,u)[t]\right)}{2},\ -\frac{\frac{d}{dt}\left(L_1(v,u)[t]\right) - \frac{d}{dt}\left(L_2(v,u)[t]\right)}{2}\right)$$

on edges (v,u) for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. Now, considering the integral flow

$$\int_0^t \frac{d}{dt}\left(\overrightarrow{G}^{L^2}[t]\right) dt = \overrightarrow{G}^{\int_0^t \frac{d}{dt}\left(L^2[t]\right)dt},$$

we immediately get a harmonic flow on time t by that $\overrightarrow{G}^{L^2}[0]$ is a such one. $\square$

Theorem 4.5 implies that for any continuity flow $\overrightarrow{G}^{L^2}[t]$ with

$$\left[A, \frac{d}{dt}\right] = \mathbf{0} \quad \text{and} \quad \left[A, \int_t^0\right] = \mathbf{0}$$

for $A \in \mathscr{A}$ there are always input or output operations O on vertices $v_1, v_2, \cdots, v_s$ of $\overrightarrow{G}$ which transforms a $\overrightarrow{G}^{L^2}[t]$ to a harmonic flow if $\overrightarrow{G}^{L^2}[0]$ is a harmonic flow, and these vertices do not dependent on $\overrightarrow{G}^{L^2}[t]$ is variable or not.

Definition 4.6 *A vertex $v \in V\left(\overrightarrow{G}^{L^2}[t]\right)$ is zero-acted on time t if $O(v) = \mathbf{0}$.*

Theorem 4.7 *Let $\overrightarrow{G}^{L^2}[t]$ be a continuity flow on time t. Then, all vertices of $\overrightarrow{G}^{L^2}[t]$ are zero-acted.*

Proof By the proof of Theorem 4.5, a vertex $v \in V\left(\overrightarrow{G}^{L^2}[t]\right)$ is zero-acted, i.e., $O(v) = \mathbf{0}$ if and only if

$$\sum_{w \in N^-(v)} \frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2} = \sum_{u \in N^+(v)} \frac{L_1^{A_{vu}}(v,u)[t] + L_2^{A_{vu}}(v,u)[t]}{2}.$$

Notice that

$$\sum_{w \in N^-(v)} L_1^{A_{wv}}(w,v)[t] = \sum_{u \in N^+(v)} L_1^{A_{vu}}(v,u)[t], \tag{4.1}$$

$$\sum_{w \in N^-(v)} L_2^{A_{wv}}(w,v)[t] = \sum_{u \in N^+(v)} L_2^{A_{vu}}(v,u)[t] \tag{4.2}$$

by definition. We naturally know that

$$\sum_{w \in N^-(v)} \frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2} = \sum_{u \in N^+(v)} \frac{L_1^{A_{vu}}(v,u)[t] + L_2^{A_{vu}}(v,u)[t]}{2}.$$

Adding (4.1) with (4.2) and divided the sum by 2, we get the result. $\square$

Notice that $O(v) = \mathbf{0}$ does not implies there are not needed action O on $v \in V\left(\overrightarrow{G}^{L^2}[t]\right)$ for transforming $\overrightarrow{G}^{L^2}[t]$ to a harmonic flow, for instance the continuity flow $\overrightarrow{C}_n^{L^2}$ shown in Fig.9. But, *how to hold on a zero-action O*? In fact, we can not realize O just one input or output action on v. In this case, O is decomposed into 2 actions, i.e., $O = O_1 + O_2$ with

$$O_1(v) = -\sum_{w \in N^-(v)} \frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2}, \quad O_2(v) = \sum_{u \in N^+(v)} \frac{L_1^{A_{vu}}(v,u)[t] + L_2^{A_{vu}}(v,u)[t]}{2}$$

and each O_1 or O_2 action on v allocates respectively

$$\left(-\frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2}, -\frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2}\right)$$

on edges (w, v), $w \in N^{-}(v)$ and

$$\left(\frac{L_1^{A_{vu}}(v,u)[t] + L_2^{A_{vu}}(v,u)[t]}{2}, \frac{L_1^{A_{vu}}(v,u)[t] + L_2^{A_{vu}}(v,u)[t]}{2}\right)$$

on edges (v, u), $u \in N^{+}(v)$, such as those shown in Fig.10.

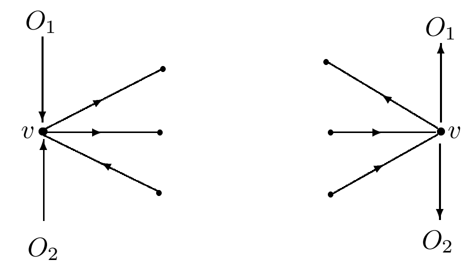


Fig.10

However, a calculation immediately enables one getting the numbers $\varpi\left(\overrightarrow{P}_n^{L^2}\right)$ and $\varpi\left(\overrightarrow{C}_n^{L^2}\right)$.

Theorem 4.8 *If $\overrightarrow{P}_n^{L^2}[t]$ and $\overrightarrow{C}_n^{L^2}[t]$ are continuity flows with $\left[A, \frac{d}{dt}\right] = \mathbf{0}$, $\left[A, \int_t^0\right] = \mathbf{0}$ for $A \in \mathscr{A}$, then $\varpi\left(\overrightarrow{P}_n^{L^2}[t]\right) = 1$ and $\varpi\left(\overrightarrow{C}_n^{L^2}[t]\right) = 1$ if $\overrightarrow{P}_n^{L^2}[0]$ and $\overrightarrow{C}_n^{L^2}[0]$ are harmonic.*

Proof This fact is an immediate conclusion by calculation. Assuming flows of $\overrightarrow{P}_n^{L^2}[t]$ shown in Fig.11 in time t with $a_i \neq -b_i$ for integers $1 \leq i \leq n$,

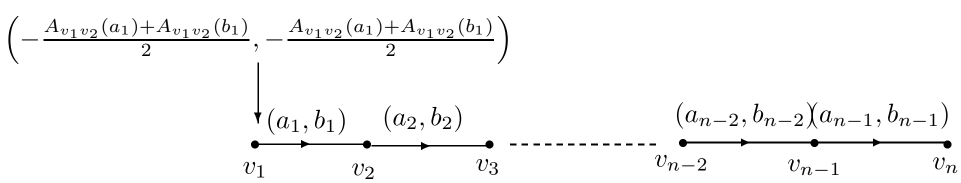


Fig.11

we can calculate flows on its edges by an input action

$$O(v_1) = \left(-\frac{A_{v_1v_2}(a_1) + A_{v_1v_2}(b_1)}{2}, -\frac{A_{v_1v_2}(a_1) + A_{v_1v_2}(b_1)}{2}\right)$$

on vertex v_1. In fact, the flow on edge (v_1, v_2) is

$$\left(a_1 - \frac{a_1 + b_1}{2}, b_1 - \frac{a_1 + b_1}{2}\right) = \left(\frac{a_1 - b_1}{2}, -\frac{a_1 - b_1}{2}\right).$$

Then, we can determine flows on edges $(v_2, v_3), \cdots, (v_{n-2}, v_{n-1}), (v_{n-1}, v_n)$ by the conservation laws on vertices $v_2, v_3, \cdots, v_{n-1}$. For example, let the flows on edge (v_2, v_3), $\cdots$, (v_{n-2}, v_{n_1}),

(v_{n-1}, v_n) be (a_2', b_2'), $\cdots$, (a_{n-2}', b_{n-2}'), (a_{n-1}', b_{n-1}'), respectively. Then, we must know that

$$A_{v_2 v_1}\left(\frac{a_1 - b_1}{2}\right) = A_{v_2 v_3}\left(a_2'\right) \quad \text{and} \quad A_{v_2 v_1}\left(-\frac{a_1 - b_1}{2}\right) = A_{v_2 v_3}\left(b_2'\right),$$

by the conservation law on vertex v_2, i.e., $A_{v_2 v_3}\left(a_2'\right) = -A_{v_2 v_3}\left(b_2'\right)$ or $A_{v_2 v_3}\left(a_2' + b_2'\right) = \mathbf{0}$. There must be $a_2' = -b_2'$ by the linearity assumption on end-operators in $\mathscr{A}$. Continuing this process on vertices $v_3, \cdots, v_{n-1}$, we finally get $a_3' = -b_3', \cdots, a_{n-2}' = -b_{n-2}'$ and $a_{n-1}' = -b_{n-1}'$, i.e., a harmonic flow on $\overrightarrow{P}_n$. Whence, $\varpi\left(\overrightarrow{P}_n^{L^2}[t]\right) = 1$.

Similarly, we know that $\varpi\left(\overrightarrow{C}_n^{L^2}[t]\right) = 1$ by a zero-action O on any vertex of $\overrightarrow{C}_n^{L^2}$. This completes the proof. $\square$

The proof of Theorem 4.8 enables one to know that a vertex v with of $\rho(v) = 2$ is not needed in the calculation of $\varpi\left(\overrightarrow{G}^{L^2}[t]\right)$. We introduce a conception following.

Definition 4.9 *If $\overrightarrow{G}$ is a graph, the topological neighborhood $N^p(v)$ on vertex $v \in V\left(\overrightarrow{G}\right)$ is defined to be all vertices in $\overrightarrow{G}$ connecting v with an induced path of $\overrightarrow{G}$. If $p(v,u)$ is such a path for $u \in N^p(v)$, the induced subgraph $\langle\ (\{v\} \bigcup V(p(u,v))) \setminus \{u\} \mid u \in N^p(v)\}\rangle$, denoted by $[v]_p$ is called the claw graph on v in $\overrightarrow{G}$.*

Clearly, if all $p(v,u) = \overrightarrow{P}_2$ in $\overrightarrow{G}$, $N^p(v) = N(v)$, and if $\overrightarrow{G}^{L^2}$ is a continuity flow with all induced paths connecting vertex $v \in \left(\overrightarrow{G}^{L^2}\right)$ balanced, then $\left(\overrightarrow{G} \setminus \{[v]_P\}\right)^{L^2}$ is also a continuity flow by the proof of Theorem 4.8. This fact enables one to get $\varpi\left(\overrightarrow{G}^{L^2}[t]\right)$ following.

Theorem 4.10 *If $\overrightarrow{G}^{L^2}[t]$ is a continuity flow on a connected graph $\overrightarrow{G}$ of order≥ 3, then there are vertices $v^1, v^2, \cdots, v^{k_0} \in V\left(\overrightarrow{G}[t]\right)$ such that*

$$\overrightarrow{G} \setminus \left\{\left[v^1\right]_p, \left[v^2\right]_p, \cdots, \left[v^{k_0}\right]_p\right\} = \left(\bigcup_{i=1}^{k_1} \overrightarrow{C}_{n_i}\right) \bigcup \left(\bigcup_{j=1}^{k_2} \overrightarrow{P}_{n_j}\right) \bigcup \left(\bigcup_{l=1}^{k_3} \overrightarrow{P}_1\right), \tag{4.3}$$

where $n_i \geq 3, n_j \geq 2, k_i \geq 0$ for integers $0 \leq i \leq 2$, $k_0 + \sum\limits_{i=1}^{k_1} n_i + \sum\limits_{j=1}^{k_2} n_j + k_3 = \left|\overrightarrow{G}\right|$ and $\varpi\left(\overrightarrow{G}^{L_2}[t]\right) \leq k_0 + k_1 + k_2$, which implies that

$$\varpi\left(\overrightarrow{G}^{L_2}[t]\right) = \min\left\{k_0 + k_1 + k_2 \middle|\ \textit{all triad } k_0, k_1, k_2 \textit{ holds with equality } (4.3)\right\}$$

if $\overrightarrow{G}^{L^2}[0]$ is harmonic.

Proof Clearly, there are only 2 continuity graphs $\overrightarrow{C}_3$ and $\overrightarrow{P}_3$ if $\left|\overrightarrow{G}\right| = 3$ and $\varpi\left(\overrightarrow{G}^{L^2}[t]\right) = 1$ by Theorem 4.8. If $\left|\overrightarrow{G}\right| = 4$, for $\forall v \in V\left(\overrightarrow{G}\right)$, we know that $\overrightarrow{G} \setminus \{[v]_p\} = \overrightarrow{P}_3$, $\overrightarrow{C}_3$, the disjoint union of $\overrightarrow{P}_2$ with $\overrightarrow{P}_1$ or 3 isolated vertices and $\varpi\left(\overrightarrow{G}^{L^2}[t]\right) = 1$ or 2, i.e., the result is true for integers $\left|\overrightarrow{G}\right| \leq 4$.

Suppose the result is true for all graphs $\overrightarrow{G}$ with $\left|\overrightarrow{G}\right| \leq k$. We prove it is also true in the case of $\left|\overrightarrow{G}\right| \leq k+1$. The proof is divided into 2 cases following.

Case 1. $\overrightarrow{G}$ is 2-connected.

In this case, for $\forall v \in V\left(\overrightarrow{G}\right)$, $\overrightarrow{G} \setminus \{[v]_P\}$ is connected. By the induction assumption, there are vertices $v^1, v^2, \cdots, v^{k_0}$ in $\overrightarrow{G}$ such that

$$\left(\overrightarrow{G} \setminus \{[v]_p\}\right) \setminus \left\{\left[v^1\right]_p, \left[v^2\right]_p, \cdots, \left[v^{k_0}\right]_p\right\} = \left(\bigcup_{i=1}^{k_1} \overrightarrow{C}_{n_i}\right) \bigcup \left(\bigcup_{j=1}^{k_2} \overrightarrow{P}_{n_j}\right) \bigcup \left(\bigcup_{l=1}^{k_3} \overrightarrow{P}_1\right),$$

i.e.,

$$\overrightarrow{G} \setminus \left\{[v]_p, \left[v^1\right]_p, \left[v^2\right]_p, \cdots, \left[v^{k_0}\right]_p\right\} = \left(\bigcup_{i=1}^{k_1} \overrightarrow{C}_{n_i}\right) \bigcup \left(\bigcup_{j=1}^{k_2} \overrightarrow{P}_{n_j}\right) \bigcup \left(\bigcup_{l=1}^{k_3} \overrightarrow{P}_1\right).$$

Applying Theorem 4.8, we immediately know that $\varpi\left(\overrightarrow{G}^{L_2}[t]\right) \leq (k_0+1)+k_1+k_2$ by the initial condition and the result follows.

Case 2. $\overrightarrow{G}$ is 1-connected.

In this case, there is a cut vertex $v \in V\left(\overrightarrow{G}\right)$ such that $\overrightarrow{G} \setminus \{[v]_P\}$ is a disjoint union of s connected blocks $\overrightarrow{B}_1, \overrightarrow{B}_2, \cdots, \overrightarrow{B}_s$, $s \geq 2$. It is obvious that $\left|\overrightarrow{B}_i\right| \leq k$. Without of loss of generality, assume $\left|\overrightarrow{B}_i\right| \geq 3$. Then there are vertices $v^{i_1}, v^{i_2}, \cdots, v^{i_{k_0}}$ in $\overrightarrow{B}_i$ such that

$$\overrightarrow{B}_i \setminus \left\{\left[v^{i_1}\right]_p, \left[v^{i_2}\right]_p, \cdots, \left[v^{i_{k_0}}\right]_p\right\} = \left(\bigcup_{i=1}^{k_1(\overrightarrow{B}_i)} \overrightarrow{C}_{n_i}\right) \bigcup \left(\bigcup_{j=1}^{k_2(\overrightarrow{B}_i)} \overrightarrow{P}_{n_j}\right) \bigcup \left(\bigcup_{l=1}^{k_3(\overrightarrow{B}_i)} \overrightarrow{P}_1\right),$$

i.e., $\varpi\left(\overrightarrow{B}_i^{L^2}[t]\right) \leq i_{k_0}\left(\overrightarrow{B}_i\right) + k_1\left(\overrightarrow{B}_i\right) + k_2\left(\overrightarrow{B}_i\right)$ by the induction assumption for integers $1 \leq i \leq s$. Whence,

$$\begin{aligned}
&\left(\overrightarrow{G} \setminus \{[v]_p\}\right) \setminus \left(\bigcup_{i=1}^{s}\left\{\left[v^{i_1}\right]_p, \left[v^{i_2}\right]_p, \cdots, \left[v^{i_{k_0}}\right]_p\right\}\right) \\
&= \left(\bigcup_{i=1}^{s} \overrightarrow{B}_i\right) \setminus \left(\bigcup_{i=1}^{s}\left\{\left[v^{i_1}\right]_p, \left[v^{i_2}\right]_p, \cdots, \left[v^{i_{k_0}}\right]_p\right\}\right) \\
&= \bigcup_{i=1}^{s}\left(\overrightarrow{B}_i \setminus \left\{\left[v^{i_1}\right]_p, \left[v^{i_2}\right]_p, \cdots, \left[v^{i_{k_0}}\right]_p\right\}\right) \\
&= \bigcup_{i=1}^{s}\left(\left(\bigcup_{i=1}^{k_1(B_i)} \overrightarrow{C}_{n_i}\right) \bigcup \left(\bigcup_{j=1}^{k_2(B_i)} \overrightarrow{P}_{n_j}\right) \bigcup \left(\bigcup_{j=1}^{k_3(B_i)} \overrightarrow{P}_1\right)\right) \\
&= \left(\bigcup_{i=1}^{k_1} \overrightarrow{C}_{n_i}\right) \bigcup \left(\bigcup_{j=1}^{k_2} \overrightarrow{P}_{n_j}\right) \bigcup \left(\bigcup_{l=1}^{k_3} \overrightarrow{P}_1\right),
\end{aligned}$$

where $k_1 = \sum_{i=1}^{s} k_1\left(\overrightarrow{B}_i\right)$, $k_2 = \sum_{i=1}^{s} k_2\left(\overrightarrow{B}_i\right)$ and $k_3 = \sum_{i=1}^{s} k_3\left(\overrightarrow{B}_i\right)$, i.e.,

$$\overrightarrow{G} \setminus \left(\{[v]_p\} \bigcup \left(\bigcup_{i=1}^{s} \left\{\left[v^{i_1}\right]_p, \left[v^{i_2}\right]_p, \cdots, \left[v^{i_{k_0}}\right]_p\right\}\right)\right)$$
$$= \left(\bigcup_{i=1}^{k_1} \overrightarrow{C}_{n_i}\right) \bigcup \left(\bigcup_{j=1}^{k_2} \overrightarrow{P}_{n_j}\right) \bigcup \left(\bigcup_{j=1}^{k_3} \overrightarrow{P}_1\right),$$

which implies that $\varpi\left(\overrightarrow{G}^{L^2}[t]\right) \leq (k_0+1)+k_1+k_2$ by the induction assumption, Theorem 4.8 and initial condition with $k_0 = \sum_{i=1}^{s} i_{k_0}\left(\overrightarrow{B}_i\right)$.

Combining Cases 1 and 2, we know $\varpi\left(\overrightarrow{G}^{L^2}[t]\right) \leq k_0+k_1+k_2$ is true for all graphs $\overrightarrow{G}$ and follows with

$$\varpi\left(\overrightarrow{G}^{L^2}[t]\right) = \min\left\{k_0+k_1+k_2 \mid \text{all triad } k_0, k_1, k_2 \text{ holds with equality (4.3)}\right\}.$$

This completes the proof. □

We immediately get a corollary on the number of $\varpi\left(\overrightarrow{G}^{L_2}[t]\right)$ by Theorem 4.9.

Corollary 4.11 *Let $\overrightarrow{G}$ be one of graphs $\overrightarrow{S}_{1,n}$, $\overrightarrow{W}_{1,n}$, $T^k(\overrightarrow{S}_{1,n})$, $T^{k,l}(\overrightarrow{W}_{1,n})$, $\overrightarrow{K}_n$, $\overrightarrow{K}_{n,m}$, $\overrightarrow{K}_{n_1,n_2,\cdots,n_s}$, $s \geq 2$, $\overrightarrow{P}_n \times \overrightarrow{P}_2$ and $\overrightarrow{C}_n \times \overrightarrow{P}_2$, where $T^k(\overrightarrow{S}_{1,n})$ is the topological subdivision of $\overrightarrow{S}_{1,n}$ by subdividing k times, $T^{k,l}(\overrightarrow{W}_{1,n})$ is the graph obtained by respectively subdividing k times on spoke edges, l times on wheel edges in $\overrightarrow{W}_{1,n}$. Then, $\varpi\left(\overrightarrow{G}^{L_2}[t]\right)$ is shown in Table 1 if $\overrightarrow{G}^{L^2}$ satisfies $\left[A, \frac{d}{dt}\right] = \mathbf{0}$, $\left[A, \int_t^0\right] = \mathbf{0}$ for $A \in \mathscr{A}$ and $\overrightarrow{G}^{L^2}[0]$ is harmonic.*

$\overrightarrow{G}$	$\overrightarrow{G}^{L^2}$	$n,\ k$	$\varpi\left(\overrightarrow{G}^{L_2}[t]\right)$
$\overrightarrow{S}_{1,n}$	$\overrightarrow{S}_{1,n}^{L^2}$	$n \geq 2$	1
$T^k\left(\overrightarrow{S}_{1,n}\right)$	$T^k\left(\overrightarrow{S}_{1,n}\right)^{L^2}$	$n \geq 3,\ k \geq 2$	1
$\overrightarrow{W}_{1,n}$	$\overrightarrow{W}_{1,n}^{L^2}$	$n \geq 3$	2
$T^{k,l}\left(\overrightarrow{W}_{1,n}\right)$	$T^{k,l}\left(\overrightarrow{W}_{1,n}\right)^{L^2}$	$n \geq 3, k \leq 2$	2
$\overrightarrow{K}_n$	$\overrightarrow{K}_n^{L^2}$	$n \geq 3$	$n-2$
$\overrightarrow{K}_{n,m}$	$\overrightarrow{K}_{n,m}^{L^2}$	$m \geq n, n \neq 2$	n
$\overrightarrow{K}_{n,m}$	$\overrightarrow{K}_{n,m}^{L^2}$	$m = n = 2$	1
$\overrightarrow{K}_{2,\cdots,2(s\ times)}$	$\overrightarrow{K}_{2,\cdots,2(s\ times)}^{L^2}$	$s \geq 3$	$2s-3$
$\overrightarrow{K}_{n_1,n_2,\cdots,n_s}$	$\overrightarrow{K}_{n_1,n_2,\cdots,n_s}^{L^2}$	$3 \leq n_1 \leq \cdots \leq n_s, s \geq 3$	$n_1+n_2+\cdots+n_{s-1}$

Table 1

For a tree $\overrightarrow{T}$, there is only one path $u-v$ connecting 2 vertices u, v in $\overrightarrow{T}$. We can get an efficient way for getting the number $\varpi\left(\overrightarrow{T}^{L^2}[t]\right)$. Let $V_3\left(\overrightarrow{T}\right)$ be all vertices v of valency≥ 3

connecting with a leaf by a path in $\overrightarrow{T}$. Clearly, $T\setminus V_3\left(\overrightarrow{T}\right)$ is still a tree, and we can recursively define

$$\begin{aligned}
&V_3^0\left(\overrightarrow{T}\right)=V_3\left(\overrightarrow{T}\right),\\
&V_3^1\left(\overrightarrow{T}\right)=V_3(\overrightarrow{T}\setminus V_3^0\left(\overrightarrow{T}\right)),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&V_3^m\left(\overrightarrow{T}\right)=V_3\left(\overrightarrow{T}\setminus\left(\bigcup_{i=0}^{m}V_3^i\left(\overrightarrow{T}\right)\right)\right),
\end{aligned}$$

where m is the minimum number such that $\overrightarrow{T}\setminus\left(\bigcup\limits_{i=0}^{m}V_3^i\left(\overrightarrow{T}\right)\right)$ is a path P or an empty set. Notice that if $\overrightarrow{T}\setminus\left(\bigcup\limits_{i=0}^{m}V_3^i\left(\overrightarrow{T}\right)\right)$ is not empty, there must be a vertex $v\in V_3^{m-1}$ adjacent to an internal vertex of P, $|P|\geq 3$. Otherwise, $\overrightarrow{T}\setminus\left(\bigcup\limits_{i=0}^{m}V_3^i\left(\overrightarrow{T}\right)\right)$ must be empty by definition, a contradiction. Denoted by $V_{\geq 3}=\bigcup\limits_{i=0}^{m}V_3^i\left(\overrightarrow{T}\right)$, $|V_{\geq 3}|=n_3$. By Theorem 4.10, we get a result on the number of $\varpi\left(\overrightarrow{T}^{L^2}\right)$ following.

Corollary 4.12 *Let $\overrightarrow{T}$ be a tree with vertices of valency≥ 3. Then*

$$\varpi\left(\overrightarrow{T}^{L^2}[t]\right)=n_3+n_\delta$$

if $\overrightarrow{T}^{L^2}$ satisfies $\left[A,\frac{d}{dt}\right]=\mathbf{0}$, $\left[A,\int_t^0\right]=\mathbf{0}$ for $A\in\mathscr{A}$ and $\overrightarrow{T}^{L^2}[0]$ is harmonic, where

$$n_\delta=\begin{cases}0, & if\ T\setminus V_{\geq 3}=\emptyset;\\ 1, & otherwise.\end{cases}$$

Proof Notice that $\overrightarrow{T}\setminus V_{\geq 3}$ is an empty set or a path, and all vertices in $V_{\geq 3}$ should be acted by input or output operations O. Otherwise, there must be vertices of valency≥ 3 in $\overrightarrow{T}\setminus\left(\bigcup\limits_{i=0}^{m}V_3^i\left(\overrightarrow{T}\right)\right)$, contradicts to the assumption on number m. Whence, we get that

$$\varpi\left(\overrightarrow{T}^{L^2}[t]\right)=n_3+n_\delta. \qquad \square$$

Let $\overrightarrow{C}_n=v_1v_2\cdots v_nv_1$ be a circuit and let $\overrightarrow{P}_m=u_1^iu_2^i\cdots u_m$ be a path disjoint with $\overrightarrow{C}_n$. Define a graph $\overrightarrow{C}_n\odot\overrightarrow{P}_m$ by identifying v_1 with u_1, and if $\overrightarrow{P}^i_{m_i}=u_1^iu_2^i\cdots u_{m_i}$, $1\leq i\leq s$ are s distinct paths and disjoint with $\overrightarrow{C}_n$, define

$$\overrightarrow{C}_n\bigodot_{i=1}^{s}\overrightarrow{P}^i_{m_i}=\left(\cdots\left(\left(\overrightarrow{C}_n\bigodot\overrightarrow{P}^1_{m_1}\right)\bigodot\overrightarrow{P}^2_{m_2}\right)\cdots\bigodot\overrightarrow{P}^s_{m_s}\right),$$

i.e., identifying $\overrightarrow{C}_n$ with paths $\overrightarrow{P}^i_{m_1}$ one by one, each with different identified vertices on $\overrightarrow{C}_n$.

Similar to Corollary 4.12 we get the following result.

Corollary 4.13 *For integers $n \geq 3, m_i \geq 2$ and $s \leq n$, there are*

$$\varpi\left(\left(\overrightarrow{C}_n \bigodot_{i=1}^{s} \overrightarrow{P}^i_{m_i}\right)^{L^2}[t]\right) = \begin{cases} s & if\ d(v_i, v_{i+1}) > 1,\ i(mod\ s); \\ \lceil \frac{s}{2} \rceil & if\ d(v_i, v_{i+1}) = 1,\ i(mod\ s) \end{cases}$$

if $\left(\overrightarrow{C}_n \bigodot_{i=1}^{s} \overrightarrow{P}^i_{m_i}\right)^{L^2}$ *satisfies* $\left[A, \frac{d}{dt}\right] = \mathbf{0}$, $\left[A, \int_t^0\right] = \mathbf{0}$ *for* $A \in \mathscr{A}$ *and* $\left(\overrightarrow{C}_n \bigodot_{i=1}^{s} \overrightarrow{P}^i_{m_i}\right)^{L^2}[0]$ *is harmonic.*

Corollary 4.14 $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) = n-1$, $\varpi\left(\left(\overrightarrow{C}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) = n$ *if* $\left(\overrightarrow{C}_n \bigodot_{i=1}^{s} \overrightarrow{P}^i_{m_i}\right)^{L^2}$ *satisfies* $\left[A, \frac{d}{dt}\right] = \mathbf{0}$, $\left[A, \int_t^0\right] = \mathbf{0}$ *for* $A \in \mathscr{A}$ *and* $\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[0]$, $\left(\overrightarrow{C}_n \times \overrightarrow{P}_2\right)^{L^2}[0]$ *both are harmonic.*

Proof Clearly, there are $2(n-2)$ vertices of valency 3, and 4 vertices of valency 2 in $\overrightarrow{P}_n \times \overrightarrow{P}_2$. Notice that there are no vertices with valency≥ 3 in graph $\overrightarrow{G} \setminus \{v^1, v^2, \cdots, v^{k_0}\}$, i.e., any vertex of valency≥ 3 should be acted itself by an action O or adjacent to acted vertices in Theorem 4.10. Whence, there are $n-2$ vertices should be acted by an input or output operation O at least. But, if there are just $n-2$ such acted vertices, there are must be a vertex of valency 3 or a circuit in the resulted subgraph by deleted these $n-2$ vertices, i.e., there are an additional vertex should be acted also. By Theorem 4.10,

$$\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) \geq n - 2 + 1 = n - 1.$$

Now, let $\overrightarrow{P}_n = v_1 v_2 \cdots v_n$ and $\overrightarrow{P}_2 = u_1 u_2$.

Case 1. If $n \equiv 0(\mathrm{mod}\ 2)$, let O act on vertices $(v_2, u_1), (v_3, u_2), (v_4, u_1), \cdots, (v_{n-2}, u_1)$ of $\overrightarrow{P}_n \times \overrightarrow{P}_2$, then $\overrightarrow{P}_n \times \overrightarrow{P}_2 \setminus \left\{[(v_2, u_1)]_p, [(v_3, u_2)]_p, [(v_4, u_1)]_p, \cdots, [(v_{n-1}, u_2)]_p\right\}$ is an empty set. Whence, $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) \leq n-1$ by Theorem 4.8. Therefore, $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) = n-1$ in this case.

Case 2. If $n \equiv 1(\mathrm{mod}\ 2)$, let O act on vertices $(v_2, u_1), (v_3, u_2), (v_4, u_1), \cdots, (v_{n-2}, u_1)$ of $\overrightarrow{P}_n \times \overrightarrow{P}_2$, then $\overrightarrow{P}_n \times \overrightarrow{P}_2 \setminus \left\{[(v_2, u_1)]_p, [(v_3, u_2)]_p, [(v_4, u_1)]_p, \cdots, [(v_{n-2}, u_1)]_p\right\}$ is $\overrightarrow{C}_4$, i.e., $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) \leq n - 2 + 1 = n - 1$ by Theorem 4.8. We get $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) = n - 1$ in this case.

Similarly, let $\overrightarrow{C}_n = v_1 v_2 \cdots v_n v_1$ and $\overrightarrow{P}_2 = u_1 u_2$. Then, there are $2n$ vertices of valency 3 in $\overrightarrow{C}_n \times \overrightarrow{P}_2$. The action O should be acted on n vertices of $\overrightarrow{C}_n \times \overrightarrow{P}_2$ at least because if O acts on vertices less than n, then there must be an integer $i, 1 \leq i \leq n$ such that (v_i, u_1), (v_i, u_2) both in the resulted graph $\overrightarrow{G}'$ by deleted claw graphs on these acted vertices in $\overrightarrow{C}_n \times \overrightarrow{P}_2$, i.e., O must acts on (v_{i-1}, u_1), (v_{i+1}, u_1) and (v_{i-1}, u_2), (v_{i+1}, u_2) $(mod\ n)$. Otherwise, one of the valence of (v_i, u_1), (v_i, u_2) must be 3 in $\overrightarrow{G}'$, i.e., there are additional vertices in $\overrightarrow{G}'$ should be acted also by Theorem 4.10. However, if O must acts on (v_{i-1}, u_1), (v_{i+1}, u_1) and (v_{i-1}, u_2), (v_{i+1}, u_2) but not on (v_i, u_1), (v_i, u_2) for $(mod\ n)$, there are must be $2\lceil \frac{n}{2} \rceil \geq n$ acted vertices, i.e., $\varpi\left(\left(\overrightarrow{C}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) \geq n$.

Let O act on vertices $(v_1, u_1), (v_2, u_1), \cdots, (v_{n-1}, u_1), (v_n, u_2)$ of $\overrightarrow{C}_n \times \overrightarrow{P}_2$. Then, the resulted graph $\overrightarrow{C}_n \times \overrightarrow{P}_2 \setminus \left\{[(v_1, u_1)]_p, [(v_2, u_1)]_p, \cdots, [(v_{n-1}, u_1)]_p, [(v_n, u_2)]_p\right\}$ is a path P_{n-1}. Similar to the proof of Theorem 4.8 we know such a path is already harmonic after the final action O, i.e., $\varpi\left(\left(\overrightarrow{C}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) \leq n$. We therefore get $\varpi\left(\left(\overrightarrow{C}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) = n$. $\square$

§5 Harmonic Flow Model with Healing in Chinese Medicine

Today, we all known that a human body maybe invaded by wind, cold, hot, humidity, dry and fire in the nature which maybe result in that a human gets sick. As we have introduced in the first section there are 12 meridians, i.e., LU, LI, ST, SP, HT, SI, BL, KI, PC, SI, GB and LR meridians on a human body such as those shown in Fig.12 by the Standard China National Standard (GB 12346-90), whose combined with du meridian (DU) and ren meridian (RN) on anterior or posterior thoracic vertebrae consist of the 14 main meridians of a human body.

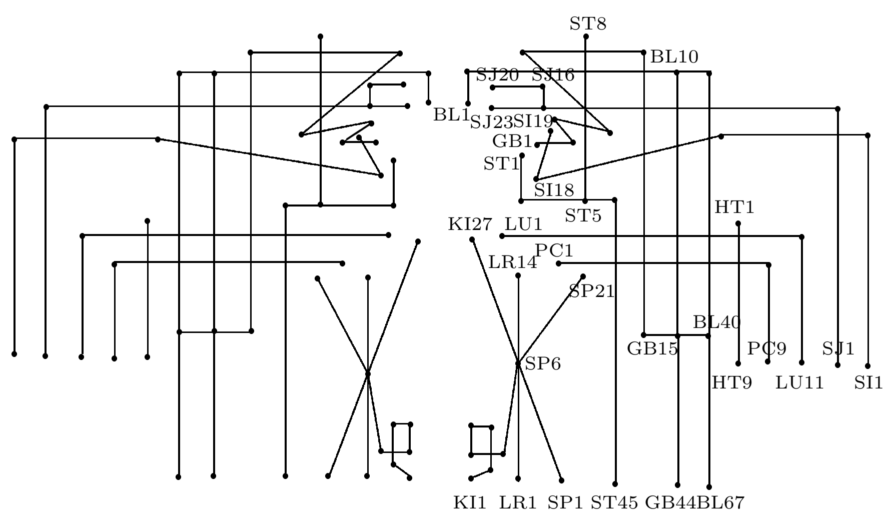


Fig.12 12 Meridians of Human Body

All of these 12 meridians are left-right symmetric with RN, DU meridians on the central axis of a human body and run with the whole life of a human, i.e., *a human is sicked if and only if there exist acupoints on meridians of body which are imbalanced.* For the recovery of a patient, the traditional Chinese doctor applies acupuncture needles inserting in acupoints on meridians for a while, then pulling out for constraint the ruler of reducing the excess with supply the insufficient by quickly or slowly and the staying time for recovery of the $\{Y^-, Y^+\}$ balance. However, it is surprised the western doctor that there are no more need of medicines unless acupuncture needles on acupoints of body for the healing of an illness such as those shown in Fig.13.

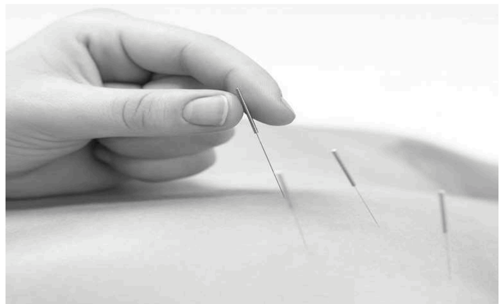


Fig.13

The traditional Chinese medicine treatment theory is the recovery of the $\{Y^-, Y^+\}$ balance on a human body which is essentially equivalent a human body to nothing else but a harmonic flow $\overrightarrow{G}^{L^2}$ over a non-connected graph $\overrightarrow{G}$ as follows:

(1) Paths: LU, LI, SP, HT, SI, KI, PC and LR meridians;

(2) Trees: GB, ST and SJ meridians;

(3) $C_n \bigodot P_{m_1} \bigodot P_{m_2}$: BL meridian.

Certainly, the 14 meridians do not run separately but conjointly in the human body. But *how do they run*? There are 2 viewpoints on this question at least in traditional Chinese medicine:

View 1.(Inner Canon of Emperor, [29]) Hand Yin meridians: *from the chest to the hand*, Hand Yang meridians: *from the hand to the head* and Foot Yang meridians: *from the head to the foot*, Foot Yin meridians: *from the foot to the chest* such as those shown in Fig.14,

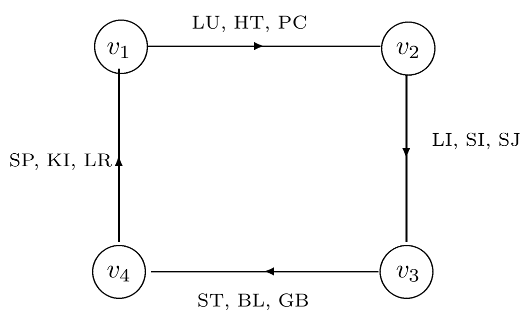


Fig.14

where v_1 =chest, v_2 =hand, v_3 =head and v_4 =foot. Furthermore, a Taoist priest Mr.Zhu spoke his seeing inside in one of danada breathing that all these meridians are with Yin and Yang in pair, i.e., $\{LU, LI\}$, $\{ST, SP\}$, $\{HT, SI\}$, $\{BL, KI\}$, $\{PC, SJ\}$, $\{GB, LR\}$, but the RN meridian and DU meridian are run respectively themselves in 2 cycles ([30]), coincident with the requirement of Chinese Qigong for getting through the RN meridian with the DU meridian.

View 2.(National Standard of 14 Meridian Pictures) The 12 meridians run from the chest to the hand, then to the head, then to the foot and then to the chest, connected respectively the first of later meridian with the end of the former meridian such as those shown in Fig.15,

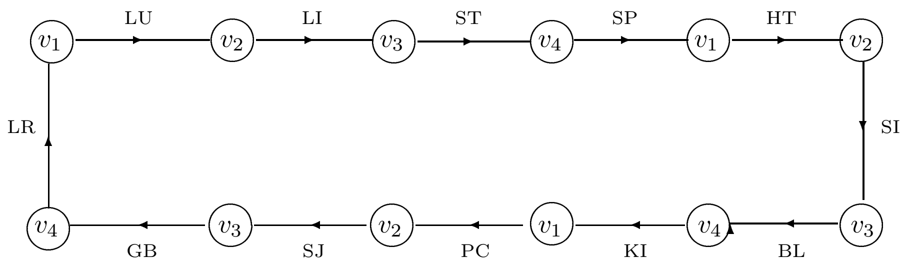


Fig.15

where v_1 =chest, v_2 =hand, v_3 =head and v_4 =foot. Therefore, there are only 2 connected graphs $\overrightarrow{G}_1, \overrightarrow{G}_2$ by View 1 and 1 connected graph $\overrightarrow{G}_3$ by View 2 such as those shown in Fig.16.

