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Harmonic Flow's Dynamics on Animals in Microscopic Level With Balance Recovery

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Abstract: Actually, different models characterize things in the world, particularly, animals dependent on the microscopic level. However, there are no a mathematical subfield characterizing animals or human beings ourselves in such a level globally unless local elements such as points or spaces in classical sciences. *Could we establish a mathematics describing animal's microscopic behaviors globally?* The answer is affirmative. In fact, an animal or a human is nothing else but a skeleton or a topological graph under the electron microscope and generally, there always exist a universal connection between things, no matter which is an organic or inorganic matter in philosophy. We have found a new kind of mathematical elements, i.e., *continuity flows* or topological graphs $\overrightarrow{G}$ with each edge labeled by a vector and 2 end-operators of Banach space $\mathscr{B}$ holding with the continuity equation at vertices which globally characterizes the dynamic behavior of self-adaptive systems. However, the 12 meridians with treatment theory in Chinese medicine indicates that there is also a *harmonic flow* model, i.e., $\overrightarrow{G}^{L^2}$ with $L^2:(v,u)\in E\left(\overrightarrow{G}\right)\to(L(v,u),-L(v,u))$, $L(v,u)\in\mathscr{B}$ on human body which alludes that the Euler-Lagrange dynamic equation is more rightful for characterizing the dynamic behavior of animals in the microscopic level. In this paper, we establish such a mathematical theory on harmonic flows with dynamics, including Banach harmonic flow space closed under action of differential, integral operators. A few well-known results such as those of Banach theorem, closed graph theorem and Hahn-Banach theorem are generalized with extended Euler-Lagrange equation and balance recovery on harmonic flows. All of these results form elementary dynamics on harmonic flows for characterizing the behavior of self-adaptive systems, particularly, the animals or human beings.

Key Words: Harmonic flow, mathematical element, Banach space, harmonic flow dynamics, Smarandache multispace, mathematical combinatorics, Chinese medicine.

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§1. Introduction

Today, as the time passed into 21st century, a fundamental question on the function of mathematics is in front of scientists, i.e., *what is the nature of mathematics on reality of things*? And *what is its the original intension, is it just the minority's intellectual game on notations*?

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Certainly not because its original intension or nature is revealing the reality of things in the world. However, this aim is forgotten along with the development of mathematics in depth for many years ([24]).

As is well known, all mathematical elements came from the understanding of things by human's 5 sensory organs such as these of hearing, sight, smell, taste or touch, and also dependent on the observing is from macroscopic to microscopic or microscopic to macroscopic. The macroscopic recognizing is elementary but basic with an essential cognition in the microscopic. For example, an animal anatomy P is shown in Fig.1 in which we know that an animal is consisting of systems. For example, let μ_1 =nervous, μ_2 =circulatory, μ_3 =immune, μ_4 =endocrine, μ_5 =digestive, μ_6 =respiratory, μ_7 =urinary and μ_8 =reproductive systems with μ_9=epithelial tissue.

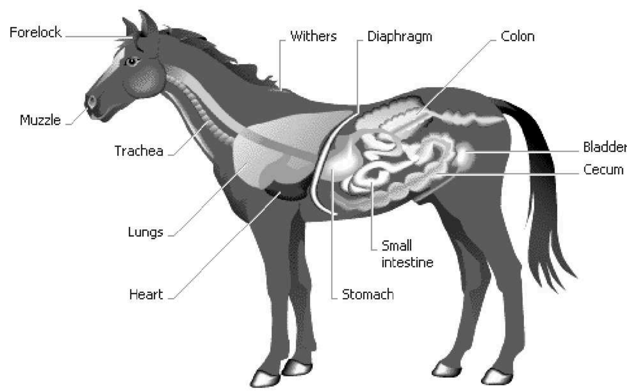

Fig.1

Whence, an animal P is understand by

$$P = \mu_1 \bigcup \mu_2 \bigcup \cdots \bigcup \mu_9 \tag{1.1}$$

in the macroscopic, which is nothing else but a *Smarandache multispace* ([8]-[10]) or *parallel universes* ([25]). But if we hold on P in the microscopic level, we know that all of its organic systems are consisted of cells, the smallest unit of life ([30]) and a cell is consisting of cytoplasm enclosed within a membrane that envelops the cell, regulates what moves in and out, maintains the electric potential of this cell and furthermore, inheres in a cytoskeleton, i.e., a stable and

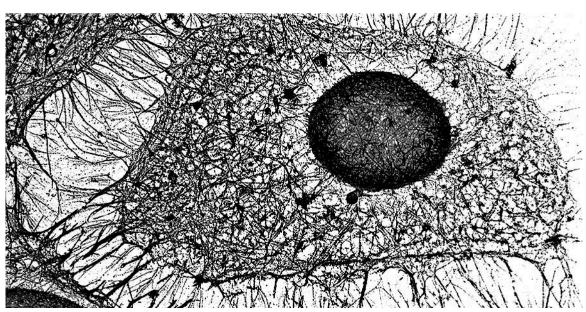

Fig.2

dynamic network of interlinking protein filaments that extend from the cell nucleus to the cell membrane, gives the cell's shape and structure such as those shown in Fig.2.

Let $\mathscr{N}(\mu_i)$ be the dynamic network of μ_i in cells at time t. Then, an animal P is underlying a complex network

$$P = \mathscr{N}(\mu_1)\bigcup\mathscr{N}(\mu_2)\bigcup\cdots\bigcup\mathscr{N}(\mu_9) \tag{1.2}$$

in the microscopic at time t, which is a complex network ([3], [4]).

Similarly, the divisibility of matter initiates human beings to search elementary constituting cells of matter, i.e., elementary particles such as those of quarks, leptons with interaction quanta including photons and other particles of mediated interactions, also with those of their antiparticles ([26]), and unmatters between a matter and its antimatter which is partially consisted of matter but others antimatter in the microscopic. Even though a free quark was never found in experiments, we can also get similar equalities (1.1) and (1.2) in theory such as those shown in Fig.3, where (a) a meson composed of a quark with an antiquark, (b) a baryon consisted of 3 quarks and (c) a particle composed of 5 quarks, respectively.

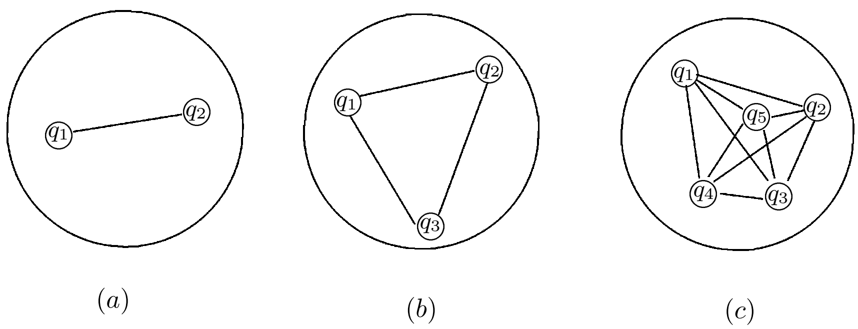

Fig.3

Notice that all these known characters on a thing P can not exist in isolation no matter which is organic or not, and the equality (1.2) is a complex network, or abstractly, a labeled graph G^L in space because they are indeed consisting of P. This fact also implies that we should find typical labeled graphs, called *continuity flows* and reviews them to be mathematical elements for revealing the reality of things ([19]) which can globally characterizes the dynamic behavior of things in the world.

Definition 1.1([22-23]) *A continuity flow* $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ *is an oriented embedded graph* $\overrightarrow{G}$ *in a topological space* $\mathscr{S}$ *associated with a mapping* $L: v \to L(v),\ (v,u) \to L(v,u)$*, 2 end-operators* $A^+_{vu}: L(v,u) \to L^{A^+_{vu}}(v,u)$ *and* $A^+_{uv}: L(u,v) \to L^{A^+_{uv}}(u,v)$ *on a Banach space* $\mathscr{B}$ *over a field* $\mathscr{F}$ *such as those shown in Fig.*4 *following*

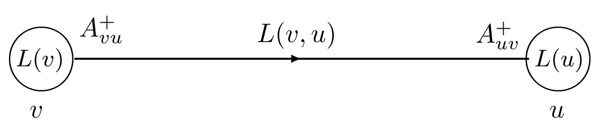

Fig.4

with $L(v,u) = -L(u,v)$*,* $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ *for* $\forall(v,u) \in E\left(\overrightarrow{G}\right)$ *holding with* *continuity equation*

$$\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u)=L(v) \quad for \quad \forall v\in V\left(\overrightarrow{G}\right)$$

and all such continuity flows are denoted by $\mathscr{G}_{\mathscr{B}}$*.*

Certainly, the continuity flows is such a mathematical element that its vertex equations maybe non-solvable ([11]-[14]). However, it indeed characterizes the reality of things, no matter what is organic or not. In fact, an independent energy system, including automobile, aircraft and animals is nothing else but a continuity flow, and there are monographs and papers published on continuity flows $\left(\overrightarrow{G};L,\mathscr{A}\right)$ with constraint conditions. For examples, the dynamic behavior of *complex network*, i.e., $A=\mathbf{1}_{\mathscr{V}}$ for $A\in\mathscr{A}$ with a number field $\mathbb{Z}$ or $\mathbb{R}$ is discussed in monographs [5] and [6]; an elementary $\overrightarrow{G}$*-flow theory*, i.e., $A=\mathbf{1}_{\mathscr{V}}$ for $A\in\mathscr{A}$ is established in [15]-[17] with applying to elementary particles; the *action flows*, i.e., x_v is a constant $\mathbf{v}_v$ dependent on v with applying to n-biological systems in [20]-[23] and an elementary theory on continuity flows is established in [22]-[23] with synchronization.