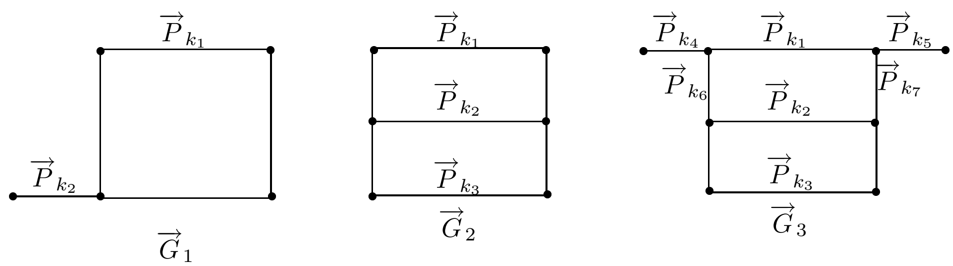


Fig.16

where $\overrightarrow{P}_k$ denotes the path of length $k-1$ with a direction and integers $k_i \geq 4$ for $1 \leq i \leq 7$.

Notice that there is always a harmonic flow $\overrightarrow{C}_n^{L^2}[0]$ for a healthy human and $\varpi\left(\overrightarrow{C}_n^{L^2}[t]\right) = 1$, $\varpi\left(\overrightarrow{P}_n^{L^2}[t]\right) = 1$ by Theorem 4.8.

Applying Theorem 4.10, we know that $\varpi\left(\overrightarrow{G}_1^{L_1^2}[t]\right) \leq 2$, $\varpi\left(\overrightarrow{G}_2^{L_2^2}[t]\right) \leq 4$ and $\varpi\left(\overrightarrow{G}_2^{L_2^2}[t]\right) \leq 7$. Thus, we acupuncture 1 needle if the imbalance appears on LU, LI, SP, HT, SI, KI, PC or LR meridian and 2 needles on original acupoints of a human body if the imbalance appears on GB, ST or SJ meridian at the early of illness, but if it is on the BL meridian or it develops serious, i.e., the imbalance is on meridians more than 3 there are needed simultaneously 3 needles on acupoints for a while at least for a patient recovery.

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Null Quaternionic Slant Helices in Minkowski Spaces

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Abstract: In this study, we give definition of null quaternionic slant helices by definition of slant helices in Euclidean 3-space. Besides, relationships between curvatures of null quaternionic curves are given for null quaternionic slant helices in Minkowski 3-space E_1^3. In other section, we give definition of null quaternionic W-slant helix in Minkowski 4-space E_1^4. We obtain that in which case curve is null quaternionic W-slant helix. Moreover, we have relationships between the curvatures of null quaternionic W-slant helix.

Key Words: Null quaternionic curves, slant helices, W-slant helices, Serret-Frenet formulae.

AMS(2010): 53A35, 53B30, 11R52, 14H45.

§1. Introduction

In differential geometry of curves, a curve of constant slope or general helix in Euclidean 3-space E^3 is defined by the property that the tangent makes a constant angle with a fixed straight line. As other definition, a necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant [8,10,11]. In 2004, Izumiya and Takeuchi defined slant helices and conical geodesic curve. In the light of definition, slant helices are adapted to more spaces and surfaces constructed on slant helices by the researchers [1,6,9,12,13,15].

The quaternions were firstly introduced by Hamilton. Later, quaternions were very popular for researcher and many books were written about them. These papers are very important for examining differential geometry of quaternionic curves [7,14]. In 1987, the Serret-Frenet formulas for a quaternionic curve in E^3 and E^4 was defined by Bharathi and Nagaraj [2] and then in 2004, Serret-Frenet formulas for quaternionic curves and quaternionic inclined curves have been defined in Semi-Euclidean space by Çöken and Tuna [3]. In 2015, they defined Serret-Frenet formulas for null quaternionic curves in semi-Euclidean spaces [4,5].

In this paper, we give the definition of null quaternionic slant helices in Minkowski 3-space E_1^3 and null quaternionic W-slant helices in Minkowski 4-space E_1^4. We examine that in which case curve is null quaternionic W-slant helix or null quaternionic slant helix. Besides, we get relationships between the curvatures of these slant helices.

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§2. Preliminaries

In this section, we give basic concepts related to the semi-real quaternions. For more detailed information, we refer ref. [5,6].

The set of semi-real quaternions is given by

$$Q = \{ q| \quad q = ae_1 + be_2 + ce_3 + d\,; \quad a,b,c,d \in IR\}$$

where $e_1, e_2, e_3 \in E_1^3$, $h(e_i, e_i) = \varepsilon(e_i), \quad 1 \leqslant i \leqslant 3$ and

$$e_i \times e_i = -\varepsilon(e_i),$$

$$e_i \times e_j = \varepsilon(e_i)\varepsilon(e_i)e_k \ \in E_1^3.$$

The multiplication of two semi real quaternions p and q are defined by

$$p \times q = S_pS_q + S_pV_q + S_qV_p + h\left(V_p, V_q\right) + V_p \wedge V_q$$

where S_p and V_p is show scaler and vectoral parts of quaternion p.

Herein, we have inner and cross products in semi-Euclidean space E_1^3. $q = ae_1+be_2+ce_3+d$ and$\alpha q = -ae_1 - be_2 - ce_3 + d$ are semi real quaternion and its conjugate, respectively and inner product h are defined by ([5])

$$h(p,q) = \frac{1}{2}\left[\varepsilon(p)\varepsilon(\alpha q)\left(p \times \alpha q\right) + \varepsilon(q)\varepsilon(\alpha p)\left(q \times \alpha p\right)\right]$$

The three-dimensional semi-Euclidean space E_1^3 is identified with the space of null spatial quaternionic curves $\left\{\gamma \in Q_{E_1^3}\middle| \quad \gamma + \alpha\gamma = 0\right\}$ in an obvious manner,

$$\gamma(s) = \sum_{i=1}^{3} \gamma_i(s)e_i, \quad 1 \leqslant i \leqslant 3.$$

where $\{l,n,u\}$ are Frenet frames of the null quaternionic curves in E_1^3 and e_2 be timelike vector. Then, the Frenet formulae are

$$\begin{bmatrix} l' \\ n' \\ u' \end{bmatrix} = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & \tau \\ -\tau & -k & 0 \end{bmatrix} \begin{bmatrix} l \\ n \\ u \end{bmatrix} \tag{2.1}$$

where k and τ are curvatures of null spatial quaternionic curve and

$$h\left(l,l\right) = h\left(n,n\right) = h\left(l,u\right) = h\left(n,u\right) = 0\ ,\ h\left(l,n\right) = h\left(u,u\right) = 1. \tag{2.2}$$

Note that l and n are null vectors and u is a spacelike vector. Herein, the quaternion product is given by ([5])

$$\begin{aligned} &\times n = -1 - u, \quad n \times l = -1 + u, \quad n \times u = -n, \quad u \times n = n \\ &u \times l = -l, \quad l \times u = l, \quad u \times u = -1\ , \quad l \times l = n \times n = 0 \end{aligned} \tag{2.3}$$

Let $\gamma(s) = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3$ be a quaternionic curve in E_1^3. An orthonormal basis of E_1^4 is $\{e_1, e_2, e_3, e_4 = 1\}$ and let e_2 be timelike vector and $\beta = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4$ be a null quaternionic curve in E_1^4 defined over the interval I and $\{L, N, U, W\}$ be the Frenet components of β in E_1^4. Then, Frenet formulae are given by

$$\begin{bmatrix} L' \\ N' \\ U' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K \\ 0 & 0 & \tau + p & p \\ \tau + p & 0 & 0 & 0 \\ p & K & 0 & 0 \end{bmatrix} \begin{bmatrix} L \\ N \\ U \\ W \end{bmatrix} \tag{2.4}$$

where K is the first curvature of β in E_1^4. Here,

$$\begin{aligned} &h(L, L) = h(N, N) = h(L, U) = h(N, U) = h(W, U) = 0 \\ &h(U, U) = h(W, W) = 1, \quad h(L, N) = -1, \quad h(N, W) = h(L, W) = 0 \end{aligned} \tag{2.5}$$

where L and N are null vectors, U and W are spacelike vectors for which the quaternion product is given by ([6])

$$\begin{aligned} &L \times N = 1 - U\ ,\ N \times L = 1 + U\ ,\ N \times U = N\ ,\ U \times N = -N \\ &U \times L = L\ ,\ L \times U = -L\ ,\ U \times U = -1\ ,\ L \times L = N \times N = 0 \end{aligned} \tag{2.6}$$

§3. Null Quaternionic Slant Helices in E_1^3

In this section, we give definition of null quaternionic slant helices by using definition of the slant helices. Besides, relationships between curvatures of null quaternionic curves are given for null quaternionic slant helices.

Definition 3.1 *Let $\gamma(s)$ be a null quaternionic curve in Minkowski 3-space. If principal normal vector u of γ makes a constant angle θ by a constant direction v , then, the curve γ is called null quaternionic slant helix.*

Theorem 3.1 *Let $\gamma(s)$ be a null quaternionic curve in E_1^3. If γ is null quaternionic slant helix in E_1^3. Then, following results are obtained:*

(1) *If $v = a\,l + b\,n + c\,u$, $a, b, c \in IR$ is spacelike and plane spanned by u and v is timelike,*

$$v = a\,l + b\,n + (\cos\theta)\ u,\ ab = \frac{1}{2}\sin^2\theta$$

(2) *If* $v = a\,l + b\,n + c\,u,\ a, b, c \in IR$ *is spacelike and plane spanned by* u *and* v *is spacelike,*

$$v = a\,l + b\,n + (\cosh\theta)\ u, \quad ab = \frac{-\sinh^2\theta}{2} = \frac{1-\cosh^2\theta}{2}$$

(3) *If* $v = a\,l + b\,n + c\,u,\ a, b, c \in IR$ *is timelike,*

$$v = a\,l + b\,n + (\sinh\theta)\ u, \quad ab = \frac{-\cosh^2\theta}{2}$$

where u *is principal normal vector of* γ *and* θ *is constant angle between principal normal vector* u *and constant direction* v*.*

Proof Let γ be null quaternionic slant helix in E_1^3. Then, principal normal vector u of γ make a constant angle θ by a constant direction v. Suppose that constant direction v is

$$v = a\,l + b\,n + c\,u, \quad a, b, c \in IR. \tag{3.1}$$

Therefore, by using the quaternionic inner product, we have

$$h(u, v) = h(u, al + bn + cu) = a\,h(u, l) + b\,h(u, n) + c\,h(u, u),$$

$$h(u, v) = c. \tag{3.2}$$

(1) If $v = a\,l + b\,n + c\,u,\ a, b, c \in IR$ is spacelike and spanned plane by u and v is timelike, since principal normal vector u is spacelike, we write that

$$h(u, v) = c = \cos\theta. \tag{3.3}$$

Since constant direction v is spacelike, we have

$$\begin{aligned} h(v, v) &= ab\ h(l, n) + ba\ h(n, l) + c^2\,h(u, u) \\ &= 2ab + c^2 \end{aligned}\ . \tag{3.4}$$

From (**??**), we obtain the desired equality

$$v = a\,l + b\,n + (\cos\theta)\ u \quad ab = \frac{1}{2}\sin^2\theta. \tag{3.5}$$

(2) If $v = a\,l + b\,n + c\,u,\ \ a, b, c \in IR$ is spacelike and spanned plane by u and v is spacelike, since principal normal vector u is spacelike, we write that

$$h(u, v) = c = \cosh\theta. \tag{3.6}$$

Since constant direction v is spacelike and from (3.4), we have the desired equality.

(3) If $v = a\,l + b\,n + c\,u,\ \ a, b, c \in IR$ is timelike, since principal normal vector u is spacelike,

we write that

$$h\,(u,v) = c = \sinh\theta. \tag{3.7}$$

Since constant direction v is timelike and from (**??**), we find the desired result. □

Theorem 3.2 *Let $\gamma(s)$ be a null quaternionic curve in E_1^3. γ is null quaternionic slant helix in E_1^3 if and only if*

$$\tau b + \kappa a = 0. \tag{3.8}$$

Proof Let γ be null quaternionic slant helix, Then, principal normal vector u of γ make a constant angle θ by a constant direction v,

$$h\,(u,v) = \text{constant}. \tag{3.9}$$

By taking the derivation of (**??**), we get

$$\begin{aligned} h\,(u',v) &= h\,(-\tau l - \kappa n\,,\, v) \\ &= h\,(-\tau l - \kappa n\,,\; a\,l + b\,n + c\,u) \end{aligned} \tag{3.10}$$

$$h\,(u',v) = -\tau b\, h\,(l,n) - \kappa a\, h\,(n,l) = 0. \tag{3.11}$$

Therefore, we obtain the equation (**??**).

Conversely, we suppose that equation (**??**) is provided. By using the Frenet formulae, from (**??**) and (**??**), we obtain that $h\,(u',v) = -\tau b - \kappa a = 0$. Thus, it is clear that

$$h\,(u,v) = \text{constant}.$$

This means that γ is null quaternionic slant helix. □

§4. Null Quaternionic W-Slant Helices in E_1^4

In this section, we obtain that in which case curve is null quaternionic W-slant helix. Besides, we have relationships between the curvatures of null quaternionic W-slant helix.

Definition 4.1 *Let $\beta(s)$ be a null quaternionic curve in Minkowski 4-space. If binormal vector W of β makes a constant angle θ by a constant direction v . Then, the curve β is called null quaternionic W-slant helix.*

Theorem 4.1 *Let $\beta(s)$ be a null quaternionic curve in E_1^4. If β is null quaternionic W-slant helix in E_1^4. Then, following results are obtained:*

(1) *If $v^* = a\,L + b\,N + c\,U + d\,W,\quad a,b,c,d \in IR$ is spacelike and plane spanned by W and v^* is timelike,*

$$v^* = a\,L + b\,N + c\,U + (\cos\theta)\;W,\quad 2ab - c^2 = -\sin^2\theta$$

(2) *If* $v^* = a\,L + b\,N + c\,U + d\,W, \quad a, b, c, d \in IR$ *is spacelike and plane spanned by* W *and* v^* *is spacelike,*

$$v^* = a\,L + b\,N + c\,U + (\cosh\theta)\ W, \quad 2ab - c^2 = \sinh^2\theta$$

(3) *If* $v^* = a\,L + b\,N + c\,U + d\,W, \quad a, b, c, d \in IR$ *is timelike,*

$$v^* = a\,L + b\,N + c\,U + (\sinh\theta)\ W, \quad 2ab - c^2 = \cosh^2\theta$$

where u *is principal normal vector of* γ *and* θ *is constant angle between principal normal vector* u *and constant direction* v*.*

Proof Let β be null quaternionic W-slant helix in E_1^4. Then, binormal vector W of β make a constant angle θ by a constant direction v^*. Suppose that constant direction v^* is

$$v^* = a\,L + b\,N + c\,U + d\,W, \quad a, b, c, d \in IR. \tag{4.1}$$

Thus, by using the quaternionic inner product, we get

$$h\,(W, v^*) = d. \tag{4.2}$$

(1) If $v^* = a\,L + b\,N + c\,U + d\,W, \quad a, b, c, d \in IR$ is spacelike and plane spanned by W and v^* is timelike, since binormal vector W is spacelike, we write that

$$h\,(W, v^*) = \cos\theta. \tag{4.3}$$

Since constant direction v is spacelike, we have

$$\begin{aligned} h\,(v^*, v^*) &= ab\ h\,(L, N) + ba\ h\,(N, L) + c^2\,h\,(U, U) + d^2\,h\,(W, W) \\ &= -2ab + c^2 + d^2 \\ &= 1 \end{aligned} \quad . \tag{4.4}$$

From (**??**), (**??**) and (**??**), the proof is completed.

(2) If $v^* = a\,L + b\,N + c\,U + d\,W, \quad a, b, c, d \in IR$ is spacelike and plane spanned by W and v^* is spacelike, since binormal vector W is spacelike, we have

$$h\,(W, v^*) = \cosh\theta. \tag{4.5}$$

Since constant direction v^* is spacelike, we have equality (**??**). From (**??**), (**??**) and (**??**), the desired result is found.

(3) If $v^* = a\,L + b\,N + c\,U + d\,W, \quad a, b, c, d \in IR$ is timelike, since binormal vector W is spacelike, we write that

$$h\,(W, v^*) = \sinh\theta. \tag{4.6}$$

Since constant direction v^* is timelike, we obtain that

$$\begin{aligned} h\left(v^*, v^*\right) &= ab\ h\left(L, N\right) + ba\ h\left(N, L\right) + c^2\, h\left(U, U\right) + d^2\, h\left(W, W\right) \\ &= -2ab + c^2 + d^2 = -1 \end{aligned} \quad . \tag{4.7}$$

From (4.2), (4.6) and (4.7), the proof is completed. □

Theorem 4.2 *Let $\beta(s) = \gamma_1\, e_1 + \gamma_2\, e_2 + \gamma_3\, e_3 + \gamma_4\, e_4$ be null quaternionic curve in E_1^4, Then, β is null quaternionic W-slant helix if and only if*

$$p\, b = -K\, a. \tag{4.8}$$

Proof Suppose that β be null quaternionic W-slant helix in E_1^4. Then, binormal vector W of β make a constant angle θ by a constant direction v^*. Constant direction v^* is

$$v^* = a\, L + b\, N + c\, U + d\, W, \quad a, b, c, d \in IR.$$

Thus, we get equation (**??**). By differentiating equation (**??**) according to arclength parameter s and using the Frenet formulae of null quaternionic curves, we have

$$h(W', v^*) + h(W, {v^*}') = 0,$$

$$h(W', v^*) = h(p\, L + K\, N,\, v^*) = 0, \tag{4.9}$$

$$p\ h(L,\, v^*) + K\ h(N,\, v^*) = 0,$$

$$p\, b\ h(L,\, N) + K\, a\, h(N,\, L) = 0, \tag{4.10}$$

$$p\, b\ = -K\, a.$$

Conversely, we have equality (**??**). Suppose that the inner product of binormal vector W of β and a constant direction v^* is function $F(s)$. By differentiating of the inner product, we get that

$$-p\, b\ - K\ a\ = F'(s).$$

From (**??**), we obtain that $F(s)$ is a constant. That is, β is the null quaternionic W-slant helix in E_1^4. □

Theorem 4.3 *If the curve β is null quaternionic W-slant helix in E_1^4. Then, curvatures of null quaternionic W-slant helix is provided that*

$$2dpK + cK\,(\tau + p) - b\, p' - a\, K' = 0. \tag{4.11}$$

Proof If the curve β is null quaternionic W-slant helix in E_1^4. Then, we have equation

(??). By differentiating (??), we obtain that

$$p'\,h(L,v^*) + p\,h(L',v^*) + K'\,h(N,v^*) + K\,h(N',v^*) = 0. \tag{4.12}$$

By substituting Frenet formulae in (4.12), we have

$$-b\,p' + p\,K\,h(W,v^*) + K'\,(-a) + K\,h\,((\tau+p)U + pW, v^*) = 0 \tag{4.13}$$

and the desired equation is found. □

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Unique Metro Domination Number of Circulant Graphs

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Abstract: A dominating set D of G which is also a resolving set of G is called a *metro dominating set*. A metro dominating set D of a graph $G(V,E)$ as a *unique metro dominating set* (in short an UMD-set) if $|N(v) \cap D| = 1$ for each vertex $v \in V - D$ and the minimum cardinality of an UMD-set of G is the *unique metro domination number* of G. In this paper, we determine unique metro domination number of circulant graphs.

Key Words: Domination, Smarandachely k-dominating set, metric dimension, metro domination, uni-metro domination.

AMS(2010): 05C20, 05C26.

§1. Introduction

All the graphs considered in this paper are simple, connected and undirected. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d(u,v)$. For a vertex v of a graph, $N(v)$ denotes the set of all vertices adjacent to v and is called open neighborhood of v. Similarly, the closed neighborhood of v is defined as $N[v] = N(v) \cup \{v\}$.

Let $G(V,E)$ be a graph. For each ordered subset $S = \{v_1, v_2, \cdots, v_k\}$ of V, each vertex $v \in V$ can be associated by a vector of distances denoted by $\Gamma(v/S) = (d(s_1,v), d(s_2,v), \cdots, d(s_k,v))$. The set S is said to be a *resolving set* of G, if $\Gamma(v/S) \neq \Gamma(u/S)$, for every $u, v \in V - S$. A resolving set of minimum cardinality is the *metric basis* and cardinality of a metric basis is the *metric dimension* of G. The k-tuple, $\Gamma(v/S)$ associated to the vertex $v \in V$ with respect to a Metric basis S, is referred as a *code generated by S* for that vertex v. If $\Gamma(v/S) = \{v_1, v_2, \cdots, v_k\}$, then $v_1, v_2, \cdots, v_k$ are called components of the code of v generated by S and in particular v_i, $1 \leq i \leq k$, is called i^{th}-component of the code of v generated by S.

A *dominating set* D of a graph $G(V,E)$ is the subset of V having the property that for each vertex $v \in V - D$ there exists a vertex u in D such that $uv \in E$. Generally, a set $D \subseteq V$ of G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S with $k \geq 1$. Clearly, a dominating set is a Smarandachely 1-dominating

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set. A dominating set D of G which is also a resolving set of G is called a *metro dominating set* or in short an $MD-set$. A metro dominating set D of a graph $G(V,E)$ as a *unique metro dominating set* (in short an UMD-set) if $|N(v)\cap D| = 1$ for each vertex $v\in V-D$ and the minimum cardinality of an UMD-set of G is the *unique metro domination number* of G, denoted by $\gamma_{u\beta}(G)$.

Metric dimensions and locating dominating sets of certain classes of graphs were studied in [1-14].

§2. Resolvability of Circulant Graphs

A graph whose vertex set is $\{v_i | i\in Z^+\}$ and two vertices v_i and v_j are adjacent if and only if $i-j \pmod n \in C$, for a given $C\subseteq Z_n$ with $0\notin C$, is called a *circulant diagraph*. If the set C has the property that $C=-C$, then the underlying graph is called *circulant graph*, and we denote it by $X_{n,\Delta}$, where $|C|=\Delta$. The set C is referred to as a connected set. The circulant graph $X_{n,\Delta}$ is a Δ-regular graph. In this paper, we consider a family of circulant graph $X_{n,3}$ with connection set $C=\{1,\frac{n}{2},n-1\}$, where n is even.

We state the following lemma whose proof follows directly by the definition of domination, and is most helpful to find UMD-sets.

Lemma 2.1 *In the circulant graph $X_{n,3}$, n is even, with connection set $C=\{1,\frac{n}{2},n-1\}$, a vertex v_i dominates v_{i-1}, v_{i+1} and $v_{i+\frac{n}{2}}$, where $i+\frac{n}{2}$ is under modulo n.*

Now we consider $G=X_{n,3}$, n is even, where the connection set $C=\{1,\frac{n}{2},n-1\}$. Let S be a dominating set of G. Then by Lemma 2.1, a vertex $v_i\in S$ can dominate at most 3 vertices in $V-S$. Hence $|V-S|\leqslant 3|S|$. Therefore,

$$|V|-|S|\leqslant 3|S| \Rightarrow 4|S|\geqslant |V| \Rightarrow |S|\geqslant \frac{n}{4}.$$

Thus we have the following lemma.

Lemma 2.2 *For any positive even integer n,*

$$\gamma(X_{n,3})\geqslant \left\lceil \frac{n}{4}\right\rceil .$$

Let R be a set of two or more vertices of the principal cycle. Consider two distinct vertices u and v of R. Let P, P' be two distinct uv-path on the principal cycle. The vertices u and v are said to be neighboring vertices if u and v are the only vertices of S contained in one of the paths P, P'. If P (or P') is the path containing only u,v from S, then the set of all vertices of $P-\{u,v\}$ is called a gap of S determined by u and v and is denoted by γ. The number of vertices in the gap is called order of the gap and is denoted by $o(\gamma)$.

Notice that it is shown that $X_{26,3}$ with a unique metro dominating set $S=\{v_i, v_{i+4}, v_{i+8}, v_{i+15}, v_{i+19}, v_{i+23}, v_{i+24}, v_{i+25}\}$ in Figure 1. We observe that v_i, v_{i+4} are neighboring vertices of S and the gap determined by v_i and v_{i+4} is of order 3. Similarly, the gap determined by v_{i+8} and v_{i+15} is of order 6. The gap between v_{i+24} and v_{i+25} is of order 0 or empty gap.

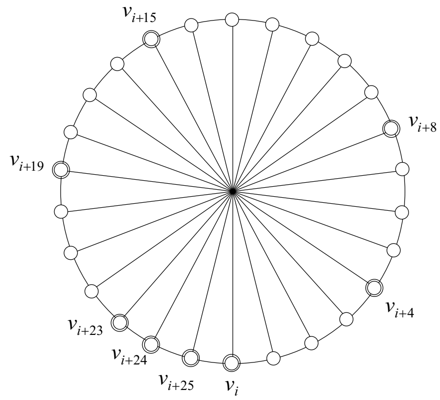

Figure 1 Circulant graph $X_{26,3}$ with $C = \{1, 13, 25\}$.

Consider $G = X_{n,3}$, n is even. Let S be a UMD-set of G. Suppose there is a gap of S of order 1. Let u, v be two neighboring vertices and w be the only vertex of the gap. Then w is dominated by both u and v and hence w is not uniquely dominated. Thus, we have

Lemma 2.1 *If γ is a gap of a UMD-set S of the graph $G = X_{n,3}$, then $0(\gamma) \neq 1$.*

In the discussion to follow, we want to find suitable gaps of a dominating set S, so that S becomes a UMD-set. Gaps of order 4 or more will introduce gaps of order 0 and thereby increase $|S|$; for, consider a gap γ of order 4 between the neighboring vertices v_i and v_{i+5}. Vertices in the gaps are $v_{i+1}, v_{i+2}, v_{i+3}$ and v_{i+4}. Vertices v_{i+1} and v_{i+4} are dominated by v_i and v_{i+5} respectively. It is therefore obvious that v_{i+2} and v_{i+3} should be dominated by $v_{i+2+\frac{n}{2}}$ and $v_{i+3+\frac{n}{2}}$. Thus, $v_{i+2+\frac{n}{2}}$ and $v_{i+3+\frac{n}{2}}$ belongs to S. The gap between them is empty. Hence $|S|$ is increased.

If all the gaps of S are of order 3, then $|S|$ is the least; for, if $v_i, v_{i+4} \in S$ and are neighboring vertices, then the gap determined by them is of order 3. As v_{i+1} and v_{i+3} are dominated by v_i and v_{i+4} respectively, the vertex v_{i+2} has to be dominated by $v_{i+2+\frac{n}{2}}$. Thus, $v_{i+2+\frac{n}{2}} \in S$.

Observe that $v_{i+\frac{n}{2}+4}$ is dominated by v_{i+4}. Therefore, we take $v_{i+\frac{n}{2}+6} \in S$, so that $v_{i+\frac{n}{2}+6}$ and $v_{i+\frac{n}{2}+2}$ are neighboring vertices of a gap of order 3. As $v_{i+\frac{n}{2}+6}$ dominates v_{i+6}, we include v_{i+8} in S so that v_{i+4} and v_{i+8} are neighboring vertices of gap of order 3. Thus, $S = \{v_i,\, v_{i+4},\, v_{i+2+\frac{n}{2}},\, v_{i+8}, \cdots\}$.

If the above set of vertices has $v_i + \frac{n}{2} - 2$ as the last vertex, then the above sequence of vertices in S terminates at v_{i-4}. In this case each vertex in S is uniquely dominating exactly 3 vertices in $V - S$. Thus by Lemma 2.1, $|S|$ is the least. This leads to the lemma.

Lemma 2.4 *A dominating set S of $X_{n,3}$ has a least $|S|$, when each gap of S is of order 3.*

Lemma 2.5 *If G is a graph of order n having a dominating set S such that every gap of S is of order 3, then $n \equiv 4(mod\ 8)$.*

Proof If every gap is of order 3, then $v_{i+\frac{n}{2}+6}$, $v_{i+\frac{n}{2}+10}, \cdots, v_{i-4}$ are in S. Hence $\frac{n}{2}+4k+2 \equiv 0 \pmod n \Rightarrow n \equiv 4 \pmod 8$. □

Note that when $n \equiv 4 \pmod 8$, there are exactly $\frac{n}{4}$ gaps of order 3 on the principal cycle. Also observe that $\frac{n}{4}$ is an odd integer. From these we conclude the following result.

Lemma 2.6 *When $n \equiv 0 (mod\ 8)$, $n > 8$, there is at least one gap of order less than 3.*

In a circulant graph $G = X_{n,3}$, let $v_i, v_{i+1}, \cdots, v_{i+(n-1)}, v_{i+n} = v_i$ (subscripts increase in anti-clockwise direction) form the principal cycle. Each vertex v_i, on the principal cycle, also lies on two other cycles:

(1) $v_i, v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}-1}, \ldots, v_{i+1}, v_i$ (Clockwise) and

(2) $v_i, v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1}, \ldots, v_{i-1}, v_i$ (anti-clockwise).

Length of these cycles is $\frac{n}{2}+1$. Hence maximum distance between any two vertices on these cycles is $\frac{1}{2}\left(\frac{n}{2}+1\right)$, if $\frac{n}{2}+1$ is even and is $\frac{1}{2}\left(\frac{n}{2}\right)$ if $\frac{n}{2}+1$ is odd. Thus we have

Lemma 2.7 *If x and y are any two vertices of $X_{n,3}$ then $d(x,y) = k \leqslant \lceil \frac{n}{4} \rceil$.*

Note that the subscripts of the vertex names are in Z_n, i.e. congruent modulo n.

Lemma 2.8 *If $n > 8$ and $n \equiv 0$ (mod 4), then for a fixed i, $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}\}$ is a resolving set.*

Proof For the cases, $n = 12, 16$ and 20, it is easy to see that the set S is a resolving set. We prove the case when $n > 20$. Suppose that x and y are vertices on the principal cycle such that $d(v_i, x) = d(v_i, y)$. Then there are four cases, in each case $k \leqslant \lceil \frac{n}{4} \rceil$.

Case 1. $x = v_{i-k}$ and $y = v_{i+k}$.

In this case we see that $d(v_{i+4}, y) = |k-4|$. If $k \leqslant \frac{n}{4}-4$, then $d(v_{i+4}, x) = k+4 \neq d(v_{i+4}, y)$; If $k > \frac{n}{4}-4$, then $d(v_{i+4}, x) = \frac{n}{2}+1-(k+4) = \frac{n}{2}-k-3 \neq d(v_{i+4}, y)$ for if $\frac{n}{2}-k-3 = k-4$, then $2k = \frac{n}{2}+1 \Rightarrow k = \frac{n}{4}+\frac{1}{2}$, which is not possible as $n \equiv 0$ modulo 4. Thus v_{i+4} resolves x and y.

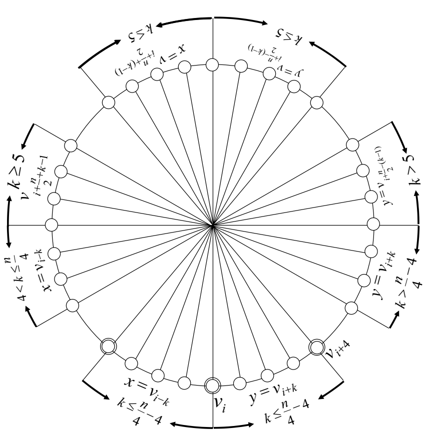

Figure 2. Circulant graph $X_{32,3}$.

Case 2. $x = v_{i+\frac{n}{2}+k-1}$ and $y = v_{i+\frac{n}{2}-(k-1)}$.

If $k \leqslant 5$, then $d(v_{i+4}, x) = 6-k$ and $d(v_{i+4}, y) = 4+k$. Now $6-k = 4+k \Rightarrow 2k = 2 \Rightarrow k = 1$ which is not possible. Hence $d(v_{i+4}, x) \neq d(v_{i+4}, y)$; If $k > 5$, then $d(v_{i+4}, x) = k-4$, which is not equal to $d(v_{i+4}, y)$. Thus v_{i+4} resolves x and y.

Case 3. $x = v_{i-k}$ and $y = v_{i+\frac{n}{2}-(k-1)}$.

If $k \leqslant \lceil \frac{n}{4} \rceil - 4$, then we have $d(v_{i+4}, x) = 4+k$ and $d(v_{i+4}, y) = 4+k$. But $d(v_{i+\frac{n}{2}+2}, x) = 3+k \neq d(v_{i+\frac{n}{2}+2}, y) = k+1$; If $\frac{n}{4} - 4 < k \leqslant \frac{n}{4}$ then $d(v_{i+4}, x) = \frac{n}{2} - k - 3 = d(v_{i+4}, y)$ and $d(v_{i+\frac{n}{2}+2}, x) \neq d(v_{i+\frac{n}{2}+2}, y)$. Hence in all the cases, $v_{i+\frac{n}{2}+2}$ resolves x and y.

Case 4. If $x = v_{i+\frac{n}{2}+k-1}$ and $y = v_{i+k}$ then $d(v_{i+4}, y) = |k-4|$.

If $k < 5$, then $d(v_{i+4}, x) = 6-k \neq d(v_{i+4}, y)$; If $k \geqslant 5$, then $d(v_{i+4}, x) = k-4 = d(v_{i+4}, y)$. However, $d(v_{i+\frac{n}{2}+2}, x) = k-3 \neq d(v_{i+\frac{n}{2}+2}, y) = k-1$. Hence $v_{i+\frac{n}{2}+2}$ resolves x and y.

Note that the Theorem 1 of Muhammad Salman et al in [6], states that for all $n \geqslant 4$ and $n \equiv 0 \pmod 4$, the metric dimension of $X_{n,3} > 2$. Hence $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}\}$ is a resolving set. Hence the lemma. □

Lemma 2.9 *For any integer $n > 8$ and $n \equiv 2 (mod\ 4)$, the set $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}, v_{i+\frac{n}{2}+6}\}$ is a resolving set.*

Proof The cases where $n \leqslant 22$ follows easily. We now suppose $n > 22$ and, x and y are two vertices of G such that $d(v_i, x) = d(v_i, y)$. Then there are four cases in each case $k \leqslant \frac{n+2}{4}$.

Case 1. $x = v_{i-k}$ and $y = v_{i+k}$.

In this case $d(v_{i+4}, y) = |k-4|$. If $k \leqslant \frac{n+2}{4} - 4$, then $d(v_{i+4}, x) = k+4 \neq d(v_{i+4}, y)$; If $k > \frac{n+2}{4} - 4$, then $d(v_{i+4}, x) = \frac{n}{2} - k - 3$. Now $d(v_{i+4}, y) = d(v_{i+4}, x) \Rightarrow k - 4 = \frac{n}{2} - k - 3 \Rightarrow k = \frac{n+2}{4}$. If $k = \frac{n+2}{4}$, $d(v_{i+\frac{n}{2}+2}, x) = k-3 \neq d(v_{i+\frac{n}{2}+2}, y) = k-1$.

Case 2. $x = v_{i+\frac{n}{2}+k-1}$ and $y = v_{i+\frac{n}{2}-(k-1)}$.

If $k \leqslant 5$, $d(v_{i+4}, x) = 6-k \neq d(v_{i+4}, y) = k+4$; except when $k = 1$. But when $k = 1$, x and y coincide. Now if $5 < k < \frac{n+2}{4}$, then $d(v_{i+4}, y) = k-4 \neq d(v_{i+4}, x) = k+4$.

Case 3. $x = v_{i-k}$ and $y = v_{i+\frac{n}{2}-(k-1)}$.

When $k \leqslant \frac{n+2}{4} - 4$, $d(v_{i+4}, x) = 4+k$ and $d(v_{i+4}, y) = 4+k$. But $d(v_{i+\frac{n}{2}+2}, x) = 3+k \neq d(v_{i+\frac{n}{2}+2}, y) = k+1$, and when $\frac{n+2}{4} - 4 < k \leqslant \frac{n+2}{4}$, $d(v_{i+4}, x) = \frac{n}{2} - k - 3$ and $d(v_{i+4}, y) = \frac{n}{2} - k - 3$. However $d(v_{i+\frac{n}{2}+2}, x) = \frac{n}{2} - k - 2$ and $d(v_{i+\frac{n}{2}+2}, y) = k+1$.

Now $\frac{n}{2} - k - 2 = k + 1 \Rightarrow 2k = \frac{n}{2} - 3 \Rightarrow k = \frac{n-6}{4}$.

If $k \neq \frac{n-6}{4}$, then $v_{i+\frac{n}{2}+2}$ resolves x and y; If $k = \frac{n-6}{4}$, then $d(v_{i+\frac{n}{2}+6}, x) = \frac{n}{2} - k - 6$ and $d(v_{i+\frac{n}{2}+6}, y) = \frac{n}{2} - k - 4$, which are not equal. Hence $v_{i+\frac{n}{2}+6}$ resolves x and y.

Case 4. $x = v_{i+\frac{n}{2}+k-1}$ and $y = v_{i+k}$.

If $k \leqslant 5$, then $d(v_{i+4}, x) = 6-k \neq d(v_{i-4}, y) = |k-4|$; If $5 < k < \frac{n+2}{4}$, then $d(v_{i+4}, x) = k-4 = d(v_{i+4}, y)$. But $d(v_{i+\frac{n}{2}+2}, x) = k-3 \neq d(v_{i+\frac{n}{2}+2}, y) = k-1$. Thus $v_{i+\frac{n}{2}+2}$ resolves x and y.

Now by the Theorem 2 of Muhammad Salman et al [6], we have for all $n \geqslant 6$ and $n \equiv 2 \pmod 6$, the metric dimension of the graph $X_{n,3} > 3$.

Hence the set $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}, v_{i+\frac{n}{2}+6}\}$ as per the above four cases becomes a resolving set. Hence the lemma. $\square$

§4. Algorithm to Extend Circulant Graphs and Resolving Sets

We give an algorithm, which constructs new circulant graph from the old one by increasing its order and extending the its resolving set to suit for the newly constructed one (as in Figure 3).

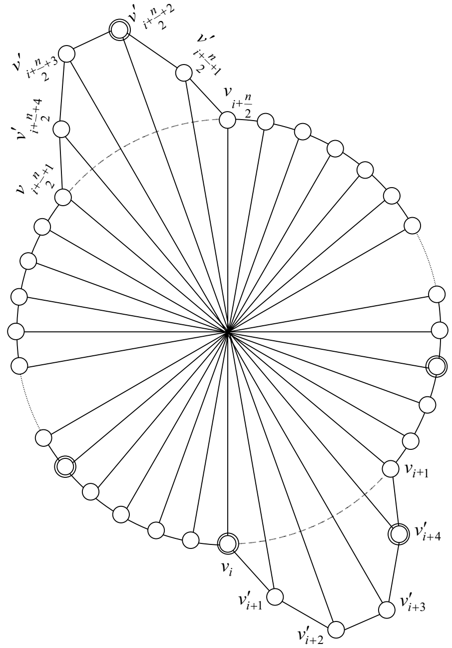

Figure 3. Algorithmic construction of $X_{n+8,3}$ from $X_{n,3}$.

Input: The graph $X_{n,3}$ and a metric basis S with $|S| = k$.

Step 1: Select two neighboring vertices on the principal cycle with a gap of 3. Let v_i and v_{i+4} be the vertices. Then v_i, v_{i+4} and $v_{i+\frac{n}{2}+2} \in S$.

Step 2: Delete the edge v_iv_{i+1}. Add four vertices v'_{i+1}, v'_{i+2}, v'_{i+3} and v'_{i+4} between the vertices v_i and v_{i+1}. Join the vertices to get the edges $v_iv'_{i+1}$, $v'_{i+1}v'_{i+2}$, $v'_{i+2}v'_{i+3}$, $v'_{i+3}v'_{i+4}$ and $v'_{i+4}v_{i+1}$.

Step 3: Delete the edge $v_{i+\frac{n}{2}}v_{i+\frac{n}{2}+1}$. Add four vertices vertices $v'_{i+\frac{n}{2}+1}$, $v'_{i+\frac{n}{2}+2}$, $v'_{i+\frac{n}{2}+3}$ and $v'_{i+\frac{n}{2}+4}$ between the vertices $v_{i\frac{n}{2}}$ and $v_{i+\frac{n}{2}+1}$. Join these vertices to get the edges $v_{i\frac{n}{2}}v'_{i+\frac{n}{2}+1}$, $v'_{i+\frac{n}{2}+1}v'_{i+\frac{n}{2}+2}$, $v'_{i+\frac{n}{2}+2}v'_{i+\frac{n}{2}+3}$, $v'_{i+\frac{n}{2}+3}v'_{i+\frac{n}{2}+4}$ and $v'_{i+\frac{n}{2}+4}v_{i+\frac{n}{2}+1}$.

Step 4: Join these 8 vertices to get the edges, $v_{i+1}v'_{i+\frac{n}{2}+1}$, $v_{i+2}v'_{i+\frac{n}{2}+2}$, $v_{i+3}v'_{i+\frac{n}{2}+3}$ and $v_{i+4}v'_{i+\frac{n}{2}+4}$.

Step 5: Add v'_{i+4} and $v_{i+\frac{n}{2}+2}$ into S.

Output: The graph $X_{n+8,3}$ and a metric basis S with $|S| = k+2$

§4. Unique Metro Domination Number of $X_{n,3}$

In this section we completely determine unique metro domination number of 3-regular circulant graphs in the form of following sequence of theorems.

Theorem 4.1 *If $n > 8$ is an even integer and $n \equiv 6$ or 4 (mod 8), then $\gamma_{\mu\beta}(X_{n,3}) = \lceil \frac{n}{4} \rceil$.*

Proof We consider two cases separately following.

Case 1. $n \equiv 4 \pmod 8$ and $n > 8$.

We first take the smallest possible n, that is $n = 12$. Define $S = \{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}\}$ for any i, $1 \leqslant i \leqslant 12$. In view of Lemma 2.8, the set S so defined is a resolving set. Further, each gap is of order 3. Hence S is uniquely dominates $V - S$. So, by Lemma 2.2, $\gamma_{\mu\beta}(X_{12,3}) \geqslant \lceil \frac{n}{4} \rceil$ and here $|S| = 3$. Thus, S is a UMD-set. Therefore, $\gamma_{\mu\beta}(X_{12,3}) = \lceil \frac{n}{4} \rceil$, when $n = 12$.

Now we apply the algorithm to construct the next cases for n. At each time algorithm increases the order by 8 and $|S|$ by 2. Hence $\gamma_{\mu\beta}(X_{n+8,3}) = \lceil \frac{n}{4} \rceil + 2 = \lceil \frac{n+8}{4} \rceil$.

Case 2. $n \equiv 6 \pmod 8$ and $n > 8$.

For the least possible $n = 14$, by Lemma 2.8, it follows that the set $\{v_i, v_{i+4}, v_{i+\frac{n}{2}+2}, v_{i+\frac{n}{2}+6}\}$ resolves $V - D$. Now from Lemma 2.2, $\gamma_{\mu\beta}(X_{n,3}) \geqslant \lceil \frac{n}{4} \rceil = 4$. There is a gap of order 0 in S. The middle vertex v_k of a gap of order 3 is dominated by $v_{k+\frac{n}{2}} \in S$. Hence S is a UMD-set.

We apply the algorithm on $X_{14,3}$. It increases $n = 14$ to $n = 22$. It also increases $|S|$ by 2. Hence $\gamma_{\mu\beta}(X_{22,3}) = \lceil \frac{n}{4} \rceil$. Repeated application of the algorithm gives the theorem. □

Theorem 4.2 *If n is any even integer, $n > 8$ and $n \equiv 0$ or 2 (mod 8), then $\gamma_{\mu\beta}(X_{n,3}) = \lceil \frac{n}{4} \rceil + 1$.*

Proof We prove the theorem in two different cases as follows:

Case 1. $n \equiv 0 \pmod 8$.

The graph $X_{16,3}$ is the graph of least possible order in this case. Invoking Lemma 2.4, consider a gap of order 3, having v_i and v_{i+4} as the neighboring vertices. Then v_i, v_{i+4}, and $v_{i+\frac{n}{2}+2} \in S$.

This leads to a gap of order 5 between v_{i+4} and $v_{i+\frac{n}{2}+2}$ in which v_{i+6}, v_{i+7} are not dominated and a gap of order 5 between $v_{i+\frac{n}{2}+2} = v_{i+10}$ and v_i in which v_{i-2} and v_{i-3} are not dominated. As $\lceil \frac{n}{4} \rceil = 4$, S contains a minimum of 4 vertices by Lemma 2.2. If $v_{i+6} \in S$, then v_{i+5} is not uniquely dominated. If $v_{i+7} \in S$, then v_{i-1} is not uniquely dominated and if $v_{i-3} \in S$, then v_{i+5} is not uniquely dominated. Hence none of $v_{i+6}, v_{i+7}, v_{i-2}, v_{i-3}$ can be included in S. If $v_{i-1} \in S$, then v_{i-2} and $v_{i+7} \in V - S$ are uniquely dominated. Similarly if $v_{i+5} \in S$, then v_{i+6} and v_{i-3} in $V - S$ are uniquely dominated. However $|S| = 5$ and is not possible to reduce it to 4. (Because each gap of order 5 can be converted to two gaps of order 2, But it will have $|S| = 5$).

Hence $\gamma_{\mu\beta}(X_{n,3}) = 5$. Application of algorithm now will increase the order by 8 and $|S|$ increases by 2. Therefore $\gamma_{\mu\beta}(X_{n,3}) = \lceil \frac{n}{4} \rceil + 1$.

Case 2. $n \equiv 2 \pmod 8$.

As in Case 1, we take v_i, v_{i+4} and $v_{i+\frac{n}{2}+2} \in S$. If $v_{i+\frac{n}{2}+6} \in S$, then it leaves a gap of order 2 between v_i and v_{i-3} and a gap of order 6 between v_{i+4} and $v_{i+\frac{n}{2}+2}$. In this gap of order 6, the vertices $v_{i+5}, v_{i+6}, v_{i+9}$ and v_{i+10} are uniquely dominated. If $v_{i+7} \in S$, then v_{i-2} and v_{i+6} are not uniquely dominated. If $v_{i+8} \in S$, then v_{i-1} and v_{i+9} are not uniquely dominated. If we include v_{i-2} and v_{i-1} in S, then the domination of all vertices in $V - S$ is unique. However $|S| = 6$. It can not be reduced to $5 = \lceil \frac{n}{4} \rceil$. If the gap of order 6 between v_{i+4} and v_{i+11} is converted into 2 gaps by including $v_{i+8} \in S$, then v_{i-1} is not uniquely dominated. Similarly including v_{i+4} fails. Hence $\gamma_{\mu\beta}(X_{n,3}) = 6$. As before we apply algorithm to conclude the rest of the theorem. □

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On Hemi-Slant Submanifold of Kenmotsu Manifold

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Abstract: We present here a brief analysis on some properties of hemi-slant submanifold of Kenmotsu manifold. After the introduction some preliminaries about this manifold have been discussed. Necessary and sufficient condition for distributions to be integrable are worked out. Some important results have been obtained in this direction. The last section emphasizes the geometry of leaves of hemi-slant submanifold of Kenmotsu manifold.

Key Words: Kenmotsu manifold, hemi-slant submanifold, integrability, leaves of distribution.

AMS(2010): 53C05, 53C15.

§1. Introduction

The notion of Kenmotsu manifold was defined by K. Kenmotsu in 1972 [9]. Then several works have been done on Kenmotsu manifold by G.Pitis [20] in 1988; J.B.Jun, U.C.De and G.Pathak [8] in 2005; C.S. Bagewadi and Venkatesha in 2007.

An interesting topic in the differential geometry is the theory of submanifolds in space endowed with additional structures [4], [5]. B.Y.Chen in 1990 initiated the study of slant manifold of an almost Hermitian manifold as a natural generalization of both holomorphic and totally real submanifolds. N.Papaghiuc have studied semi-invariant submanifolds in a Kenmotsu manifold [17], [18]. He also studied the geometry of leaves on a semi-invariant $\xi^{\perp}$-submanifolds in a Kenmotsu manifolds [18]. Afterwords in 1994, N.Papaghiuc introduced a class of submanifolds in an almost Hermitian manifold, called the semi-slant submanifolds, which includes the class of proper CR-submanifolds and slant submanifolds. Then in 1996, A. Lotta extended the notion of slant immersions in the setting of almost contact metric manifold. Later slant submanifolds of K-contact and Sasakian manifolds have been characterized by Cabrerizo, Carriazo and Fernandez in some papers (1999-2002).

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The idea of hemi-slant submanifold was given by Carriazo as a particular class of bislant submanifolds and he called them anti slant submanifolds. After him B.Sahin in 2009 mentioned anti-slant submanifolds as hemi-slant submanifolds.

§2. Preliminaries

Let $\tilde{M}^{(2n+1)}(\phi,\xi,\eta,\tilde{g})$ be an almost contact Riemannian manifold where ϕ is a tensor field of type $(1,1)$, ξ is a vector field and η is a 1-form and $\tilde{g}$ is the induced Riemannian metric on $\tilde{M}$ satisfying

$$\eta\circ\phi=0,\quad \phi\xi=0,\quad \eta(\xi)=1, \tag{2.1}$$

$$\phi^2X=-X+\eta(X)\xi, \tag{2.2}$$

$$\tilde{g}(X,\xi)=\eta(X), \tag{2.3}$$

$$\tilde{g}(\phi X,Y)=-\tilde{g}(X,\phi Y), \tag{2.4}$$

$$\tilde{g}(\phi X,\phi Y)=\tilde{g}(X,Y)-\eta(X)\eta(Y), \tag{2.5}$$

for all vector fields X,Y on $\tilde{M}$. Now if

$$(\tilde{\nabla}_X\phi)Y=-\eta(Y)\phi X-g(X,\phi Y)\xi, \tag{2.6}$$

$$\tilde{\nabla}_X\xi=X-\eta(X)\xi, \tag{2.7}$$

$\tilde{\nabla}$ is the Riemannian connection of $\tilde{g}$, then$(\tilde{M},\phi,\xi,\eta,\tilde{g})$ is called a Kenmotsu manifold.

In Kenmotsu manifold the following relations hold [9]:

$$(\tilde{\nabla}_X\eta)Y=\tilde{g}(\phi X,\phi Y), \tag{2.8}$$

$$\eta(R(X,Y)Z)=-\tilde{g}(Y,Z)\eta(X)+\tilde{g}(X,Z)\eta(Y), \tag{2.9}$$

$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X, \tag{2.10}$$

$$R(\xi,X)Y=\eta(Y)X-\tilde{g}(X,Y)\xi, \tag{2.11}$$

$$S(X,\xi)=-2n\eta(X), \tag{2.12}$$

$$(\tilde{\nabla}_ZR)(X,Y)\xi=\tilde{g}(Z,X)Y-\tilde{g}(Z,Y)X-R(X,Y)Z, \tag{2.13}$$

where R is the Riemannian curvature tensor and S is the Ricci tensor.

Let M be a submanifold of $\tilde{M}$ with Kenmotsu structure $(\phi,\xi,\eta,\tilde{g})$ with induced metric g and let ∇ is the induced connection on the tangent bundle TM and $\nabla^{\perp}$ is the induced connection on the normal bundle $T^{\perp}M$ of M.

The Gauss and Weingarten formulae are characterized by

$$\tilde{\nabla}_XY=\nabla_XY+h(X,Y), \tag{2.14}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.15}$$

for any $X, Y \in TM$, $N \in T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N via

$$g(A_N X, Y) = g(h(X, Y), N). \tag{2.16}$$

The mean curvature H is denoted by

$$H = \frac{1}{k}\sum_{i=1}^{k} h(e_i, e_i), \tag{2.17}$$

where k is the dimension of M and $\{e_1, e_2, e_3, \cdots, e_k\}$ is the local orthonormal frame on M. For any $X \in \Gamma(TM)$ we can write,

$$\phi X = TX + FX, \tag{2.18}$$

where TX is the tangential component and FX is the normal component of ϕX. Similarly for any $V \in \Gamma(T^{\perp}M)$ we can put

$$\phi V = tV + fV, \tag{2.19}$$

where tV denote the tangential component and fV denote the normal component of ϕV. The covariant derivatives of the tensor fields T, F, t and f are defined as

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.20}$$

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{2.21}$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla^{\perp}_X V, \tag{2.22}$$

$$(\nabla_X f)V = \nabla^{\perp}_X fV - f\nabla^{\perp}_X V, \tag{2.23}$$

for all $X, Y \in TM$and for all $V \in T^{\perp}M$. A submanifold M is said to be invariant if F is identically zero, i.e., $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is said to be anti-invariant if T is identically zero, i.e., $\phi X \in \Gamma(T^{\perp}M)$ for any $X \in \Gamma(TM)$.

A submanifold M of $\tilde{M}$ is called totally umbilical if

$$h(X, Y) = g(X, Y)H, \tag{2.24}$$

for any $X, Y \in \Gamma(TM)$, where H is the mean curvature. A submanifold M is said to be totally geodesic if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is minimal if $H = 0$ on M.

Now to study slant submanifolds let M be a Riemannian manifold,isometrically immersed in an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ and ξ be tangent to M.Then the tangent bundle TM decomposes as $TM = D\oplus < \xi >$ where D is the orthogonal distribution to ξ.Now for each nonzero vector X tangent to M at x,such that X is not proportional to ξ_x,we denote the angle between ϕX and D_x by $\theta(X)$.M is said to be slant submanifold if the angle $\theta(X)$ is constant,which is independent of the choice of $x \in M$ and $X \in T_xM - < \xi_x >$. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of M in $\tilde{M}$. If $\theta = 0$ the submanifold is invariant submanifold, if $\theta = \pi/2$ then it is anti-invariant submanifold and if $\theta \neq 0, \pi/2$ then it is proper slant submanifold.

According to A. Lotta [16], when M is a proper slant submanifold of $\tilde{M}$ with slant angle θ then

$$T^2X = -cos^2\theta(X - \eta(X)\xi), \tag{2.25}$$

for all $X \in \Gamma(TM)$.

Cabrerizo et. al. [2] extended the above result into a characterization for a slant submanifold in a contact metric manifold.

Theorem 2.1 *Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then M is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that $T^2 = -\lambda(I - \eta \otimes \xi)$. Furthermore, in such case, if θ is the slant angle of M, then $\lambda = cos^2\theta$.*

This theorem has the following consequences:

$$g(TX, TX) = cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \tag{2.26}$$

$$g(FX, FY) = sin^2\theta(g(X, Y) - \eta(X)\eta(Y)) \tag{2.27}$$

for all $X, Y \in \Gamma(TM)$.

§3. Hemi-slant Submanifolds of Kenmotsu Manifold

A.Carriazo [3] introduced hemi-slant submanifolds as a special case of bislant submanifolds and he called them pseudo-slant submanifolds.

Definition 3.1([10]) *A submanifold M of a Kenmotsu manifold $\tilde{M}$ is said to be a hemi-slant submanifold if there exist two orthogonal complementary distributions D^θ and $D^\perp$ satisfying the following properties from:*

(1) $TM = D^\theta \oplus D^\perp \oplus < \xi >$;

(2) D^θ *is a slant distribution with slant angle* $\theta \neq \pi/2$;

(3) $D^\perp$ *is totally real i.e.,* $\phi D^\perp \subseteq T^\perp M$.

A hemi-slant submanifold is called proper hemi-slant submanifold if $\theta \neq 0, \frac{\pi}{2}$.

It is clear from above that CR-submanifolds and slant submanifolds are hemi-slant submanifolds with slant angle $\theta = \frac{\pi}{2}$ and $D^\theta = 0$, respectively.

In the rest of the paper, we use M as hemi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. If we denote the dimensions of the distribution $D^\perp$ and D^θ by m_1 and m_2 respectively, then we have the following cases:

(i) If $m_2 = 0$ then M is anti-invariant submanifold;

(ii) If $m_1 = 0$ and $\theta = 0$, then M is an invariant submanifold;

(iii) If $m_1 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold with slant angle θ;

(iv) If $m_1 m_2 \neq 0$ and $\theta \in (0, \frac{\pi}{2})$, then M is a proper hemi-slant submanifold.

Suppose M to be a hemi-slant submanifold of a Kenmotsu manifold manifold $\tilde{M}$, then for any $X \in TM$, we put

$$X = P_1 X + P_2 X + \eta(X)\xi \tag{3.1}$$

where P_1 and P_2 are projection maps on the distribution $D^\perp$ and D^θ. Now operating ϕ on both sides of above equation, we arrive at

$$\phi X = \phi P_1 X + \phi P_2 X + \eta(X)\phi\xi.$$

Using (2.1) and (2.18) we have

$$TX + FX = FP_1 X + TP_2 X + FP_2 X.$$

Comparing we get

$$TX = TP_2 X,\ FX = FP_1 X + FP_2 X.$$

If we denote the orthogonal complement of ϕTM in $T^\perp M$ by μ, then the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = F(D^\perp) \ \oplus \ F(D^\theta) \ \oplus \ < \mu >, \tag{3.2}$$

as $F(D^\perp)$ and $F(D^\theta)$ are orthogonal distributions. Now $g(Z, W) = 0$ for each $Z \in D^\perp$ and $W \in D^\theta$. Thus by (2.5) and (2.18) we obtain

$$g(FZ, FX) = g(\phi Z, \phi X) = g(Z, X) = 0,$$

which shows that the distributions $F(D^\perp)$ and $F(D^\theta)$ are mutually perpendicular. In fact, the decomposition (3.2) is an orthogonal direct decomposition.

In this section we will derive some results on involved distributions of a hemi-slant submanifold, which play a crucial role from a geometrical point of view.

Theorem 3.1 *Let M be a hemi-slant submanifold of Kenmotsu manifold $\tilde{M}$ then*

$$A_{\phi W} Z = A_{\phi Z} W + \eta(Z)\phi W - \eta(W)\phi Z$$

for all $Z, W \in D^\perp$.

Proof On using (2.16) we get

$$\begin{aligned} g(A_{\phi W} Z, X) &= g(h(Z, X), \phi W) = -g(\phi h(Z, X), W) \\ &= -g(\phi \tilde{\nabla}_X Z, W) + g(\phi \nabla_X Z, W) = -g(\phi \tilde{\nabla}_X Z, W) \\ &= -g(\tilde{\nabla}_X \phi Z - (\tilde{\nabla}_X \phi) Z, W) \\ &= -g(\tilde{\nabla}_X \phi Z, W) + g((\tilde{\nabla}_X \phi) Z, W). \end{aligned}$$

Again using (2.6) and (2.15) we obtain

$$g(A_{\phi W}Z, X) = -g(-A_{\phi Z}X + \nabla^{\perp}_{X}\phi Z, W) + g(-\eta(Z)\phi X - g(X, \phi Z)\xi, W)$$

After some steps of calculations we get

$$g(A_{\phi W}Z, X) = g(h(W, X), \phi Z) + \eta(Z)g(\phi W, X) - g(\phi Z, X)\eta(W)$$

$$= g(A_{\phi Z}W + \eta(Z)\phi W - \eta(W)\phi Z, X).$$

Hence the theorem. □

Theorem 3.2 *Let M be a hemi-slant submanifold of Kenmotsu manifold $\tilde{M}$. Then the distribution $D^{\theta} \oplus D^{\perp}$ is integrable if and only if $g([X, Y], \xi) = 0$ for all $X, Y \in D^{\theta} \oplus D^{\perp}$.*

Proof For $X, Y \in D^{\theta} \oplus D^{\perp}$ we have

$$g([X, Y], \xi) = g(\tilde{\nabla}_X Y, \xi) - g(\tilde{\nabla}_Y X, \xi) = -g(\tilde{\nabla}_X \xi, Y) + g(\tilde{\nabla}_Y \xi, X)$$

$$= -g(X - \eta(X)\xi, Y) + g(Y - \eta(Y)\xi, X) = 0.$$

Since $TM = D^{\theta} \oplus D^{\perp} \oplus < \xi >$, therefore $[X, Y] \in D^{\theta} \oplus D^{\perp}$. So, $D^{\theta} \oplus D^{\perp}$ is integrable.

Conversely, let $D^{\theta} \oplus D^{\perp}$ is integrable. Then for all $X, Y \in D^{\theta} \oplus D^{\perp}, [X, Y] \in D^{\theta} \oplus D^{\perp}$. As $TM = D^{\theta} \oplus D^{\perp} \oplus < \xi >$,therefore $g([X, Y], \xi) = 0$. □

Theorem 3.3 *Let M be a hemi-slant submanifold of Kenmotsu manifold $\tilde{M}$. Then the anti-invariant distribution $D^{\perp}$ is integrable if and only if $\eta(Z)FW = \eta(W)FZ$ for any $Z, W \in D^{\perp}$.*

Proof For $Z, W \in D^{\perp}$ we have from (2.6),

$$(\tilde{\nabla}_Z \phi)W = -\eta(W)\phi Z - g(Z, \phi W)\xi. \tag{3.3}$$

After some steps of calculations and using Gauss and Weingarten formula we can obtain

$$-A_{FW}Z + \nabla^{\perp}_{Z}FW - T\nabla_Z W - F\nabla_Z W - th(Z, W) - fh(W, Z) \tag{3.4}$$

$$= -\eta(W)TZ - \eta(W)FZ - g(Z, TW + FW)\xi.$$

Comparing the tangential components, we have

$$-A_{FW}Z - T\nabla_Z W - th(Z, W) = -\eta(W)TZ - g(Z, TW)\xi. \tag{3.5}$$

Interchanging Z and W, we get

$$-A_{FZ}W - T\nabla_W Z - th(W, Z) = -\eta(Z)TW - g(W, TZ)\xi. \tag{3.6}$$

Subtracting equation (3.6) from (3.5) and using the fact that h is symmetric we get

$$A_{FW}Z - A_{FZ}W + T(\nabla_Z W - \nabla_W Z) = \eta(W)TZ + g(Z, TW)\xi - \eta(Z)TW - g(W, TZ)\xi. \quad (3.7)$$

Notice that $D^{\perp}$ is integrable iff $[Z, W] \in D^{\perp}$. Now $D^{\perp}$ is anti-invariant, i.e. $\phi D^{\perp} \subseteq T^{\perp}M$. Hence $T(Z) = 0,\ T(W) = 0,\ T[Z, W] = 0$.

Again from (4.7)

$$A_{FW}Z - A_{FZ}W + T[Z, W] = 0. \quad (3.8)$$

So $D^{\perp}$ is integrable $\Longleftrightarrow A_{FW}Z - A_{FZ}W = 0$. By Theorem 3.1 we get the result. □

Theorem 3.4 *Let M be a hemi-slant submanifold of Kenmotsu manifold $\tilde{M}$. Then the slant distribution D^{θ} is integrable if and only if*

$$P_1(\nabla_X TY - \nabla_Y TX + R(\xi, TX)Y - R(\xi, TY)X) = 0 \quad (3.9)$$

for any $X, Y \in D^{\theta}$.

Proof We denote by P_1 and P_2 the projections on $D^{\perp}$ and D^{θ} respectively. For any vector fields $X, Y \in D^{\theta}$, we have from (2.6),

$$(\tilde{\nabla}_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi. \quad (3.10)$$

On applying (2.14), (2.15), (2.18) and (2.19) we get

$$(\tilde{\nabla}_X \phi)Y = \nabla_X TY + h(X, TY) - A_{FY}X + \nabla^{\perp}_X FY - (T\nabla_X Y + F\nabla_X Y) - (th(X, Y) + fh(X, Y)). \quad (3.11)$$

Therefore from (3.10) and (3.11)

$$\begin{aligned} \nabla_X TY \quad &+ \quad h(X, TY) - A_{FY}X + \nabla^{\perp}_X FY - (T\nabla_X Y + F\nabla_X Y) \\ &- \quad (th(X, Y) + fh(X, Y)) = -\eta(Y)(T + F)X - g(X, (T + F)Y)\xi. \end{aligned} \quad (3.12)$$

Comparing the tangential components

$$\nabla_X TY - A_{FY}X - T\nabla_X Y - th(X, Y) = -\eta(Y)TX - g(X, TY)\xi. \quad (3.13)$$

Interchanging X and Y and subtracting from above equation we obtain

$$\nabla_X TY - \nabla_Y TX - A_{FY}X + A_{FX}Y - T\nabla_X Y + T\nabla_Y X \quad (3.14)$$

$$= -\eta(Y)TX + \eta(X)TY - g(X, TY)\xi + g(Y, TX)\xi.$$

From (2.11) we get

$$\nabla_X TY - \nabla_Y TX - A_{FY}X + A_{FX}Y - T\nabla_X Y + T\nabla_Y X = -R(\xi, TX)Y + R(\xi, TY)X. \quad (3.15)$$

Since $X, Y \in D^\theta$, $FX = 0$ and $FY = 0$. Hence applying P_1 to both sides of above equation we conclude our theorem. □

Theorem 3.5 *Let M be a hemi-slant submanifold of Kenmotsu manifold $\tilde{M}$. If the leaves of $D^\perp$ are totally geodesic in M, then*

$$g(h(Z,X),FW) + g(th(Z,W),X) = 0 \tag{3.16}$$

for all $X \in D^\theta$ and $Z, W \in D^\perp$.

Proof From (2.6), (2.14) and (2.15) we get

$$\nabla_Z \phi W + h(Z,\phi W) - A_{FW}Z + \nabla^\perp_Z FW - \phi\nabla_z W - \phi h(Z,W) = -\eta(W)\phi Z - g(Z,\phi W)\xi. \tag{3.17}$$

Comparing the tangential components and on taking inner product with $X \in D^\theta$, we obtain

$$-g(A_{FW}Z,X) - g(th(Z,W),X) - g(T\nabla_Z W,X) = 0. \tag{3.18}$$

The leaves of $D^\perp$ are totally geodesic in M if for $Z, W \in D^\perp$,$\nabla_Z W \in D^\perp$.So $T\nabla_Z W = 0$. Hence we have

$$-g(A_{FW}Z,X) - g(th(Z,W),X) = 0. \tag{3.19}$$

This completes the proof. □

Theorem 3.6 *Let M be a totally umbilical hemi-slant submanifold of Kenmotsu manifold $\tilde{M}$. Then at least one of the following holds:*

(*i*) *dim* $D^\perp = 1$;

(*ii*) $H \in \mu$;

(*iii*) M *is proper hemi-slant submanifold.*

Proof In a Kenmotsu manifold for any $z \in D^\perp$ we have from (2.6),

$$(\tilde{\nabla}_Z \phi)Z = -\eta(Z)\phi Z - g(Z,\phi Z)\xi. \tag{3.20}$$

Using (2.14) and (2.18) we obtain

$$\tilde{\nabla}_Z FZ - \phi(\nabla_Z Z + h(Z,Z)) = -\eta(Z)FZ - g(Z,FZ)\xi. \tag{3.21}$$

Since $Z \in D^\perp, TZ = 0$. Now from (2.15),(2.18) and (2.19)

$$-A_{FZ}Z + \nabla^\perp_Z FZ - F\nabla_Z Z - th(Z,Z) - fh(Z,Z) = -\eta(Z)FZ - g(Z,FZ)\xi. \tag{3.22}$$

Comparing the tangential components

$$-A_{FZ}Z - th(Z,Z) = 0. \tag{3.23}$$

Taking inner product with $W \in D^{\perp}$,we get on using the fact that M is totally umbilical submanifold

$$g(g(Z,W)H,FZ)+g(tg(Z,Z)H,W)=0. \tag{3.24}$$

After some brief calculations we get

$$g(Z,W)g(H,FZ)=0. \tag{3.25}$$

Hence either $g(Z,W)=0$ or $g(H,FZ)=0$. If $g(Z,W)=0$ then either $Z=0$ or $Z=W$. As Z is arbitrary taken from $D^{\perp}$, so if $Z=0$ then $D^{\perp}=0$. And if $Z=W$ then dim $D^{\perp}=1$. Now, if $g(H,FZ)=0$, then $H \in \mu$. □

§4. An Example of Hemi-slant Submanifold of a Kenmotsu Manifold

Let us consider a 9-dimensional submanifold M of $\Re^9$ defined by [7]

$$(u_1,-\sqrt{2}u_2,u_2 sin\theta_1,u_2 cos\theta_1,scos\theta_2,-cos\theta_2,ssin\theta_2,-sin\theta_2,z).$$

The independent vector fields

$$e_1=\frac{\partial}{\partial x_1},$$

$$e_2=-\sqrt{2}\frac{\partial}{\partial y_1}\ +\ sin\theta_1\frac{\partial}{\partial x_2}\ +\ cos\theta_1\frac{\partial}{\partial y_2},$$

$$e_3=cos\theta_2\frac{\partial}{\partial x_3}\ +\ sin\theta_2\frac{\partial}{\partial x_4},$$

$$e_4=-ssin\theta_2\frac{\partial}{\partial x_3}\ +\ sin\theta_2\frac{\partial}{\partial x_3}+scos\theta_2\frac{\partial}{\partial x_4}\ -\ cos\theta_2\frac{\partial}{\partial y_4},$$

$$e_5=\xi=\frac{\partial}{\partial z}$$

span the tangent bundle of M. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(\frac{\partial}{\partial x_i})=\frac{\partial}{\partial y_i},\phi(\frac{\partial}{\partial y_j})=-\frac{\partial}{\partial x_j},\phi(\frac{\partial}{\partial z})=0\ \ 1\leq i,j\leq 4.$$

For any vector field

$$X=\lambda_i\frac{\partial}{\partial x_i}+\mu_j\frac{\partial}{\partial y_j}+\gamma\frac{\partial}{\partial z},$$

$$Y=\lambda_i'\frac{\partial}{\partial x_i}+\mu_j'\frac{\partial}{\partial y_j}+\gamma'\frac{\partial}{\partial z}\in\Gamma(T\Re^9)$$

where $i,j\in\{1,2,3,4\}$.

After calculations we have

$$\phi^2 X = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_j},$$

$$-X + \eta(X)\xi = -\lambda_i \frac{\partial}{\partial x_i} - \mu_j \frac{\partial}{\partial y_j}$$

$$g(\phi X, \phi Y) = \lambda_i \lambda'_i + \mu_j \mu'_j$$

Again

$$\phi(X,Y) = \lambda_i \lambda'_i + \mu_j \mu'_j + \gamma\gamma'$$

$$\eta(X)\eta(Y) = \gamma\gamma'.$$

Therefore we can see that $\phi^2 X = -X + \eta(X)\xi$ .Moreover equation (2.1) and (2.5) are also satisfied. Hence (ϕ, η, ξ, g) is an almost contact structure.

By direct calculation we can infer $D^\theta = span\{e_1, e_2\}$ ia a slant distribution with slant angle $\theta = cos^{-1}(\frac{\sqrt{6}}{3})$. Since ϕe_3 and ϕe_4 are orthogonal to M, $D^\perp = span\{e_3, e_4\}$ is an anti-invariant distribution. Thus M is a $5-$ dimensional proper semi-slant submanifold of $\Re^9$ with (ϕ, η, ξ, g).

Let $\tilde{\nabla}$ be the Levi-Civita connection on $\Re^9$. $[e_1, e_2]f = 0$. By similar calculation we get $[e_i, e_j] = 0,\ i, j \in \{1, 2, 3, 4, 5\}$. We can also calculate that

$$g(e_1, e_1) = g(e_3, e_3) = 1, g(e_2, e_2) = 3,$$

$$g(e_4, e_4) = s^2 + 1, g(e_5, e_5) = 1,$$

$$g(e_i, e_j) = 0 for\ i \neq j.$$

By using Koszul formula for g we can find the values of $\nabla_{e_i} e_j$ and verify (2.6) and (2.7). Therefore (ϕ, η, ξ, g) is a Kenmotsu manifold.

Let $z', w' \in D^\perp$ so $z' = \lambda_3 e_3 + \lambda_4 e_4, w' = \mu_3 e_3 + \mu_4 e_4$ for some $\lambda_3, \lambda_4, \mu_3, \mu_4$.

$\eta(z')\phi(w') = g(\lambda_3 e_3 + \lambda_4 e_4, e_5) \times \{-\mu_4 sin\theta_2 \frac{\partial}{\partial x_3} + \mu_4 cos\theta_2 \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_3}(\mu_3 cos\theta_2 - s\mu_4 sin\theta_2) + \frac{\partial}{\partial y_4}(\mu_3 sin\theta_2 + s cos\theta_2 \mu_4)\} = 0$. Similarly we compute $\eta(w')\phi(z') = 0$ which indicates $\eta(z')\phi(w') = \eta(w')\phi(z')$.

Now $g([e_3, e_4], e_5) = g([e_3, e_4], e_1) = g([e_3, e_4], e_2) = 0$. Therefore $[e_3, e_4] \in D^\perp$. Hence $D^\perp$ is integrable.

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The Number of Rooted Nearly 2-Regular Loopless Planar Maps

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Abstract: This paper investigates the enumeration of rooted nearly 2-regular loopless planar maps and presents some formulae for such maps with the valency of the root-face, the numbers of nonrooted vertices and inner faces as three parameters.

Key Words: Loopless map, nearly 2-regular map, enumerating function, functional equation, Lagrangian inversion.

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§1. Introduction

Since Tutte's papers on enumerating planar maps in [21–23] were published in the early 1960's, the enumerative theory has been developed greatly up to now. Eulerian maps have played a crucial role in enumerative map theory. In particular, Tutte's sum-free formula [22] for the number of eulerian planar maps, all of whose vertices are labelled and contain a distinguished edge-end, with a given sequence of (even) vertex valencies was an essential stepin obtaining his ground-breaking formula for counting rooted planar maps by number of edges [23]. Several new results on the subject have been published since then (see, e.g. [1–20, 24]).

Here we deal with the enumeration of rooted nearly 2-regular loopless planar maps with the valency of the root-face, the numbers of nonrooted vertices and inner faces as three parameters. Several explicit expressions of its enumerating functions are obtained and one of them is summation-free.

A map is a connected graph cellularly embedded on a surface. A map is rooted if an edge and a direction along that edge are distinguished. If the root is the oriented edge from u to v, then u is the root-vertex while the face on the right side of the edge as seen by an observer on the edge facing away from u is defined as the root-face. A map is called Eulerian if all the valencies of its vertices are even. A nearly 2-regular map is a rooted map such that all vertices probably except the root-vertex are of valency 2. It is clear that a nearly 2-regular map is also an Eulerian map. In this paper, maps are always rooted and planar.

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For a set of some maps $\mathscr{M}$, the enumerating function discussed in this paper is defined as

$$f_{\mathscr{M}}(x,y,z)=\sum_{M\in\mathscr{M}} x^{l(M)}y^{p(M)}z^{q(M)}, \tag{1}$$

where $l(M),p(M)$ and $q(M)$ are the root-face valency, the number of nonrooted vertices and the number of inner faces of M, respectively.

Furthermore, we introduce some other enumerating functions for $\mathscr{M}$ as follows:

$$\begin{aligned} g_{\mathscr{M}}(x,y)&=\sum_{M\in\mathscr{M}} x^{l(M)}y^{n(M)},\\ h_{\mathscr{M}}(y,z)&=\sum_{M\in\mathscr{M}} y^{p(M)}z^{q(M)},\\ H_{\mathscr{M}}(y)&=\sum_{M\in\mathscr{M}} y^{n(M)}, \end{aligned} \tag{2}$$

where $l(M),p(M)$ and $q(M)$ are the same in (1) and $n(M)$ is the number of edges of M, that is

$$\begin{aligned} g_{\mathscr{M}}(x,y)=&f_{\mathscr{U}}(x,y,y), \quad h_{\mathscr{M}}(y,z)=f_{\mathscr{M}}(1,y,z),\\ H_{\mathscr{M}}(y)=&g_{\mathscr{M}}(1,y)=h_{\mathscr{M}}(y,y)=f_{\mathscr{M}}(1,y,y). \end{aligned} \tag{3}$$

For the power series $f(x)$, $f(x,y)$ and $f(x,y,z)$, we employ the following notations:

$$\partial_x^l f(x), \quad \partial_{(x,y)}^{(l,p)}f(x,y) \quad and \quad \partial_{(x,y,z)}^{(l,p,q)}f(x,y,z)$$

to represent the coefficients of x^l in $f(x)$, x^ly^p in $f(x,y)$ and $x^ly^pz^q$ in $f(x,y,z)$, respectively. Terminologies and notations not explained here can be found in [10].

§2. Functional Equations

In this section we will set up the functional equations satisfied by the enumerating functions for rooted nearly 2-regular loopless planar maps.

Let $\mathscr{E}$ be the set of all rooted nearly 2-regular loopless planar maps with convention that the vertex map ϑ is in $\mathscr{E}$ for convenience. For any $M\in\mathscr{E}-\vartheta$, it is obvious that each edge of M is contained in only one circuit. The circuit containing the root-edge is called the root circuit of M, and denoted by $C(M)$.

It is clear that the length of the root circuit is no more than the root-face valency, and

$$\mathscr{E}=\mathscr{E}_0+\bigcup_{i\geq 2}\mathscr{E}_i, \tag{4}$$

where

$$\mathscr{E}_i=\{M \mid M\in\mathscr{E}, \text{the length of } C(M) \text{ is } i\} \tag{5}$$

and $\mathscr{E}_0$ is only consist of the vertex map ϑ.

It is easy to see that the enumerating function of $\mathscr{E}_0$ is

$$f_{\mathscr{E}_0}(x,y,z)=1. \tag{6}$$

For any $M\in\mathscr{E}_i$ $(i\geq 2)$, the root circuit divides $M-C(M)$ into two domains, the inner domain and outer domain. The submap of M in the outer domain is a general map in $\mathscr{E}$, while the submap of M in the inner domain does not contribute the valency of the root-face of M. Thus, the enumerating function of $\mathscr{E}_i$ is

$$f_{\mathscr{E}_i}(x,y,z)=x^iy^{i-1}zhf, \tag{7}$$

where $h=h_{\mathscr{E}}(y,z)=f_{\mathscr{E}}(1,y,z)$.