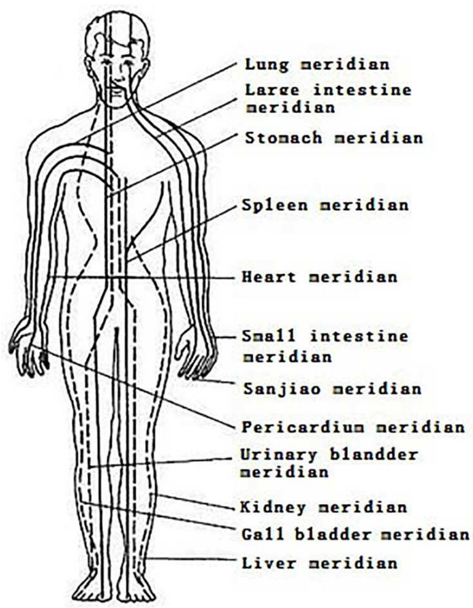

Fig.5

However, all of these results can not immediately characterize the regulatory or recovery mechanism of animals, particularly, the human body which means that we should furthermore find typical continuity flows for animals. It should be noted that preserving the balance Yin (Y^-) with Yang (Y^+) of a human body is the fundamental ruler, and there are 12 meridians in a human body which which completely reflects the physical condition in traditional Chinese medicine, i.e., the lung meridian of hand-TaiYin (LU), the large intestine meridian of hand YangMing (LI), the stomach meridian of foot-YangMing (ST), the spleen meridian of foot-TaiYin (SP), the heart meridian of hand-ShaoYin (HT), the small intestine meridian of hand-TaiYang (SI), the urinary blandder meridian of foot-TaiYang (BL), the kidney meridian of foot-ShaoYin (KI), the pericardium meridian of hand-JueYin (PC), the sanjiao meridian of hand-ShaoYang (SJ), the gall bladder meridian of foot-ShaoYang (GB), the liver meridian of foot-JueYin (LR) in Standard China National Standard (GB 12346-90), i.e., the *Body Model for Both Meridian and Extraordinary Points of China*, such as those in Fig.5, and similarly, the 12 meridians on animals such as the gall bladder meridian on a horse is shown in Fig.6.

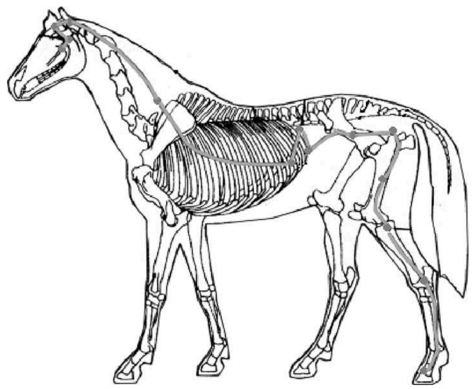

Fig.6

By the treatment theory in the traditional Chinese medicine ([29]), if there is a point on one of the 12 meridians in which $\{Y^-, Y^+\}$ are imbalance, this person must be in illness, and in turn, there must be points on the 12 meridians in which $\{Y^-, Y^+\}$ are imbalance for a patient, and the main duties of a doctor is to find out which points on which meridians are imbalance with Y^- more than Y^+ or Y^+ more than Y^-, and then by the natural ruler of the universe in traditional Chinese culture, i.e., *reducing the excess with supply the insufficient*, the doctor regulates these meridians by acupuncture or drugs so that points balance in $\{Y^-, Y^+\}$ again. This treatment theory naturally induced a subclass of continuity flows, called *harmonic flow* labeling each edge of a topological graph $\overrightarrow{G}$ by a 2-tuple vectors $(\mathbf{v}, -\mathbf{v})$ following.

Definition 1.2 *A harmonic flow* $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ *is an oriented embedded graph* $\overrightarrow{G}$ *in a topological space* $\mathscr{S}$ *associated with a mapping* $L : v \to (L(v), -L(v))$ *for* $v \in E\left(\overrightarrow{G}\right)$ *and* $L : (v, u) \to (L(v, u), -L(v, u))$*, 2 end-operators* A^+_{vu}*,* A^+_{uv} *with*

$$A^+_{vu} : (L(v, u), -L(v, u)) \to \left(L^{A^+_{vu}}(v, u), -L^{A^+_{vu}}(v, u)\right),$$
$$A^+_{uv} : (L(v, u), -L(v, u)) \to \left(L^{A^+_{uv}}(v, u), -L^{A^+_{uv}}(v, u)\right),$$

$L(v, u) = -L(u, v)$ *for* $\forall (v, u) \in E\left(\overrightarrow{G}\right)$ *on a Banach space* $\mathscr{B}$ *holding with continuity equation*

$$\sum_{u \in N_G(v)} L^{A^+_{vu}}(v, u) = L(v)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$*, and all such harmonic flows are denoted by* $\mathscr{G}^{\pm}_{\mathscr{B}}$*.*

Clearly, a harmonic flow is naturally a continuity flow because of

$$\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u)+\sum_{u\in N_G(v)}\left(-L^{A^+_{vu}}(v,u)\right)=L(v)-L(v)=\mathbf{0}$$

for $\forall v\in\left(\overrightarrow{G}\right)$ and in fact, it is balanced at every where on $\overrightarrow{G}$ such as those shown in Fig.7, where $a,b,c\in\mathbb{R}$ hold with $a=b+c$.

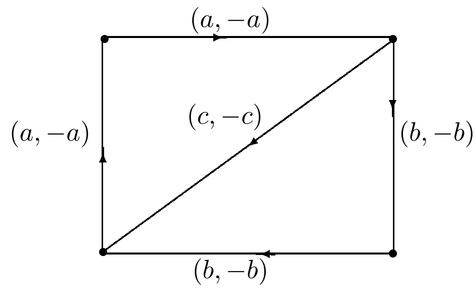

Fig.7

In this paper, we always assume that all end-operators in $\mathscr{A}$ are both *linear* and *continuous*. In this case, the result following on linear operators of Banach space is well-known.

Theorem 1.3([3]) *Let $\mathscr{B}_1,\mathscr{B}_2$ be Banach spaces over a field $\mathbb{F}$ with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then, a linear operator $\mathbf{T}:\mathscr{B}_1\to\mathscr{B}_2$ is continuous if and only if it is bounded, or equivalently,*

$$\|\mathbf{T}\|:=\sup_{\mathbf{0}\neq\mathbf{v}\in\mathscr{B}_1}\frac{\|\mathbf{T}(\mathbf{v})\|_2}{\|\mathbf{v}\|_1}<+\infty.$$

The main purpose of this paper is to establish the dynamic theory on harmonic flows globally, an open problem for establishing graph dynamics in [7] including Banach harmonic flow space closed under action of differential, integral operators. A few well-known results such as those of Banach theorem, closed graph theorem and Hahn-Banach theorem are generalized with extended Euler-Lagrange equation and balance recovery on harmonic flows which is motivated by traditional Chinese medicine. We denote a continuity flow $\overrightarrow{G}^L$ with $L:(v,u)\to(L_1(v,u),L_2(v,u))$ by $\overrightarrow{G}^{L^2}$ for emphasizing L^2 mapping edges to $\mathscr{B}\times\mathscr{B}$, where $L_1(v,u),L_2(v,u)\in\mathscr{B}$ and all 2-tuple flows $\overrightarrow{G}^{L^2}$ with $L^2:E\left(\overrightarrow{G}\right)\to\mathscr{B}\times\mathscr{B}$ by $\mathscr{G}_{\mathscr{B}^2}$.

For terminologies and notations not mentioned here, we follow references [1] for mechanics, [3] for functional analysis, [4] for biological mathematics, [5]-[6] for complex network, [10] for combinatorial geometry, and [9], [27] for Smarandache systems and multispaces.

§2. Banach Harmonic Flow Spaces

2.1 Commutative Rings over Graphs

Let $\mathscr{G}$ be a closed family of graphs $\overrightarrow{G}$ under the union operation and let $\mathscr{B}$ be a linear space $(\mathscr{B};+,\cdot)$, or furthermore, a commutative ring $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$. For $\forall\overrightarrow{G}^{L^2},\overrightarrow{G'}^{L'^2}\in\mathscr{G}_{\mathscr{B}^2}$, define

$$\overrightarrow{G}^{L^2} + \overrightarrow{G'}^{L'^2} = \left(\overrightarrow{G} \setminus \overrightarrow{G'}\right)^{L^2} \bigcup \left(\overrightarrow{G} \bigcap \overrightarrow{G'}\right)^{L^2+L'^2} \bigcup \left(\overrightarrow{G'} \setminus \overrightarrow{G}\right)^{L'^2}, \quad (2.1)$$

$$\overrightarrow{G}^{L^2} \cdot \overrightarrow{G'}^{L'^2} = \left(\overrightarrow{G} \setminus \overrightarrow{G'}\right)^{L^2} \bigcup \left(\overrightarrow{G} \bigcap \overrightarrow{G'}\right)^{L^2 \cdot L'^2} \bigcup \left(\overrightarrow{G'} \setminus \overrightarrow{G}\right)^{L'^2} \quad (2.2)$$

and

$$\lambda \cdot \overrightarrow{G}^{L^2} = \overrightarrow{G}^{\lambda \cdot L^2}, \quad (2.3)$$

where $\lambda \in \mathscr{F}$ and

$$\begin{aligned}
&L^2 : (v,u) \to (L_1(v,u), L_2(v,u)), \quad L'^2 : (v,u) \to (L'_1(v,u), L'_2(v,u)),\\
&L^2 + L'^2 : (v,u) \to (L_1(v,u) + L'_1(v,u),\ L_2(v,u) + L'_2(v,u)),\\
&L^2 \ \cdot \ L'^2 : (v,u) \to (L_1(v,u) \cdot L'_1(v,u), L_2(v,u) \cdot L'_2(v,u)),\\
&\lambda \cdot L^2(v,u) = (\lambda \cdot L_1(v,u), \lambda \cdot L_2(v,u))
\end{aligned}$$

with substituting end-operator $A : (v,u) \to A^+_{vu}(v,u) + (A')^+_{vu}(v,u)$ or $A : (v,u) \to A^+_{vu}(v,u) \cdot (A')^+_{vu}(v,u)$ for $(v,u) \in E\left(\overrightarrow{G} \bigcap \overrightarrow{G'}\right)$ in $\overrightarrow{G}^{L^2} + \overrightarrow{G'}^{L'^2}$ or $\overrightarrow{G}^{L^2} \cdot \overrightarrow{G'}^{L'^2}$ and $L_1(v,u), L_2(v,u), L'_1(v,u), L'_2(v,u) \in \mathscr{B}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ or $E\left(\overrightarrow{G'}\right)$.

Define

$$L^\circ_{kl}(e) = \begin{cases} L^2_k(e), & \text{if } e \in E\left(\overrightarrow{G}_k \setminus \overrightarrow{G}_l\right)\\ L^2_l(e), & \text{if } e \in E\left(\overrightarrow{G}_l \setminus \overrightarrow{G}_k\right)\\ L^2_k(e) \circ L^2_l(e) & \text{if } e \in E\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)\end{cases}, \quad (2.4)$$

and

$$L^\circ_{kls}(e) = \begin{cases} L^2_k(e), & \text{if } e \in E\left(\overrightarrow{G}_k \setminus \left(\overrightarrow{G}_l \bigcup \overrightarrow{G}_s\right)\right)\\ L^2_l(e), & \text{if } e \in E\left(\overrightarrow{G}_l \setminus \left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_s\right)\right)\\ L^2_s(e), & \text{if } e \in E\left(\overrightarrow{G}_s \setminus \left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)\right)\\ L^\circ_{kl}(e), & \text{if } e \in E\left(\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right) \setminus \overrightarrow{G}_s\right)\\ L^\circ_{ks}(e), & \text{if } e \in E\left(\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_s\right) \setminus \overrightarrow{G}_l\right)\\ L^\circ_{ls}(e), & \text{if } e \in E\left(\left(\overrightarrow{G}_l \bigcap \overrightarrow{G}_s\right) \setminus \overrightarrow{G}_k\right)\\ L^2_k(e)^\circ L^2_l(e)^\circ L^2_s(e) & \text{if } e \in E\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l \bigcap \overrightarrow{G}_s\right)\end{cases}, \quad (2.5)$$

where $\circ$ is the operation $+$, $-$ or $\cdot$ and $\overrightarrow{G}_k, \overrightarrow{G}_l, \overrightarrow{G}_s \in \mathscr{G}$.

Clearly, if $\overrightarrow{G}^{L^2}, \overrightarrow{G'}^{L'^2} \in \mathscr{G}_{\mathscr{B}^2}$ with linear end-operators A^+_{vu}, A^+_{uv}, then $\overrightarrow{G}^{L^2} + \overrightarrow{G'}^{L'^2}$, $\overrightarrow{G}^{L^2} \cdot \overrightarrow{G'}^{L'^2}$ and $\lambda \cdot \overrightarrow{G}^{L^2} \in \mathscr{G}^2_{\mathscr{B}}$, i.e., $\mathscr{G}_{\mathscr{B}^2}$ is closed under operations (2.1)-(2.3). Furthermore, for $\forall \overrightarrow{G}_k, \overrightarrow{G}_l, \overrightarrow{G}_s \in \mathscr{G}$ calculation shows the operations "+" and "·" satisfy

(1) commutative, i.e., $\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}=\overrightarrow{G}_l^{L_l^2}+\overrightarrow{G}_k^{L_k^2}$ and $\overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_l^{L_l^2}=\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_k^{L_k^2}$ because of

$$\begin{aligned}\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2} &= \left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_k^2}\bigcup\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{L_k^2+L_l^2}\bigcup\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2}\\ &= \left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2}\bigcup\left(\overrightarrow{G}_l\bigcap\overrightarrow{G}_k\right)^{L_l^2+L_k^2}\bigcup\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_l^2}\\ &= \overrightarrow{G}_l^{L_l^2}+\overrightarrow{G}_k^{L_k^2},\\ \overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_l^{L_l^2} &= \left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_k^2}\bigcup\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{L_k^2\cdot L_l^2}\bigcup\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2}\\ &= \left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2}\bigcup\left(\overrightarrow{G}_l\bigcap\overrightarrow{G}_k\right)^{L_l^2\cdot L_k^2}\bigcup\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_k^2}\\ &= \overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_k^{L_k^2}.\end{aligned}$$

(2) associative, i.e., $\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)+\overrightarrow{G}_s^{L_s^2}=\overrightarrow{G}_k^{L_k^2}+\left(\overrightarrow{G}_l^{L_l^2}+\overrightarrow{G}_s^{L_s^2}\right)$ and $\left(\overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_l^{L_l^2}\right)\cdot\overrightarrow{G}_s^{L_s^2}=\overrightarrow{G}_k^{L_k^2}\cdot\left(\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_s^{L_s^2}\right)$, and distributive, i.e., $\overrightarrow{G}_s^{L_s^2}\cdot\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)=\overrightarrow{G}_s^{L_s^2}\cdot\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_s^{L_s^2}\cdot\overrightarrow{G}_l^{L_l^2}$ and $\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)\cdot\overrightarrow{G}_s^{L_s^2}=\overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_s^{L_s^2}+\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_s^{L_s^2}$ if $\mathscr{B}$ is furthermore a commutative ring $(\mathscr{B};+,\cdot)$ because of

$$\begin{aligned}\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)+\overrightarrow{G}_s^{L_s^2} &= \left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)^{L_{kl}^+}+\overrightarrow{G}_s^{L_s^2}=\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\bigcup\overrightarrow{G}_s\right)^{L_{kls}^+}\\ &= \overrightarrow{G}_k^{L_k^2}+\left(\overrightarrow{G}_l\bigcup\overrightarrow{G}_s\right)^{L_{ls}^+}=\overrightarrow{G}_k^{L_k^2}+\left(\overrightarrow{G}_l^{L_l^2}+\overrightarrow{G}_s^{L_s^2}\right),\\ \left(\overrightarrow{G}_k^{L_k^2}\ \cdot\ \overrightarrow{G}_l^{L_l^2}\right)\cdot\overrightarrow{G}_s^{L_s^2} &= \left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)^{L_{kl}^{\cdot}}\cdot\overrightarrow{G}_s^{L_s^2}=\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\bigcup\overrightarrow{G}_s\right)^{L_{kls}^{\cdot}}\\ &= \overrightarrow{G}_k^{L_k^2}\cdot\left(\overrightarrow{G}_l\bigcup\overrightarrow{G}_s\right)^{L_{ls}^{\cdot}}=\overrightarrow{G}_k^{L_k^2}\cdot\left(\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_s^{L_s^2}\right)\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{G}_s^{L_s^2}\cdot\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right) &= \overrightarrow{G}_s^{L_s^2}\cdot\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)^{L_{kl}^{\cdot}}=\left(\overrightarrow{G}_s\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)\right)^{L_{s(kl)}^{\cdot}}\\ &= \left(\overrightarrow{G}_s\bigcup\overrightarrow{G}_k\right)^{L_{sk}^{\cdot}}\bigcup\left(\overrightarrow{G}_s\bigcup\overrightarrow{G}_l\right)^{L_{sl}^{\cdot}}=\overrightarrow{G}_s^{L_s^2}\cdot\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_s^{L_s^2}\cdot\overrightarrow{G}_l^{L_l^2}.\end{aligned}$$

Similarly, we can check that $\left(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right)\cdot\overrightarrow{G}_s^{L_s^2}=\overrightarrow{G}_k^{L_k^2}\cdot\overrightarrow{G}_s^{L_s^2}+\overrightarrow{G}_l^{L_l^2}\cdot\overrightarrow{G}_s^{L_s^2}$.

(3) There are a unique zero flow $\mathbf{O}$, i.e., $\mathbf{O}(v,u)=\{\mathbf{0},\mathbf{0}\}$ in $(\mathscr{G}_{\mathscr{B}^2};+)$ and a unique unit zero $\mathbf{1}$, i.e., $\mathbf{1}(v,u)=\{\mathbf{1},\mathbf{1}\}$ for $\forall(v,u)\in E\left(\overrightarrow{\mathscr{G}}\right)$ in $(\mathscr{G}_{\mathscr{B}^2};\cdot)$ such that $\mathbf{O}+\overrightarrow{G}^{L^2}=\overrightarrow{G}^{L^2}+\mathbf{O}=\overrightarrow{G}^{L^2}$ and $\mathbf{1}\cdot\overrightarrow{G}^{L^2}=\overrightarrow{G}^{L^2}\cdot\mathbf{1}=\overrightarrow{G}^{L^2}$;

(4) For $\forall\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}^2}$ there is a unique flow $\overrightarrow{G}^{-L^2}$ such that $\overrightarrow{G}^{L^2}+\overrightarrow{G}^{-L^2}=\mathbf{O}$;

(5) A scalar multiplication "$\cdot$ defined by (2.3) associating a flow $\overrightarrow{G}^{L^2}$ in $\mathscr{G}_{\mathscr{B}^2}$ and a scalar $\alpha\in\mathscr{F}$ with a flow $\alpha\cdot\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}^2}$ in such a way that

(*a*) $1\cdot\overrightarrow{G}^{L^2}=\overrightarrow{G}^{L^2}$;

(*b*) $(\alpha_1\alpha_2)\cdot\overrightarrow{G}^{L^2}=\alpha_1(\alpha_2\cdot\overrightarrow{G}^{L^2})$ for $\alpha_1,\alpha_2\in\mathscr{F}$;

(*c*) $\alpha\cdot(\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2})=\alpha\cdot\overrightarrow{G}_k^{L_k^2}+\alpha\cdot\overrightarrow{G}_l^{L_l^2}$ for $\alpha\in\mathscr{F}$;

(*d*) $(\alpha_1+\alpha_2)\cdot\overrightarrow{G}^{L^2}=\alpha_1\cdot\overrightarrow{G}^{L^2}+\alpha_2\cdot\overrightarrow{G}^{L^2}$ for $k_1,k_2\in\mathscr{F}$.

In conclusion, we know that $(\mathscr{G}_{\mathscr{B}^2},+)$ and $(\mathscr{G}_{\mathscr{B}^2},\cdot)$ are respectively a commutative group, a commutative semigroup with unit if $\mathscr{B}$ is a commutative ring, and $(\mathscr{G}_{\mathscr{B}^2},+,\cdot)$ is a linear space if $\mathscr{B}$ is so. We therefore get the following result.