Theorem 2.1 *The enumerating function $f=f_{\mathscr{E}}(x,y,z)$ satisfies the following equation:*

$$f=\left(1-\frac{x^2yzh}{1-xy}\right)^{-1}, \tag{8}$$

where $h=h_{\mathscr{E}}(y,z)=f_{\mathscr{E}}(1,y,z)$.

Proof By (4), (6) and (7), we have

$$\begin{aligned} f=&1+\sum_{i\geq 2}x^iy^{i-1}zhf\\ =&1+\frac{x^2yzhf}{1-xy}, \end{aligned}$$

which is equivalent to the theorem. □

Let $y=z$ in (8). Then we have

Corollary 2.1 *The enumerating function $g=g_{\mathscr{E}}(x,y)$ satisfies the following equation:*

$$g=\left(1-\frac{x^2y^2H}{1-xy}\right)^{-1}, \tag{9}$$

where $H=H_{\mathscr{E}}(y)=g_{\mathscr{E}}(1,y)$.

Let $x=1$ in (8). Then we obtain

Corollary 2.2 *The enumerating function $h=h_{\mathscr{E}}(y,z)$ satisfies the following equation:*

$$yzh^2+(y-1)h-y+1=0. \tag{10}$$

Further, let $y=z$ in (10). Then we have

Corollary 2.3 *The enumerating function $H = H_{\mathscr{E}}(y)$ satisfies the following equation:*

$$y^2H^2 + (y-1)H - y + 1 = 0. \tag{11}$$

§3. Enumeration

In this section we will find the explicit formulae for enumerating functions $f = f_{\mathscr{E}}(x,y,z), g = g_{\mathscr{E}}(x,y), h = h_{\mathscr{E}}(y,z)$ and $H = H_{\mathscr{E}}(y)$ by using Lagrangian inversion.

By (10) we have

$$h = \frac{(1-y)\left(1-\sqrt{1-\frac{4yz}{1-y}}\right)}{2yz}. \tag{12}$$

Let

$$y = \frac{\theta}{1+\theta}, \quad z = \eta(1-\theta\eta). \tag{13}$$

By substituting (13) into (12), one may find that

$$h = \frac{1}{1-\theta\eta}. \tag{14}$$

By (13) and (14), we have the following parametric expression of $h = h_{\mathscr{E}}(y,z)$:

$$\begin{aligned} y =& \frac{\theta}{1+\theta}, \quad z = \eta(1-\theta\eta), \\ h =& \frac{1}{1-\theta\eta} \end{aligned} \tag{15}$$

and from which we get

$$\Delta_{(\theta,\eta)} = \begin{vmatrix} \frac{1}{1+\theta} & 0 \\ * & \frac{1-2\theta\eta}{1-\theta\eta} \end{vmatrix} = \frac{1-2\theta\eta}{(1+\theta)(1-\theta\eta)}. \tag{16}$$

Theorem 3.1 *The enumerating function $h = h_{\mathscr{E}}(y,z)$ has the following explicit expression:*

$$h_{\mathscr{E}}(y,z) = 1 + \sum_{p\geq 1}\sum_{q=1}^{p} \frac{(2q)!(p-1)!}{q!(q+1)!(p-q)!(q-1)!} y^p z^q. \tag{17}$$

Proof By employing Lagrangian inversion with two parameters, from (15) and (16) one may find that

$$
\begin{aligned}
h_{\mathscr{E}}(y,z) =& \sum_{p,q\geq 0} \partial^{(p,q)}_{(\theta,\eta)} \frac{(1+\theta)^{p-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+2}} y^p z^q \\
=& 1+\sum_{p,q\geq 1}\left[\partial^{(p,q)}_{(\theta,\eta)}\frac{(1+\theta)^{p-1}}{(1-\theta\eta)^{q+2}} - 2\partial^{(p-1,q-1)}_{(\theta,\eta)}\frac{(1+\theta)^{p-1}}{(1-\theta\eta)^{q+2}}\right] y^p z^q \\
=& 1+\sum_{p\geq 1}\sum_{q=1}^{p}\frac{(2q)!}{q!(q+1)!}\partial_\theta^{p-q}(1+\theta)^{p-1} y^p z^q = 1+\sum_{p\geq 1}\sum_{q=1}^{p}\frac{(2q)!(p-1)!}{q!(q+1)!(p-q)!(q-1)!} y^p z^q,
\end{aligned}
$$

which is just the theorem. □

In what follows we present a corollary of Theorem 3.1.

Corollary 3.1 *The enumerating function $H = H_{\mathscr{E}}(y)$ has the following explicit expression:*

$$
H_{\mathscr{E}}(y) = 1+\sum_{n\geq 2}\sum_{q=1}^{\lfloor \frac{n}{2}\rfloor}\frac{(2q)!(n-q-1)!}{q!(q+1)!(n-2q)!(q-1)!} y^n. \tag{18}
$$

Proof It follows immediately from (17) by putting $y = z$ and $n = p + q$. □

Now, let

$$
x = \frac{\xi(1+\theta)}{1+\xi\theta}. \tag{19}
$$

By substituting (15) and (19) into Equ. (8), one may find that

$$
f = \frac{1}{1-\frac{\xi^2\theta\eta(1+\theta)}{1+\xi\theta}}. \tag{20}
$$

By (15), (19) and (20), we have the parametric expression of the function $f = f_{\mathscr{E}}(x, y, z)$ as follows:

$$
\begin{aligned}
x =& \frac{\xi(1+\theta)}{1+\xi\theta}, \quad y = \frac{\theta}{1+\theta}, \\
z =& \eta(1-\theta\eta), \quad f = \frac{1}{1-\frac{\xi^2\theta\eta(1+\theta)}{1+\xi\theta}}.
\end{aligned} \tag{21}
$$

According to (21), we have

$$
\Delta_{(\xi,\theta,\eta)} = \begin{vmatrix} \frac{1}{1+\xi\theta} & * & 0 \\ 0 & \frac{1}{1+\theta} & 0 \\ 0 & * & \frac{1-2\theta\eta}{1-\theta\eta} \end{vmatrix} = \frac{1-2\theta\eta}{(1+\xi\theta)(1+\theta)(1-\theta\eta)}. \tag{22}
$$

Theorem 3.2 *The enumerating function $f = f_{\mathscr{E}}(x,y,z)$ has the following explicit expression:*

$$
\begin{aligned}
f_{\mathscr{E}}(x,y,z) =& 1 + \sum_{p\geq 1}\sum_{q=1}^{p}\sum_{l=2}^{p+q}\sum_{k=\max\{1,\lceil\frac{l+q-p}{2}\rceil\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \frac{k}{q}\binom{2q-k-1}{q-k}\binom{l-k-1}{l-2k} \\
&\times \binom{p-l+k-1}{p-q-l+2k} x^l y^p z^q. \qquad (23)
\end{aligned}
$$

Proof By using Lagrangian inversion with three variables, from (21) and (22) one may find that

$$
\begin{aligned}
f_{\mathscr{E}}(x,y,z) =& \sum_{l,p,q\geq 0} \partial^{(l,p,q)}_{(\xi,\theta,\eta)} \frac{(1+\xi\theta)^{l-1}(1+\theta)^{p-l-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+1}\left[1-\frac{\xi^2\theta\eta(1+\theta)}{(1+\xi\theta)}\right]} x^l y^p z^q \\
=& \sum_{l,p,q\geq 0}\sum_{k=0}^{\min\{\lfloor\frac{l}{2}\rfloor,p,q\}} \partial^{(l-2k,p-k,q-k)}_{(\xi,\theta,\eta)} \frac{(1+\xi\theta)^{l-k-1}}{(1-\theta\eta)^{q+1}} \\
&\times (1+\theta)^{p-l+k-1}(1-2\theta\eta) x^l y^p z^q \\
=& 1 + \sum_{p,q\geq 1}\sum_{l=2}^{p+q}\sum_{k=\max\{1,l-p\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \binom{l-k-1}{l-2k} \\
&\times \partial^{(p-l+k,q-k)}_{(\theta,\eta)} \frac{(1+\theta)^{p-l+k-1}(1-2\theta\eta)}{(1-\theta\eta)^{q+1}} x^l y^p z^q \\
=& 1 + \sum_{p,q\geq 1}\sum_{l=2}^{p+q}\sum_{k=\max\{1,l-p\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \binom{l-k-1}{l-2k} \\
&\times \Bigg[\partial^{(p-l+k,q-k)}_{(\theta,\eta)} \frac{(1+\theta)^{p-l+k-1}}{(1-\theta\eta)^{q+1}} \\
&- 2\partial^{(p-l+k-1,q-k-1)}_{(\theta,\eta)} \frac{(1+\theta)^{p-l+k-1}}{(1-\theta\eta)^{q+1}}\Bigg] x^l y^p z^q \\
=& 1 + \sum_{p\geq 1}\sum_{q=1}^{p}\sum_{l=2}^{p+q}\sum_{k=\max\{1,\lceil\frac{l+q-p}{2}\rceil\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \frac{(2q-k-1)!k}{(q-k)!q!}\binom{l-k-1}{l-2k} \\
&\times \partial^{p-q-l+2k}_{\theta}(1+\theta)^{p-l+k-1} x^l y^p z^q \\
=& 1 + \sum_{p\geq 1}\sum_{q=1}^{p}\sum_{l=2}^{p+q}\sum_{k=\max\{1,\lceil\frac{l+q-p}{2}\rceil\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \frac{(2q-k-1)!k}{(q-k)!q!}\binom{l-k-1}{l-2k} \\
&\times \binom{p-l+k-1}{p-q-l+2k} x^l y^p z^q,
\end{aligned}
$$

which is what we wanted. □

Finally, we give a corollary of Theorem 3.2.

Corollary 3.2 *The enumerating function $g = g_{\mathscr{E}}(x, y)$ has the following explicit expression:*

$$
\begin{aligned}
g_{\mathscr{E}}(x,y) =& 1 + \sum_{n\geq 2}\sum_{l=2}^{n}\sum_{q=1}^{\lfloor \frac{n}{2}\rfloor}\sum_{k=\max\{1,\lceil\frac{l+2q-n}{2}\rceil\}}^{\min\{\lfloor\frac{l}{2}\rfloor,q\}} \frac{k}{q}\binom{2q-k-1}{q-k}\binom{l-k-1}{l-2k} \\
&\times \binom{n-q-l+k-1}{n-2q-l+2k} x^l y^n. \qquad (24)
\end{aligned}
$$

Proof It follows soon from (23) by putting $y = z$ and $n = p + q$. □

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A Note on Common Fixed Points for (ψ, α, β)-Weakly Contractive Mappings in Generalized Metric Space

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Abstract: In this paper, we establish a common fixed point theorem for mappings satisfying a (ψ, α, β)-weakly contractive condition in generalized metric space. Presented theorems extend and generalize many existing results in the literature. We prove the main results for four self mappings using any two continuous mappings.

Key Words: Fixed point theory, generalized metric space, (ψ, α, β)-weakly contractive mappings, common fixed point.

AMS(2010): 47H10, 54H25.

§1. Introduction

Fixed point theory is an important part of mathematics. Moreover, its well known that the contraction mapping principle, which is introduced by S. Banach in 1922.

During the last few decades, this theorem has undergone various generalizations either by relaxing the condition on contractivity or withdrawing the requirement of completeness or sometimes even both.

In 2000, Branciari [1] obtained a very interesting generalization of metric space by changing the structure of the space. He, replaced the triangle inequality of a metric space by an inequality involving three terms instead of two called quadrilateral inequality. He, proved the Banach fixed point theorem in such space. Recently, many fixed point results have been established for this interesting space ([3],[7],[8],[9]). As such, any metric space is a generalized metric space, but the converse is not true [1].

Recently, many researchers have interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems. Choudhury and Kundu [2] established the (ψ, α, β)-weakly contraction principal to coincidence point and common fixed point results in partially ordered metric spaces. In a recent paper Isik and Turkoglu [3] proved common fixed point for (ψ, α, β)-weakly contractive mappings in generalized metric spaces for two mappings.

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The aim of this paper to prove a common fixed point for (ψ, α, β) -weakly contractive mappings in generalized metric space of four self mappings.

§2. Preliminaries

Definition 2.1([1]) *Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a generalized metric on X if the following conditions are satisfied:*

(1) $d(x, y) = 0$ *iff* $x = y$ *for all* x, y *in* X*;*

(2) $d(x, y) = d(y, x)$ *for all* x, y *in* X*;*

(3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ *for all* x, y, u, v *in* X*.*

The pair (X, d) is called a generalized metric space abbreviated to g.m.s .

Definition 2.2([1]) *Let (X,d) be a g.m.s and let (x_n) be a sequence in X and $x \in X$.*

(1) (x_n) *is a g.m.s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$;*

(2) x_n *is a g.m.s. Cauchy sequence if and only if for each $\in > 0$ there exists a natural number $n(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n > m > n(\epsilon)$;*

(3) (X, d) *is called a complete g.m.s if every g.m.s. Cauchy sequence is g.m.s. convergent in X;*

We denote by Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

$(\psi 1)$ *ψ is continuous and monotone non decreasing;*

$(\psi 2)$ *$\psi(t) = 0$ if and only if t=0.*

We denote by ϕ the set of function $\alpha : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

$(\alpha 1)$ *α is continuous;*

$(\alpha 2)$ *$\alpha(t) = 0$ if and only if t=0.*

We denote by Γ the set of function $\beta : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

$(\beta 1)$ *β is lower semi continuous;*

$(\beta 2)$ *$\beta(t) = 0$ if and only if t=0.*

Definition 2.3([3]) *Let A and B be mappings from a metric space (X, d) into itself. A and B are said to be weakly compatible mapping if they commute at their coincidence point i.e, $Ax = Bx$ for some x in X implies $ABx = BAx$.*

Lemma 2.1([3]) *Let a_n be a sequence of non negative real numbers. If*

$$\psi(a_{n+1}) \leq \alpha(a_n) - \beta(a_n) \tag{A}$$

for all $n \in N$, where $\psi \in \Psi$, $\alpha \in \phi$, $\beta \in \Gamma$ and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \ for \ all \ t > 0, \tag{B}$$

then the following hold:

(1) $a_{n+1} \leq a_n$ *if* $a_n > 0$*;*

(2) $a_n \to 0$ *as* $n \to \infty$.

Proof (1) Let, if possible $a_n < a_{n+1}$ for some $n \in N$ then using the monotone property of Ψ and from (A) we have, $\psi(a_n) \leq \psi(a_{n+1}) \leq \alpha(a_n) - \beta(a_n)$, which implies that $a_n = 0$ by (B) a contradiction with $a_n > 0$. Therefore, for all $n \in N$, $a_{n+1} \leq a_n$.

(2) By (1) the sequence a_n is non-increasing, hence there is $a \geq 0$ such that $a_n \to a$ as $n \to \infty$ letting $n \to \infty$ in (A), using the lower semi continuity of β and the continuities of Ψ and α, we obtain $\psi(a) \leq \alpha(a) - \beta(a)$, which by (B) implies that $a = 0$. □

§3. Main Results

Theorem 3.1 *Let (X, d) be a Hausdorff and complete g.m.s. and let f, g, h and J be four mappings of X into itself and $f(X) \subset h(X)$, $g(X) \subset J(X)$. Without loss of generality, assume h, J are continuous, f and J, g and h both are compatible satisfying the following condition*

$$\psi(d(fx, gy) \leq \alpha(M(x, y)) - \beta(M(x, y)), \tag{1}$$

where $M(x, y) = \max\{d(Jx, hy), d(fx, Jx), d(gy, hy), d(fx, hy)\}$ for all $x, y \in X$, where $\psi \subset \Psi$, $\alpha \subset \phi$ and $\beta \subset \Gamma$ satisfying condition (B). Then, f, g, h and J have a unique common fixed point in X.

Proof Notice that $f(X) \subset h(X)$ and $g(X) \subset J(X)$. Let x_0 be any point in X. Then, there exists sequences (x_n) and (y_n) such that $y_n = fx_n = hx_{n+1}$, $y_{n+1} = gx_{n+1} = Jx_{n+2}$, $n = 0, 1, 2, 3, \cdots$. Now,

$$\psi(d(y_n, y_{n+1})) = \psi(d(fx_n, gx_{n+1})) \leq \alpha(M(x, y)) - \beta(M(x, y)), \tag{2}$$

where,

$$\begin{aligned} M(x, y) &= \max\{d(Jx_n, hx_{n+1}), d(fx_n, Jx_n), d(gx_{n+1}, hx_{n+1}), d(fx_n, hx_{n+1})\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), d(y_n, y_n)\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), 0\} \end{aligned}$$

If possible, let $d(y_n, y_{n+1}) > 0$ and $d(y_n, y_{n+1}) > d(y_{n-1}, y_n)$. Then from (2) we get that

$$\psi(d(y_n, y_{n+1})) \leq \alpha(d(y_n, y_{n+1})) - \beta(d(y_n, y_{n+1})). \tag{3}$$

By Lemma 2.1, each number in the sequence y_n is non negative and real. Hence there exists $a \geq 0$ such that $y_n \to a$ as $n \to \infty$ in (3). Using the lower semi continuity of β and the continuities of ψ and α, we obtain $\psi(a) \leq \alpha(a) - \beta(a)$. However, (B) implies $a = 0$, which is a contradiction. So $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$.

From (2) we get that

$$\psi(d(y_n, y_{n+1})) \leq \alpha(d(y_{n-1}, y_n)) - \beta(d(y_{n-1}, y_n)).$$

According to Lemma 2.1 we know that $\psi(a_{n+1}) \leq \alpha(a_n) - \beta(a_n)$ for all $n \in N$, where $\psi \subset \Psi$, $\alpha \subset \phi$ and $\beta \subset \Gamma$ and $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$. Thus, $a_{n+1} \leq a_n$ if $a_n > 0$ and $a_n \to \infty$. Therefore,

$$d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n) \text{ and } \lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{4}$$

Now, we prove that y_n is a g.m.s Cauchy sequence. If y_n is not a g.m.s Cauchy sequence, then there exists $\epsilon > 0$, for which we can find a sub sequence y_{n_k} and y_{m_k} of y_n with $m_k > n_k > k$ such that

$$d(y_{n_k}, y_{m_k}) \geq \epsilon \tag{5}$$

and corresponding to n_k, we can choose m_k in such a way that it is the smallest integer with $m_k > n_k$ satisfying (4) and

$$d(y_{n_k}, y_{m_{k-1}}) < \epsilon. \tag{6}$$

Applying (5) and (6) and the rectangular inequality, we get that

$$\epsilon \leq d(y_{n_k}, y_{m_k}) \leq d(y_{n_k}, y_{n_{k-1}}) + d(y_{n_{k-1}}, y_{m_{k-1}}) + d(y_{m_{k-1}}, y_{m_k}). \tag{7}$$

Applying (4) we get that

$$\epsilon \leq d(y_{n_{k-1}}, y_{m_{k-1}}) \tag{8}$$

and

$$d(y_{n_{k-1}}, y_{m_{k-1}}) \leq d(y_{n_{k-1}}, y_{n_k}) + d(y_{n_k}, y_{m_k}) + d(y_{m_k}, y_{m_{k-1}}).$$

By (4) and (6) we have that

$$d(y_{n_{k-1}}, y_{m_{k-1}}) \leq \epsilon. \tag{9}$$

From (8) and (9) we know that

$$d(y_{n_{k-1}}, y_{m_{k-1}}) = \epsilon. \tag{10}$$

Applying (4) and (10) in (7) we get that

$$\epsilon \leq d(y_{n_k}, y_{m_k}) \leq \epsilon, \text{ i.e., } d(y_{n_k}, y_{m_k}) = \epsilon. \tag{11}$$

Now,

$$\psi(d(y_{n_k}, y_{m_k})) = \psi(d(fx_{n_k}, gx_{m_k})) \leq \alpha(M(x, y)) - \beta(M(x, y)),$$

where,

$$\begin{aligned} M(x,y) &= \max\{d(Jx_{n_k}, hx_{m_k}), d(fx_{n_k}, Jx_{n_k}), d(gx_{m_k}, hx_{m_k}), d(fx_{n_k}, hx_{m_k})\} \\ &= \max\{d(y_{n_k}, y_{m_k}), d(y_{n_k}, y_{n_{k-1}}), d(y_{m_k}, y_{m_k}), d(y_{n_k}, y_{m_k})\} \\ &= \max\{\epsilon, 0, 0, \epsilon\} \ \text{(by (11) and (4))}. \end{aligned}$$

we therefore get that $\psi(\epsilon) \leq \alpha(\epsilon) - \beta(\epsilon)$ and $\epsilon = 0$ by lemma 2.1, which is a contradiction as we assume that $\epsilon > 0$. Then it follows that y_n is a g.m.s Cauchy sequence and hence y_n is convergent in the complete g.m.s space (X, d). Let $lim_{n\to\infty} y_n = z$, i.e.,

$$\lim_{n\to\infty} fx_n = lim_{n\to\infty} hx_{n+1} = lim_{n\to\infty} gx_{n+1} = lim_{n\to\infty} Jx_{n+2} = z. \tag{12}$$

Notice that J is a continuous function, we know that

$$\psi(d(fJx_n, gx_{n+1})) \leq \alpha(M(x,y)) - \beta(M(x,y)), \tag{13}$$

where,

$$M(x,y) = \max\{d(JJx_n, hx_{n+1}), d(Jfx_n, JJx_n), d(gx_{n+1}, hx_{n+1}), d(Jfx_n, hx_{n+1})\}$$

Let $n \to \infty$ in the above. We get that

$$\begin{aligned} M(x,y) &= \max\{d(Jz,z), d(Jz,Jz), d(z,z), d(Jz,z)\} \\ &= \max\{d(Jz,z), 0, 0, d(Jz,z)\} \\ &= d(Jz,z). \end{aligned}$$

Let $n \to \infty$ on both sides in (12). We know that $\psi(d(Jz,z)) \leq \alpha(d(Jz,z)) - \beta(d(Jz,z))$. By Lemma 2.1 we have $d(Jz,z) = 0$, i.e.,

$$Jz = z. \tag{14}$$

Now,

$$\psi(d(fz, gx_{n+1})) \leq \alpha(M(x,y)) - \beta(M(x,y)), \tag{15}$$

where,

$$M(x,y) = \max\{d(Jz, hx_{n+1}), d(fz, Jz), d(gx_{n+1}, hx_{n+1}), d(fz, hx_{n+1})\}$$

by (1) and (14). Now, taking $n \to \infty$ on above we get that

$$\begin{aligned} M(x,y) &= \max\{d(z,z), d(z,z), d(z,z), d(fz,z)\} \\ &= \max\{0,0,0,d(fz,z)\} = d(fz,z). \end{aligned}$$

Similarly, let $n \to \infty$ in (14) on both sides we get that

$$\psi(d(fz,z)) \leq \alpha(d(fz,z)) - \beta(d(fz,z)).$$

Applying Lemma 2.1 we get that $d(fz, z) = 0$, i.e.,

$$fz = z. \tag{16}$$

So, from (14) and (16) we know that

$$Jz = fz = z. \tag{17}$$

Similarly as h is continuous in X we can prove

$$hz = gz = z. \tag{18}$$

From (17) and (18) we get that

$$Jz = fz = hz = gz = z. \tag{19}$$

So z is a common fixed point of f, g, h and J.

Now, we prove that z is unique. If $w(\neq z)$ is another fixed point. Notice that

$$\psi(d(z, w) = \psi(d(fz, gw)) \leq \alpha(M(x, y)) - \beta M(x, y)), \tag{20}$$

where,

$$\begin{aligned} M(x, y) &= \max\{d(Jz, hw), d(fz, Jz), d(gw, hw), d(fz, hw)\} \\ &= \max\{d(z, w), d(z, z), d(w, w), d(z, w)\} = max(d(z, w), 0, 0, d(z, w)), \end{aligned}$$

which enables us to get that $\psi(d(z, w) \leq \alpha(d(z, w)) - \beta(d(z, w))$ from (20). By Lemma 2.1 we get $d(z, w) = 0$, i.e., $z = w$. Thus the fixed point z is unique. □

§4. Conclusion

The main result is an extension of the result [3] to the set of generalized metric space. This paper is also a generalization of many existing results in this literature.

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Z_k-Magic Labeling of Cycle of Graphs

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Abstract: For any non-trivial Abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling $f : E(G) \rightarrow A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An *A-magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k and these graphs are referred as *k-magic* graphs. In this paper we prove that the graphs such as cycle of generalized peterson, shell, wheel, closed helm, double wheel, triangular ladder, flower and lotus inside a circle are Z_k-magic graphs and also prove that if G is Z_k-magic graph and n is even then $C(n.G)$ is Z_k-magic.

Key Words: A-magic labeling, Z_k-magic labeling, Z_k-magic graph, cycle of graphs, Smarandachely A-magic.

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§1. Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to vertices or edges or both subject to certain conditions. A detailed survey was done by Gallian in [6]. If the labels of edges are distinct positive integers and for each vertex v the sum of the labels of all edges incident with v is the same for every vertex v in the given graph then the labeling is called a magic labeling. Sedláček [8] introduced the concept of A-magic graphs. A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all vertices. Low and Lee [7] examined the A-magic property of the resulting graph obtained from the product of two A-magic graphs. Shiu, Lam and Sun [9] proved that the product and composition of A-magic graphs were also A-magic.

For any non-trivial Abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling $f : E(G) \rightarrow A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. Otherwise, it is said to be *Smarandachely A-magic*, i.e., $|\{f^+(v), v \in V(G)\}| \geq 2$. An *A-magic* graph G is said to

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be Z_k-magic graph if the group A is Z_k, the group of integers modulo k. These Z_k-magic graphs are referred to as *k-magic graphs.* Shiu and Low [10] determined all positive integers k for which fans and wheels have a Z_k-magic labeling with a magic constant 0. Motivated by the concept of A-magic graph in [8] and the results in [7], [9] and [10] Jeyanthi and Jeya Daisy [1]-[5] proved that some standard graphs admit Z_k-magic labeling. Let G be a graph with n vertices $\{u_1, u_2, \ldots u_n\}$ and consider n copies of G as $G_1, G_2, \ldots G_n$ with vertex set $V(G_i) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq n\}$. The cycle of graph G is denoted by $C(n.G)$ is obtained by identifying the vertex u_1^j of G_j with u_i of G for $1 \leq i \leq n, 1 \leq j \leq n$. In this paper we study the Z_k-magic labeling of some cycle of graphs and also prove that if G is Z_k-magic graph and n is even then $C(n.G)$ is Z_k-magic. We use the following definitions in the subsequent section.

Definition 1.1 *A generalized peterson graph $P(n,m)$, $n \geq 3, 1 \leq m < \frac{n}{2}$ is a 3 regular graph with $2n$ vertices $\{u_1, u_2, \cdots u_n, v_1, v_2 \ldots v_n\}$ and edges $(u_iv_i), (u_iu_{i+1}), (v_iv_{i+m})$ for all $1 \leq i \leq n$, where the subscripts are taken modulo n.*

Definition 1.2 *A shell S_n is the graph obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the apex.*

Definition 1.3 *The wheel W_n is obtained by joining the vertices $v_1, v_2, \cdots v_n$ of a cycle C_n to an extra vertex v called the centre.*

Definition 1.4 *The closed helm CH_n is the graph obtained from a helm H_n by joining each pendent vertex to form a cycle.*

Definition 1.5 *A double wheel graph DW_n of size n can be composed of $2C_n + K_1$, that is it consists of two cycles of size n, where the vertices of the two cycles are all connected to a common hub.*

Definition 1.6 *The triangular ladder graph TL_n, $n \geq 2$ is obtained by completing the ladder $P_2 \times P_n$ by adding the edges $v_{1,j}v_{2,j+1}$ for $1 \leq j \leq n$. The vertex set of the ladder is $\{v_{1,j}, v_{2,j} : 1 \leq j \leq n\}$.*

Definition 1.7 *The flower Fl_n is the graph obtained from a helm H_n by joining each pendent vertex to the central vertex of the helm.*

Definition 1.8 *A lotus inside a circle LC_n is a graph obtained from the cycle $C_n : u_1, u_2, \ldots u_n, u_1$ and a star $K_{1,n}$ with the central vertex v_0 and the end vertices $v_1, v_2, \cdots, v_n$ by joining each u_i and $u_{i+1}(mod\ n)$.*

§2. Main Results

Theorem 2.1 *Let G be a Z_k-magic graph with magic constant b then $C(n.G)$ is Z_k-magic if n is even.*

Proof For any integer $b \in Z_k$. Let $v_1, v_2, \cdots v_n$ be the vertices of the cycle C_n. Let G be any Z_k-magic graph with magic constant b. Therefore $f^+(v) \equiv\ b(mod\ k)$ for all $v \in V(G)$.

For any integer $a \in Z_k - \{0\}$, define the edge labeling $g : E(C(n.G)) \rightarrow Z_k - \{0\}$ as follows:

$$g(v_i v_{i+1}) = \begin{cases} a & \text{for } i \text{ is odd,} \\ k-a & \text{for } i \text{ is even,} \end{cases}$$

and $g(e) = f(e)$ for other $e \in E(C(n.G))$. Then the induced vertex labeling $g^+ : V(C(n.G)) \rightarrow Z_k$ is $g^+(v) \equiv\ b(mod\ k)$ for all $v \in V(C(n.G))$. Hence g^+ is constant and it is equal to $b(mod\ k)$. Notice that $C(n.G)$ admits Z_k-magic labeling when n is even, then it is therefore a Z_k-magic graph. □

Theorem 2.2 *Let G be a Z_k-magic graph with magic constant b then $C(n.G)$ is Z_k-magic if k is even.*

Proof For any integer $b \in Z_k$. Let $v_1, v_2, \cdots v_n$ be the vertices of the cycle C_n. Let G be any Z_k-magic graph with magic constant b. Therefore $f^+(v) \equiv\ 0(mod\ k)$ for all $v \in V(G)$.

For any integer $a \in Z_k - \{0\}$, define the edge labeling $g : E(C(n.G)) \rightarrow Z_k - \{0\}$ to be $g(v_i v_{i+1}) = \frac{k}{2}$ for $1 \leq i \leq n-1$, $g(v_n v_1) = \frac{k}{2}$, $g(e) = f(e)$ for other $e \in E(C(n.G))$.

Then the induced vertex labeling $g^+ : V(C(n.G)) \rightarrow Z_k$ is $g^+(v) \equiv\ b(mod\ k)$ for all $v \in V(C(n.G))$. Hence g^+ is constant and it is equal to $b(mod\ k)$. Since $C(n.G)$ admits Z_k-magic labeling when k is even, then it is a Z_k-magic graph. □

Theorem 2.3 *The graph $C(n.C_r)$ is Z_k-magic except r is even, n is odd and k is odd.*

Proof Let the vertex set and the edge set of $C(n.C_r)$ be $V(C(n.C_r)) = \{v_i^j : 1 \leq i \leq r,\ 1 \leq j \leq n\}$ and $E(C(n.C_r)) = \{v_i^j v_{i+1}^j : 1 \leq i \leq r-1,\ i \leq j \leq n\} \bigcup \{v_r^j v_1^j : 1 \leq j \leq n\} \bigcup \{v_1^j v_1^{j+1} : 1 \leq j \leq n-1\} \bigcup \{v_1^n v_1^1\}$.

Case 1. r is odd.

For any integer $a \in Z_k - \{0\}$, define the edge labeling $f : E(C(n.C_r)) \rightarrow Z_k - \{0\}$ as follows:

$$f(v_i^j v_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd, } 1 \leq j \leq n, \\ a & \text{for } i \text{ is even, } 1 \leq j \leq n, \end{cases}$$

$f(v_1^j v_1^{j+1}) = a$ for $1 \leq j \leq n-1$,

$f(v_1^n v_1^1) = a$.

Then the induced vertex labeling $f^+ : V(C(n.C_r)) \rightarrow Z_k$ is $f^+(v) \equiv\ 0(mod\ k)$ for all $v \in V(C(n.C_r))$.

Case 2. r is even.

Subcase 2.1 n is even.

The cycle C_r is Z_k-magic with magic constant zero when r is even. Therefore by theorem 2.1 it is Z_k-magic.

Subcase 2.2 n is odd and k is even.

By Theorem 2.2 it is Z_k-magic.

Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.C_r)$ admits Z_k-magic labeling, then it is a Z_k-magic graph. □

The example for Z_{15}-magic labeling of $C(5.C_7)$ is shown in Figure 1.

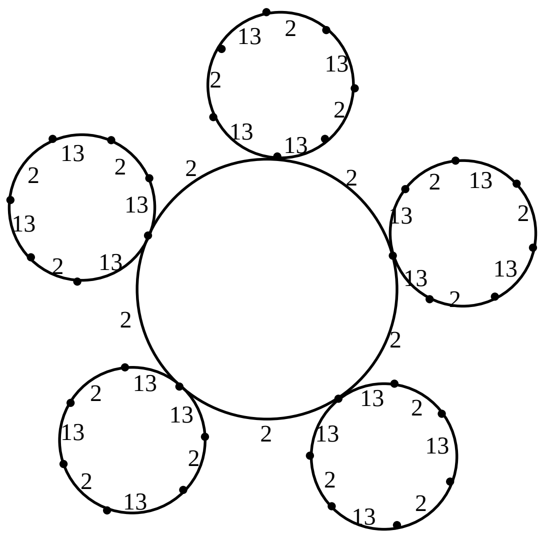

Figure 1 Z_{15}-magic labeling of $C(5.C_7)$

Conjecture 2.4 *The graph $C(n.C_r)$ is not Z_k-magic when r is even, n is odd and k is odd.*

Observation 2.1 The graph $C(n.C_{n_1}, C_{n_2} \cdots, C_{n_l})$ is Z_k-magic when $n_1, n_2, \cdots n_l$ are odd.

Theorem 2.5 *The cycle of generalized peterson graph $C(n.P(r,m))$ is Z_k-magic except r is even, n is odd and k is odd.*

Proof Let the vertex set and the edge set of $C(n.P(r,m))$ be respectively $V(C(n.P(r,m))) = \{u_i^j, v_i^j : 1 \le i \le r, 1 \le j \le n\}$ and $E(C(n.P(r,m))) = \{u_i^j v_i^j : 1 \le i \le r, 1 \le j \le n\} \bigcup \{u_i^j u_{i+1}^j : 1 \le i \le r-1, 1 \le j \le n\} \bigcup \{u_r^j u_1^j : 1 \le j \le n\} \bigcup \{v_i^j v_{i+m}^j : 1 \le i \le r, 1 \le j \le n\}$ where the subscripts are taken modulo r.

Case 1. r is odd.

For any integer a such that $k > 3a$, define the edge labeling $f : E(C(n.P(r,m))) \to Z_k - \{0\}$ as follows:

$$f(v_i^j v_{i+m}^j) = a \text{ for } 1 \le i \le r,\ 1 \le j \le n,$$
$$f(u_i^j v_i^j) = k - 2a \text{ for } 1 \le i \le r,\ 1 \le j \le n,$$
$$f(u_i^j u_{i+1}^j) = \begin{cases} 3a & \text{for } i \text{ is odd},\ 1 \le j \le n, \\ k - a & \text{for } i \text{ is even},\ 1 \le j \le n, \end{cases}$$

$f(u_1^j u_1^{j+1}) = k - 2a$ for $1 \le j \le n-1$ and $f(u_1^n u_1^1) = k - 2a$.

Then the induced vertex labeling $f^+ : V(C(n.P(r,m))) \to Z_k$ is $f^+(v) \equiv\ 0(mod\ k)$ for all $v \in V(C(n.P(r,m)))$.

Case 2. r is even.

Subcase 2.1 n is even.

The graph $P(r,m)$ is Z_k-magic with magic constant zero. Therefore by theorem 2.1 it is Z_k-magic.

Subcase 2.2 n is odd and k is even.

By theorem 2.2 it is Z_k-magic in this case.

Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.P(r,m))$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example for Z_5-magic labeling of $C(5.P(5,2))$ is shown in Figure 2.

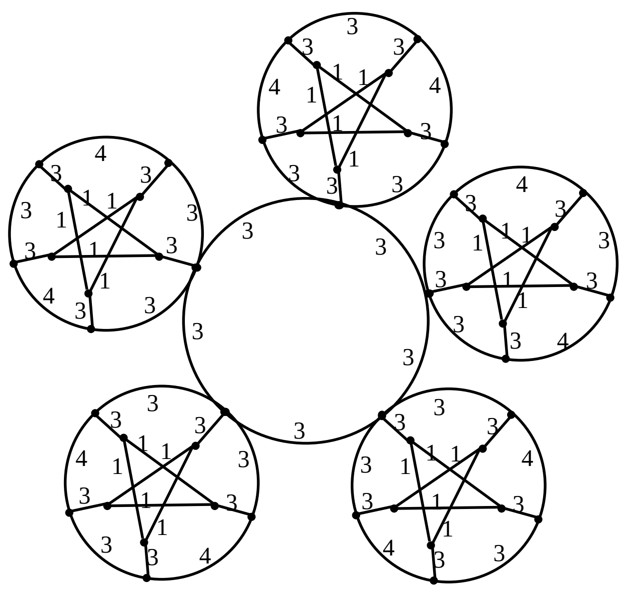

Figure 2 Z_5-magic labeling of $C(5.P(5,2))$

Conjecture 2.6 *The cycle of generalized peterson graph $C(n.P(r,m))$ is not Z_k-magic when r is even, n is odd and k is odd.*

Theorem 2.7 *The cycle of shell graph $C(n.S_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.S_r)$ be respectively $V(C(n.S_r)) = \{v_i^j : 1 \le i \le r,\ 1 \le j \le n\}$ and $E(C(n.S_r)) = \{v_i^j v_{i+1}^j : 1 \le i \le r-1,\ 1 \le j \le n\} \bigcup \{v_r^j v_1^j : 1 \le j \le n\} \bigcup \{v_1^j v_{i+2}^j : 1 \le i \le r-3,\ 1 \le j \le n\} \bigcup \{v_1^j v_1^{j+1} : 1 \le j \le n-1\} \bigcup \{v_1^n v_1^1\}$. For any integer a such that $k > (r-2)a$, define the edge labeling

$f : E(C(n.S_r)) \to Z_k - \{0\}$ as follows:

$f(v_1^j v_{i+2}^j) = 2a$ for $1 \le i \le r-3,\ 1 \le j \le n$,
$f(v_1^j v_2^j) = f(v_r^j v_1^j) = a$ for $1 \le j \le n$,
$f(v_i^j v_{i+1}^j) = k - a$ for $2 \le i \le r-1, 1 \le j \le n$,
$f(u_1^j u_1^{j+1}) = k - (r-2)a$ for $1 \le j \le n-1$,
$f(u_1^n u_1^1) = k - (r-2)a$.