Theorem 2.1 *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a linear space $(\mathscr{B};+,\cdot)$, then, all pair flows $(\mathscr{G}_{\mathscr{B}^2};+,\cdot)$ is a linear space, and furthermore, a commutative ring if $\mathscr{B}$ is a commutative ring $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$.*

2.2 Banach Harmonic Flow Space

For $\forall\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}^2}$ with $L^2(e)=(L_1(e),L_2(e))$, $e\in E\left(\overrightarrow{G}\right)$ define

$$\left\|\overrightarrow{G}^{L^2}\right\|=\sum_{e\in E\left(\overrightarrow{G}\right)}\left(\|L_1(e)\|+\|L_2(e)\|\right), \tag{2.6}$$

where $\mathscr{B}$ is a Banach space $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$ with a norm $\|\cdot\|$. Then, for $\forall\overrightarrow{G}^{L^2},\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}_{\mathscr{B}^2}$ we are easily know that

(1) $\left\|\overrightarrow{G}^{L^2}\right\|\geq 0$ and $\left\|\overrightarrow{G}^{L^2}\right\|=0$ if and only if $L_1(e)=\mathbf{0}$ and $L_2(e)=\mathbf{0}$, i.e., $\overrightarrow{G}^{L^2}=\mathbf{O}$;

(2) $\left\|\overrightarrow{G}^{\xi L^2}\right\|=|\xi|\left\|\overrightarrow{G}^{L^2}\right\|$ for any scalar $\xi\in\mathscr{F}$;

(3) $\left\|\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right\|\leq\left\|\overrightarrow{G}_k^{L_k^2}\right\|+\left\|\overrightarrow{G}_l^{L_l^2}\right\|$ because of

$$\begin{aligned}\left\|\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2}\right\| &= \sum_{e\in E\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)}\left(\|L_{k1}(e)\|+\|L_{k2}(e)\|\right)+\sum_{e\in E\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)}\left(\|L_{l1}(e)\|+\|L_{l2}(e)\|\right)\\ &\quad+\sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\|L_{k1}(e)+L_{l1}(e)\|+\|L_{k2}(e)+L_{l2}(e)\|\right)\\ &\leq \sum_{e\in E\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)}\left(\|L_{k1}(e)\|+\|L_{k2}(e)\|\right)+\sum_{e\in E\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)}\left(\|L_{l1}(e)\|+\|L_{l2}(e)\|\right)\\ &\quad+\sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\|L_{k1}(e)\|+\|L_{k2}(e)\|+\|L_{l1}(e)\|+\|L_{l2}(e)\|\right)\\ &= \left\|\overrightarrow{G}_k^{L_k}\right\|+\left\|\overrightarrow{G}_l^{L_l}\right\|\end{aligned}$$

by $\|\mathbf{v}_1+\mathbf{v}_2\|\leq\|\mathbf{v}_1\|+\|\mathbf{v}_2\|$ for $\forall\mathbf{v}_1,\mathbf{v}_2\in\mathscr{B}$. Therefore, $\|\cdot\|$ is also a norm on $\mathscr{G}_{\mathscr{B}^2}$.

Furthermore, if $\mathscr{B}$ is a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, for $\forall\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}_{\mathscr{B}^2}$, define

$$\left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right\rangle=\sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\langle L_{k1}(e),L_{l1}(e)\rangle+\langle L_{k2}(e),L_{l2}(e)\rangle\right) \tag{2.7}$$

Clearly, $\left\langle\overrightarrow{G}_k^{L_k^2}\bigcup\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle=\left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}^{L^2}\right\rangle+\left\langle\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle$ if $E\left(\overrightarrow{G}_k\right)\bigcap E\left(\overrightarrow{G}_l\right)=\emptyset$ and

$\left\langle\overrightarrow{G}_k^{L_{k1}^2+L_{k2}^2},\overrightarrow{G}^{L^2}\right\rangle=\left\langle\overrightarrow{G}_k^{L_{k1}^2},\overrightarrow{G}^{L^2}\right\rangle+\left\langle\overrightarrow{G}_k^{L_{k2}^2},\overrightarrow{G}^{L^2}\right\rangle$ by definition (2.9), and we are easily know also that

(1) For $\forall\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}^2}$,

$$\left\langle\overrightarrow{G}^{L^2},\overrightarrow{G}^{L^2}\right\rangle=\sum_{e\in E\left(\overrightarrow{G}\right)}\left(\langle L_1(e),L_1(e)\rangle+\langle L_2(e),L_2(e)\rangle\right)\geq 0$$

and $\left\langle\overrightarrow{G}^{L^2},\overrightarrow{G}^{L^2}\right\rangle=0$ if and only if $L_1(e)=\mathbf{0}$, $L_2(e)=\mathbf{0}$, i.e., $\overrightarrow{G}^{L^2}=\mathbf{O}$.

(2) For $\forall\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}_{\mathscr{B}^2}$, $\left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right\rangle=\overline{\left\langle\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}_k^{L_k^2}\right\rangle}$ because of

$$\begin{aligned}\left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right\rangle &= \sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\langle L_{k1}(e),L_{l1}(e)\rangle+\langle L_{k2}(e),L_{l2}(e)\rangle\right)\\ &= \sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\overline{\langle L_{l1}(e),L_{k1}(e)\rangle}+\overline{\langle L_{l2}(e),L_{k2}(e)\rangle}\right)\\ &= \overline{\left\langle\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}_k^{L_k^2}\right\rangle}\end{aligned}$$

for $\langle\mathbf{v}_1,\mathbf{v}_2\rangle=\overline{\langle\mathbf{v}_2,\mathbf{v}_1\rangle}$, $\mathbf{v}_1,\mathbf{v}_2$ in Hilbert space $\mathscr{B}$.

(3) For $\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}_{\mathscr{B}^2}$ and $\lambda\in\mathscr{F}$, $\left\langle\overrightarrow{G}_k^{L_k^2},\lambda\overrightarrow{G}_l^{L_l^2})\right\rangle=\lambda\left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right\rangle$ because of

$$\begin{aligned}\left\langle\overrightarrow{G}_k^{L_k^2},\lambda\overrightarrow{G}_l^{L_l^2}\right\rangle &= \left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{\lambda L_l^2}\right\rangle\\ &= \sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\langle L_{k1}(e),\lambda L_{l1}(e)\rangle+\langle L_{k2}(e),\lambda L_{l2}(e)\rangle\right)\\ &= \sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\lambda\left(\langle L_{k1}(e),L_{l1}(e)\rangle+\langle L_{k2}(e),L_{l2}(e)\rangle\right)\\ &= \lambda\sum_{e\in E\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)}\left(\langle L_{k1}(e),L_{l1}(e)\rangle+\langle L_{k2}(e),L_{l2}(e)\rangle\right)\\ &= \lambda\left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\right\rangle.\end{aligned}$$

by definition (2.7).

(4) For $\overrightarrow{G}^{L^2},\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}_{\mathscr{B}^2}$, $\left\langle\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle=\left\langle\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}^{L^2}\right\rangle+\left\langle\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle$ because of

$$\begin{aligned}\left\langle\overrightarrow{G}_k^{L_k^2}+\overrightarrow{G}_l^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle &= \left\langle\left(\overrightarrow{G}_k\bigcup\overrightarrow{G}_l\right)^{L_{kl}^+},\overrightarrow{G}^{L^2}\right\rangle\\ &= \left\langle\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{L_k^2}\bigcup\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{L_k^2+L_l^2}\bigcup\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{L_l^2},\overrightarrow{G}^{L^2}\right\rangle\end{aligned}$$

$$\begin{aligned}
&= \left\langle \left(\overrightarrow{G}_k \setminus \overrightarrow{G}_l\right)^{L_k^2}, \overrightarrow{G}^{L^2} \right\rangle + \left\langle \left(\overrightarrow{G}_l \setminus \overrightarrow{G}_k\right)^{L_l^2}, \overrightarrow{G}^{L^2} \right\rangle \\
&\quad + \left\langle \left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)^{L_k^2+L_l^2}, \overrightarrow{G}^{L^2} \right\rangle \\
&= \left\langle \left(\overrightarrow{G}_k \setminus \overrightarrow{G}_l\right)^{L_k^2}, \overrightarrow{G}^{L^2} \right\rangle + \left\langle \left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)^{L_k^2}, \overrightarrow{G}^{L^2} \right\rangle \\
&\quad + \left\langle \left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right)^{L_l^2}, \overrightarrow{G}^{L^2} \right\rangle + \left\langle \left(\overrightarrow{G}_l \setminus \overrightarrow{G}_k\right)^{L_l^2}, \overrightarrow{G}^{L^2} \right\rangle \\
&= \left\langle \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}^{L^2} \right\rangle + \left\langle \overrightarrow{G}_l^{L_l^2}, \overrightarrow{G}^{L^2} \right\rangle .
\end{aligned}$$

by definition (2.9). Whence, $(\mathscr{G}_{\mathscr{B}^2}; +, \cdot)$ is also an inner product space and a normed space with

$$\left\| \overrightarrow{G}^{L^2} \right\| = \sqrt{\left\langle \overrightarrow{G}^{L^2}, \overrightarrow{G}^{L^2} \right\rangle}$$

for $\forall \overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}^2}$.

Definition 2.2 *For $\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}_{\mathscr{B}^2}$, the distance between $\overrightarrow{G}_k^{L_k^2}$, $\overrightarrow{G}_l^{L_l^2}$ is defined by*

$$d\left(\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2}\right) = \left\| \overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2} \right\| = \left\| \overrightarrow{G}_k^{L_k^2} + \overrightarrow{G}_l^{-L_l^2} \right\|. \tag{2.8}$$

Clearly, $(\mathscr{G}_{\mathscr{B}^2}; +, \cdot)$ is also a distance space by Definition 2.2 with previous properties $(1)-(3)$ or $(1)-(4)$ of Banach or Hilbert space, respectively.

Definition 2.3 *A sequence $\forall \overrightarrow{G}_1^{L_1^2}, \overrightarrow{G}_2^{L_2^2}, \cdots, \overrightarrow{G}_n^{L^2}$ in $\mathscr{G}_{\mathscr{B}^2}$ is called Cauchy sequence if for any number $\varepsilon > 0$, there always exists an integer $N(\varepsilon)$ such that*

$$\left\| \overrightarrow{G}_n^{L_k^2} - \overrightarrow{G}_l^{L_l^2} \right\| < \varepsilon$$

for integers $k, l \geq N(\varepsilon)$.

Let $\left\{ \overrightarrow{G}_n^{L_n^2} \right\}$ be a Cauchy sequence of $\mathscr{G}_{\mathscr{B}^2}$ and $\overrightarrow{\Pi} = \bigcup\limits_{\overrightarrow{G} \in \mathscr{G}} \overrightarrow{G}$. Notice that $\mathscr{G}$ is closed under operation union by assumption. We know that $\overrightarrow{\Pi} \in \mathscr{G}$ is finite and embed each $\overrightarrow{G}_n^{L_n^2}$ into a subflows $\overrightarrow{\Pi}^{\widehat{L^2}} \in \mathscr{G}_{\mathscr{B}^2}$ by defining

$$\widehat{L}_n^2(e) = \begin{cases} L_n^2(e) & \text{if} \quad e \in E\left(\overrightarrow{G}_n\right), \\ \{\mathbf{0}, \mathbf{0}\} & \text{if} \quad e \in E\left(\overrightarrow{\Pi} \setminus \overrightarrow{G}_n\right). \end{cases}$$

Clearly,

$$\begin{aligned}
\left\| \overrightarrow{G}^{L_k^2} - \overrightarrow{G}^{L_l^2} \right\| &= \left\| \overrightarrow{\Pi}^{\widehat{L}_k^2} - \overrightarrow{\Pi}^{\widehat{L}_l^2} \right\| = \left\| \overrightarrow{\Pi}^{\widehat{L}_k^2 - \widehat{L}_l^2} \right\| \\
&= \sum_{e \in E\left(\overrightarrow{\Pi}\right)} \left(\left\| \widehat{L}_{k1}(e) - \widehat{L}_{l1}(e) \right\| + \left\| \widehat{L}_{k2}(e) - \widehat{L}_{l2}(e) \right\| \right).
\end{aligned}$$

Now, for $\forall\varepsilon > 0$ if $\left\|\overrightarrow{G}^{L_k^2} - \overrightarrow{G}^{L_l^2}\right\| \leq \varepsilon$ for integers $k, l \geq N(\varepsilon)$ then there must be $\left\|\widehat{L}_{k1}(e) - \widehat{L}_{l1}(e)\right\| \leq \varepsilon$ and $\left\|\widehat{L}_{k2}(e) - \widehat{L}_{l2}(e)\right\| \leq \varepsilon$ for integers $k, l \geq N(\varepsilon)$, i.e., $\left\{\widehat{L}_n^2\right\}$ is a Cauchy sequence for $\forall e \in E\left(\overrightarrow{\Pi}\right)$, which is convergent in $\mathscr{B}$ by assumption. Without loss of generality, let $\lim\limits_{n\to\infty} \widehat{L}_n^2 = (L_{01}, L_{02}) = L_0^2$. Then, $\lim\limits_{n\to\infty} \overrightarrow{G}_n^{L_n^2} = \lim\limits_{n\to\infty} \overrightarrow{\Pi}^{\widehat{L}_n^2} = \overrightarrow{\Pi}^{\lim\limits_{n\to\infty} \widehat{L}_n^2} = \overrightarrow{\Pi}^{L_0^2}$, i.e., $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ is convergent in $\mathscr{G}_{\mathscr{B}^2}$ by definition. We therefore get the result following.

Theorem 2.4 *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a Banach space $(\mathscr{B}; +, \cdot)$, then, $\mathscr{G}_{\mathscr{B}^2}$ with linear operators A_{vu}^+, A_{uv}^+ for $\forall(v, u) \in E\left(\bigcup\limits_{G\in\mathscr{G}} \overrightarrow{G}\right)$ is a Banach space, and furthermore, if $\mathscr{B}$ is a Hilbert space, $\mathscr{G}_{\mathscr{B}^2}$ is a Hilbert space too.*

We have known that all continuity flows $\overrightarrow{G}^L$ form a Banach or Hilbert space $\mathscr{G}_{\mathscr{B}}$ respect to that $\mathscr{B}$ is a Banach or Hilbert space in [24] and [25]. By definition, $\left(\overrightarrow{G}_i^L, \overrightarrow{G}_j^L\right) \in \mathscr{G}_{\mathscr{B}}^2$ for $\overrightarrow{G}_i^L, \overrightarrow{G}_j^L \in \mathscr{G}_{\mathscr{B}}$. Notice that a harmonic flow $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}^{\pm}$ is isomorphic to a continuity flow $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$ because flows $(L^{A_{vu}^+}(v, u), -L^{A_{vu}^+}(v, u))$ is isomorphic to $L^{A_{vu}^+}(v, u)$ for $\forall(v, u) \in E\left(\overrightarrow{G}\right)$. Thus, all harmonic flows $\mathscr{G}_{\mathscr{B}}^{\pm}$ is in fact a Banach or Hilbert subspace of $\mathscr{G}_{\mathscr{B}}^2$ by Theorem 2.4.

Theorem 2.5 *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a Banach space $(\mathscr{B}; +, \cdot)$, then, all harmonic flows $\mathscr{G}_{\mathscr{B}}$ with linear operators A_{vu}^+, A_{uv}^+ for $\forall(v, u) \in E\left(\bigcup\limits_{G\in\mathscr{G}} \overrightarrow{G}\right)$ under operations $+$ and $\cdot$ form a Banach or Hilbert space respect to that $\mathscr{B}$ is a Banach or Hilbert space with inclusions*

$$\mathscr{G}_{\mathscr{B}}^{\pm} \subset \mathscr{G}_{\mathscr{B}}^2 \subset \mathscr{G}_{\mathscr{B}^2}.$$

2.3 Operators on Banach Harmonic Flow Space

Definition 2.7 *Let $\mathbf{T} : \mathscr{G}_{\mathscr{B}}^{\pm} \to \mathscr{B}^{\pm}$ be an operator on Banach harmonic flow space $\mathscr{G}_{\mathscr{B}}^{\pm}$ over a field $\mathscr{F}$. Then, $\mathbf{T}$ is linear if*

$$\mathbf{T}\left(\lambda\overrightarrow{G}_k^{L_k^2} + \mu\overrightarrow{G}_l^{L_l^2}\right) = \lambda\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right) + \mu\mathbf{T}\left(\overrightarrow{G}_l^{L_l^2}\right)$$

for $\forall\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ and $\lambda, \mu \in \mathscr{F}$, is continuous at $\overrightarrow{G}_0^{L_0^2}$ if there always exist a number $\delta(\varepsilon)$ for $\forall\epsilon > 0$ such that

$$\left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right) - \mathbf{T}\left(\overrightarrow{G}_0^{L_0^2}\right)\right\| < \varepsilon$$

if $\left\|\overrightarrow{G}^{L^2} - \overrightarrow{G}_0^{L_0^2}\right\| < \delta(\varepsilon)$, bounded if $\left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\| \leq \xi\left\|\overrightarrow{G}^{L^2}\right\|$ for $\forall\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ with a constant $\xi \in [0, \infty)$ and furthermore, a contractor if

$$\left\|\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right) - \mathbf{T}\left(\overrightarrow{G}_l^{L_l^2}\right)\right\| \leq \xi\left\|\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right\|$$

for $\forall\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ with $\xi \in [0, 1)$.

Theorem 2.8(Fixed Harmonic Flow Theorem) *If* $\mathbf{T}:\mathscr{G}_{\mathscr{B}}^{\pm}\rightarrow\mathscr{G}_{\mathscr{B}}^{\pm}$ *is a linear continuous contractor, then there is a uniquely harmonic flow* $\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}}^{\pm}$ *such that*

$$\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)=\overrightarrow{G}^{L^2}.$$

Proof Let $\overrightarrow{G}^{L_0^2}\in\mathscr{G}_{\mathscr{B}}^{\pm}$ be a harmonic flow. We define a sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ by

$$\begin{aligned}
&\overrightarrow{G}_1^{L_1^2}=\mathbf{T}\left(\overrightarrow{G}^{L_0^2}\right),\\
&\overrightarrow{G}_2^{L_2^2}=\mathbf{T}\left(\overrightarrow{G}_1^{L_1^2}\right)=\mathbf{T}^2\left(\overrightarrow{G}^{L_0^2}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\overrightarrow{G}_n^{L_n}=\mathbf{T}\left(\overrightarrow{G}_{n-1}^{L_{n-1}^2}\right)=\mathbf{T}^n\left(\overrightarrow{G}^{L_0^2}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

We prove the sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ is a Cauchy sequence in $\mathscr{G}_{\mathscr{B}}^{\pm}$. By assumption $\mathbf{T}$ is a contractor, there is a constant $\xi\in[0,1)$ such that $\left\|\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right)-\mathbf{T}\left(\overrightarrow{G}_l^{L_l^2}\right)\right\|\leq\xi\left\|\overrightarrow{G}_k^{L_k^2}-\overrightarrow{G}_l^{L_l^2}\right\|$ for $\forall\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}_{\mathscr{B}}^{\pm}$. We therefore know that

$$\begin{aligned}
\left\|\overrightarrow{G}_{m+1}^{L_{m+1}^2}-\overrightarrow{G}_m^{L_m^2}\right\| &= \left\|\mathbf{T}\left(\overrightarrow{G}_m^{L_m^2}\right)-\mathbf{T}\left(\overrightarrow{G}_{m-1}^{L_{m-1}^2}\right)\right\|\leq\xi\left\|\overrightarrow{G}_m^{L_m^2}-\overrightarrow{G}_{m-1}^{L_{m-1}^2}\right\|\\
&= \left\|\mathbf{T}\left(\overrightarrow{G}_{m-1}^{L_{m-1}^2}\right)-\mathbf{T}\left(\overrightarrow{G}_{m-2}^{L_{m-2}^2}\right)\right\|\leq\xi^2\left\|\overrightarrow{G}_{m-1}^{L_{m-1}^2}-\overrightarrow{G}_{m-2}^{L_{m-2}^2}\right\|\\
&\leq \cdots\leq\xi^m\left\|\overrightarrow{G}_1^{L_1^2}-\overrightarrow{G}^{L_0^2}\right\|,
\end{aligned}$$

where $m\geq1$ is an integer. Applying the triangle inequality, we generally know that

$$\begin{aligned}
\left\|\overrightarrow{G}_m^{L_m^2}-\overrightarrow{G}_n^{L_n^2}\right\| &\leq \left\|\overrightarrow{G}_m^{L_m^2}-\overrightarrow{G}_{m-1}^{L_{m-1}^2}\right\|++\cdots+\left\|\overrightarrow{G}_{n-1}^{L_{n-1}^2}-\overrightarrow{G}_n^{L_n^2}\right\|\\
&\leq \left(\xi^m+\xi^{m-1}+\cdots+\xi^{n-1}\right)\times\left\|\overrightarrow{G}_1^{L_1^2}-\overrightarrow{G}^{L_0^2}\right\|\\
&= \frac{\xi^{n-1}-\xi^m}{1-\xi}\times\left\|\overrightarrow{G}_1^{L_1^2}-\overrightarrow{G}^{L_0^2}\right\|\leq\frac{\xi^{n-1}}{1-\xi}\times\left\|\overrightarrow{G}_1^{L_1^2}-\overrightarrow{G}^{L_0^2}\right\|
\end{aligned}$$

with $m\geq n$ and $0<\xi<1$. Consequently, $\left\|\overrightarrow{G}_m^{L_m^2}-\overrightarrow{G}_n^{L_n^2}\right\|\rightarrow0$ if $m,n\rightarrow\infty$. Whence, $\left\{\overrightarrow{G}_n^{L_n}\right\}$ is a Cauchy sequence convergent to a harmonic flow $\overrightarrow{G}^{L^2}\in\mathscr{G}_{\mathscr{B}}^{\pm}$ because of

$$\begin{aligned}
&\lim_{n\rightarrow\infty}\left(\sum_{u\in N_{\overrightarrow{G}_n}(v)}L_n^{A_{vu}^+}(v,u),-\sum_{u\in N_{\overrightarrow{G}_n}(v)}L_n^{A_{vu}^+}(v,u)\right)\\
&=\left(\lim_{n\rightarrow\infty}\sum_{u\in N_{\overrightarrow{G}_n}(v)}L_n^{A_{vu}^+}(v,u),-\lim_{n\rightarrow\infty}\sum_{u\in N_{\overrightarrow{G}_n}(v)}L_n^{A_{vu}^+}(v,u)\right)=(L(v),-L(v))
\end{aligned}$$

for $\forall v \in V\left(\overrightarrow{G}\right)$. Notice that

$$\begin{aligned}\left\|\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\| &\leq \left\|\overrightarrow{G}^{L^2} - \overrightarrow{G}_m^{L_m^2}\right\| + \left\|\overrightarrow{G}_m^{L_m^2} - \mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\| \\ &\leq \left\|\overrightarrow{G}^{L^2} - \overrightarrow{G}_m^{L_m^2}\right\| + \xi\left\|\overrightarrow{G}_{m-1}^{L_{m-1}^2} - \overrightarrow{G}^{L^2}\right\|.\end{aligned}$$

Thus, if $m \to \infty$, we get that $\left\|\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\| = 0$, i.e., $\mathbf{T}\left(\overrightarrow{G}^{L^2}\right) = \overrightarrow{G}^{L^2}$.