Then the induced vertex labeling $f^+ : V(C(n.S_r)) \to Z_k$ is $f^+(v) \equiv\ 0(mod\ k)$ for all $v \in V(C(n.S_r))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.S_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example for Z_7-magic labeling of $C(5.S_5)$ is shown in Figure 3.

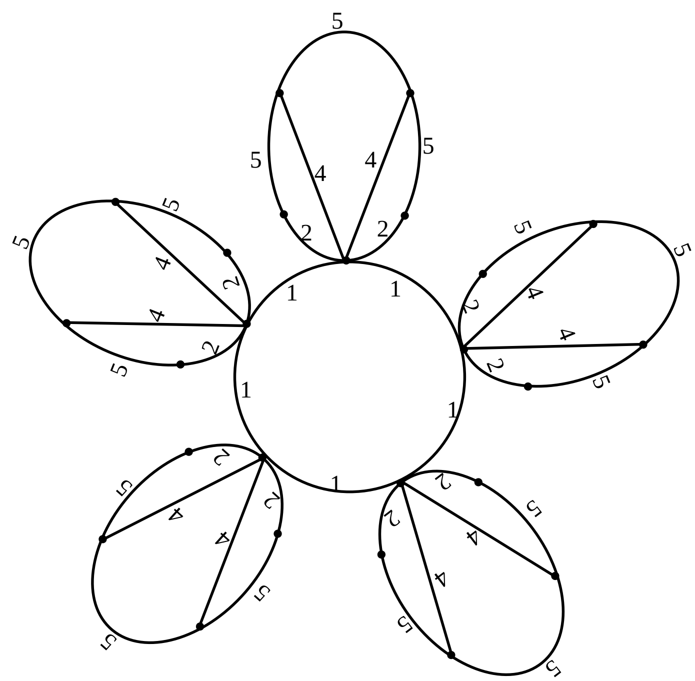

Figure 3 Z_7-magic labeling of $C(5.S_5)$

Theorem 2.8 *The cycle of wheel graph $C(n.W_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.W_r)$ be respectively $V(C(n.W_r)) = \{w_j, u_i^j :\ 1 \le i \le r,\ 1 \le j \le n\}$ and $E(C(n.W_r)) = \{u_i^j u_{i+1}^j :\ 1 \le i \le r-1,\ 1 \le j \le n\} \bigcup \{u_r^j u_1^j :\ 1 \le j \le n\} \bigcup \{w_j u_i^j :\ 1 \le i \le r,\ 1 \le j \le n\} \bigcup \{u_1^j u_1^{j+1} :\ 1 \le j \le n-1\} \bigcup \{u_1^n u_1^1\}$. For any integer a such that $k > 2(r-1)a$, define the edge labeling $f : E(C(n.W_r)) \to Z_k - \{0\}$ as follows:

$f(w_j u_i^j) = 2a$ for $2 \le i \le r,\ 1 \le j \le n$,
$f(w_j u_1^j) = k - 2(r-1)a$ for $1 \le j \le n$,
$f(u_i^j u_{i+1}^j) = k - a$ for $1 \le i \le r-1, 1 \le j \le n$,

$f(u_r^j u_1^j) = k - a$ for $1 \leq j \leq n$,
$f(u_1^j u_1^{j+1}) = ra$ for $1 \leq j \leq n-1$,
$f(u_1^n u_1^1) = ra$.

Then the induced vertex labeling $f^+ : V(C(n.W_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.W_r))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.W_r)$ admits Z_k-magic labeling , the cycle of wheel graph $C(n.W_r)$ is a Z_k-magic graph. □

The example of Z_{13}-magic labeling of $C(5.W_7)$ is shown in Figure 4.

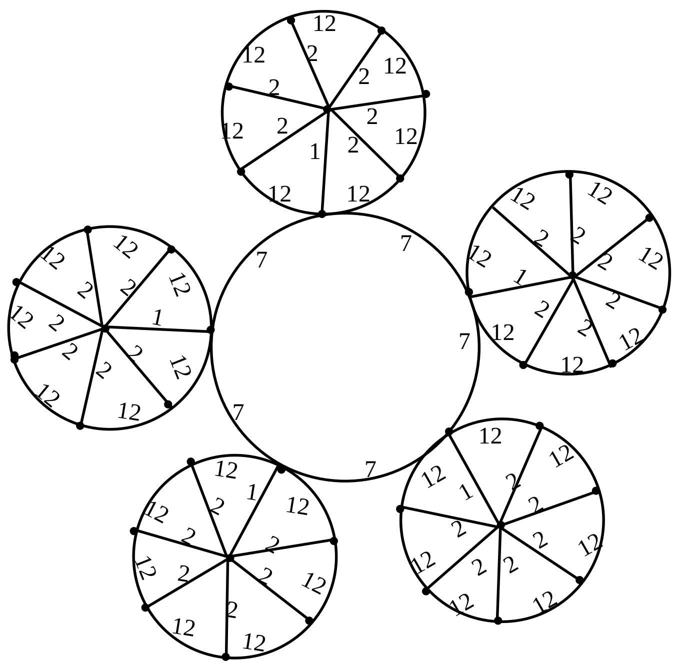

Figure 4 Z_{13}-magic labeling of $C(5.W_7)$

Theorem 2.9 *The cycle of closed helm graph $C(n.CH_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.CH_r)$ be respectively $V(C(n.CH_r)) = \{w_j, u_i^j, x_i^j : 1 \leq i \leq r,\ 1 \leq j \leq n\}$ and $E(C(n.CH_r)) = \{u_i^j u_{i+1}^j : 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{u_r^j u_1^j : 1 \leq j \leq n\} \bigcup \{x_i^j x_{i+1}^j : 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{x_r^j x_1^j : 1 \leq j \leq n\} \bigcup \{w_j x_i^j : 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{x_i^j u_i^j : 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\} \bigcup \{u_1^n u_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > (r+1)a$, define the edge labeling $f : E(C(n.CH_r)) \to Z_k - \{0\}$ as follows:

$$f(u_i^j u_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd, } 1 \leq j \leq n, \\ 2a & \text{for } i \text{ is even, } 1 \leq j \leq n, \end{cases}$$

$$f(x_i^j x_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd}, \ 1 \le j \le n, \\ a & \text{for } i \text{ is even}, \ 1 \le j \le n, \end{cases}$$

$f(w_j x_1^j) = k - (r-1)a$ for $1 \le j \le n$,
$f(w_j x_i^j) = a$ for $2 \le i \le r, 1 \le j \le n$,
$f(x_1^j u_1^j) = (r+1)a$ for $1 \le j \le n$,
$f(x_i^j u_i^j) = k - a$ for $2 \le i \le r, 1 \le j \le n$,
$f(u_1^j u_1^{j+1}) = k - \frac{(r-1)a}{2}$ for $1 \le j \le n-1$,
$f(u_1^n u_1^1) = k - \frac{(r-1)a}{2}$.

Then the induced vertex labeling $f^+ : V(C(n.CH_r)) \to Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(C(n.CH_r))$.

Case 2. r is even.

For any integer a such that $k > ra$, define the edge labeling $f : E(C(n.CH_r)) \to Z_k - \{0\}$ as follows:

$$f(u_i^j u_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd}, \ 1 \le j \le n, \\ 2a & \text{for } i \text{ is even}, \ 1 \le j \le n, \end{cases}$$

$$f(x_i^j x_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd}, \ 1 \le j \le n, \\ a & \text{for } i \text{ is even}, \ 1 \le j \le n, \end{cases}$$

$f(w_j x_1^j) = k - (r-1)a$ for $1 \le j \le n$,
$f(w_j x_i^j) = a$ for $2 \le i \le r, 1 \le j \le n$,
$f(x_1^j u_1^j) = (r-1)a$ for $1 \le j \le n$,
$f(x_i^j u_i^j) = k - a$ for $2 \le i \le r, 1 \le j \le n$,
$f(u_1^j u_1^{j+1}) = k - \frac{ra}{2}$ for $1 \le j \le n-1$,
$f(u_1^n u_1^1) = k - \frac{ra}{2}$.

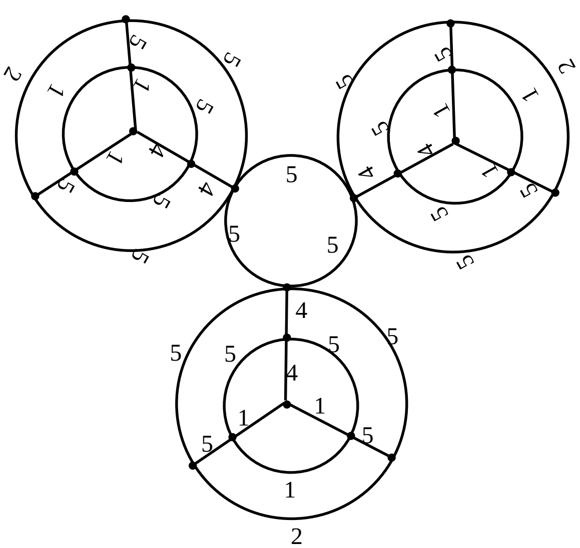

Figure 5 Z_6-magic labeling of $C(3.CH_3)$

Then the induced vertex labeling $f^+ : V(C(n.CH_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.CH_r))$.

Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.CH_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_6-magic labeling of $C(3.CH_3)$ is shown in Figure 5.

Theorem 2.10 *The cycle of double wheel graph $C(n.DW_r)$ is Z_k-magic except r is even, n is odd and k is odd.*

Proof Let the vertex set and the edge set of $C(n.DW_r)$ be respectively $V(C(n.DW_r)) = \{v_j, u_i^j :\ 1 \le i \le r,\ 1 \le j \le n\}$ and $E(C(n.DW_r)) = \{v_iv_i^j, v_iu_i^j :\ 1 \le i \le r,\ 1 \le j \le n\} \bigcup \{v_i^jv_{i+1}^j :\ 1 \le i \le r-1,\ 1 \le j \le n\} \bigcup \{v_r^jv_1^j :\ 1 \le j \le n\} \bigcup \{u_i^ju_{i+1}^j :\ 1 \le i \le r-1,\ 1 \le j \le n\} \bigcup \{u_r^ju_1^j :\ 1 \le j \le n\} \bigcup \{u_1^ju_1^{j+1} :\ 1 \le j \le n-1\} \bigcup \{u_1^nu_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > 3a$, define the edge labeling $f : E(C(n.DW_r)) \to Z_k - \{0\}$ as follows:

$f(v_iv_i^j) = 2a$ for $1 \le i \le r,\ 1 \le j \le n$,

$f(v_iu_i^j) = k - 2a$ for $1 \le i \le r,\ 1 \le j \le n$,

$f(v_i^jv_{i+1}^j) = k - a$ for $1 \le i \le r-1, 1 \le j \le n$,

$$f(u_i^ju_{i+1}^j) = \begin{cases} 3a & \text{for } i \text{ is odd},\ 1 \le j \le n, \\ k-a & \text{for } i \text{ is even},\ 1 \le j \le n, \end{cases}$$

$f(v_r^jv_1^j) = k - a$ for $1 \le j \le n$,

$f(u_1^ju_1^{j+1}) = k - 2a$ for $1 \le j \le n-1$,

$f(u_1^nu_1^1) = k - 2a$.

Case 2. r is even.

Subcase 2.1 n is even.

The graph DW_r is Z_k-magic with magic constant zero. Therefore by theorem 2.1 it is Z_k-magic.

Subcase 2.2 n is odd and k is even.

By Theorem 2.2 it is Z_k-magic in this case.

Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.DW_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_9-magic labeling of $C(5.DW_3)$ is shown in Figure 6.

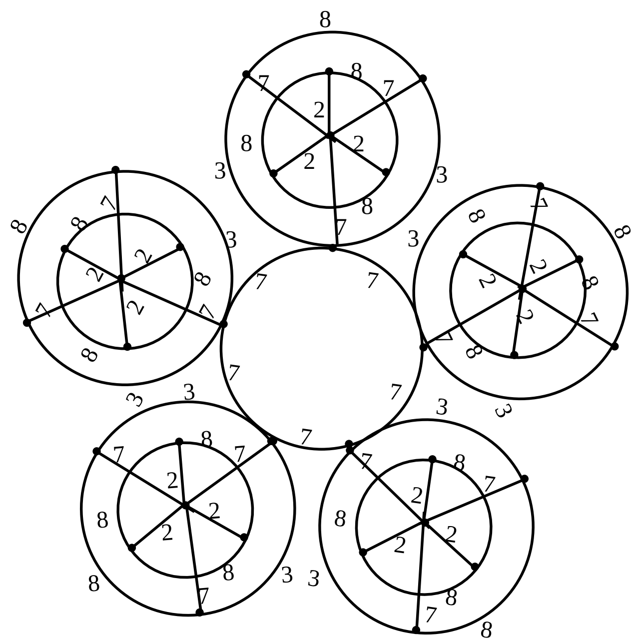

Figure 6 Z_9-magic labeling of $C(5.DW_3)$

Conjecture 2.11 *The cycle of double wheel graph $C(n.DW_r)$ is not Z_k-magic when r is even, n is odd and k is odd.*

Obsevation 2.2 The graph $C(n.DW_{n_1}, DW_{n_2} \ldots, DW_{n_l})$ is Z_k-magic when $n_1, n_2, \ldots n_l$ are odd.

Theorem 2.12 *The cycle of triangular ladder graph $C(n.TL_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.TL_r)$ be respectively $V(C(n.TL_r)) = \{u_i^j, v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(C(n.TL_r)) = \{u_i^j u_{i+1}^j, v_i^j v_{i+1}^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \bigcup \{u_i^j v_{i+1}^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \bigcup \{u_i^j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \bigcup \{v_1^j v_1^{j+1} : 1 \leq j \leq n-1\} \cup \{v_1^n v_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > 2a$, define the edge labeling $f : E(C(n.TL_r)) \to Z_k - \{0\}$ as follows:

$$f(u_i^j u_{i+1}^j) = \begin{cases} 2a & \text{for } i \text{ is odd}, \ 1 \leq j \leq n, \\ k-a & \text{for } i \text{ is even}, \ 1 \leq j \leq n, \end{cases}$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} k-a & \text{for } i \text{ is odd}, \ 1 \leq j \leq n, \\ a & \text{for } i \text{ is even}, \ 1 \leq j \leq n, \end{cases}$$

$f(u_1^j v_1^j) = k-a$ for $1 \leq j \leq n$,

$f(u_i^j v_i^j) = a$ for $2 \leq i \leq r, 1 \leq j \leq n$,

$f(u_1^j v_2^j) = k - a$ for $1 \leq j \leq n$,

$f(u_i^j v_{i+1}^j) = k - 2a$ for $2 \leq i \leq r-1, 1 \leq j \leq n$,

$f(v_1^j v_1^{j+1}) = a$ for $1 \leq j \leq n-1$,

$f(v_1^n v_1^1) = a$.

Then the induced vertex labeling $f^+ : V(C(n.TL_r)) \rightarrow Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(C(n.TL_r))$.

Case 2. r is even.

For any integer a such that $k > 2a$, define the edge labeling $f : E(C(n.TL_r)) \rightarrow Z_k - \{0\}$ as follows:

$$f(u_i^j u_{i+1}^j) = \begin{cases} 2a & \text{for } i \text{ is odd, } 1 \leq j \leq n, \\ k - 2a, & \text{for } i \text{ is even, } 1 \leq j \leq n, \end{cases}$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} k - a & \text{for } i \text{ is odd}, i \neq (r-1)\ 1 \leq j \leq n, \\ a & \text{for } i \text{ is even, } 1 \leq j \leq n, \end{cases}$$

$f(v_{r-1}^j v_r^j) = a$ for $1 \leq j \leq n$,

$f(u_1^j v_1^j) = k - a$ for $1 \leq j \leq n$,

$f(u_i^j v_i^j) = a$ for $2 \leq i \leq r-2, 1 \leq j \leq n$,

$f(u_{r-1}^j v_{r-1}^j) = k - a$ for $1 \leq j \leq n$,

$f(u_r^j v_r^j) = a$ for $1 \leq j \leq n$,

$f(u_i^j v_{i+1}^j) = k - a$ for $1 \leq i \leq r-2, 1 \leq j \leq n$,

$f(u_{r-1}^j v_r^j) = a$ for $1 \leq j \leq n$,

$f(v_1^j v_1^{j+1}) = a$ for $1 \leq j \leq n-1$,

$f(v_1^n v_1^1) = a$.

Then the induced vertex labeling $f^+ : V(C(n.TL_r)) \rightarrow Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(C(n.TL_r))$. Hence f^+ is constant and it is equal to $0 (mod\ k)$. Since $C(n.TL_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_{10}-magic labeling of $C(5.TL_5)$ is shown in Figure 7.

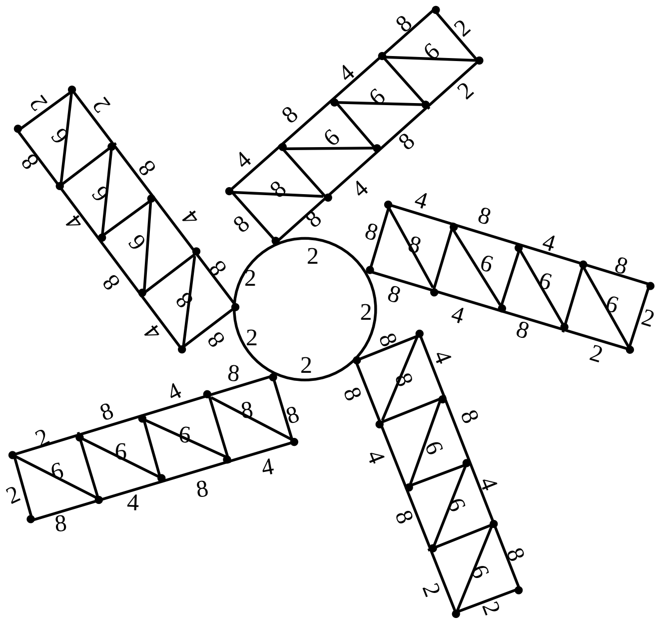

Figure 7 Z_{10}-magic labeling of $C(5.TL_5)$

Theorem 2.13 *The cycle of flower graph $C(n.Fl_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.Fl_r)$ be respectively $V(C(n.Fl_r)) = \{v_j, v_i^j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(C(n.Fl_r)) = \{v_j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \bigcup \{v_i^j u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \bigcup \{v_j u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \bigcup \{v_i^j v_{i+1}^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \bigcup \{v_r^j v_1^j : 1 \leq j \leq n\} \bigcup \{v_1^j v_1^{j+1} : 1 \leq j \leq n-1\} \bigcup \{v_1^n v_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > 3a$, define the edge labeling $f : E(C(n.Fl_r)) \rightarrow Z_k - \{0\}$ as follows:

$f(v_j v_i^j) = a$ for $1 \leq i \leq r,\ 1 \leq j \leq n$,
$f(v_i^j u_i^j) = a$ for $1 \leq i \leq r,\ 1 \leq j \leq n$,
$f(u_i^j v_j) = k - a$ for $1 \leq i \leq r,\ 1 \leq j \leq n$,

$$f(v_i^j v_{i+1}^j) = \begin{cases} a & \text{for } i \text{ is odd},\ 1 \leq j \leq n, \\ k - 3a & \text{for } i \text{ is even},\ 1 \leq j \leq n, \end{cases}$$

$f(v_1^j v_1^{j+1}) = k - 2a$ for $1 \leq j \leq n-1$,
$f(v_1^n v_1^1) = k - 2a$.

Then the induced vertex labeling $f^+ : V(C(n.Fl_r)) \rightarrow Z_k$ is $f^+(v) \equiv 0 (mod\ k)$ for all $v \in V(C(n.Fl_r))$.

Case 2. r is even.

For any integer a such that $k > 2a$, define the edge labeling $f : E(C(n.Fl_r)) \rightarrow Z_k - \{0\}$ as follows:

$f(v_j v_1^j) = 2a$ for $1 \leq j \leq n$,
$f(v_j v_i^j) = a$ for $2 \leq i \leq r,\ 1 \leq j \leq n$,
$f(v_1^j u_1^j) = 2a$ for $1 \leq j \leq n$,
$f(v_i^j u_i^j) = a$ for $2 \leq i \leq r,\ 1 \leq j \leq n$,
$f(u_1^j v_j) = k - 2a$ for $1 \leq j \leq n$,
$f(u_i^j v_j) = k - a$ for $2 \leq i \leq r,\ 1 \leq j \leq n$,
$f(v_i^j v_{i+1}^j) = k - a$ for $1 \leq i \leq r-1,\ 1 \leq j \leq n$,
$f(v_n^j v_1^j) = k - a$,
$f(v_1^j v_1^{j+1}) = k - a$ for $1 \leq j \leq n-1$,
$f(v_1^n v_1^1) = k - a$.

Then the induced vertex labeling $f^+ : V(C(n.Fl_r)) \to Z_k$ is $f^+(v) \equiv\ 0(mod\ k)$ for all $v \in V(C(n.Fl_r))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.Fl_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_5-magic labeling of $C(3.Fl_3)$ is shown in Figure 8.

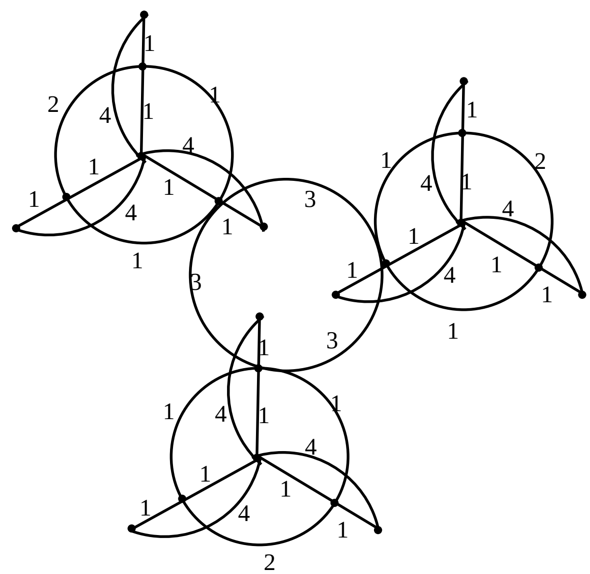

Figure 8 Z_5-magic labeling of $C(3.Fl_3)$

Theorem 2.14 *The cycle of lotus inside a circle graph $C(n.LC_r)$ is Z_k-magic.*

Proof Let the vertex set and the edge set of $C(n.LC_r)$ be respectively $V(C(n.LC_r)) = \{v_j, v_i^j, u_i^j\ :\ 1 \leq i \leq r,\ 1 \leq j \leq n\}$ and $E(C(n.LC_r)) = \{v_j v_i^j\ :\ 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{v_i^j u_i^j\ :\ 1 \leq i \leq r,\ 1 \leq j \leq n\} \bigcup \{u_i^j v_{i+1}^j\ :\ 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{u_r^j v_1^j\ :\ 1 \leq j \leq n\} \bigcup \{u_i^j u_{i+1}^j\ :\ 1 \leq i \leq r-1,\ 1 \leq j \leq n\} \bigcup \{u_r^j u_1^j\ :\ 1 \leq j \leq n\} \bigcup \{u_1^j u_1^{j+1}\ :\ 1 \leq j \leq n-1\} \bigcup \{u_1^n u_1^1\}$.

Case 1. r is odd.

For any integer a such that $k > (r-1)a$, define the edge labeling $f : E(C(n.LC_r)) \to Z_k - \{0\}$ as follows:

$f(v_jv_1^j) = k-(n-1)a$ for $1 \leq j \leq n$,

$f(v_jv_i^j) = a$ for $2 \leq i \leq r,\ 1 \leq j \leq n$,

$f(v_1^ju_1^j) = (r-2)a$ for $1 \leq j \leq n$,

$f(v_i^ju_i^j) = k-2a$ for $2 \leq i \leq r, 1 \leq j \leq n$,

$f(u_i^jv_{i+1}^j) = a$ for $1 \leq i \leq r-1, 1 \leq j \leq n$,

$f(u_r^jv_1^j) = a$ for $1 \leq j \leq n$,

$$f(u_i^ju_{i+1}^j) = \begin{cases} 2a & \text{for } i \text{ is odd},\ 1 \leq j \leq n, \\ k-a & \text{for } i \text{ is even},\ 1 \leq j \leq n, \end{cases}$$

$f(u_1^ju_1^{j+1}) = k - \frac{(r+3)a}{2}$ for $1 \leq j \leq n-1$,

$f(u_1^nu_1^1) = k - \frac{(r+3)a}{2}$.

Then the induced vertex labeling $f^+ : V(C(n.LC_r)) \to Z_k$ is
$f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.LC_r))$.

Case 2. r is even.

For any integer a such that $k > (r-1)a$, define the edge labeling $f : E(C(n.LC_r)) \to Z_k - \{0\}$ as follows:

$f(v_jv_1^j) = k-(n-1)a$ for $1 \leq j \leq n$,

$f(v_jv_i^j) = a$ for $2 \leq i \leq r,\ 1 \leq j \leq n$,

$f(v_1^ju_1^j) = (r-2)a$ for $1 \leq j \leq n$,

$f(v_i^ju_i^j) = k-2a$ for $2 \leq i \leq r, 1 \leq j \leq n$,

$f(u_i^jv_{i+1}^j) = a$ for $1 \leq i \leq r-1, 1 \leq j \leq n$,

$f(u_r^jv_1^j) = a$ for $1 \leq j \leq n$,

$$f(u_i^ju_{i+1}^j) = \begin{cases} 2a & \text{for } i \text{ is odd},\ 1 \leq j \leq n, \\ k-a & \text{for } i \text{ is even},\ 1 \leq j \leq n, \end{cases}$$

$f(u_1^ju_1^{j+1}) = k - \frac{ra}{2}$ for $1 \leq j \leq n-1$,

$f(u_1^nu_1^1) = k - \frac{ra}{2}$.

Then the induced vertex labeling $f^+ : V(C(n.LC_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C(n.LC_r))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $C(n.LC_r)$ admits Z_k-magic labeling , then it is a Z_k-magic graph. □

The example of Z_8-magic labeling of $C(5.LC_5)$ is shown in Figure 9.

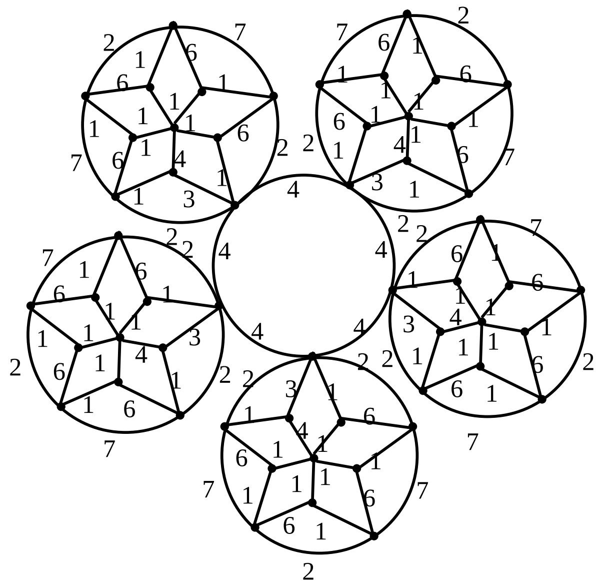

Figure 9 Z_8-magic labeling of $C(5.LC_5)$

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Topological Efficiency Index of Some Composite Graphs

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Abstract: In this paper, we study the behavior of a new graph invariants ρ for some composite graphs such as splice, link and rooted product of two given graphs.

Key Words: Wiener index, topological efficiency index, composite graph.

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§1. Introduction

Throughout this paper, we consider only simple connected graphs. We use $d_G(v)$ to denote the degree of a vertex v in G. Let $d_G(u,v)$ denote the distance between two vertices u and v in G and let $w_v(G)$ denote the sum of all distance of vertices of G from v, that is, $w_v(G) = \sum\limits_{u\in V(G)} d_G(v,u)$ with $\underline{w}(G) = \min \{w_v(G) : v \in V(G)\}$.

The topological indices (also known as the molecular descriptors) had been received much attention in that past decades, and they have been found to be useful in structure-activity relationships (SAR) and pharmaceutical drug design in organic chemistry see, [2, 3, 7]. Many researchers also were devoted to study their graphical properties. Indeed, the topological index of a graph G can be viewed as a graph invariant under the isomorphism of graphs, that is, for some topological index $TI, TI(G) = TI(H)$ if $G \cong H$.

One of the most thoroughly studied topological indices was the Wiener index which was proposed by Wiener in 1947 [8]. This index has been shown to posses close relation with the graph distance, which is an important concept in pure graph theory. It is also well correlated with many physical and chemical properties of a variety of classes of chemical compounds. For more details, see [1, 4, 5, 6, 9].

The Wiener index of a graph G, denoted by $W(G)$, is defined as

$$W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v) = \frac{1}{2}\sum_{v\in V(G)} w_v(G).$$

Hossein-zadeh et al. [12] proposed a new graph descriptor ρ, called topological efficiency

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index based on minimal vertex contribution $\underline{w}$ defined for a connected graph G as

$$\rho(G) = \frac{2W(G)}{|V(G)|\,\underline{w}(G)}.$$

The topological efficiency index of C_{66} fullerence graph is computed in [13]. In [12], the topological efficiency of some product graphs such as Cartesian product, join, corona product, Hierarchical product, composition are given. In this sequence, here we study the behavior of a new graph invariants ρ for some composite graphs such as splice, link and rooted product of two given graphs are obtained.

§2. Splice Graph

For given vertices $x \in V(G_1)$ and $y \in V(G_2)$ *the splice* of G_1 and G_2 by vertices x and y, which is denoted by $S(G_1, G_2)(x, y)$, is defined by identifying the vertices x and y in the union of G_1 and G_2, see Figure 1. The various topological indices of splice graph are studied in [10, 14].

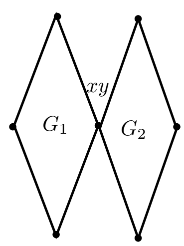

Figure 1 The splice of C_4 and C_4

The proof of the following lemma is easily followed from the structure of splice of graphs G_1 and G_2.

Lemma 2.1 *Let G_1 andG_2 are two connected graphs with $x \in V(G_1)$ and $y \in V(G_2)$. Then*

(i) $|V(S(G_1, G_2)(x, y))| = |V(G_1)| + |V(G_2)| - 1$ *and* $|E(S(G_1, G_2)(x, y))| = |E(G_1)| + |E(G_2)|$*;*

(ii) If $V(G_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ *and* $V(G_2) = \{v_1, v_2, \cdots, v_{n_2}\}$*, then*

$$\begin{aligned} d_{S(G_1,G_2)(x,y)}(u_i, u_j) &= d_{G_1}(u_i, u_j), \\ d_{S(G_1,G_2)(x,y)}(u_i, v_j) &= d_{G_1}(u_i, x) + d_{G_2}(v_j, y), \\ d_{S(G_2,G_2)(x,y)}(v_i, v_j) &= d_{G_2}(v_i, v_j). \end{aligned}$$

Theorem 2.2 *Let G_1 and G_2 be a connected graph with n_1 and n_2 vertices. For vertices $x \in V(G_1)$ and $y \in V(G_2)$, consider $S(G_1, G_2)(x, y)$. Then*

$$\underline{w}(S(G_1, G_2)(x, y)) = n_1 w_{u_k}(G_2) + n_2 w_{u_k}(G_1).$$

Proof Let $V(G_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \cdots, v_{n_2}\}$. For our convenience we denote $S(G_1, G_2)(x, y)$ by G and v_{xy} by the identifying the vertices x and y.

Now we compute the sum of distances between a fixed vertex to all other vertices of G.

Case 1. Let $u_k \neq v_{xy} \in V(G_1)$. Then by Lemma 2.1, for a vertex $u_i \in V(G_1)$, $d_G(u_k, u_i) = d_{G_1}(u_k, u_i)$ and for a vertex $v_j \in V(G_2)$, we have, $d_G(u_k, v_j) = d_{G_1}(u_k, x) + d_{G_2}(y, v_j)$. Thus

$$\begin{aligned} w_{u_k}(G) &= \sum_{i=1}^{n_1-1} d_G(u_k, u_i) + \sum_{j=1}^{n_2} d_G(u_k, v_j) \\ &= w_{u_k}(G_1) + w_y(G_2) + (n_2 - 1) d_{G_1}(u_k, x). \end{aligned}$$

Case 2. Let $v_k \neq v_{xy} \in V(G_2)$. Then by Lemma 2.1, for a vertex $v_j \in V(G_2)$, $d_G(v_k, v_j) = d_{G_2}(v_k, v_j)$ and for a vertex $u_i \in V(G_1)$, we get $d_G(v_k, u_i) = d_{G_2}(v_k, y) + d_{G_1}(x, u_i)$. Thus

$$\begin{aligned} w_{v_k}(G) &= \sum_{j=1}^{n_2-1} d_G(v_k, v_j) + \sum_{i=1}^{n_1} d_G(v_k, u_i) \\ &= w_{v_k}(G_2) + w_x(G_1) + (n_1 - 1) d_{G_2}(v_k, y). \end{aligned}$$

Case 3. Let $v_{xy} \in V(G)$. Then by Lemma 2.1, for a vertices $u_i \in V(G_1)$ and $v_j \in V(G_2)$, $d_G(v_{xy}, u_i) = d_{G_1}(x, u_i)$ and $d_G(v_{xy}, v_j) = d_{G_2}(y, v_j)$ Therefore

$$w_{v_{xy}}(G) = w_x(G_1) + w_y(G_2).$$

From Cases 1 and 3, we know that

$$w_{u_k} - w_{v_{xy}} = w_{u_k}(G_1) + w_y(G_2) + (n_2 - 1) d_{G_1}(u_k, x) - (w_x(G_1) + w_y(G_2)) > 0.$$

From Cases 2 and 3, we get that

$$w_{v_k} - w_{v_{xy}} = w_{v_k}(G_2) + w_x(G_1) + (n_1 - 1) d_{G_2}(v_k, y) - (w_x(G_1) + w_y(G_2)) > 0.$$

Therefore, by the above discussion and the definition of $\underline{w}(G)$, we have that

$$\underline{w}(G) = w_x(G_1) + w_y(G_2). \qquad \square$$

From [10] that the Wiener index of the splice graph of G_1 and G_2 is given by the formula

$$W(S(G_1, G_2))(x, y) = W(G_1) + W(G_2) + (|V(G_1)| - 1) w_{v_{xy}}(G_2) + (|V(G_2)| - 1) w_{v_{xy}}(G_1).$$

Using Theorem 2.2 and $W(S(G_1, G_2))(x, y)$, we obtain the ρ value of splice of G_1 and G_2.

Theorem 2.3 *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices. Then*

$$\rho(S(G_1, G_2)(x, y)) = \frac{2\left(W(G_1) + W(G_2) + (n_1 - 1) w_{v_{xy}}(G_2) + (n_2 - 1) w_{v_{xy}}(G_1)\right)}{(n_1 + n_2 - 1)\left(w_x(G_1) + w_y(G_2)\right)}.$$

§3. Link Graph

A *Link* of G_1 and G_2 by the vertices x and y, which is denoted by $L(G_1 \sim G_2)(x, y)$, is defined as the graph obtained by joining x and y by an edge in the union of G_1 and G_2 graph, see Figure2. The various topological indices of link graph are studied in [10].

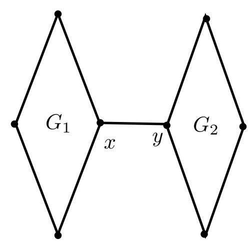

Figure 2 The link of C_4 and C_4

The proof of the following Lemma is easily follows from the structure of link of the graphs G_1 and G_2.

Lemma 3.1 *Let G_1 andG_2 are two connected graphs with $x \in V(G_1)$ and $y \in V(G_2)$. Then*

(i) $|V(L(G_1 \sim G_2)(x, y))| = |V(G_1)| + |V(G_2)|$ *and* $|E(L(G_1 \sim G_2)(x, y))| = |E(G_1)| + |E(G_2)| + 1$;

(ii) If $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ *and* $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$, *then*

$$\begin{aligned} d_{L(G_1 \sim G_2)(x,y)}(u_i, u_j) &= d_{G_1}(u_i, u_j), \\ d_{L(G_1 \sim G_2)(x,y)}(u_i, v_j) &= d_{G_1}(u_i, x) + d_{G_2}(v_j, y) + 1, \\ d_{L(G_1 \sim G_2)(x,y)}(v_i, v_j) &= d_{G_2}(v_i, v_j). \end{aligned}$$

Theorem 3.2 *Let G_1 and G_2 be a connected graph with n_1 and n_2 vertices. For vertices $x \in V(G_1)$ and $y \in V(G_2)$, consider $L(G_1 \sim G_2)(x, y)$. Then*

$$\underline{w}(G) = \begin{cases} w_x(G_1) + w_y(G_2) + n_2, & \text{if } n_1 > n_2. \\ w_x(G_1) + w_y(G_2) + n_1, & \text{if } n_1 < n_2. \end{cases}$$

Proof Let $V(G_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \cdots, v_{n_2}\}$. For our convenience we denote $L(G_1 \sim G_2)(x, y)$ by G. We compute the sum of the distances between a fixed vertex in G to all other vertices of G.

Case 1. Let $u_k \neq x \in V(G_1)$. Then by Lemma 3.1, for a vertex $u_i \in V(G_1)$, $d_G(u_k, u_i) = d_{G_1}(u_k, u_i)$ and for a vertex $v_j \in V(G_2)$,$d_G(u_k, v_j) = d_{G_1}(u_k, x) + d_{G_2}(y, v_j) + 1$. Hence

$$w_{u_k}(G) = \sum_{i=1}^{n_1-1} d_G(u_k, u_i) + \sum_{j=1}^{n_2} d_G(u_k, v_j) = w_{u_k}(G_1) + w_y(G_2) + n_2(d_{G_1}(u_k, x) + 1).$$

Case 2. Let $x \in V(G_1)$. Then by Lemma 3.1, for a vertex $u_i \in V(G_1)$, $d_G(x,u_i) = d_{G_1}(x,u_i)$ and for a vertex $v_j \in V(G_2), d(x,v_j) = d_{G_2}(y,v_j) + 1$. Thus

$$w_x(G) \quad = \quad \sum_{i=1}^{n_1-1} d_G(x,u_i) + \sum_{j=1}^{n_2} d_G(x,v_j) = w_x(G_1) + w_y(G_2) + n_2.$$

Case 3. Let $v_k \neq y \in V(G_2)$. Then by Lemma 3.1, then for a vertex $v_j \in V(G_2)$, $d_G(v_k,v_j) = d_{G_2}(v_k,v_j)$ and for a vertex $u_i \in V(G_1)$,$d_G(v_k,u_i) = d_{G_2}(v_k,y) + d_{G_1}(x,u_i) + 1$. Hence

$$w_{v_k}(G) \quad = \quad w_{v_k}(G_2) + w_x(G_1) + n_1(d_{G_2}(v_k,y) + 1).$$

Case 4. Let $y \in V(G_2)$. Then by Lemma 3.1, for a vertex $v_j \in V(G_2)$, $d_G(y,v_j) = d_{G_2}(y,v_j)$ and for a vertex $u_i \in V(G_1), d(y,u_i) = d_{G_1}(x,u_i) + 1$. Thus

$$w_y(G) \quad = \quad \sum_{j=1}^{n_2-1} d_G(y,v_j) + \sum_{i=1}^{n_1} d_G(y,u_i) = w_y(G_2) + w_x(G_1) + n_1.$$

From Cases 1 and 2, we obtain:

$$w_{u_k}(G) - w_x(G) = w_{u_k}(G_1) + w_y(G_2) + n_2(d_{G_1}(u_k,x) + 1) - (w_x(G_1) + w_y(G_2) + n_2) > 0$$

and

$$w_{v_k}(G) - w_y(G) = w_{v_k}(G_2) + w_x(G_1) + n_1(d_{G_2}(v_k,y) + 1) - (w_x(G_1) + w_y(G_2) + n_1) > 0.$$

From the above discussion and the definition of $\underline{w}(G)$, we have

$$\underline{w}(G) = \begin{cases} w_x(G_1) + w_y(G_2) + n_2, & \text{if } n_1 > n_2. \\ w_x(G_1) + w_y(G_2) + n_1, & \text{if } n_1 < n_2. \end{cases}$$

This completes the proof. □

Recall [10] from that the Wiener index of the link of G_1 and G_2 is given by the formula

$$W(L(G_1 \sim G_2)(x,y)) = W(G_1) + W(G_2) + |V(G_1)|\, w_y(G_2) + |V(G_2)|\, w_x(G_1) + |V(G_1)|\, |V(G_2)|\,.$$

Using Theorem3.2 and $W(L(G_1 \sim G_2))(x,y)$, we obtain the ρ value of the link graph of G_1 and G_2.