If there is another harmonic flow $\overrightarrow{G}'^{L'^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$ holding with $\mathbf{T}\left(\overrightarrow{G}'^{L'^2}\right) = \overrightarrow{G}^{L^2}$, by

$$\left\|\overrightarrow{G}^{L^2} - \overrightarrow{G}'^{L'^2}\right\| = \left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right) - \mathbf{T}\left(\overrightarrow{G}'^{L'^2}\right)\right\| \leq \xi\left\|\overrightarrow{G}^{L^2} - \overrightarrow{G}'^{L'^2}\right\|,$$

it is true only in the case of $\overrightarrow{G}^{L} = \overrightarrow{G}'^{L'}$, i.e., $\overrightarrow{G}^{L^2}$ is unique. □

Theorem 2.9 *A linear operator* $\mathbf{T} : \overrightarrow{\mathscr{G}}^{\pm} \to \mathscr{G}^{\pm}_{\mathscr{B}}$ *is continuous if and only if it is bounded.*

Proof If $\mathbf{T}$ is bounded, then

$$\left\|\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right) - \mathbf{T}\left(\overrightarrow{G}_l^{L_l^2}\right)\right\| = \left\|\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right)\right\| \leq \xi\left\|\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right\|$$

for a constant $\xi \in [0, \infty)$ and $\forall \overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$ by definition. Let $\delta(\varepsilon) = \frac{\varepsilon}{\xi}$ with $\xi \neq 0$. Clearly, $\left\|\mathbf{T}\left(\overrightarrow{G}^{L} - \overrightarrow{G}^{L_0}\right)\right\| < \varepsilon$ if $\left\|\overrightarrow{G}^{L} - \overrightarrow{G}^{L_0}\right\| < \delta(\varepsilon)$, i.e., $\mathbf{T}$ is continuous. If $\xi = 0$ then it is obvious that $\mathbf{T}$ is bounded.

Now, if $\mathbf{T}$ is continuous but unbounded, there must be a sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ in $\mathscr{G}^{\pm}_{\mathscr{B}}$ such that $\left\|\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right)\right\| \geq n\left\|\overrightarrow{G}_n^{L_n^2}\right\|$. Let $\overrightarrow{G^*}_n^{L_n^2} = \frac{\overrightarrow{G}_n^{L_n^2}}{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|}$. Then $\left\|\overrightarrow{G^*}_n^{L_n^2}\right\| = \dfrac{1}{n}$, which implies that $\left\|\mathbf{T}\left(\overrightarrow{G^*}_n^{L_n^2}\right)\right\| = \frac{1}{n} \to 0$ if $n \to \infty$. However, by definition

$$\left\|\mathbf{T}\left(\overrightarrow{G^*}_n^{L_n^2}\right)\right\| = \left\|\mathbf{T}\left(\frac{\overrightarrow{G}_n^{L_n^2}}{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|}\right)\right\| = \frac{\left\|\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right)\right\|}{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|} \geq \frac{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|}{n\left\|\overrightarrow{G}_n^{L_n^2}\right\|} = 1,$$

a contradiction. Thus, such a sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ can not exists in $\mathscr{G}^{\pm}_{\mathscr{B}}$ and $\mathbf{T}$ is bounded. □

The following results generalize the Banach inverse mapping theorem, closed graph theorem in classical Banach space to Banach harmonic flow space.

Theorem 2.10(Banach) *Let* $\mathbf{T} : \mathscr{G}^{\pm}_{\mathscr{B}_1} \to \mathscr{G}^{\pm}_{\mathscr{B}_2}$ *be a linear continuous operator with Banach spaces* $\mathscr{B}_1$ *and* $\mathscr{B}_2$*. If* $\mathbf{T}$ *is bijective then its inverse operator* $\mathbf{T}^{-1}$ *is continuous.*

Proof Clearly, the inverse operator $\mathbf{T}^{-1}$ exists by the assumption that $\mathbf{T}$ is bijective. For integers $n \in \mathbb{Z}^+$, let $O_n = \left\{\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}_1} \middle| \left\|\overrightarrow{G}^{L^2}\right\| \leq n\right\}$ and $M_n = \mathbf{T}(O_n)$. Notice that $\bigcup\limits_{n=1}^{\infty} O_n = \mathscr{G}^{\pm}_{\mathscr{B}_1}$. Whence, $\mathscr{G}^{\pm}_{\mathscr{B}_2} = \bigcup\limits_{n=1}^{\infty} \mathbf{T}(O_n)$. We prove that $\mathbf{T}^{-1}$ is continuous which follows by

3 claims following.

Claim 1. there is an integer n_0 such that the closure of M_{n_0} is closed, i.e., $Cl\left(M_{n_0}\right) = M_{n_0}$ in sphere $B\left(\overrightarrow{G}_0^{L_0^2}, r_0\right) = \left\{\overrightarrow{G}^{L^2} \left| \left\| \overrightarrow{G}^{L^2} - \overrightarrow{G}_0^{L_0^2} \right\| \leq r_0 \right.\right\} \subset \mathscr{G}_{\mathscr{B}_2}^{\pm}$.

If Claim 1 is not true, there always exists a closed sphere $\mathbb{B}'_n$ in the interior of closed sphere $\mathbb{B}_n$ of $\mathscr{G}_{\mathscr{B}_2}^{\pm}$ such that $B_n \bigcap M_n = \emptyset$ for integers $n \geq 1$. Now let $\mathbb{B}_0$ be a closed sphere of $\mathscr{G}_{\mathscr{B}_2}^{\pm}$. Then there is a closed sphere $\mathbb{B}_1 \subset \mathbb{B}_0$ with $\mathbb{B}_1 \bigcap M_1 = \emptyset$. Similarly, there is a closed sphere $\mathbb{B}_2 \subset \mathbb{B}_1$ with $\mathbb{B}_2 \bigcap M_2 = \emptyset$. Continuing this process, we get a sequence $\{\mathbb{B}_n\}$ of closed spheres with $\mathbb{B}_n \supset \mathbb{B}_{n+1}$.

Without loss of generality, assume the diameter $\text{Diam}\left(\mathbb{B}_n\right) \to 0$ with $\mathbb{B}_n \neq \mathbb{B}_{n+1}$ as $n \to \infty$. We can always choose harmonic flow $\overrightarrow{G}_n^{L_n^2} \in \mathbb{B}_n - \mathbb{B}_{n-1}$ and get a harmonic flow sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ of $\mathscr{G}_{\mathscr{B}_2}^{\pm}$. Clearly, $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ is a Cauchy sequence for $d\left(\overrightarrow{G}_n^{L_n^2}, \overrightarrow{G}_m^{L_m^2}\right) \leq \text{Diam}\left(\mathbb{B}_m\right) \to 0$ as $m \to \infty$ if $n \geq m$. Thus, there is a harmonic flow $\overrightarrow{G}_\infty^{L_\infty^2} \in \bigcap\limits_{n\geq 1} \mathbb{B}_n$ but $\overrightarrow{G}_\infty^{L_\infty^2} \notin \bigcup\limits_{n\geq 1} M_n$, a contradiction to $\mathscr{G}_{\mathscr{B}_2}^{\pm} = \bigcup\limits_{n=1}^{\infty} \mathbf{T}\left(O_n\right)$.

Define $\lambda_0 = \frac{r_0}{n_0}$ and $\mathbb{B}_{\lambda_0} = \left\{\overrightarrow{G}^{L^2} \left| \left\| \overrightarrow{G}^{L^2} \right\| \leq \lambda_0 \right.\right\}$. By Claim 1, $M_1 \subset \mathbb{B}_{\lambda_0}$.

Claim 2. $Cl\left(M_1\right) = M_1$.

Clearly, if $\overrightarrow{G}^{L^2} \in \mathbb{B}_{\lambda_0}$ then $\overrightarrow{G}_0^{L_0} + n_0\overrightarrow{G}^{L^2}, \overrightarrow{G}_0^{L_0} - n_0\overrightarrow{G}^{L^2} \in B\left(\overrightarrow{G}_0^{L_0^2}, r_0\right)$. Whence, there are sequence $\left\{\overrightarrow{G}_k^{L_k^2}\right\}$ and $\left\{\overrightarrow{G'}_k^{L_k^2}\right\}$ in $\mathbb{B}_{n_0}$ such that

$$\lim_{k\to\infty} \mathbf{T}\left(\overrightarrow{G}_k^{L_k^2}\right) = \overrightarrow{G}_0^{L_0} + n_0\overrightarrow{G}^{L^2} \quad \text{and} \quad \lim_{k\to\infty} \mathbf{T}\left(\overrightarrow{G'}_k^{L_k^2}\right) = \overrightarrow{G}_0^{L_0} + n_0\overrightarrow{G}^{L^2}.$$

Whence, we get that

$$\mathbf{T}\left(\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G'}_k^{L_k^2}\right) = 2n_0\overrightarrow{G}^{L^2}, i.e., \quad \mathbf{T}\left(\frac{\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G'}_k^{L_k^2}}{2n_0}\right) = \overrightarrow{G}^{L^2}.$$

Clearly, $\frac{\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G'}_k^{L_k^2}}{2n_0} \in O_1$. We know that $Cl\left(M_1\right) = M_1$.