Theorem 3.3 *Let G_i be a graph with n_i vertices, $i = 1, 2$. Then*

$$\rho(L(G_1 \sim G_2))(x,y) = \begin{cases} \dfrac{2\Big(W(G_1)+W(G_2)+n_1 w_y(G_2)+n_2 w_x(G_1)+n_1 n_2\Big)}{(n_1+n_2)\Big(w_x(G_1)+w_y(G_2)+n_2\Big)}, & \text{if } n_1 > n_2. \\ \dfrac{2\Big(W(G_1)+W(G_2)+n_1 w_y(G_2)+n_2 w_x(G_1)+n_1 n_2\Big)}{(n_1+n_2)\Big(w_x(G_1)+w_y(G_2)+n_1\Big)}, & \text{if } n_1 < n_2. \end{cases}$$

§4. Rooted Product

The *rooted product* $G_1\{G_2\}$, is obtained by taking one copy of $G_1$and $|V(G_1)|$ copies of a rooted graph G_2 and by identifying the root vertex v_i of the i^{th} copy of G_2 with the i^{th} vertex of G_1, $i=1,2,\cdots,|V(G_1)|$, one can observe that $|E(G_1\{G_2\})|=|E(G_1)|+|V(G_1)|\,|E(G_2)|$, and $|V(G_1\{G_2\})|=|V(G_1)|\,|V(G_2)|$, see Figure 3 for details. The i^{th} copy of G_2 is denoted by $G_{2,i}$. The various topological indices of the rooted product are studied in [11].

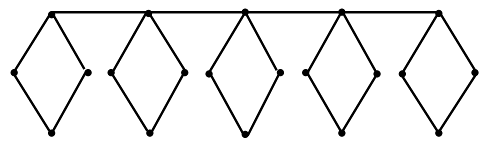

Figure 3 The rooted graph $P_5\{C_4\}$

The proof of the following lemma is easily follows from the structure of the rooted product of the graphs G_1 and G_2.

Lemma 4.1 *Let G_1 be a simple graph and G_2 be a rooted graph with $u_i$as its root. Then for a vertex u_k of $G_1\{G_2\}$ such that $u_k\in V(G_1)$, we have $\delta_{G_1\{G_2\}}(u_k)=\delta_{G_1}(u_k)\delta_{G_2}(u_i)$, and for a vertex v_k of $G_1\{G_2\}$ such that $v_k\notin V(G_1)$ we have $\delta_{(G_1\{G_2\})}(v_k)=\delta_{G_2}(v_0)$ where v_0 is the corresponding vertex in G_2 as v_j of $G_{2,j}$. Moreover*

(i) If $u_k,u_j\in V(G_1)$, then $d_{(G_1\{G_2\})}(u_k,u_j)=d_{G_1}(u_k,u_j)$;

(ii) If $u_k\in V(G)$, $v_{k_i}\in V(G_{2,i})$, where $i=1,2,\ldots,|V(G_1)|$, then $d_{(G_1\{G_2\})}(u_k,v_{k_i})=d_{G_1}(u_k,u_i)+d_{G_{2,i}}(u_i,v_{k_i})=d_{G_1}(u_k,u_i)+d_{G_2}(u,v_0)$, where u_i is the root of $G_{2,i}$,u is the root of G_2 and v_0 is the corresponding vertex in G_2 as v_{k_i} of $G_{2,i}$;

(iii) If $v_{0_i},v_{k_i}\in V(G_{2,i})$, where $i=1,2,\ldots,|V(G_1)|$, then $d_{(G_1\{G_2\})}(v_{0_i},v_{k_i})=d_{G_2}(v_0,v_k)$,where v_0 and v_k are the corresponding vertices in G_2 as v_k and v_{0_i} of $G_{2,i}$;

(iv) If $v_{k_i}\in V(G_{2,i}),v_{k_j}\in V(G_{2,j})$ and $1\le i<j\le|V(G_1)|$, then $d_{(G_1\{G_2\})}(v_{k_i},v_{k_j})=d_{G_{2,i}}(v_{k_i},u_i)+d_{G_{2,j}}(v_{k_j},u_j)+d_{G_1}(u_i,u_j)=d_{G_2}(v_0,u)+d_{G_2}(v_n,u)+d_{G_1}(u_i,u_j)$, where u_i is root of $G_{2,i}$ and $u_j$is the root of $G_{2,j}$. Also v_0 and v_n are the corresponding vertices in G_2 as v_{k_i} of $G_{2,i}$ and v_{k_j} of $V(G_{2,j})$, respectively.

Theorem 4.2 *Let G_1 be a graph with n_1 vertices and G_2 be a rooted graph on n_2 vertices with root vertex v_i. Then*

$$\underline{w}(G_1\{G_2\})=n_2\underline{w}(G_1)+n_1\underline{w}(G_2).$$

Proof Let $V(G_1)=\{u_1,u_2,\cdots,u_{n_1}\}$ and $V(G_2)=\{v_1,v_2,\ldots,v_{n_2}\}$. Let v_i be the root vertex of G_2. We consider the following two cases to find the sum of the distance from a fixed vertex to all the vertices in $G_1\{G_2\}$.

Case 1. Let $u_k\in V(G_1)$. Then by Lemma 4.1, we have$d_{G_1\{G_2\}}(u_k,u_i)=d_{G_1}(u_k,u_i)$, if $u_i\in V(G_1)$ and $d_{G_1\{G_2\}}(u_k,v_k)=d_{G_2}(v_i,v_k)$ if $v_k\in V(G_{2,k})$. Moreover, for $v_k\in V(G_{2,i})$,

$$d_{G_1\{G_2\}}(u_k,v_k)=d_{G_1}(u_k,u_i)+d_{G_2}(v_i,v_k).$$

Hence $w_{u_k}(G)=n_2w_{u_k}(G_1)+n_1w_{v_i}(G_2)$.

Case 2. We consider 3 cases for discussion: (1) $v_k \neq v_i \in V(G_2)$. In this case, by Lemma 4.1, for a vertex $v_s \in V(G_2)$, $d_{G_1\{G_2\}}(v_k, v_s) = d_{G_2}(v_k, v_s)$. Thus $w_{v_k}(G_1\{G_2\}) = w_{v_k}(G_2)$; (2) $u_k \in V(G_1)$. In this case, $d_{G_1\{G_2\}}(v_k, u_k) = d_{G_2}(v_k, v_i) + d_{G_1}(u_i, u_k)$. Thus $w_{v_k}(G_1\{G_2\}) = (n_1-1)d_{G_{2,i}}(v_k, v_i) + w_{u_i}(G_1)$; (3) $v_r \in V(G_{2,r})$. In this case, $d_{(G_1\{G_2\})}(v_k, v_r) = d_{G_2}(v_k, v_i) + d_{G_1}(u_i, u_r) + d_{G_2}(v_i, v_r)$. Thus$w_{v_k}(G_1\{G_2\}) = (n_1-1)(n_2-1)d_{G_{2,i}}(v_k, v_i) + (n_1-1)w_{v_i}(G_2) + (n_2-1)w_{u_i}(G_1)$.

The total contribution of $v_k \in V(G_2)$ is

$$w_{v_k}(G_1\{G_2\}) = w_{v_k}(G_2) + w_{v_i}(G_2)n_2 + d_{G_{2,i}}(v_k, v_i)\Big[(n_1-1) + (n_1-1)(n_2-1)\Big] + (n_1-1)w_{v_i}(G_2).$$

From Cases 1 and 2 we have

$$w_{v_k}(G_1\{G_2\}) - w_{u_k}(G_1\{G_2\}) > 0.$$

Hence

$$\underline{w}(G_1\{G_2\}) = n_2\underline{w}(G_1) + n_1\underline{w}(G_2).$$ □

From [11] that the Wiener index of the rooted product of G_1 and G_2 is given by the formula

$$W(G_1\{G_2\}) = |V(G_2)|^2 W(G_1) + |V(G_1)| W(G_2) + \left(|V(G_1)|^2 - |V(G_2)|\right)|V(G_2)| w_{v_i}(G_2),$$

where v_i is a root-vertex of G_2. Using Theorem4.2 and $W(G_1\{G_2\})$, we obtain the ρ value of rooted product of G_1 and G_2.

Theorem 4.3 *Let $G_1$and G_2 be two graphs with n_1 and n_2 be a number of vertices in G_1 and G_2 Then*

$$\rho(G_1\{G_2\}) = \frac{2\left(n_2^2 W(G_1) + n_1 W(G_2) + (n_1^2 - n_2)n_2 w_{v_i}(G_2)\right)}{(n_1 n_2)\left(n_1\underline{w}(G_2) + n_2\underline{w}(G_1)\right)}.$$

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Total Domination Stable Graphs

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Abstract: In this paper, we study the total domination number and total domination polynomials of some graph and its square. We discuss nonzero real total domination roots of these graphs. We also investigate whether all the total domination roots of some graphs lying left half plane or not.

Key Words: Total dominating set, Smarandachely total k-dominating set, total domination number, total domination polynomial, total domination root, stable.

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§1. Introduction

Let $G(V, E)$ be a simple finite graph. The order of G is the number of vertices of G. A set $S \subseteq V$ is a total dominating set if every vertex $v \in V$ is adjacent to at least one vertex in S. Generally, a set $D \subseteq V$ of G is said to be a *Smarandachely total k-dominating set* if each vertex of G is dominated by at least k vertices of S with $k \geq 1$. Clearly, a total dominating set is a Smarandachely total 1-dominating set. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of the total dominating sets in G. Let $\mathcal{D}_t(G, i)$ be the family of total dominating sets of G with cardinality i and let $d_t(G, i) = |\mathcal{D}_t(G, i)|$. The polynomial

$$D_t(G, x) = \sum_{i=\gamma_t(G)}^{|V(G)|} d_t(G, i)x^i$$

is defined as total domination polynomial of G. For more information on this polynomial the reader may refer to [8]. A root of $D_t(G, x)$ is called a total domination root of G. It is easy to see that the total domination polynomial is monic with no constant term. Consequently, 0 is a root of every total domination polynomial (in fact, 0 is a root whose multiplicity is the total domination number of the graph).

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§2. Main Results

2.1 $\mathtt{d}_t$-Number

In this section we find the number of real roots of the total domination polynomial of some graphs. We are already find out total domination polynomials of complete partite graphs [3] and square of some graphs (The square of a graph G is the graph with the same set of vertices as G and an edge between two vertices if and only if there is a path of length at most two between them, and that graph is denoted by G^2.). We are interested to find the number of real total domination roots of graphs. We define $\mathtt{d}_t$-number of a graph G as follows:

Definition 2.1 *Let G be a graph. The number of distinct real total domination roots of the graph G is called $\mathtt{d}_t$-number of G and is denoted by $\mathtt{d}_t(G)$.*

Theorem 2.1 *For any graph G, $\mathtt{d}_t(G) \geq 1$.*

Proof It follows from the fact that 0 is a total domination root of any graph. □

Theorem 2.2 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$\mathtt{d}_t(G) \leq \sum_{i=1}^{m} \mathtt{d}_t(G_i) - m + 1.$$

Proof It follows from the fact that $D_t(G, x) = \prod_{i=1}^{m} D_t(G_i, x)$. □

Theorem 2.3 *If G and H are isomorphic, then $\mathtt{d}_t(G) = \mathtt{d}_t(H)$.*

Proof It follows from the fact that if G, H are isomorphic then $D_t(G, x) = D_t(H, x)$. □

Theorem 2.4 *For $n \geq 2$ the $\mathtt{d}_t$-number of the complete graph K_n is 1 for even n and 2 for odd n.*

Proof We have the total domination polynomial of K_n is

$$D_t(K_n, x) = (1 + x)^n - nx - 1.$$

From the above equation it follows that $D_t(K_n, y - 1) = y^n - ny + n - 1.$. Clearly, $y = 1$ is a double root of $D_t(K_n, y - 1)$. By De Gua's rule for imaginary roots, there are at least $n - 2$ complex roots if n is even and there are at least $n - 3$ complex roots if n is odd. This give the result. □

Theorem 2.5 *For all m, n the $\mathtt{d}_t$-number of the complete bipartite graph $K_{m,n}$ is*

$$\mathtt{d}_t(K_{m,n}) = \begin{cases} 1 & if\ both\ m\ and\ n\ are\ odd, \\ 2 & otherwise. \end{cases}$$

Proof We have the total domination polynomial of $K_{m,n}$ is

$$D_t(K_{m,n},x) = [(1+x)^m - 1][(1+x)^n - 1]. \tag{1}$$

The result follows from the transformation $y = 1+x$ in equation (1). □

Theorem 2.6 *For $m,n \geq 2$ the $\mathtt{d}_t$-number of the complete partite graph $K_{n[m]}$ is*

$$\mathtt{d}_t(K_{n[m]}) = \begin{cases} 2 & if\ n\ is\ even, \\ 1 & if\ m\ is\ even\ and\ n\ is\ odd, \\ 2 & if\ both\ m\ and\ n\ are\ odd. \end{cases}$$

Proof We have

$$D_t(K_{n[m]},x) = (1+x)^{mn} - m(1+x)^n + m - 1. \tag{2}$$

From the equation (2), it follows that $D_t(K_{n[m]}, y-1) = y^{mn} - my^n + m - 1$. To find the real roots of $y^{mn} - my^n + m - 1 = 0$, it is enough to find the real roots of $f_m(z) = z^m - mz + m - 1 = 0$. Clearly, $z = 1$ is a double root of $f_m(z)$. If m is even, then by De Gua's rule for imaginary roots, there are at least $m-2$ complex roots. Therefore $z = 1$ is the only real root of $f_m(z)$. But $y^n = 1$ has exactly two real solutions, namely $y = \pm 1$ for even n and has exactly one solution, namely $y = 1$ for odd n. If m is odd, then by De Gua's rule for imaginary roots, there are at least $m-3$ complex roots. By the intermediate value theorem, $f_m(z)$ has at least one real root in $(-3,-1)$. So the roots of $f_m(z)$ are 1 and $c \in (-3,-1)$. But $y^n = c$ has a real solution only for odd n and that solution is unique. Therefore $K_{n[m]}$ has only one nonzero real total domination root for even n and if m is even and n is odd, then $K_{n[m]}$ has no nonzero real total domination root. Finally, if both m and n are odd $K_{n[m]}$ has exactly one nonzero total domination root. □

Theorem 2.7 *For all n the $\mathtt{d}_t$-number of the star graph S_n is 1 if n is odd and 2 if n is even.*

Proof We have the total domination polynomial of S_n is

$$D_t(S_n,x) = x((1+x)^n - 1). \tag{3}$$

The result follows from the transformation $y = 1+x$ in equation (3). □

The corona $H \circ G$ of two graphs H and G is the graph formed from one copy of H and $|V(H)|$ copies of G, where the i^{th} vertex of H is adjacent to every vertex in the i^{th} copy of G.

Theorem 2.8 *Let G be a graph of order n without isolated vertices and let H be any graph. Then the total dominating number $\gamma_t(G \circ H) = n$.*

Theorem 2.9 *Let G be a graph of order n without isolated vertices. Then the total domination* *polynomial of* $G \circ \overline{K_m}$ *is*

$$D_t(G \circ \overline{K_m}, x) = x^n(1+x)^{mn}.$$

Proof By above theorem we have $\gamma_t(G \circ \overline{K_m}) = n$. If S is a total dominating set of $G \circ \overline{K_m}$, then $V(G) \subset S$. Therefore $d_t(G \circ \overline{K_m}, n) = 1$ and for $n+1 \le i \le n(m+1)$,

$$d_t(G \circ \overline{K_m}, i) = \binom{mn}{i-n}. \qquad \square$$

Theorem 2.10 *Let G be a graph of order n. Then the total domination polynomial of $K_1 \circ G$ is*

$$D_t(K_1 \circ G, x) = D_t(G, x) + x\left((1+x)^n - 1\right).$$

Proof It follows from the facts that total dominating sets of G is a total dominating sets of $K_1 \circ G$ and any set of vertices of $K_1 \circ G$ containing the vertex of K_1 is also a total dominating set. $\square$

The Dutch-windmill graph G_3^n is the graph obtained by selecting one vertex in each of n triangles and identifying them.

Corollary 2.1 *The total domination polynomial of the Dutch-windmill graph G_3^n is*

$$D_t(G_3^n, x) = x^{2n} + x\left((1+x)^{2n} - 1\right).$$

Proof It follows from the fact that G_3^n and $K_1 \circ nK_2$ are isomorphic. $\square$

Theorem 2.11 *For all n the $\mathtt{d}_t$-number of the Dutch windmill graph G_3^n is greater than or equal to* 2.

Proof We have the total domination polynomial of the Dutch windmill graph G_3^n is

$$D_t(G_3^n, x) = x^{2n} + x((1+x)^{2n} - 1).$$

Consider,

$$\begin{aligned} D_t(G_3^n, -\ln n) &= (-\ln n)^{2n} + (-\ln n)((1-\ln n)^{2n} - 1) \\ &= (\ln n)^{2n}\left(1 - \ln n(\tfrac{1-\ln n}{\ln n})^{2n} + \ln n \tfrac{1}{(\ln n)^{2n}}\right). \end{aligned}$$

From Theorem **??**, we have $D_t(G_3^n, -\ln n) > 0$ for large n. Next we show that $D_t(G_3^n, -n) < 0$. Consider $f(x) = x^{2n-1} + (2n+1)x^{2n-2} + \binom{2n}{2}x^{2n-3} + \ldots + 2n$. Then

$$\begin{aligned} f(-n) &= (-1)^{2n-1}n^{2n-1} + (2n+1)n^{2n-2} + (-1)^{2n-3}\tbinom{2n}{2}n^{2n-3} + \ldots + 2n \\ &= (-1)^{2n-1}n^{2n-1}\left(1 - \tfrac{2n+1}{n} + \tfrac{\binom{2n}{2}}{n^2} - \ldots - \tfrac{2n}{n^{2n-1}}\right). \end{aligned}$$

But for sufficiently large n,

$$1 - \frac{2n+1}{n} + \frac{\binom{2n}{2}}{n^2} - \ldots - \frac{2n}{n^{2n-1}} < 0.$$

That is, $D_t(G_3^n, -n) < 0$ for sufficiently large n. By the intermediate value theorem, for sufficiently large n, $D_t(G_3^n, x)$ has a real root in the interval $(-n, -\ln n)$. Therefore the Dutch windmill graph G_3^n has at least two real total domination root and hence $\mathtt{d}_t(G_3^n) \geq 2$. $\square$

Theorem 2.12 *For all n, $\mathtt{d}_t((K_n \circ K_1)^2) = 1$.*

Proof We have $D_t((K_n \circ K_1)^2, y-1) = y^{2n} - y^n - ny + n$. Let $f(y) = y^{2n} - y^n - ny + n$. Since the number of variations of the signs of the coefficients of $f(y)$ is 2, by Descartes rule, it has at most two positive real roots. Clearly, $y = 1$ is a double root of $f(y)$. Now consider, $f(-y)$.

Case 1. n is odd.

$f(-y) = y^{2n} + y^n + ny + n$. There is no sign changes, $f(y)$ has no negative real roots. Therefore the only possible real root of $D_t((K_n \circ K_1)^2, x)$ is zero.

Case 2. n is even.

$f(-y) = y^{2n} - y^n + ny + n$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 2, by Descartes rule, it has at most two negative real roots. We claim that $f(-y)$ has no positive real roots. Let $z > 0$ be a real root of $f(-y)$. Then $z^{2n} - z^n + nz + n = 0$. That is, $z^{2n} - z^n = -n(z+1)$. If $z \geq 1$, $z^{2n} - z^n \geq 0$, but right side is negative. Therefore $z \geq 1$ is not possible. If $0 < z < 1$, then $-1 \leq z^{2n} - z^n \leq 0$, but right side is greater than -1. Therefore $0 < z < 1$ is also not possible.

In both cases the only possible real roots of $D_t((K_n \circ K_1)^2, x)$ is zero. Hence we get the result. $\square$

A spider graph Sp_{2n+1} is the graph obtained by subdividing each edges once in the star graph $K_{1,n}$.

Theorem 2.13 *The total domination polynomial of the spider graph Sp_{2n+1} is*

$$D_t(Sp_{2n+1}, x) = x^n \left((1+x)^{n+1} - 1\right).$$

Proof Let v, $V = \{v_1, v_2, \cdots, v_n\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of Sp_{2n+1} such that v is adjacent to v_i for every $i = 1, 2, \cdots, n$ and v_i and u_i are adjacent for every $i = 1, 2, \cdots, n$. It is clear that the total dominating sets of Sp_{2n+1} are exactly the sets of vertices of Sp_{2n+1} properly containing V. Hence $\gamma(Sp_{2n+1}) = n+1$ and $d_t(Sp_{2n+1}, n+i) = \binom{n+1}{i}$ for $i = 1, 2, \cdots, n+1$. $\square$

Theorem 2.14 *For $n \geq 2$, the $\mathtt{d}_t$-number of the spider graph Sp_{2n+1} is 1 for even n and 2 for odd n.*

Proof By Theorem 2.13 we have the total domination polynomial of the spider graph Sp_{2n+1} is

$$D_t(Sp_{2n+1}, x) = x^n \left((1+x)^{n+1} - 1\right). \tag{4}$$

The result follows from the transformation $y = 1 + x$ in (4). $\square$

The lollipop graph $L_{n,1}$ is the graph obtained by joining a complete graph K_n to a path P_1, with a bridge.

Theorem 2.15 *The total domination polynomial of the lollipop graph $L_{n,1}$ is*

$$D_t(L_{n,1}, x) = x\left((1+x)^n - 1\right).$$

Proof Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of the complete graph K_n and v be the path P_1 and let v is adjacent to v_1. Clearly the total dominating sets of $L_{n,1}$ are exactly the set of vertices of $L_{n,1}$ properly containing v_1. Therefore , $\gamma_t(L_{n,1}) = 2$ and $d_t(L_{n,1}, i) = \binom{n}{i-1}$ for $2 \le i \le n+1$. $\square$

Theorem 2.16 *The $\mathtt{d}_t$-number of the lollipop graph $L_{n,1}$ is 1 for odd n and 2 for even n.*

Proof By Theorem 2.15 we have the total domination polynomial of the lollipop graph $L_{n,1}$ is

$$D_t(L_{n,1}, x) = x\left((1+x)^n - 1\right). \tag{5}$$

The result follows from the transformation $y = 1 + x$ in equation (5). $\square$

The bipartite cocktail party graph B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$.

Theorem 2.17 *The total domination polynomial of the bipartite Cocktail party graph B_n is*

$$D_t(B_n, x) = \left((1+x)^n - nx - 1\right)^2.$$

Proof Let $V = \{v_1, v_2, \cdots, v_n\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of B_n such that every vertex v_i in V and every vertex u_i in U are adjacent if $i \neq j$. The total dominating set S of B_n are exactly the set of vertices of B_n such that S contains at least two v_i and at least two u_i. Note that sets of this form are of size greater than or equal to 4. Therefore $\gamma_t(B_n) = 4$. Also for $4 \le i \le n$, $d_t(B_n, i) = \binom{2n}{i} - 2\binom{n}{i} - 2\binom{n}{i-1}$, $d_t(B_n, n+1) = \binom{2n}{n+1} - 2n$ and for $n+2 \le i \le 2n$, $d_t(B_n, i) = \binom{2n}{i}$. $\square$

Theorem 2.18 *For $n \ge 2$ the $\mathtt{d}_t$-number of the bipartite cocktail party graph B_n is 1 for even n and 2 for odd n.*

Proof The proof is similar to the proof of Theorem 2.4. $\square$

Theorem 2.19 *For $n \ge 3$, the total domination polynomial of square of the bipartite cocktail party graph B_n is*

$$D_t(B_n^2, x) = (1+x)^{2n} - n(1+x)^2 + (n-1).$$

Proof Let $V = \{v_1, v_2, \cdots, v_n\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of B_n such that every vertex v_i in V and every vertex u_i in U are adjacent if $i \neq j$. Clearly, any subset of vertices of B_n of cardinality 2 forms a total dominating set of B_n^2 excluding $\{v_i, u_i\}$ for all $i = 1, 2, \cdots, n$.

Therefore $\gamma_t(B_n^2)=2$, $d_t(B_n^2,2)=\binom{2n}{2}-n$ and $d_t(B_n^2,i)=\binom{2n}{i}$ for all $3\le i\le 2n$. $\square$

Theorem 2.20 *The* $\mathtt{d}_t$*-number of the square of the bipartite cocktail party graph* B_n *is* 2 *for* $n\ge 3$.

Proof We have $D_t(B_n^2,y-1)=y^{2n}-ny^2+n-1$. Then by De Gua's rule for imaginary roots, there are at least $2n-4$ complex roots. Clearly, $y=1$ and $y=-1$ are double roots of $D_t(B_n^2,y-1)$. Therefore $x=0$ and $x=-2$ are the only real roots. $\square$

The generalized barbell graph $B_{m,n,1}$ is the simple graph obtained by connecting two complete graphs K_m and K_n by a path P_1.

Theorem 2.21 *For* $m\le n$*, the total domination polynomial of generalized barbell graph* $B_{m,n,1}$ *is*

$$D_t(B_{m,n,1},x)=\left[(1+x)^m-(m-1)x-1\right]\left[(1+x)^n-(n-1)x-1\right].$$

Proof Let $V=\{v_1,v_2,\cdots,v_m\}$ and $U=\{u_1,u_2,\cdots,u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i\ne j$ every vertices V are adjacent, every vertices U are adjacent and v_m and u_n is adjacent. The only two element total dominating set of $B_{m,n,1}$ is $\{v_m,u_n\}$. Therefore $\gamma_t(B_{m,n,1})=2$ and $d(B_{m,n,1},2)=1$. Also observe that for $2\le i\le 2n$, a subset S of vertices $B_{m,n,1}$ of cardinality i is not a total domination set if and only if (i) $S\subset V$ or (ii) $S\subset U$ or (iii) S contains one element from $V-\{v_n\}$ and $i-1$ elements from U or (iv)S contains one element from $U-\{u_n\}$ and $i-1$ elements from V. Therefore

$$d_t(B_{m,n,1},i)=\begin{cases}1 & \text{if } i=2\\ \binom{m+n}{i}-\binom{n}{i}-\binom{m}{i}-(n-1)\binom{m}{i-1}-(m-1)\binom{n}{i-1} & \text{if } 3\le i\le m\\ \binom{m+n}{m+1}-\binom{n}{m+1}-(n-1)-(m-1)\binom{n}{m} & \text{if } i=m+1\\ \binom{m+n}{i}-\binom{n}{i}-(m-1)\binom{n}{i-1} & \text{if } m+2\le i\le n\\ \binom{m+n}{n+1}-(m-1) & \text{if } i=n+1\\ \binom{m+n}{i} & \text{if } n+2\le i\le m+n\end{cases}.$$

Hence

$$D_t(B_{m,n,1},x)=\left[(1+x)^m-(m-1)x-1\right]\left[(1+x)^n-(n-1)x-1\right].\quad\square$$

Theorem 2.22 *For* $m,n\ge 2$; $m\ne n$*, the* $\mathtt{d}_t$*-number of the generalized barbell graph* $B_{m,n,1}$ *is*

$$\mathtt{d}_t(B_{m,n,1})=\begin{cases}3 & \textit{if both } m \text{ and } n \text{ are even},\\ 5 & \textit{if both } m \text{ and } n \text{ are odd},\\ 4 & \textit{if } m \text{ and } n \text{ have opposite parity}.\end{cases}$$

Proof By Theorem 2.21 we have the total domination polynomial of generalized barbell graph $B_{m,n,1}$ is

$$D_t(B_{m,n,1},x)=\left[(1+x)^m-(m-1)x-1\right]\left[(1+x)^n-(n-1)x-1\right].$$

Since there is no real number satisfying both the following equations

$$(1+x)^m - (m-1)x - 1 = 0$$
$$(1+x)^n - (n-1)x - 1 = 0$$

simultaneously. So it is enough to show that $f(x) = x^n - (n-1)x + n - 2$ has exactly one nonzero real root for even n and has exactly two nonzero real roots for odd n. Clearly $x = 1$ is a simple root of $f(x)$. For even n, by De Gua's rule for imaginary roots, there are at least $n-2$ complex roots. Therefore the remaining root is real number different from 1. For odd n by De Gua's rule for imaginary roots, there are at least $n-3$ complex roots. Observe that $f(-1) > 0$ and $f(-2) < 0$. By the intermediate value theorem, we have $f(x)$ has a root in the interval $(-2,-1)$. Therefore the remaining roots real numbers different from 1. It remains to show that $f(x)$ has no double roots. Suppose $a \in \mathbb{R}$ is a double root of $f(x)$. Then

$$a^n - (n-1)a + n - 2 = 0, \tag{6}$$

$$na^{n-1} - (n-1) = 0. \tag{7}$$

Solving these equations we get $a = \frac{n(n-2)}{(n-1)^2}$. This implies that $a \geq 0$, a contradiction, since $a < 0$. So we have the result. □

The $n-$barbell graph $B_{n,1}$ is the simple graph obtained by connecting two copies of complete graph K_n by a bridge.

Corollary 2.2 *The total domination polynomial of the n-barbell graph $B_{n,1}$ is*

$$D_t(B_{n,1}) = \left((1+x)^n - (n-1)x - 1\right)^2.$$

Proof It follows from the fact that the n-barbell graph $B_{n,1}$ and the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

Corollary 2.3 *For $n \geq 2$ the $\mathtt{d}_t$-number of the n-barbell graph $B_{n,1}$, is*

$$\mathtt{d}_t(B_{n,1}) = \begin{cases} 2 & \text{if } n \text{ is even}, \\ 3 & \text{if } n \text{ is odd}. \end{cases}$$

2.2 Total Domination Stable Graphs

In this section we introduce $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs. We obtained some examples of $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs.

Definition 2.2 *Let $G = (V(G), E(G))$ be a graph. The graph G is said to be a total domination stable graph or simply $\mathtt{d}_t$-stable graph if all the nonzero total domination roots lie in the left open half-plane, that is, if real part of the nonzero total domination roots are negative. If G is not $\mathtt{d}_t$-stable graph, then G is said to be a total domination unstable graph or simply $\mathtt{d}_t$-unstable* *graph.*

Theorem 2.23 *If G and H are isomorphic graphs then G is $\mathtt{d}_t$-stable if and if H is $\mathtt{d}_t$-stable.*

Proof It follows from the fact that if G and H are isomorphic graphs then $D_t(G,x) = D_t(H,x)$. □

Corollary 2.4 *If G and H are isomorphic graphs then G is $\mathtt{d}_t$-unstable if and if H is $\mathtt{d}_t$-unstable.*

Theorem 2.24 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}_t$-stable if and if each G_i is $\mathtt{d}_t$-stable.*

Proof It follows from the fact that $D_t(G,x) = \prod\limits_{i=1}^{m} D_t(G_i,x)$. □

Corollary 2.5 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}_t$-unstable if and if one of the G_i is $\mathtt{d}_t$-unstable.*

Theorem 2.25 *Let G be a graph of order n without isolated vertices. Then $G \circ \overline{K_m}$ is $\mathtt{d}_t$-stable for all m, n.*

Proof We have the total domination polynomial of $G \circ \overline{K_m}$ is

$$D_t(G \circ \overline{K_m}, x) = x^n (1+x)^{mn}.$$

Therefore $\mathbb{Z}(D_t(G \circ \overline{K_m}, x)) = \{0, -1\}$, hence $G \circ \overline{K_m}$ is $\mathtt{d}_t$-stable for all m, n. □

Theorem 2.26 *The spider graph Sp_{2n+1} is $\mathtt{d}_t$-stable for all n.*

Proof We have the total domination polynomial of the spider graph Sp_{2n+1} is

$$D_t(Sp_{2n+1}, x) = x^n \left((1+x)^{n+1} - 1\right).$$

Therefore

$$\mathbb{Z}(D_t(Sp_{2n+1}, x)) = \left\{\exp\left(\frac{2k\pi i}{n+1}\right) - 1 | k = 0, 1, \ldots, n\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the spider graph Sp_{2n+1} is $\mathtt{d}_t$-stable for all n. □

Theorem 2.27 *The lollipop graph $L_{n,1}$ is $\mathtt{d}_t$-stable for all n.*

Proof We have the total domination polynomial of the lollipop graph $L_{n,1}$ is

$$D_t(L_{n,1}, x) = x\left((1+x)^n - 1\right).$$

Therefore

$$\mathbb{Z}(D_t(L_{n,1}, x)) = \left\{\exp\left(\frac{2k\pi i}{n}\right) - 1 | k = 0, 1, \ldots, n\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the lollipop graph $L_{n,1}$ is $\mathtt{d}_t$-stable for all n. □

Theorem 2.28 *The bi-star graph $B_{(m,n)}$ is $\mathtt{d}_t$-stable for all m, n.*

Proof We have the total domination polynomial of the bi-star graph $B_{(m,n)}$ is

$$D_t(B_{(m,n)}, x) = x^2(1+x)^{m+n}.$$

Therefore

$$\mathbb{Z}(D_t(B_{(m,n)}, x) = \{0, -1\},$$

hence the bi-star graph $B_{(m,n)}$ is $\mathtt{d}_t$-stable for all m, n. □

Corollary 2.6 *The corona graph $K_2 \circ \overline{K_n}$ is $\mathtt{d}_t$-stable for all n.*

Proof It follows from the fact that the corona graph $K_2 \circ \overline{K_n}$ and the bi-star graph $B_{(n,n)}$ are isomorphic. □

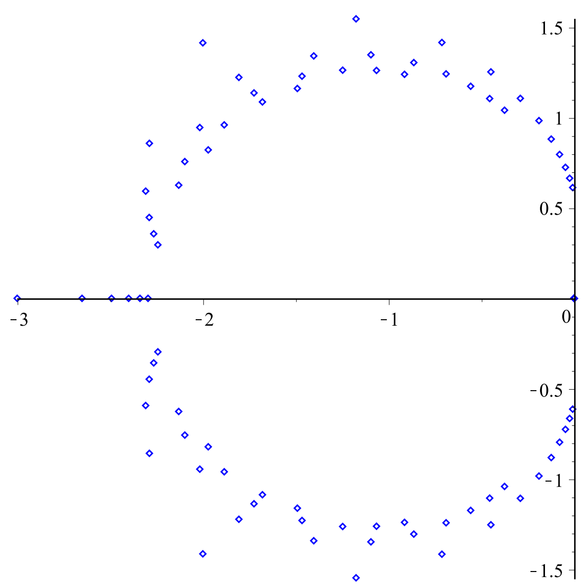

Figure 1 Total domination roots of K_n for $1 \le n \le 14$.

Remarks 2.1 Using maple, we find that the complete graph K_n is $\mathtt{d}_t$-stable for $1 \le n \le 14$ and is $\mathtt{d}_t$-unstable for $15 \le n \le 30$. We have the total domination polynomial of K_n is

$$D_t(K_n, x) = (1+x)^n - nx - 1.$$

Put $y = 1 + x$ and consider $f(y) = y^n - ny + n - 1$. Then $y = 1$ is a double root of $f(y)$.

Therefore $f(y) = (y-1)^2 g(y)$, where

$$g(y) = y^{n-2} + 2y^{n-3} + 3y^{n-4} + \ldots + (n-2)y + n - 1.$$

We have if $f(z) = a_n z^n + a_{n-1} z^n + \ldots + a_0$ is a polynomial with real coefficient satisfying $a_0 \geq a_1 \geq \ldots \geq a_n > 0$ then no roots of $f(z)$ lie in $\{z \in \mathbb{C} : |z| < 1\}$ [6]. Therefore all the roots z of $g(y)$ satisfy $|z| > 1$. This implies that all the nonzero roots of $D_t(K_n, x)$ are out side the unit circle centered at $(-1, 0)$. So we conjectured that the complete graph K_n is not $\mathtt{d}_t$-stable for all but finite values of n.

The total domination roots of the complete graph K_n for $1 \leq n \leq 14$ and $1 \leq n \leq 30$ are shown in Figures 1 and 2 respectively.

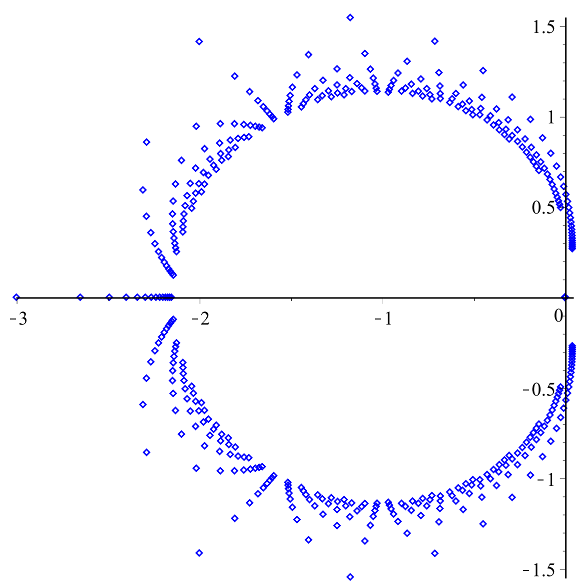

Figure 2 Total domination roots of K_n for $1 \leq n \leq 30$.

We use the following definitions and results to prove some graphs which are $\mathtt{d}_t$-unstable. These definitions and theorems are taken from [12].

Definition 2.3 *If $f_n(x)$ is a family of complex polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathbb{Z}(f_n(x))$, $\mathbb{Z}(f_n(x))$ is the set of the roots of the family $f_n(x)$.*

Now, a family $f_n(x)$ of polynomials is a recursive family of polynomials if $f_n(x)$ satisfy a homogeneous linear recurrence

$$f_n(x) = \sum_{i=1}^{k} a_i(x) f_{n-i}(x), \tag{8}$$

where the $a_i(x)$ are fixed polynomials, with $a_k(x) \neq 0$. The number k is the order of the recurrence. We can form from equation (8) its associated characteristic equation

$$\lambda^k - a_1(x)\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_k(x) = 0 \tag{9}$$

whose roots $\lambda = \lambda(x)$ are algebraic functions, and there are exactly k of them counting multiplicity.

If these roots, say $\lambda_1(x), \lambda_2(x), \cdots, \lambda_k(x)$, are distinct, then the general solution to equation (8) is known to be

$$f_n(x) = \sum_{i=1}^{k} \alpha_i(x)\lambda_i(x)^n \tag{9}$$

with the usual variant if some of the $\lambda_i(x)$ are repeated. The functions

$$\alpha_1(x), \alpha_2(x), \cdots, \alpha_k(x)$$

are determined from the initial conditions, that is, the k linear equations in the α_i obtained by letting $n = 0, 1, \cdots, k-1$ in equation (10) or its variant. The details are available in [12]. Beraha, Kahane and Weiss [12] proved the following results on recursive families of polynomials and their roots.

Theorem 2.29 *If $f_n(x)$ is a recursive family of polynomials, then a complex number z is a limit of roots of $f_n(x)$ if and only if there is a sequence (z_n) in $\mathbb{C}$ such that $f_n(z_n) = 0$ for all n and $z_n \to z$ as $n \to \infty$.*

Theorem 2.30 *Under the non-degeneracy requirements that in equation* (10) *no $\alpha_i(x)$ is identically zero and that for no pair $i \neq j$ is it true that $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some complex number ω of unit modulus, then $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if and only if either*

(1) *two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or*

(2) *for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$.*

Corollary 2.7(see [2]) *Suppose $f_n(x)$ is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \ldots + \alpha_k(x)\lambda_k(x)^n \tag{11}$$

where the $\alpha_i(x)$ and the $\lambda_i(x)$ are fixed non-zero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then the limits of roots of $f_n(x)$ are exactly those z satisfying (i) or (ii) of Theorem 2.30.

Remark 2.2 *We have the total domination polynomial of G_3^n is*

$$D_t(G_3^n, x) = x(1+x)^{2n} - x + x^{2n}.$$

Rewrite $D_t(G_3^n, x)$ *as*

$$\begin{aligned} D_t(G_3^n, x) = f_{2n}(x) &= x(1+x)^{2n} + (-x)(1)^{2n} + (1)x^{2n}. \\ &= \alpha_1\lambda_1^{2n} + \alpha_2\lambda_2^{2n} + \alpha_3\lambda_3^{2n}, \end{aligned}$$

where $\alpha_1 = x$, $\lambda_1 = 1+x$, $\alpha_2 = -x$, $\lambda_2 = 1$, $\alpha_3 = 1$ *and* $\lambda_3 = x$. *Clearly* α_1,α_2 *and* α_3 *are not identically zero and* $\lambda_i \neq \omega\lambda_j$ *for* $i \neq j$ *and any complex number* ω *of modulus* 1. *Therefore the initial conditions of Theorem* 2.30 *are satisfied. Now, applying part(i) of Theorem* 2.30, we consider the following four different cases:

(1) $|\lambda_1| = |\lambda_2| = |\lambda_3|$;

(2) $|\lambda_1| = |\lambda_2| > |\lambda_3|$;

(3) $|\lambda_1| = |\lambda_3| > |\lambda_2|$;

(4) $|\lambda_2| = |\lambda_3| > |\lambda_1|$.

Case 1. Assume that $|1+x| = |1| = |x|$. Then $|x-(-1)| = |x-0|$ implies that x lies on the vertical line $z = -\frac{1}{2}$, $|x-(-1)| = 1$ implies that x lies on the unit circle centered at $(-1, 0)$ and $1 = |x-0|$ implies that x lies on the unit circle centered at the origin. Therefore the two points of intersection, $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ are the limits of roots.

Case 2. Assume that $|1+x| = |1| > |x|$. Then $|x-(-1)| = 1$ implies that x lies on the unit circle centered at $(-1, 0)$, $|x-(-1)| > |x-0|$ implies that x lies to the right of the vertical line $z = -\frac{1}{2}$. Therefore the complex numbers x that satisfy $|x-(-1)| = 1$ and $\mathcal{R}(x) > -\frac{1}{2}$ are the limits of roots.

Case 3. Assume that $|1+x| = |x| > |1|$. Then $|x-(-1)| = |x-0|$ implies that x lies on the vertical line $z = -\frac{1}{2}$ and $|x-0| > 1$ implies that x lies outside the unit circle centered at the origin. Therefore the complex numbers x that satisfy $|x| > 1$ and $\mathcal{R}(x) > -\frac{1}{2}$ are the limits of roots.

Case 4. Assume that $|1| = |x| > |1+x|$. Then $1 = |x-0|$ implies that x lies on the unit circle centered at the origin and $|x-0| > |x-(-1)|$ implies that x lies to the left of the vertical line $z = -\frac{1}{2}$. Therefore the complex numbers x that satisfy $|x| = 1$ and $\mathcal{R}(x) < -\frac{1}{2}$ are the limits of roots.

The union of the curves and points above yield that, the limits of roots of the total domination polynomial of the Dutch windmill graph G_3^n consists of the part of the circle $|z| = 1$ with real part at most $-\frac{1}{2}$, the part of the circle $|z+1| = 1$ with real part at least $-\frac{1}{2}$ and the part of the line $\mathcal{R}(z) = -\frac{1}{2}$ with modulus at least 1. So we conjectured that the Dutch windmill graph G_3^n is $\mathtt{d}_t$-stable for all n.

The total domination roots of the Dutch windmill graph G_3^n for $1 \leq n \leq 30$ are shown in Figure 3.

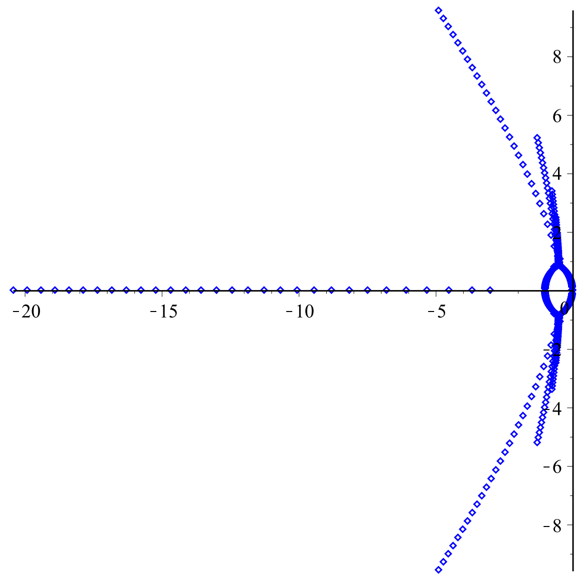

Figure 3 Total domination roots of G_3^n for $1 \leq n \leq 30$.

Remark 2.3 We have the total domination polynomial of B_n is

$$D_t(B_n, x) = ((1+x)^n - nx - 1)^2.$$

Because of the same reason as mentioned in Remark 2.1, we conjectured that the bipartite cocktail party graph B_n is not a $\mathtt{d}_t$-stable for all but finite values of n.

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Centered Triangular Mean Graphs

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Abstract: A graph $G = (V, E)$ with p vertices and q edges is said to have centered triangular mean labeling if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from S, where S is a set of non-negative integers in such a way that for each edge $e = uv$, $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ and the resulting edge labels are the first q centered triangular numbers. A graph that admits a centered triangular mean labeling is called centered triangular mean graph. In this paper, we prove that the graphs P_n, $K_{1,n}$, $B_{m,n}$, coconut tree, caterpillar $S(n_1, n_2, n_3, \ldots, n_m)$, $St(n_1, n_2, n_3, \ldots, n_m)$, $Bt(\underbrace{n, n, \ldots, n}_{\text{m times}})$ and $P_m@P_n$ are centered triangular mean graphs.

Key Words: Mean labeling, Smarndache mean m-labelling, triangular mean labeling, triangular mean graph, centered triangular mean labeling, centered triangular mean graph.

AMS(2010): 05C78.

§1. Introduction

By a graph, we mean a finite, simple and undirected one. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. For various graph theoretic notations and terminology we follow Harary [2] and for number theory Burton[1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and a detailed survey on graph labeling is available in [3]. The concept of mean labeling was introduced by Somasundaram et al.[6]. A graph $G = (V, E)$ with p vertices and q edges is called a mean graph if there is an injective function f that maps $V(G)$ to $\{0, 1, 2, \cdots, q\}$ such that each edge uv is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd and the resulting labels of the edges are distinct and are $0, 1, 2, \cdots, q$. Generally, if each edge uv is labeled with $\frac{f(u)+f(v)}{m}$ if f maps $V(G)$ to $\{0, 1, 2, \cdots, q, 2, 4, \cdots, 2q, \cdots, m, 2m, 4m \cdots, qm\}$ such that $\frac{f(u)+f(v)+m-k}{m}$ if $f(u) + f(v) \equiv k(mod m)$ the resulting labels of the edges are distinct and are $0, 1, 2, \cdots, q$, such a labeling f is called a *Smarandache mean m-labeling* and G a *Smarandache m-mean graph*, where $m \geq 2$ is an integer. Particularly, a Smarandache 2-mean

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graph is nothing else but a mean graph. Several papers have been published on mean labeling and its variations.

Seenivasan et al.[5] introduced the concept of triangular mean labeling in 2007. A triangular mean labeling of a graph $G=(V,E)$ with p vertices and q edges is an injection $f: V(G) \longrightarrow \{0,1,2,3,\cdots,T_q\}$ such that for each edge $e=uv$, the edge labels are defined as $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ such that the values of the edges are the first q triangular numbers. A graph that admits a triangular mean labeling is called triangular mean graph.

Recently, the number theory has a strong impact on graph theory. A triangular number[1] is a number obtained by adding all positive numbers less than or equal to a given positive integer n. If the n^{th} triangular number[1] is denoted by T_n, then $T_n = \frac{1}{2}n(n+1)$. A centered triangular number is a centered figurative number that represents a triangle with a dot in the center and all other dots surrounding the center in successive triangular layers. If the n^{th} centered triangular number is denoted by cT_n, then $cT_n = \frac{1}{2}(3n^2-3n+2)$. The first few centered triangular numbers are $1,\ 4,\ 10,\ 19,\ 31,\ 46,\ 64,\ 85,\cdots$.

The figurative representations of the first four centered triangular numbers are shown in Figure 1.

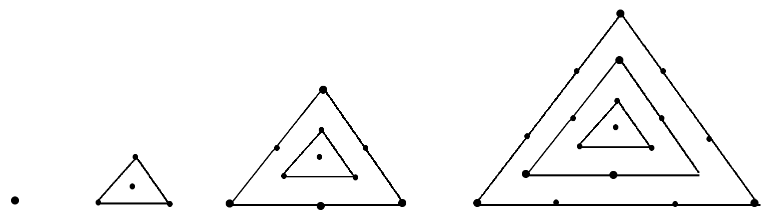

Figure 1

The notion of centered triangular sum labeling was due to Murugesan et al.[4] in 2013. Motivated by the results in [4] and [5] and using the centered triangular concept in number theory [1] we define a new labeling called centered triangular mean labeling. A graph $G=(V,E)$ with p vertices and q edges is said to have centered triangular mean labeling if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from S, where S is a set of non-negative integers such that for each edge $e=uv$, $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$, and the resulting labels of the edges are the first q centered triangular numbers. A graph that admits a centered triangular mean labeling is called centered triangular mean graph. In this paper, we prove that the graph $P_n,\ K_{1,n},\ B_{m,n}$, coconut tree, caterpillar $S(n_1,n_2,\ n_3,\cdots,n_m)$, $St(n_1,n_2,\ n_3,\cdots,n_m)$, $Bt(\underbrace{n,n,\cdots,n}_{\text{m times}})$ and $P_m@P_n$ are centered triangular mean graphs. We use the following definitions in the subsequent sequel.

Definition 1.1 *The bistar $B_{m,n}$ is a graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2.*

Definition 1.2 *A caterpillar is a tree with a path P_m: $v_1,v_2,\cdots,v_m$, called spine with leaves(pendant vertices) known as feet attached to the vertices of the spine by edges known as legs. If every spine vertex v_i is attached with n_i number of leaves then the caterpillar is* *denoted by $S(n_1, n_2, \cdots, n_m)$.*

Definition 1.3 *The shrub $St(n_1, n_2, \cdots, n_m)$ is a graph obtained by connecting a vertex v_0 to central vertex of each of m number of stars.*

Definition 1.4 *The banana tree $Bt(n_1, n_2, \cdots, n_m)$ is a graph obtained by connecting a vertex v_0 to one leaf of each of m number of stars.*

Definition 1.5 *The graph $P_m@P_n$ is obtained from P_m and m copies of P_n by identifying one pendant vertex of the i^{th} copy of P_n with i^{th} vertex of P_m where P_m is a path of length of $m-1$.*

§2. Centered Triangular Mean Graphs

Theorem 2.1 *The path $P_n (n \geqslant 1)$ is a centered triangular mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n. Define $f : V(P_n) \longrightarrow S$ as follows:

$f(v_1) = 0,$

$f(v_j) = 2(cT_{j-1} - cT_{j-2} + cT_{j-3} - \cdots + (-1)^j cT_1)$ for $2 \leqslant j \leqslant n$.

Let $e_i = v_i v_{i+1} (1 \leqslant i \leqslant n-1)$ be the edges of P_n. For each vertex label f, the induced edge label f^* is defined to be $f^*(e_i) = cT_i$ for $1 \leqslant i \leqslant n-1$. Then f is a centered triangular mean labeling. Hence P_n is a centered triangular mean graph. □

The centered triangular mean labeling of P_5 is given in Figure 2.

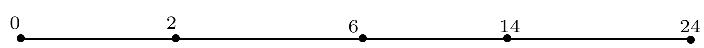

Figure 2

Theorem 2.2 *The star graph $K_{1,n} (n \geq 1)$ admits centered triangular mean labeling.*

Proof Let v be the apex vertex and let $v_1, v_2, \cdots, v_n$ be the pendant vertices of the star $K_{1,n}$. Define $f : V(K_{1,n}) \longrightarrow S$ to be $f(v) = 0, \quad f(v_j) = 2cT_j$ for $1 \leq j \leq n$.

For each vertex label f, the induced edge label f^* is defined to be $f^*(vv_j) = cT_j$ for $1 \leq j \leq n$. Then the induced edge labels are the centered triangular numbers $cT_1,\ cT_2,\ cT_3, \cdots, cT_n$. Hence $K_{1,n}$ is a centered triangular mean graph. □

The centered triangular mean labeling of $K_{1,8}$ is shown in Figure 3.

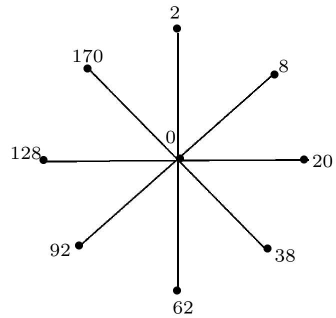

Figure 3

Theorem 2.3 *The bistar $B_{m,n}(m \geqslant 1,\ n \geqslant 1)$is a centered triangular mean graph.*

Proof Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and $E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Define $f : V(B_{m,n}) \longrightarrow S$ as follows:

$f(u) = 0,\ f(v) = 2,$
$f(u_i) = 2cT_{i+1}$ for $1 \leq i \leq m,$
$f(v_j) = 2(cT_{m+j+1} - 1)$ for $1 \leq j \leq n.$

For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(uv) = cT_1,$
$f^*(uu_i) = cT_{i+1}$ for $1 \leqslant i \leqslant m,$
$f^*(vv_j) = cT_{m+j+1}$ for $1 \leqslant j \leqslant n.$

Then the induced edge labels are the first $m+n+1$ centered triangular numbers. Hence $B_{m,n}$ is a centered triangular mean graph. □

The centered triangular labeling of $B_{4,5}$ is given in Figure 4.

Figure 4

Theorem 2.4 *The coconut tree $T(n, m)$, obtained by identifying the central vertex of the star $K_{1,m}$ with a pendant vertex of a path P_n, is a centered triangular mean graph.*

Proof Let $u_0, u_1, u_2, \cdots, u_n$ be the vertices of a path, having path length $n(n \geqslant 1)$ and $v_1,\ v_2, \cdots, v_m$ be the pendant vertices being adjacent with u_0. Define $f : V(T(n, m)) \longrightarrow S$ as follows:

$f(u_0) = 0,$
$f(u_1) = 2cT_{m+1},$
$f(v_i) = 2cT_i$ for $1 \leqslant i \leqslant m,$

$f(u_j) = 2(cT_{m+j} - cT_{m+j-1} + cT_{m+j-2} - \ldots + (-1)^{j+1}cT_{m+1})$ for $2 \leqslant j \leqslant n$.

For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(u_0v_i) = cT_i$ for $1 \leqslant i \leqslant m$,
$f^*(u_0u_1) = cT_{m+1}$,
$f^*(u_ju_{j+1}) = cT_{m+j+1}$ for $1 \leqslant j \leqslant n-1$.

Then the induced edge labels are the first $m+n$ centered triangular numbers. Hence, coconut tree admits centered triangular mean labeling. □

The centered triangular mean labeling of $T(3,7)$ is given in Figure 5.

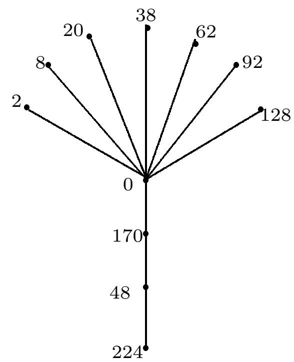

Figure 5

Theorem 2.5 *The caterpillar $S(n_1, n_2, \ldots, n_m)$ is a centered triangular mean graph.*

Proof Let $v_1, v_2, \cdots, v_m$ be the vertices of the path P_m and $v_i^j(1 \leqslant i \leqslant n_j$, $1 \leqslant j \leqslant m)$ be the pendant vertices joined with $v_j(1 \leqslant j \leqslant m)$ by an edge. Then

$$V(S(n_1, n_2, \cdots, n_m)) = \{v_j, v_i^j : 1 \leqslant i \leqslant n_j \ 1 \leqslant j \leqslant m\}$$
$$E(S(n_1, n_2, \cdots, n_m)) = \{v_tv_{t+1}, v_jv_i^j : 1 \leqslant t \leqslant m-1,\ 1 \leqslant i \leqslant n_j,\ 1 \leqslant j \leqslant m\}.$$

We define $f : V(S(n_1, n_2, \ldots, n_m)) \longrightarrow S$ as follows:

$f(v_1) = 0$, $f(v_j) = 2(cT_{j-1} - cT_{j-2} + cT_{j-3} - \cdots + (-1)^j cT_1)$ for $2 \leqslant j \leqslant m$,
$f(v_i^1) = 2cT_{m-1+i}$ for $1 \leqslant i \leqslant n_1$,
$f(v_i^j) = 2cT_{m-1+n_1+n_2+\cdots+n_{j-1}+i} + (-1)^{j-1}2(cT_1 - cT_2 + cT_3 - \cdots + (-1)^j cT_{j-1})$ for $1 \leqslant i \leqslant n_j$ and $2 \leqslant j \leqslant m$.

For the vertex label f, the induced edge label f^* is defined as follows:

$f^*(v_jv_{j+1}) = cT_j$ for $1 \leqslant j \leqslant m-1$,
$f^*(e_i^1) = cT_{m-1+i}$ for $1 \leqslant i \leqslant n_1$,
$f^*(e_i^j) = cT_{m-1+n_1+n_2+\cdots+n_{j-1}+i}$ for $1 \leqslant j \leqslant n_j$ and $2 \leqslant j \leqslant m$.

Then the edge labels are the centered triangular numbers

$$cT_1, cT_2, \cdots, cT_{m-1}, cT_m, \cdots, cT_{m-1+n_1+n_2,\cdots+n_m}$$

and also the vertex labels are different. Hence $S(n_1, n_2, \cdots, n_m)$ is a centered triangular mean graph. $\square$

The centered triangular mean labeling of $S(3,5,4,6)$ is shown in Figure 6.

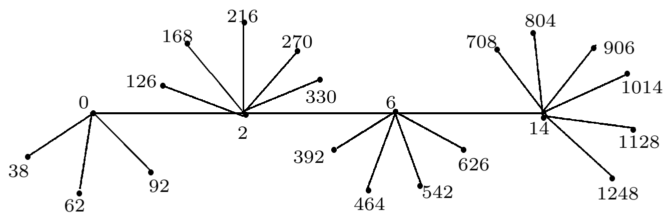

Figure 6

Theorem 2.6 *The shrub $St(n_1, n_2, \ldots, n_m)$ is a centered triangular mean graph.*

Proof let $v_0, v_j, u_i^j (1 \leqslant j \leqslant m,\ 1 \leqslant i \leqslant n_j)$ be the vertices of $St(n_1, n_2, \cdots, n_m)$. Then $E(St(n_1, n_2, \cdots, n_m)) = \{v_0v_j,\ v_ju_i^j \text{ for } 1 \leqslant i \leqslant n_j \text{ and } 1 \leqslant j \leqslant m\}$. Define $f : V(St(n_1, n_2, \cdots, n_m)) \longrightarrow S$ as follows:

$f(v_0) = 0,$
$f(v_j) = 2cT_j \quad \text{for } 1 \leqslant j \leqslant m,$
$f(u_i^j) = 2(cT_{m+n_1+n_2+\cdots+n_{j-1}+i} - cT_j)$ for $1 \leqslant i \leqslant n_j$ and $1 \leqslant j \leqslant m$.

Let $e_i^j = v_ju_i^j$ for $1 \leqslant i \leqslant n_j$ and $1 \leqslant j \leqslant m$. For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(v_0v_j) = cT_j$ for $1 \leqslant j \leqslant m$,
$f^*(e_i^j) = cT_{m+n_1+n_2+\cdots+n_{j-1}+i}$ for $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant n_j$.

Then f is a centered triangular mean labeling of $St(n_1, n_2, \cdots, n_m)$. $\square$

The centered triangular mean labeling of $St(4,5,4)$ is shown in Figure 7.

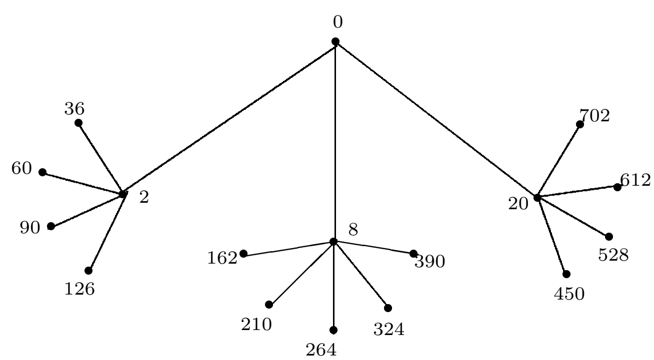

Figure 7

Theorem 2.7 *The banana tree $Bt(\underbrace{n, n, \ldots, n}_{m\ times})$ is a centered triangular mean graph.*

Proof Let $v_0, v_j, u_i^j (1 \leqslant j \leqslant m,\ 1 \leqslant i \leqslant n)$ be the vertices of $Bt(\underbrace{n, n, \cdots, n}_{\text{m times}})$. Then

$E(Bt(\underbrace{n, n, \ldots, n}_{\text{m times}})) = \{v_0 u_1^j, v_i u_i^j \text{ for } 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant m\}$. Define $f : V(Bt(\underbrace{n, n, \ldots, n}_{\text{m times}})) \longrightarrow S$ as follows:

$$f(v_0) = 0, \quad f(v_j) = 2(cT_{m+j} - cT_j) \quad \text{for } 1 \leqslant j \leqslant m,$$
$$f(u_1^j) = 2cT_j \text{ for } 1 \leqslant j \leqslant m,$$
$$f(u_i^j) = 2(cT_{2m+(j-1)(n-1)+i-1} - cT_{m+j} + cT_j) \text{ for } 2 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant m.$$

Let $e_i^j = v_j u_i^j$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. For each vertex label f, the induced edge label f^* is defined as follows:

$$f^*(v_0 u_1^j) = cT_j \text{ for } 1 \leqslant j \leqslant m,$$
$$f^*(v_j u_1^j) = cT_{m+j} \text{ for } 1 \leqslant j \leqslant m,$$
$$f^*(e_i^j) = cT_{2m+(j-1)(n-1)+(i-1)} \text{ for } 1 \leqslant j \leqslant m \text{ and } 2 \leqslant i \leqslant n.$$

Therefore f is a centered triangular mean labeling of $Bt(\underbrace{n, n, \cdots, n}_{\text{m times}})$. □

The centered triangular mean labeling of $Bt(\underbrace{3, 3, \cdots, 3}_{\text{8 times}})$ is shown in Figure 8.

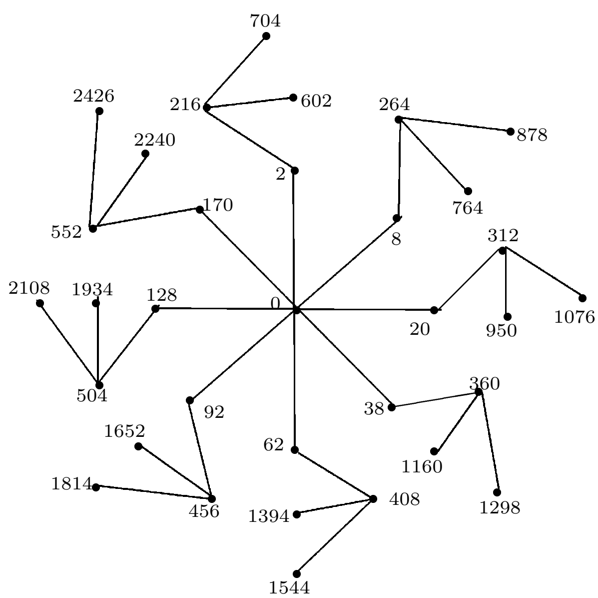

Figure 8

Theorem 2.8 *The graph $P_m@P_n$ is a centered triangular mean graph.*

proof Let $\{v_j, u_j^i,\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$ be the vertices of $P_n@P_m$ with $v_j = u_j^1$, $(1 \leqslant j \leqslant m)$. Then $E(P_n@P_m) = \{\ v_j v_{j+1}, u_j^i u_j^{i+1}\ :\ 1 \leqslant j \leqslant m-1,\ 1 \leqslant i \leqslant n-1\}$. Define $f : V(P_n@P_m) \longrightarrow S$ as follows:

$$f(u_1^1) = 0, \quad f(u_j^1) = 2(cT_{j-1} - cT_{j-2} + cT_{j-3} - \ldots + (-1)^j cT_1) \text{ for } 2 \leqslant j \leqslant m,$$

$f(u_1^2) = 2cT_m$, $f(u_j^2) = 2(cT_{m+j-1} - cT_{j-1} + cT_{j-2} - \ldots + (-1)^{j-1}cT_1)$ for $2 \leqslant j \leqslant m$,

$$\begin{aligned} f(u_j^i) &= 2(cT_{(i-1)m+j-1} - cT_{(i-2)m+j-1} + cT_{(i-3)m+j-1} - \cdots \\ &\quad +(-1)^{i-1}cT_{m+j-1}) + (-1)^{i-1}2(cT_{j-1} - cT_{j-2} + cT_{j-3} - \ldots + (-1)^j cT_1) \end{aligned}$$

for $1 \leqslant j \leqslant m$, $3 \leqslant i \leqslant n$. For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(v_j v_{j+1}) = cT_j$ for $1 \leqslant j \leqslant m-1$,
$f^*(u_j^1 u_j^2) = cT_{m+j-1}$ for $1 \leqslant j \leqslant m$,
$f^*(u_j^i u_j^{i+1}) = cT_{mi+j-1}$ for $1 \leqslant j \leqslant m$ and $2 \leqslant i \leqslant n-1$.

Therefore f is a centered triangular labeling of $P_n@P_m$. □

The centered triangular mean labeling of $P_4@P_4$ is shown in Figure 9.

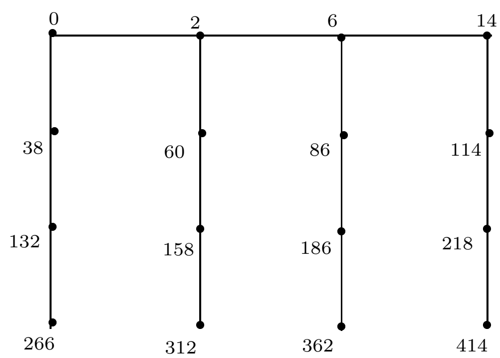

Figure 9

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New Families of Odd Mean Graphs

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Abstract: Let $G(V,E)$ be a graph with p vertices and q edges. A graph G is said to have an odd mean labeling if there exists a function $f : V(G) \to \{0,1,2,\ldots,2q-1\}$ satisfying f is $1-1$ and the induced map $f^* : E(G) \to \{1,3,5,\ldots,2q-1\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

is a bijection. A graph that admits an odd mean labeling is called an odd mean graph. In this paper, we discuss the construction of two kinds of odd mean graphs. Here we prove that $(P_n;S_1)$ for $n \geq 1$, $(P_{2n};S_m)$ for $m \geq 2, n \geq 1$, $(P_m;C_n)$ for $n \equiv 0(mod\ 4)$ and $m \geq 1$, $(P_m;Q_3)$ for $m \geq 1$, $[P_m;C_n]$ for $n \equiv 0(mod\ 4)$ and $m \geq 1$, $[P_m;Q_3]$ for $m \geq 1$ and $[P_m;C_n^{(2)}]$ for $n \equiv 0(mod\ 4)$ and $m \geq 1$ are odd mean graphs.

Key Words: Labeling, Smarandache m-module labeling, odd mean labeling, odd mean graph.

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§1. Introduction

All graphs considered here are finite, undirected and simple graph. We follow the basic notations and terminologies of graph theory as in [3]. Given a graph G, the symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of the graph G, respectively. For terminologies and notations, we follow the reference [7] with some of them mentioned in the following.

A path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. The graph $P_2 \times P_2 \times P_2$ is called a cube and is denoted by Q_3.

Let C_n be a cycle with fixed vertex v and $(P_m;C_n)$ the graph obtained from m copies of C_n and the path $P_m : u_1u_2\cdots u_m$ by joining u_i with the vertex v of the i^{th} copy of C_n by means of an edge, for $1 \leq i \leq m$.

Let Q_3 be a cube with fixed vertex v and $(P_m;Q_3)$ the graph obtained from m copies of Q_3 and the path $P_m : u_1u_2\cdots u_m$ by joining u_i with the vertex v of the i^{th} copy of Q_3 by means of an edge, for $1 \leq i \leq m$.

Let S_m be a star with vertices $v_0, v_1, v_2, \cdots, v_m$ and let $(P_{2n};S_m)$ be the graph obtained

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from $2n$ copies of S_m and the path $P_{2n} : u_1u_2\cdots u_{2n}$ by joining u_j with the vertex v_0 of the j^{th} copy of S_m by means of an edge, for $1 \leq j \leq 2n, (P_n; S_1)$ the graph obtained from n copies of S_1 and the path $P_n : u_1u_2\cdots u_n$ by joining u_j with the vertex v_0 of the j^{th} copy of S_1 by means of an edge, for $1 \leq j \leq n$.

Suppose $C_n : v_1v_2\cdots v_nv_1$ be a cycle of length n. Let $[P_m; C_n]$ be the graph obtained from m copies of C_n with vertices $v_{1_1}, v_{1_2}, \cdots, v_{1_n}, v_{2_1}, \cdots, v_{2_n}, \cdots, v_{m_1} \ldots, v_{m_n}$ and joining v_{i_j} and $v_{(i+1)_j}$ by means of an edge, for some j and $1 \leq i \leq m-1$.

Let Q_3 be a cube and $[P_m; Q_3]$ the graph obtained from m copies of Q_3 with vertices $v_{1_1}, v_{1_2}, \cdots, v_{1_8}, v_{2_1}, v_{2_2}, \cdots, v_{2_8}, \cdots, v_{m_1}, v_{m_2}, \cdots, v_{m_8}$ and the path $P_m : u_1u_2\cdots u_m$ by adding the edges $v_{1_1}v_{2_1}, v_{2_1}v_{3_1}, \cdots, v_{m-1_1}v_{m_1} (i.e) v_{i_1}v_{i+1_1}$, $1 \leq i \leq m-1$.

Let $C_n^{(2)}$ be a friendship graph. Define $\left[P_m; C_n^{(2)}\right]$ to be the graph obtained from m copies of $C_n^{(2)}$ and the path $P_m : u_1, u_2, \cdots, u_m$ by joining u_i with the center vertex of the i^{th} copy of $C_n^{(2)}$ for $1 \leq i \leq m$.

The graceful labelings of graphs was first introduced by Rosa in 1961 [1] and R.B. Gnanajothi introduced odd graceful graphs [2]. The concept of mean labeling was first introduced and studied by S. Somasundaram and R. Ponraj [8]. Further some more results on mean graphs are discussed in [6, 7, 10, 11]. The concept of odd mean labeling was introduced and studied by K. Manickam and M. Marudai [4]. Some more results on odd mean graphs are discussed in [9, 12, 13].

In [4], R. Vasuki et al. introduced the concept of even vertex odd mean labeling and studied even vertex odd meanness of some standard graphs. In [5], some construction of even vertex odd mean graphs are discussed and proved that $(P_n; S_1)$ for $n \geq 1, (P_{2n}; S_m)$ for $m \geq 2, n \geq 1, (P_m; C_n)$ for $n \equiv 0(mod\ 4), m \geq 1, [P_m; C_n]$ for $n \equiv 0(mod\ 4), m \geq 1$ and $[P_m; C_n^{(2)}]$ for $n \equiv 0(mod\ 4), m \geq 1$ are even vertex odd mean graphs.

A graph G with p vertices and q edges is said to have an even vertex odd mean labeling if there exists an injective function $f : V(G) \to \{0, 2, 4, \cdots, 2q-2, 2q\}$ such that the induced map $f^* : E(G) \to \{1, 3, 5, \cdots, 2q-1\}$ defined by

$$f^*(uv) = \frac{f(u) + f(v)}{2}$$

is a bijection. A graph that admits an even vertex odd mean labeling is called an even vertex odd mean graph. Generally, if there exists an injective function $f : V(G) \to \{0, m, 2m, \cdots, mq - m, mq\}$ such that the induced map $f^* : E(G) \to \{m-1, 2m-1, 3m-1, \cdots, mq-1\}$ defined by

$$f^*(uv) = \frac{f(u) + f(v)}{m}$$

is a bijection, G is said to have a *Smarandache m-module labeling*, where $m \geq 1$ is an integer. Clearly, a Smarandache 2-module labeling is an even vertex odd mean labeling on G.

A graph G is said to have an odd mean labeling if there exists a function $f : V(G) \to \{0, 1, 2, \cdots, 2q-1\}$ satisfying f is $1-1$ and the induced map $f^* : E(G) \to \{1, 3, 5, \cdots, 2q-1\}$

defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

is a bijection. A graph that admits an odd mean labeling is called an odd mean graph [4]. An odd mean labeling of $K_{2,5}$ is given in Figure 1.

In this paper, we prove that, the graphs $(P_n; S_1)$ for $n \geq 1$, $(P_{2n}; S_m)$ for $m \geq 2, n \geq 1$, $(P_m; C_n)$ for $n \equiv 0(mod\ 4)$ and $m \geq 1$, $(P_m; Q_3)$ for $m \geq 1$, $[P_m; C_n]$ for $n \equiv 0(mod\ 4)$ and $m \geq 1$, $[P_m; Q_3]$ for $m \geq 1$ and $\left[P_m; C_n^{(2)}\right]$ for $n \equiv 0(mod\ 4)$ and $m \geq 1$ are odd mean graphs.

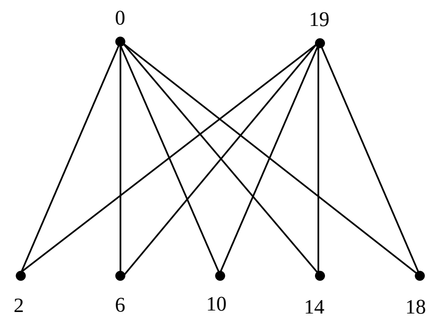


Figure 1

§2. **Odd Mean Graphs** $(P_m; G)$

Let G be a graph with fixed vertex v and let $(P_m; G)$ be the graph obtained from m copies of G and the path $P_m : u_1u_2 \cdots u_m$ by joining u_i with the vertex v of the i^{th} copy of G by means of an edge, for $1 \leq i \leq m$. For example, $(P_4; C_4)$ is shown in Figure 2.