Let $O_{\frac{1}{2^n}} = \left\{\overrightarrow{G}^{L^2} \left| \left\| \overrightarrow{G}^{L^2} \right\| \leq \frac{1}{2^n} \right.\right\}$ and $\mathbb{B}_{\frac{\lambda_0}{2^n}} = \left\{\overrightarrow{G}^{L^2} \left| \left\| \overrightarrow{G}^{L^2} \right\| \leq \frac{1}{2^n} \right.\right\}$ for integers $n \geq 1$. We are easily know that $Cl\left(\mathbf{T}\left(O_{\frac{1}{2^n}}\right)\right) = \mathbf{T}\left(O_{\frac{1}{2^n}}\right)$ in closed sphere $\mathbb{B}_{\frac{\lambda_0}{2^n}}$ by Claim 2.

Claim 3. $\mathbf{T}\left(O_1\right) \supset \mathbb{B}_{\frac{\lambda_0}{2}}$.

In fact, let $\overrightarrow{G}^{L^2} \in \mathbb{B}_{\frac{\lambda_0}{2}}$. Notice that $Cl\left(\mathbf{T}\left(O_{\frac{1}{2}}\right)\right) = \mathbf{T}\left(O_{\frac{1}{2}}\right)$ in $\mathbb{B}_{\frac{\lambda_0}{2}}$. There is $\overrightarrow{G}_1^{L_1^2} \in O_{\frac{1}{2}}$ such that $\left\| \overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2}\right) \right\| \leq \frac{\lambda_0}{2^2}$, i.e., $\overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2}\right) \in \mathbb{B}_{\frac{\lambda_0}{2^2}}$. Similarly, by $Cl\left(\mathbf{T}\left(O_{\frac{1}{2^2}}\right)\right) = \mathbf{T}\left(O_{\frac{1}{2^2}}\right)$ in $\mathbb{B}_{\frac{\lambda_0}{2^2}}$ we know that there is $\overrightarrow{G}_2^{L_2^2} \in O_{\frac{1}{2^2}}$ such that $\left\| \overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2}\right) - \mathbf{T}\left(\overrightarrow{G}_2^{L_1^2}\right) \right\| \leq \frac{\lambda_0}{2^3}$, i.e., $\left\| \overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2} + \overrightarrow{G}_2^{L_1^2}\right) \right\| \leq \frac{\lambda_0}{2^3}$. Continuing this process, we generally know that there is a harmonic flow sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ with $\overrightarrow{G}_n^{L_n^2} \in O_{\frac{1}{2}}$ for integers $n \geq 1$ such that

$$\left\| \overrightarrow{G}^{L^2} - \mathbf{T}\left(\overrightarrow{G}_1^{L_1^2} + \overrightarrow{G}_2^{L_2^2} + \cdots + \overrightarrow{G}_n^{L_n^2}\right) \right\| \leq \frac{\lambda_0}{2^{n+1}} \tag{2.9}$$

by mathematical induction. Notice that

$$\left\|\sum_{i=1}^{n}\overrightarrow{G}_i^{L_i^2}\right\| \leq \sum_{i=1}^{n}\left\|\overrightarrow{G}_i^{L_i^2}\right\| \leq \sum_{i=1}^{n}\frac{1}{2^n} = 1.$$

We therefore know that $\sum\limits_{i=1}^{n}\overrightarrow{G}_i^{L_i^2}$ is convergent in O_1. Denoted by $\overrightarrow{G}_\Sigma^{L^2} = \sum\limits_{i=1}^{n}\overrightarrow{G}_i^{L_i^2}$, i.e., $\overrightarrow{G}_\Sigma^{L^2} \in O_1$. By the continuous assumption of $\mathbf{T}$, we get immediately that $\overrightarrow{G}^{L^2} = \mathbf{T}\left(\overrightarrow{G}_\Sigma^{L^2}\right)$ by letting $n \to \infty$ in (2.9), which implies that $\mathbf{T}(O_1) \supset \mathbb{B}_{\frac{\lambda_0}{2}}$, i.e., $O_1 \supset \mathbf{T}^{-1}\left(\mathbb{B}_{\frac{\lambda_0}{2}}\right)$.

Now we prove $\mathbf{T}^{-1}$ is bounded. Let $\mathbf{O} \neq \overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$. Clearly, $\frac{\lambda_0\overrightarrow{G}^{L^2}}{2\left\|\overrightarrow{G}^{L^2}\right\|} \in \mathbb{B}_{\frac{\lambda_0}{2}}$, we know that

$$\mathbf{T}^{-1}\left(\frac{\lambda_0\overrightarrow{G}^{L^2}}{2\left\|\overrightarrow{G}^{L^2}\right\|}\right) \in O_1, \quad \text{i.e.,} \quad \left\|\mathbf{T}^{-1}\left(\frac{\lambda_0\overrightarrow{G}^{L^2}}{2\left\|\overrightarrow{G}^{L^2}\right\|}\right)\right\| \leq 1.$$

Whence we get that

$$\left\|\mathbf{T}^{-1}\left(\overrightarrow{G}^{L^2}\right)\right\| \leq \frac{2}{\lambda_0}\left\|\overrightarrow{G}^{L^2}\right\|,$$

i.e., $\mathbf{T}^{-1}$ is bounded. Applying Theorem 2.9 we know that $\mathbf{T}^{-1}$ is continuous. □

Definition 2.11 *Let $\mathbf{T} : \mathscr{G}_{\mathscr{B}_1}^{\pm} \to \mathscr{G}_{\mathscr{B}_2}^{\pm}$ be a linear continuous operator with Banach spaces $\mathscr{B}_1$, $\mathscr{B}_2$. The graph of $\mathbf{T}$ in $\mathscr{G}_{\mathscr{B}_2}^{\pm}$ is defined by*

$$\text{Grap}\mathbf{T} = \left\{\left(\overrightarrow{G}^{L^2}, \mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right)\Big|\ \overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}_1}^{\pm}\right\}$$

and $\mathbf{T}$ is closed if $Cl\,(\text{Grap}\mathbf{T}) = \text{Grap}\mathbf{T}$, i.e., a closed subspace.

Theorem 2.12 *Let $\mathbf{T} : \mathscr{G}_{\mathscr{B}_1}^{\pm} \to \mathscr{G}_{\mathscr{B}_2}^{\pm}$ be a linear operator with Banach spaces $\mathscr{B}_1$, $\mathscr{B}_2$. Then $\mathbf{T}$ is closed if and only if for any harmonic flow sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\} \in \mathscr{G}_{\mathscr{B}_1}^{\pm}$ with $\lim\limits_{n\to\infty}\overrightarrow{G}_n^{L_n^2} = \overrightarrow{G}_0^{L_0^2} \in \mathscr{G}_{\mathscr{B}_1}^{\pm}$, $\lim\limits_{n\to\infty}\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right) = \overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}_2}^{\pm}$ and $\mathbf{T}\left(\overrightarrow{G}_0^{L_0^2}\right) = \overrightarrow{G}^{L^2}$.*

Proof For $\left(\overrightarrow{G}_0^{L_0^2}, \overrightarrow{G}^{L^2}\right) \in Cl\,(\text{Grap}\mathbf{T})$, there is a harmonic flow sequence $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ such that $\left(\overrightarrow{G}_n^{L_n^2}, \mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right)\right) \to \left(\overrightarrow{G}_0^{L_0^2}, \overrightarrow{G}^{L^2}\right)$ as $n \to \infty$ by definition. We therefore get that $\overrightarrow{G}_n^{L_n^2} \to \overrightarrow{G}_0^{L_0^2}$ and $\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right) \to \overrightarrow{G}^{L^2}$ as $n \to \infty$. If $\mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right) = \overrightarrow{G}^{L^2}$, then $\left(\overrightarrow{G}_0^{L_0^2}, \overrightarrow{G}^{L^2}\right) \in \text{Grap}\mathbf{T}$. We know that Grap$\mathbf{T}$ is a closed subspace, i.e., $\mathbf{T}$ is a closed operator.

Conversely, if $\mathbf{T}$ is a closed operator, let $\left\{\overrightarrow{G}_n^{L_n^2}\right\}$ be a harmonic flow sequence in $\mathscr{G}_{\mathscr{B}_1}^{\pm}$ with $\left(\overrightarrow{G}_n^{L_n^2}, \mathbf{T}\left(\overrightarrow{G}_n^{L_n^2}\right)\right) \to \left(\overrightarrow{G}_0^{L_0^2}, \overrightarrow{G}^{L^2}\right) \in \text{Grap}\mathbf{T}$ as $n \to \infty$ by definition. Whence, $\mathbf{T}\left(\overrightarrow{G}_0^{L_0^2}\right) = \overrightarrow{G}^{L^2}$. This completes the proof. □

Theorem 2.13(Closed Graph Theorem) *If $\mathbf{T} : \mathscr{G}_{\mathscr{B}_1}^{\pm} \to \mathscr{G}_{\mathscr{B}_2}^{\pm}$ is a closed linear operator with Banach spaces $\mathscr{B}_1$, $\mathscr{B}_2$, then $\mathbf{T}$ is continuous.*

Proof Notice that $\mathscr{G}_{\mathscr{B}_1}^{\pm}\bigoplus\mathscr{G}_{\mathscr{B}_2}^{\pm}$ with norm $\left\|\overrightarrow{G}^{L^2}\right\| + \left\|\overrightarrow{G'}^{L^2}\right\|$ for $\forall\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}_1}^{\pm}, \overrightarrow{G'}^{L^2} \in \mathscr{G}_{\mathscr{B}_2}^{\pm}$ is also a Banach space with subspace Grap$\mathbf{T}$ by definition. Define $\widehat{\mathbf{T}}:\left(\overrightarrow{G}^{L^2},\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right)\rightarrow\overrightarrow{G}^{L^2}$ for $\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}_1}$. Clearly, $\widehat{\mathbf{T}}$ is bijective from Grap$\mathbf{T}$ to $\mathscr{G}^{\pm}_{\mathscr{B}_1}$. By Theorem 4.10, we know that $\widehat{\mathbf{T}}^{-1}$ is continuous or bounded, i.e.,

$$\left\|\widehat{\mathbf{T}}^{-1}\left(\overrightarrow{G}^{L^2}\right)\right\|=\left\|\left(\overrightarrow{G}^{L^2},\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right)\right\|=\left\|\overrightarrow{G}^{L^2}\right\|+\left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\|\leq\left\|\widehat{\mathbf{T}}^{-1}\right\|\left\|\overrightarrow{G}^{L^2}\right\|.$$

We therefore get that $\left\|\mathbf{T}\left(\overrightarrow{G}^{L^2}\right)\right\|\leq\left\|\widehat{\mathbf{T}}^{-1}\right\|\left\|\overrightarrow{G}^{L^2}\right\|$, i.e., $\mathbf{T}$ is bounded and continuous by Theorem 2.9. □

Notice that harmonic flow spaces $\mathscr{G}^{\pm}_{\mathscr{B}_1}$ and $\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ are both labeled graph families. A harmonic flow space $\mathscr{G}^{\pm}_{\mathscr{B}_1}$ is *isomorphic* to $\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ if there is a linear continuous operator $\mathbf{T}:\mathscr{G}^{\pm}_{\mathscr{B}_1}\rightarrow\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ of bijection with $\mathbf{T}:\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}_1}\rightarrow\overrightarrow{G'}^{L'^2}\in\mathscr{G}^{\pm}_{\mathscr{B}_2}$ such that

$$\mathbf{T}\left(A^{+}_{vu},(L(v,u),-L(v,u)),A^{+}_{uv}\right)=\left(A'^{+}_{vu},(L'(v,u),-L'(v,u)),A'^{+}_{uv}\right)$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. The following result characterizes isomorphic harmonic flow spaces.