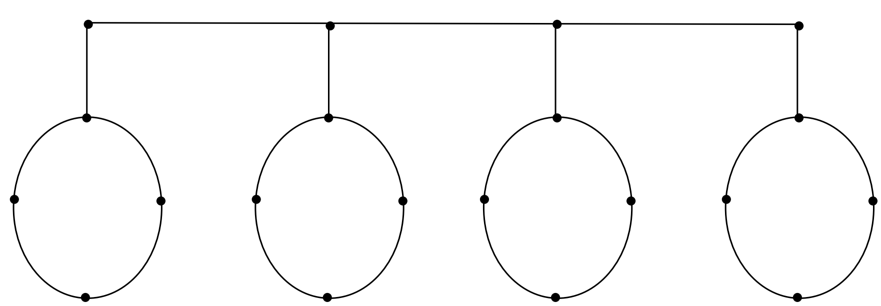


Figure 2

Theorem 2.1 *$(P_{2n}; S_m)$ is an odd mean graph, $m \geq 2, n \geq 1$.*

Proof Let $u_1, u_2, \cdots, u_{2n}$ be the vertices of P_{2n}. Let $v_{0_j}, v_{1_j}, v_{2_j}, \cdots, v_{m_j}$ be the vertices of the j^{th} copy of S_m, where v_{0_j} is the center vertex, $1 \leq j \leq 2n$. Define $f : V(P_{2n}; S_m) \rightarrow$

$\{0, 1, 2, \cdots, 2q-2, 2q-1 = 4n(m+2)-3\}$ as follows:

$$f(u_j) = \begin{cases} (2m+4)(j-1)+2, & 1 \le j \le 2n \text{ and } j \text{ is odd} \\ (2m+4)j-4, & 1 \le j \le 2n \text{ and } j \text{ is even} \end{cases}$$

$$f(v_{0_j}) = \begin{cases} (2m+4)(j-1), & 1 \le j \le 2n \text{ and } j \text{ is odd} \\ (2m+4)j-3, & 1 \le j \le 2n \text{ and } j \text{ is even} \end{cases}$$

$$f(v_{i_j}) = \begin{cases} (2m+4)(j-1)+4i+2, & 1 \le i \le m, 1 \le j \le 2n \text{ and } j \text{ is odd} \\ (2m+4)(j-2)+4i, & 1 \le i \le m, 1 \le j \le 2n \text{ and } j \text{ is even.} \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is obtained as follows:

$$f^*(u_j u_{j+1}) = (2m+4)j-1, \quad 1 \le j \le 2n-1$$

$$f^*(u_j v_{0_j}) = \begin{cases} (2m+4)(j-1)+1, & 1 \le j \le 2n \text{ and } j \text{ is odd} \\ (2m+4)j-3, & 1 \le j \le 2n \text{ and } j \text{ is even} \end{cases}$$

$$f^*(v_{0_j} v_{i_j}) = \begin{cases} (2m+4)(j-1)+2i+1, & 1 \le i \le m, 1 \le j \le 2n \\ & \text{and } j \text{ is odd} \\ (2m+4)(j-1)+2i-1, & 1 \le i \le m, 1 \le j \le 2n \text{ and } j \text{ is even.} \end{cases}$$

It can be verified that f is an odd mean labeling and hence $(P_{2n}; S_m)$ is an odd mean graph for $n \ge 1$ and $m \ge 2$. For example, an odd mean labeling of $(P_6; S_4)$ is shown in Figure 3. $\square$

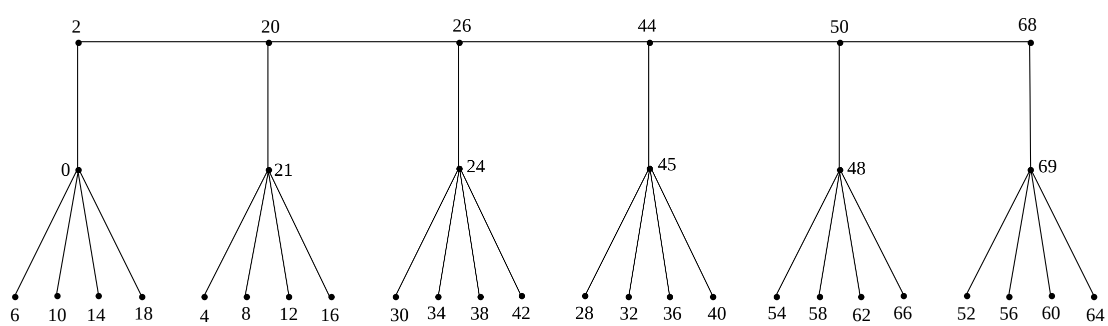


Figure 3

Theorem 2.2 *The graph $(P_n; S_1)$ $n \ge 1$ is an odd mean graph.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of P_n. Let v_{0_j} and v_{1_j} be the vertices. Define

$f : V(P_n; S_1) \to \{0, 1, 2, \cdots, 2q-2, 2q-1 = 6n-3\}$ as follows:

$$f(u_j) = \begin{cases} 6j-6, & 1 \le j \le n \text{ and } j \text{ is odd} \\ 6j-3, & 1 \le j \le n \text{ and } j \text{ is even} \end{cases}$$

$$f(v_{0_j}) = 6j-4, \quad 1 \le j \le n$$

$$f(v_{1_j}) = \begin{cases} 6j-3, & 1 \le j \le n \text{ and } j \text{ is odd} \\ 6j-7, & 1 \le j \le n \text{ and } j \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows:

$$f^*(u_j u_{j+1}) = 6j-1, \quad 1 \le j \le n-1$$

$$f^*(u_j v_{0_j}) = \begin{cases} 6j-5, & 1 \le j \le n \text{ and } j \text{ is odd} \\ 6j-3, & 1 \le j \le n \text{ and } j \text{ is even} \end{cases}$$

$$f^*(v_{0_j} v_{1_j}) = \begin{cases} 6j-3, & 1 \le j \le n \text{ and } j \text{ is odd} \\ 6j-5, & 1 \le j \le n \text{ and } j \text{ is even.} \end{cases}$$

Thus, f is an odd mean labeling. Hence, $(P_n; S_1)$ is an odd mean graph for any $n \ge 1$. For example, an odd mean labeling of $(P_8; S_1)$ is shown in Figure 4. $\square$

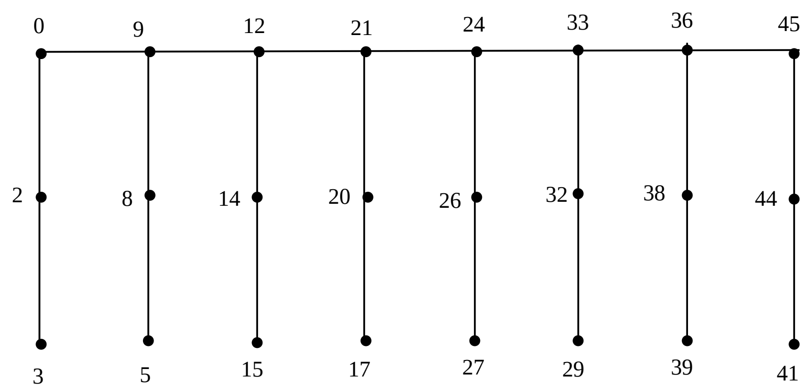


Figure 4

Theorem 2.3 *$(P_m; C_n)$ is an odd mean graph, for $n \equiv 0 (mod\ 4)$ and $m \ge 1$.*

Proof Let $v_{i_1}, v_{i_2}, \cdots, v_{i_n}$ be the vertices in the i^{th} copy of C_n, $1 \le i \le m$ and $u_1, u_2, \cdots, u_m$ be the vertices of P_m. In $(P_m; C_n)$, u_i is joined with v_{i_1} by an edge, for each $i, 1 \le i \le m$. Define $f : V(P_m; C_n) \to \{0, 1, 2, \cdots, 2q-1 = (2n+4)m-3\}$ as follows:

$$f(u_i) = \begin{cases} 2(n+2)(i-1) & \text{if } i \text{ is odd and } 1 \le i \le m \\ 2(n+2)i-3 & \text{if } i \text{ is even and } 1 \le i \le m \end{cases}$$

and for $1 \le i \le m$, i is odd,

$$f(v_{i_j}) = \begin{cases} 2(n+2)(i-1)+2j, & 1 \le j \le \frac{n}{2} \\ 2(n+2)(i-1)+2j+3, & \frac{n}{2}+1 \le j \le n \text{ and } j \text{ is odd} \\ 2(n+2)(i-1)+2j, & \frac{n}{2}+2 \le j \le n \text{ and } j \text{ is even} \end{cases}$$

and for $1 \le i \le m$, i is even,

$$f(v_{i_j}) = \begin{cases} 2(n+2)i-2(j+1), & 1 \le j \le \frac{n}{2} \\ 2(n+2)i-2(j+3), & \frac{n}{2}+1 \le j \le n \text{ and } j \text{ is odd} \\ 2(n+2)i-2(j+1), & \frac{n}{2}+2 \le j \le n \text{ and } j \text{ is even.} \end{cases}$$

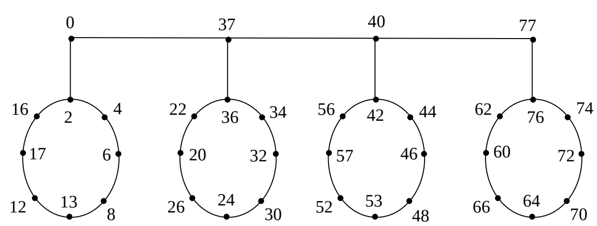
The induced edge labels are obtained by $f^*(u_iu_{i+1}) = 2i(n+2)-1$ for integers $1 \le i \le m-1$, and for $1 \le i \le m$, i is odd, $f^*(v_{i_n}v_{i_1}) = 2(n+2)(i-1)+n+1$, $f^*(u_iv_{i_1}) = 2(n+2)(i-1)+1$,

$$f^*(v_{i_j}v_{i_{(j+1)}}) = \begin{cases} 2(n+2)(i-1)+2j+1, & 1 \le j \le \frac{n}{2}-1 \\ 2(n+2)(i-1)+2j+3, & \frac{n}{2} \le j \le n-1 \end{cases}$$

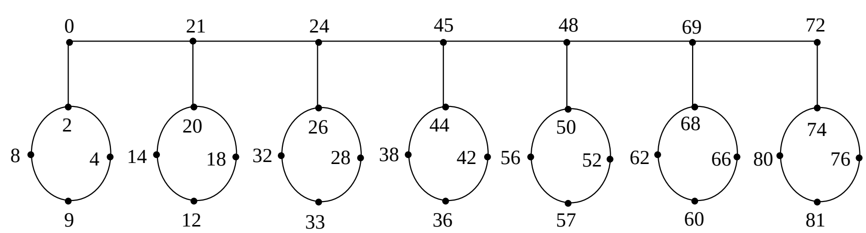
and for $1 \le i \le m$, i is even, $f^*(v_{i_n}v_{i_1}) = 2(n+2)i-n-3$, $f^*(u_iv_{i_1}) = 2(n+2)i-3$,

$$f^*(v_{i_j}v_{i_{(j+1)}}) = \begin{cases} 2(n+2)i-(2j+3), & 1 \le j \le \frac{n}{2}-1 \\ 2(n+2)i-(2j+5), & \frac{n}{2} \le j \le n-1 \end{cases}$$

Thus, f is an odd mean labeling and hence $(P_m; C_n)$ is an odd mean graph for $n \equiv 0 (mod\, 4)$, $m \ge 1$. For example, an odd mean labeling of $(P_4; C_8)$ and $(P_7; C_4)$ are shown in Figure 5. □



$(P_4; C_8)$



$(P_7; C_4)$

Figure 5

Theorem 2.4 $(P_m; Q_3), m \ge 1$ *is an odd mean graph.*

Proof For $1 \leq j \leq 8$, let v_{i_j} be the vertices in the i^{th} copy of $Q_3, 1 \leq i \leq m$ and $u_1, u_2, \cdots, u_m$ be the vertices of P_m.

The vertices of $(P_m; Q_3)$ are denoted as in Figure 6.

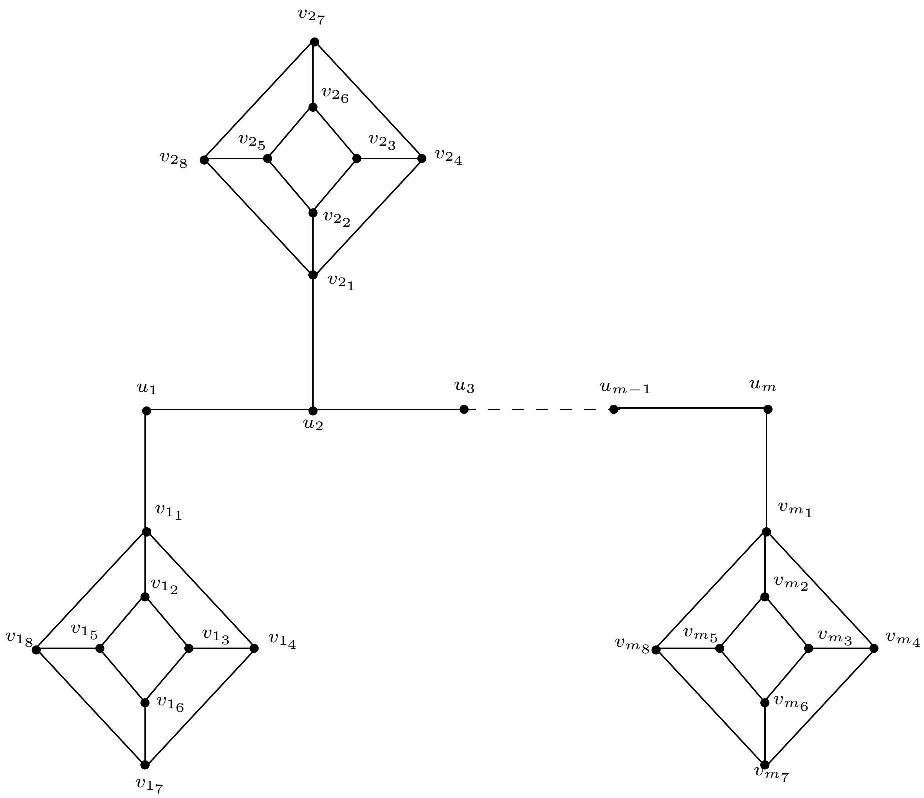


Figure 6

Define $f : V(P_m; Q_3) \to \{0, 1, 2, \cdots, 2q-2, 2q-1 = 28m-3\}$ as follows:

$$f(u_i) = \begin{cases} 28i - 28, & 1 \leq i \leq m \text{ and } i \text{ is odd} \\ 28i - 3, & 1 \leq i \leq m \text{ and } i \text{ is even,} \end{cases}$$

when i is odd,

$$f(v_{i_j}) = (28i - 28) + 2j, \quad 1 \leq i \leq m, \quad j = 1, 2, 4$$
$$f(v_{i_3}) = 28i - 18, \quad 1 \leq i \leq m$$
$$f(v_{i_j}) = 28i - 20 + 2j, \quad 1 \leq i \leq m, \quad j = 5, 6, 8$$
$$f(v_{i_7}) = 28i - 3, \quad 1 \leq i \leq m$$

and when i is even,

$$f(v_{i_j}) = 28i - (2j+2), \quad 2 \leq i \leq m, \quad 1 \leq j \leq 3$$
$$f(v_{i_4}) = 28i - 14, \quad 2 \leq i \leq m$$
$$f(v_{i_j}) = 28i - (2j+10), \quad 2 \leq i \leq m, \quad 5 \leq j \leq 7$$
$$f(v_{i_8}) = 28i - 30, \quad 2 \leq i \leq m.$$

The label of the edge u_iu_{i+1} is $28i-1, 1 \leq i \leq m-1$ and for $1 \leq i \leq m$, the label of the edge

$$u_iv_{i_1} = \begin{cases} 28i-27 & \text{if } i \text{ is odd} \\ 28i-3 & \text{if } i \text{ is even} \end{cases}$$

The label of the edges of the i^{th} copy of Q_3 are $28i-3, 28i-5, \ldots, 28i-25$ if i is odd and $28i-5, 28i-7, \ldots, 28i-27$ if i is even. Thus, $(P_m; Q_3), m \geq 1$ is an odd mean graph. For example, an odd mean labeling of $(P_5; Q_3)$ is shown in Figure 7. □

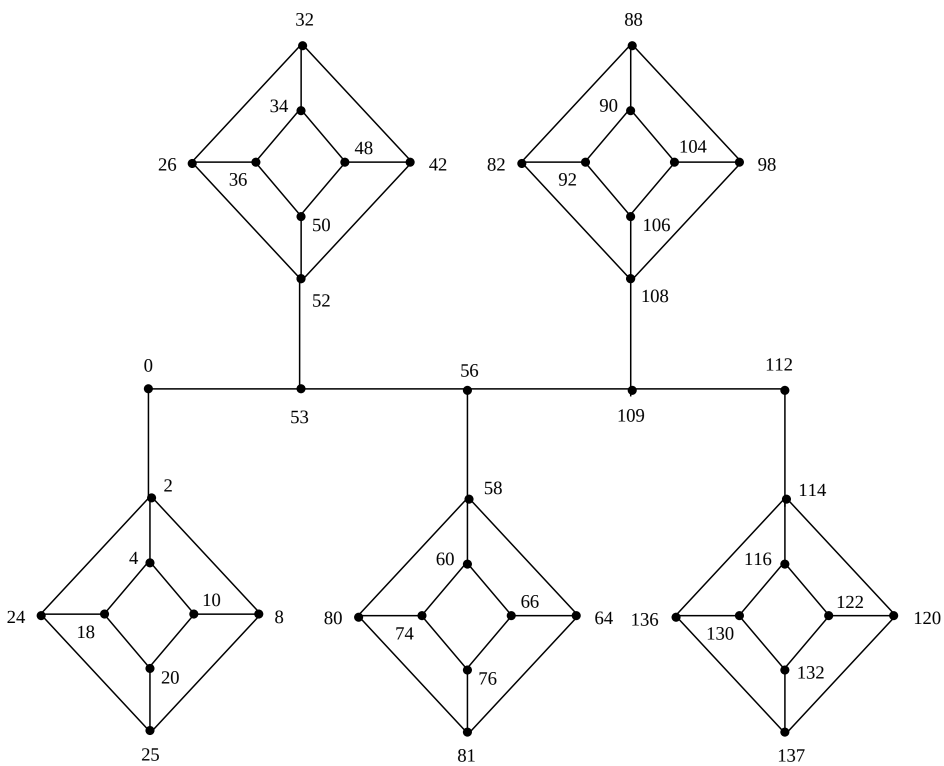


Figure 7

§3. Odd Mean Graphs $[P_m; G]$

Let G be a graph with fixed vertex v and let $[P_m; G]$ be the graph obtained from m copies of G by joining v_{i_j} and $v_{(i+1)_j}$ by means of an edge for some j and $1 \leq i \leq m-1$. For example, $[P_5; C_4]$ is shown in Figure 8.

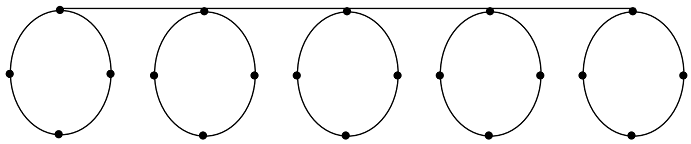


Figure 8

Theorem 3.1 *$[P_m; C_n]$ is an odd mean graph, for $n \equiv 0(mod\ 4)$ and $m \geq 1$.*

Proof Let $u_1, u_2, \cdots, u_m$ be the vertices of P_m. Let $v_{i_1}, v_{i_2}, \cdots, v_{i_n}$ be the vertices of the i^{th} copy of C_n, $1 \leq i \leq m$ and joining $v_{i_j}(= u_i)$ and $v_{(i+1)_j}(= u_{i+1})$ by means of an edge, for some j. Define $f : V([P_m; C_n]) \rightarrow \{0, 1, 2, \cdots, 2q-2, 2q-1 = (2n+2)m-3\}$ as follows:

for $1 \leq i \leq m$ and i is odd,

$$f(v_{i_j}) = \begin{cases} 2(n+1)(i-1)+2j-2, & 1 \leq j \leq \frac{n}{2} \\ 2(n+1)(i-1)+2j+1, & \frac{n}{2}+1 \leq j \leq n \text{ and } j \text{ is odd} \\ 2(n+1)(i-1)+2j-2, & \frac{n}{2}+2 \leq j \leq n \text{ and } j \text{ is even} \end{cases}$$

and for $1 \leq i \leq m$ and i is even,

$$f(v_{i_1}) = 2(n+1)i - 3$$

$$f(v_{i_j}) = \begin{cases} 2(n+1)i-2j, & 2 \leq j \leq \frac{n}{2} \\ 2(n+1)i-2(j+2), & \frac{n}{2}+1 \leq j \leq n \text{ and } j \text{ is odd} \\ 2(n+1)i-2j, & \frac{n}{2}+2 \leq j \leq n \text{ and } j \text{ is even.} \end{cases}$$

The induced edge labels are obtained $f^*(v_{i_1}v_{(i+1)_1}) = 2(n+1)i-1, 1 \leq i \leq m-1$ and for $1 \leq i \leq m$, i is odd,

$$f^*(v_{i_j}v_{i_{(j+1)}}) = \begin{cases} 2(n+1)(i-1)+2j-1, & 1 \leq j \leq \frac{n}{2}-1 \\ 2(n+1)(i-1)+2j+1, & \frac{n}{2} \leq j \leq n-1 \end{cases}$$

$$f^*(v_{i_n}v_{i_1}) = 2(n+1)i-(n+3)$$

and for $1 \leq i \leq m$, i is even,

$$f^*(v_{i_j}v_{i_{(j+1)}}) = \begin{cases} 2(n+1)i-2j-1, & 1 \leq j \leq \frac{n}{2}-1 \\ 2(n+1)i-2j-3, & \frac{n}{2} \leq j \leq n-1 \end{cases}$$

$$f^*(v_{i_n}v_{i_1}) = 2(n+1)i-(n+1).$$

Thus, f is an odd mean labeling and hence $[P_m; C_n]$ is an odd mean graph for $n \equiv 0(mod\ 4)$ and $m \geq 1$. For example, an odd mean labeling of $[P_5; C_8]$ is shown in Figure 9. □

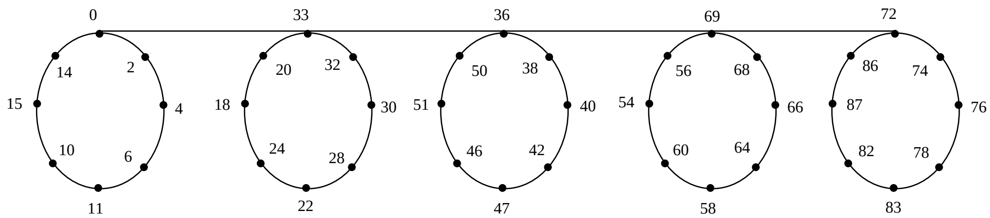


Figure 9

Theorem 3.2 *$[P_m; Q_3]$ is an odd mean graph.*

Proof For $1 \leq j \leq 8$, Let v_{i_j} be the vertices in the i^{th} copy of $Q_3, 1 \leq i \leq m$. Then $f : V([P_m; Q_3]) \rightarrow \{0, 1, 2, \cdots, 2q-2, 2q-1 = 26m-3\}$ as follows:

when i is odd,

$$f(v_{i_j}) = 26i + 2j - 28, \quad 1 \leq i \leq m, \quad j = 1, 2, 4$$
$$f(v_{i_3}) = 26i - 18, \quad 1 \leq i \leq m$$
$$f(v_{i_j}) = 26i + 2j - 20, \quad 1 \leq i \leq m, \quad j = 5, 6, 8$$
$$f(v_{i_7}) = 26i - 3, \quad 1 \leq i \leq m,$$

when i is even,

$$f(v_{i_1}) = 26i - 3, \quad 2 \leq i \leq m$$
$$f(v_{i_j}) = 26i - 2j, \quad 2 \leq i \leq m, \quad 2 \leq j \leq 3$$
$$f(v_{i_4}) = 26i - 12, \quad 2 \leq i \leq m$$
$$f(v_{i_j}) = 26i - 2j - 8, \quad 2 \leq i \leq m, \quad 5 \leq j \leq 7$$
$$f(v_{i_8}) = 26i - 28, \quad 2 \leq i \leq m.$$

The label of the edge $v_{i_1}v_{(i+1)_1}$ is $26i-1, 1 \leq i \leq m-1$. The label of the edges of the cube are $26i-3, 26i-5, \cdots, 26i-25, 1 \leq i \leq m$. For example, an odd mean labeling of $[P_4; Q_3]$ is shown in Figure 10. □

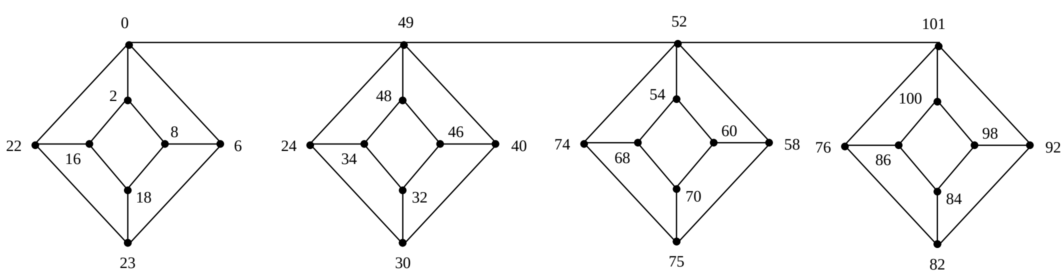


Figure 10

Theorem 3.3 *$\left[P_m; C_n^{(2)}\right]$ is an odd mean graph for $n \equiv 0 (mod\ 4)$ and $m \geq 1$.*

Proof Let $u_1, u_2, \cdots, u_m$ be the vertices of P_m and each vertex $u_i, 1 \leq i \leq m$ is attached with the common vertex in the i^{th} copy of $C_n^{(2)}$. Let v'_{i_j} and v''_{i_j} for $1 \leq j \leq n$ be the vertices in the i^{th} copy of $C_n^{(2)}$ in which $v'_{i_1} = v''_{i_1}, 1 \leq i \leq m$. Define $f : V\left(\left[P_m; C_n^{(2)}\right]\right) \rightarrow \{0, 1, 2, \ldots, 2q-2, 2q-1 = (4n+2)m-3\}$ as follows:

for $1 \leq i \leq m$,

$$f(v'_{i_j}) = \begin{cases} (4n+2)i - 2(n+j), & 1 \leq j \leq 2 \\ (4n+2)(i-1) + 2j - 6, & 3 \leq j \leq \frac{n}{2} + 2 \\ (4n+2)(i-1) + 2j - 3, & \frac{n}{2} + 3 \leq j \leq n \text{ and } j \text{ is odd} \\ (4n+2)(i-1) + 2j - 6, & \frac{n}{2} + 3 \leq j \leq n \text{ and } j \text{ is even} \end{cases}$$

$$f(v''_{i_j}) = \begin{cases} (4n+2)i - 2(n-j+2), & 2 \leq j \leq \frac{n}{2} \\ (4n+2)i - 2n + 2j - 1, & \frac{n}{2} + 1 \leq j \leq n \text{ and } j \text{ is odd} \\ (4n+2)i - 2(n-j+2), & \frac{n}{2} + 2 \leq j \leq n \text{ and } j \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows:

for $1 \leq i \leq m$,

$$f^*(v'_{i_1} v'_{i_2}) = (4n+2)i - (2n+3)$$
$$f^*(v'_{i_2} v'_{i_3}) = (4n+2)i - 3(n+1)$$
$$f^*(v'_{i_j} v'_{i_{(j+1)}}) = \begin{cases} (4n+2)(i-1) + 2j - 5, & 3 \leq j \leq \frac{n}{2} + 1 \\ (4n+2)(i-1) + 2j - 3, & \frac{n}{2} + 2 \leq j \leq n \end{cases}$$
$$f^*(v''_{i_j} v''_{i_{(j+1)}}) = \begin{cases} (4n+2)i - 2n + 2j - 5, & 1 \leq j \leq \frac{n}{2} - 1 \\ (4n+2)i - 2n + 2j - 1, & \frac{n}{2} \leq j \leq n \end{cases}$$
$$f^*(v''_{i_n} v''_{i_1}) = (4n+2)i - (n+3).$$

Thus, f is an odd mean labeling. Hence, $\left[P_m; C_n^{(2)}\right]$ is an odd mean graph for $n \equiv 0(mod\, 4)$ and $m \geq 1$. For example, an odd mean labeling of $\left[P_4; C_8^{(2)}\right]$ is shown in Figure 11. □

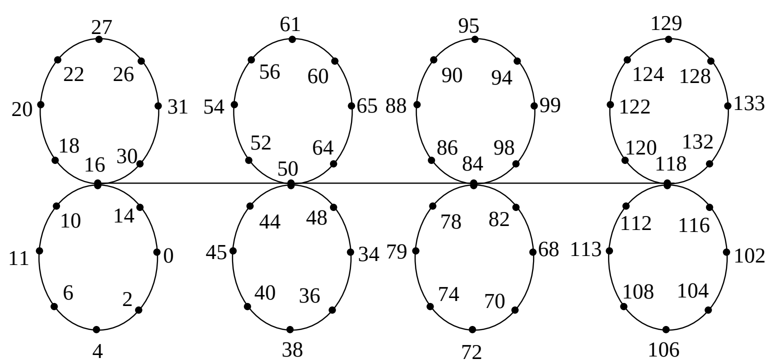


Figure 11

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F-Root Square Mean Labeling of Some Graphs

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Abstract: A function f is called $F-$ *root square mean labeling* of a graph $G(V,E)$ with p vertices and q edges if $f: V(G) \to \{1,2,\ldots,q+1\}$ is injective and the induced function f^* is defined as $f^*(uv) = \left\lfloor \sqrt{\frac{f(u)^2+f(v)^2}{2}} \right\rfloor$ for all $uv \in E(G)$ is bijective. A graph that admits a $F-$root square mean labeling is called a $F-$*root square mean graph.* In this paper, we study the $F-$root mean square meanness of triangular snake,$A(T_n)$, $D(T_n)$, quadrilateral snake, $A(Q_n)$, $D(Q_n)$.

Key Words: Triangular snake, double triangular snake, quadrilateral snake, double quadrilateral snake, $F-$root square mean labeling, Smarandache m-root mean labeling, $F-$root square mean graph.

AMS(2010): 05C12.

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [3]. For a detailed survey on graph labeling, we refer [2].

The concept of root square mean labeling was introduced and studied by S.S.Sandhya et. al [4]. Motivated by the works of so many others in the area of graph labeling, the concept of $F-$root square mean labeling was introduced by S.Arockiaraj et.al. [1].

A function f is called $F-$ *root square mean labeling* of a graph $G(V,E)$ with p vertices and q edges if $f: V(G) \to \{1,2,\cdots,q+1\}$ is injective and the induced function f^* is defined as

$$f^*(uv) = \left\lfloor \sqrt{\frac{f^2(u)+f^2(v)}{2}} \right\rfloor$$

for all $uv \in E(G)$ is bijective. Generally, if $f: V(G) \to \{m, 2m, \cdots, qm+1\}$ and

$$f^*(uv) = \left\lfloor \sqrt[m]{\frac{f^m(u)+f^m(v)}{m}} \right\rfloor$$

for all $uv \in E(G)$ is bijective, then f is called a *Smarandache m-root mean labeling*, where

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$m \geq 1$ is an integer. Clearly, a Smarandache 2-root mean labeling is nothing else but the $F-$ root square mean labeling of G. A graph that admits a $F-$root square mean labeling is called a *$F-$root square mean graph.*

In this paper, we study the $F-$root mean square meanness of triangular snake, $A(T_n)$, $D(T_n)$, quadrilateral snake, $A(Q_n)$ and $D(Q_n)$.

§2. Main Results

Theorem 2.1 *The triangular snake $T_n, n \geq 2$ is a $F-$root square mean graph.*

Proof Let $\{u_i, 1 \leq i \leq n-1, v_i, 1 \leq i \leq n\}$ be the vertices and $\{e_i, 1 \leq i \leq n-1, a_i, 1 \leq i \leq 2(n-1)\}$ the edges of T_n. First we label $f : V(G) \to \{1, 2, \cdots, q+1\}$ on vertices of T_n by $f(u_i) = 3i-2$ if $1 \leq i \leq n-1$ and $f(v_i) = 3i-1$ if $1 \leq i \leq n$. Then the induced edge labels are $f^*(e_i) = 3i-1$ if $1 \leq i \leq n-1$, and if $1 \leq i \leq 2(n-1)$,

$$f^*(a_i) = \begin{cases} \frac{3i-1}{2} & i \text{ is odd} \\ \frac{3i}{2} & i \text{ is even} \end{cases}$$

Hence, f is a $F-$root square mean labeling of the graph T_n. Thus the graph triangular snake $T_n, n \geq 2$ is a $F-$root square mean graph. □

Theorem 2.2 *The alternative triangular snake $A(T_n), n \geq 4$ is a $F-$root square mean graph.*

Proof Let $\{u_i, 1 \leq i \leq \frac{n}{2}, v_i, 1 \leq i \leq n\}$ be the vertices and $\{e_i, 1 \leq i \leq n-1, a_i, 1 \leq i \leq n\}$ the edges of $A(T_n)$. First we label $f : V(G) \to \{1, 2, \cdots, q+1\}$ of vertices of $A(T_n)$ by $f(u_i) = 4i-2$ if $1 \leq i \leq \frac{n}{2}$ and for $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} 2i-1 & i \text{ is odd} \\ 2i & i \text{ is even} \end{cases}$$

Then the induced edge labels are respectively $f^*(e_i) = 2i$ if $1 \leq i \leq n-1$ and $f^*(a_i) = 2i-1$ if $1 \leq i \leq n$. Hence, f is a $F-$root square mean labeling of the graph $A(T_n)$. Thus the graph Alternative triangular snake $A(T_n), n \geq 4$ is a $F-$root square mean graph. □

Theorem 2.3 *The double triangular snake $D(T_n), (n \geq 2)$ is a $F-$root square mean graph.*

Proof Let $\{u_i, 1 \leq i \leq n, v_i^{'}, v_i, 1 \leq i \leq n-1\}$ be the vertices and $\{a_i, 1 \leq i \leq n-1, b_i, c_i, 1 \leq i \leq 2(n-1)$ the edges of $D(T_n)$. First we label $f : V(G) \to \{1, 2, \cdots, q+1\}$ of vertices by $f(u_1) = 1,\ f(u_2) = 4,\ f(u_i) = 5i-4$ if $3 \leq i \leq n$ and $f(v_1) = 6,\ f(v_2) = 10,\ f(v_i) = 5i-1$ if $3 \leq i \leq n-14,\ f(u_1^{'}) = 2, f(u_2^{'}) = 8,\ f(u_i^{'}) = 5i-3$ if $3 \leq i \leq n-1$. Then the induced edge labels are $f^*(a_1) = 2$, $f^*(a_i) = 5i-2$ for $2 \leq i \leq n-1$, $f^*(b_1) = 1, f^*(b_2) = 3$ for $3 \leq i \leq 2n-2$,

$$f^*(b_i) = \begin{cases} \frac{5i-3}{2} & i \text{ is odd} \\ \frac{5i-1}{2} & i \text{ is even} \end{cases}$$

$f^*(c_1) = 4, f^*(c_2) = 5$ and

$$f^*(c_i) = \begin{cases} \frac{5i-1}{2} & i \text{ is odd} \\ \frac{5i}{2} & i \text{ is even} \end{cases}$$

for $3 \leq i \leq 2n-2$. Hence, f is a $F-$root square mean labeling of the graph $D(T_n)$. Thus the graph double triangular snake $D(T_n), (n \geq 2)$ is a $F-$root square mean graph. □

Theorem 2.4 *The quadrilateral snake Q_n is a $F-$root square mean graph.*

Proof Let $\{u_i, 1 \leq i \leq 2(n-1), v_i, 1 \leq i \leq n\}$ be the vertices and $\{a_i, 1 \leq i \leq n-1, b_i, 1 \leq i \leq 2(n-1), c_i, 1 \leq i \leq n-1\}$ the edges of Q_n. First we define $f : V(G) \to \{1, 2, \ldots, q+1\}$ of vertices by $f(u_1) = 1, f(u_2) = 2$,

$$f(u_i) = \begin{cases} 2i & i \text{ is odd} \\ 2i-1 & i \text{ is even} \end{cases}$$

for $3 \leq i \leq 2(n-1)$, $f(v_1) = 3$, $f(v_i) = 4i-3$ for $2 \leq i \leq n$. Then the induced edge labels are respectively $f^*(a_1) = 1$, $f^*(a_i) = 4i-2$ for $2 \leq i \leq n-1$ and $f^*(b_1) = 2, f^*(b_2) = 3$,

$$f^*(b_i) = \begin{cases} 2i-1 & i \text{ is odd} \\ 2i & i \text{ is even} \end{cases}$$

for $3 \leq i \leq 2(n-1)$, $f^*(c_1) = 4$, $f^*(c_i) = 4i-1$ for $2 \leq i \leq n-1$. Hence, f is a $F-$root square mean labeling of the graph Q_n. Thus the graph quadrilateral snake Q_n is a $F-$root square mean graph. □

Theorem 2.5 *The alternative quadrilateral snake $A(Q_n)$ is a $F-$root square mean graph.*

Proof Let $\{u_i, 1 \leq i \leq n, v_i, 1 \leq i \leq n\}$ be the vertices and $\{a_i, 1 \leq i \leq n-1, b_i, 1 \leq i \leq n, c_i, 1 \leq i \leq \frac{n}{2}\}$ the edges of $A(Q_n)$. We define $f : V(G) \to \{1, 2, \cdots, q+1\}$ of vertices by $f(u_1) = 1, f(u_2) = 2$,

$$f(u_i) = \begin{cases} \frac{5i-1}{2} & i \text{ is odd} \\ \frac{5i-4}{2} & i \text{ is even} \end{cases}$$

for $3 \leq i \leq n$ and $f(v_1) = 3$,

$$f(v_i) = \begin{cases} \frac{5i}{2} & i \text{ is odd} \\ \frac{5i-3}{2} & i \text{ is even} \end{cases}$$

for $2 \leq i \leq n$. Then the induced edge labels are respectively $f^*(a_1) = 1$, $f^*(a_i) = 5i-3$ for

$2 \leq i \leq n-1$, $f^*(b_1) = 2, f^*(b_2) = 3$,

$$f^*(b_i) = \begin{cases} \frac{5i-3}{2} & i \text{ is odd} \\ \frac{5i-2}{2} & i \text{ is even} \end{cases}$$

for $3 \leq i \leq 2(n-1)$, $f^*(c_1) = 4$ and $f^*(c_i) = 5i-2$ for $2 \leq i \leq n-1$. Hence, f is a $F-$root square mean labeling of the graph $A(Q_n)$. Thus the graph alternative quadrilateral snake $A(Q_n)$ is a $F-$root square mean graph. □

Theorem 2.6 *The double quadrilateral snake $D(Q_n), (n \geq 3)$ is a $F-$root square mean graph.*

Proof Let $\{u_i, 1 \leq i \leq 2(n-1), v_i, 1 \leq i \leq n\}$ be the vertices and $\{a_i, 1 \leq i \leq n-1, b_i, 1 \leq i \leq 2(n-1), c_i, 1 \leq i \leq n-1\}$ the edges of $D(Q_n)$. We label define $f : V(G) \to \{1, 2, \cdots, q+1\}$ of vertices by $f(u_1) = 1, f(u_2) = 4$,

$$f(u_i) = \begin{cases} 2i & i \text{ is odd} \\ 2i-1 & i \text{ is even} \end{cases}$$

for $3 \leq i \leq 2n-2$ and $f(v_1) = 3$, $f(v_i) = 4i-3$ for $2 \leq i \leq n$. Then the induced edge labels are respectively $f^*(a_1) = 1$, $f^*(a_i) = 4i-2$ for $2 \leq i \leq n-1$ and $f^*(b_1) = 2, f^*(b_2) = 3$,

$$f^*(b_i) = \begin{cases} 2i-1 & i \text{ is odd} \\ 2i & i \text{ is even} \end{cases}$$

for $3 \leq i \leq 2n-2$, $f^*(c_1) = 4, f^*(c_2) = 5$ and $f^*(c_i) = 4i-1$ for $2 \leq i \leq n-1$. Hence, f is a $F-$root square mean labeling of the graph $D(Q_n)$. Thus the graph double quadrilateral snake $D(Q_n), (n \geq 3)$ is a $F-$root square mean graph. □

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Today, the greatest crisis facing human beings is not the poverty or unfair allocation of natural resources but the greed with ignorance, and hopefully to govern the world by their own understanding or a realization dependent on local or partial perception of the nature such as those the overuse of resources, genetic engineering and the abusing of farm chemicals, internet, $\cdots$, etc., and out of the crisis needs the human self-awareness, i.e., abandoning their arrogance and developing harmoniously with the nature.

By Linfan MAO, a Chinese mathematician, philosophical critic.

Author Information

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[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

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[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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