Theorem 2.14 *A harmonic flow spaces $\mathscr{G}^{\pm}_{\mathscr{B}_1}$ is isomorphic to $\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ with $\mathbf{T}:\overrightarrow{G}^{L^2}\rightarrow\overrightarrow{G'}^{L'^2}$ if and only if $\mathscr{G}=\mathscr{G}'$ and $\mathscr{B}_1$ is isomorphic to $\mathscr{B}_2$.*

Proof Clearly, if $T:\mathscr{B}_1\rightarrow\mathscr{B}_2$ is an isomorphism and $\mathscr{G}=\mathscr{G}'$, there is an identical mapping $id:G\in\mathscr{G}\rightarrow G\in\mathscr{G}'$. We are easily know that the operator $\mathbf{T}=T\circ id:\mathscr{G}^{\pm}_{\mathscr{B}_1}\rightarrow\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ with $\mathbf{T}:\overrightarrow{G}^{L^2}\rightarrow\overrightarrow{G'}^{L'^2}$ is an isomorphism between $\mathscr{G}^{\pm}_{\mathscr{B}_1}$ and $\mathscr{G}'^{\pm}_{\mathscr{B}_2}$.

Conversely, if $\mathbf{T}:\mathscr{G}^{\pm}_{\mathscr{B}_1}\rightarrow\mathscr{G}'^{\pm}_{\mathscr{B}_2}$ is an isomorphism with $\mathbf{T}:\overrightarrow{G}^{L^2}\rightarrow\overrightarrow{G'}^{L'^2}$, we know that

$$\begin{aligned}\mathbf{T}\ &:\left(A^{+}_{vu},(L(v,u),-L(v,u)),A^{+}_{uv}\right)\in\overrightarrow{G}^{L^2}\rightarrow\left(A'^{+}_{v'u'},(L'(v',u'),-L'(v',u')),A'^{+}_{u'v'}\right)\in\overrightarrow{G}^{L^2},\\ \mathbf{T}^{-1}&:\left(A'^{+}_{v'u'},(L'(v',u'),-L'(v',u')),A'^{+}_{u'v'}\right)\in\overrightarrow{G}^{L^2}\rightarrow\left(A^{+}_{vu},\{L(v,u),-L(v,u)\},A^{+}_{uv}\right)\in\overrightarrow{G}^{L^2}\end{aligned}$$

which naturally induces

$$\mathbf{T}_v:\left\{L(v,u),u\in N_{\overrightarrow{G}}(v)\right\}\rightarrow\left\{L'(v',u'),u'\in N_{\overrightarrow{G}}(v')\right\},$$

i.e., an isomorphism $\mathbf{T}_v:v'\in V\left(\overrightarrow{G}\right)\rightarrow v'\in V\left(\overrightarrow{G'}\right)$ preserving the adjacency of vertices. We therefore know that $\overrightarrow{G}$ and $\overrightarrow{G'}$ are isomorphic, i.e., $\mathscr{G}=\mathscr{G}'$.

Notice that an isomorphism $\mathbf{T}$ is linear continuous. By Theorem 4.10 we know that $\mathbf{T}^{-1}$ is continuous also. Thus, $\mathbf{T}$, $\mathbf{T}^{-1}$ induce operators $\mathbf{T}_{vu}:\{L(v,u)\in\mathscr{B}_1\}\rightarrow\{L'(v',u')\in\mathscr{B}_2\}$, $\mathbf{T}^{-1}_{vu}:\{L'(v',u')\in\mathscr{B}_2\}\rightarrow\{L(v,u)\in\mathscr{B}_1\}$ for edges $(v,u)\in E\left(\overrightarrow{G}\right)$, $(v',u')\in E\left(\overrightarrow{G'}\right)$ and both of them are bijective. Consequently, $\mathbf{T}_{vu}$ is also linear continuous with a continuously inverse $\mathbf{T}^{-1}_{vu}$, i.e., preserving the topology on $\mathscr{B}_1$ and $\mathscr{B}_2$. Whence, $\mathbf{T}_{vu}$ is an isomorphisms between Banach spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ for $(v,u)\in E\left(\overrightarrow{G}\right)$ by definition. □

Certainly, there maybe existed more than one norm on a harmonic flow space $\mathscr{G}^{\pm}_{\mathscr{B}}$, We need to distinguish them by the equivalence following.

Definition 2.15 *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms in $\mathscr{G}^{\pm}_{\mathscr{B}}$. If there are positive numbers K_1, K_2 such that*

$$K_1\left\|\overrightarrow{G}^{L^2}\right\|_1 \leq \left\|\overrightarrow{G}^{L^2}\right\|_2 \leq K_2\left\|\overrightarrow{G}^{L^2}\right\|_1$$

for $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$, then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be equivalent on $\mathscr{G}^{\pm}_{\mathscr{B}}$.

Theorem 2.16 *Let $\|\cdot\|_1$, $\|\cdot\|_2$ be norms defining Banach spaces on $\mathscr{G}^{\pm}_{\mathscr{B}}$. If there is a positive number K such that $\left\|\overrightarrow{G}^{L^2}\right\|_2 \leq K\left\|\overrightarrow{G}^{L^2}\right\|_1$ for $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*

Proof Denoted by $\mathscr{G}^{\pm}_{\mathscr{B}_1}$, $\mathscr{G}^{\pm}_{\mathscr{B}_2}$ the Banach spaces with norm $\|\cdot\|_1$ or $\|\cdot\|_2$, respectively. Define an operator $\mathbf{I}: \mathscr{G}^{\pm}_{\mathscr{B}_1} \rightarrow \mathscr{G}^{\pm}_{\mathscr{B}_2}$ by $\mathbf{I}\left(\overrightarrow{G}^{L^2}\right) = \overrightarrow{G}^{L^2}$ for $\forall \overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}_1}$. Clearly, $\mathbf{I}$ is linear and bijective.

Now, if $\left\|\overrightarrow{G}^{L^2}\right\|_2 \leq K\left\|\overrightarrow{G}^{L^2}\right\|_1$ for $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}_2}$, then $\mathbf{I}$ is bounded. Applying Theorem 2.10 we know that $\mathbf{I}^{-1}$ is continuous, i.e., bounded by Theorem 2.9. Whence, $\left\|\mathbf{I}^{-1}\left(\overrightarrow{G}^{L^2}\right)\right\|_1 \leq \left\|\mathbf{I}^{-1}\right| \left\|\overrightarrow{G}^{L^2}\right\|_2$, i.e., $\left\|\overrightarrow{G}^{L^2}\right\|_1 \leq \left\|\mathbf{I}^{-1}\right| \left\|\overrightarrow{G}^{L^2}\right\|_2$ by definition. We get that

$$\frac{1}{\left\|\mathbf{I}^{-1}\right|}\left\|\overrightarrow{G}^{L^2}\right\|_1 \leq \left\|\overrightarrow{G}^{L^2}\right\|_2 \leq K\left\|\overrightarrow{G}^{L^2}\right\|_1. \qquad \square$$

Notice that the far or near degree of $\overrightarrow{G}^{L^2_k}_k$ and $\overrightarrow{G}^{L^2_l}_l$ is measured by the sum of norms on edges in Definition 2.2. Sometimes, we also need it to measure by the residue norms on vertices such as the synchronization of complex networks, i.e., the conception following.

Definition 2.17 *For $\forall \overrightarrow{G}^{L^2_k}_k, \overrightarrow{G}^{L^2_l}_l \in \mathscr{G}^{\pm}_{\mathscr{B}}$, the distance $D\left(\overrightarrow{G}^{L^2_k}_k, \overrightarrow{G}^{L^2_l}_l\right)$ between vertices of $\overrightarrow{G}^{L^2_k}_k$ and $\overrightarrow{G}^{L^2_l}_l$ is defined by the sum of vertices norms of $\overrightarrow{G}^{L^2_k}_k - \overrightarrow{G}^{L^2_l}_l = \overrightarrow{G}^{L^2_k}_k + \overrightarrow{G}^{-L^2_l}_l$, i.e.,*

$$D\left(\overrightarrow{G}^{L^2_k}_k, \overrightarrow{G}^{L^2_l}_l\right) = \sum_{v \in V\left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)} \left\|L^-_{kl_1}(v)\right\|, \qquad (2.10)$$

where,

$$L^-_{kl}(v) = \begin{cases} L^2_k(v) \text{ or } L^2_l(v) & \text{if } v \in V\left(\overrightarrow{G}_k \setminus \overrightarrow{G}_l\right) \text{ or } V\left(\overrightarrow{G}_k \setminus \overrightarrow{G}_l\right) \\ L^2_k(v) - L^2_l(v) & \text{if } v \in V\left(\overrightarrow{G}_k \bigcap \overrightarrow{G}_l\right) \end{cases}.$$

Clearly, $\left(\mathscr{G}^{\pm}_{\mathscr{B}}; D\right)$ is not a distance space because we have $D\left(\overrightarrow{G}^{L^2_k}_k, \overrightarrow{G}^{L^2_l}_l\right) = 0$ if the residue flows on vertices in $\overrightarrow{G}^{L^2_l}_k$, $\overrightarrow{G}^{L^2_l}_l$ are a constant. However, we can measure the near degree of $\overrightarrow{G}^{L^2_k}_k$ and $\overrightarrow{G}^{L^2_l}_l$ by norms on edges, i.e., it is stronger than that on vertex for harmonic flows.

Theorem 2.18 *For $\forall \overrightarrow{G}^{L^2_k}_k, \overrightarrow{G}^{L^2_l}_l \in \mathscr{G}^{\pm}_{\mathscr{B}}$, if all end-operators on $\overrightarrow{G}^{L^2_k}_k$ and $\overrightarrow{G}^{L^2_l}_l$ are linear continuous, then there exists a constant $c > 0$ such that*

$$D\left(\overrightarrow{G}^{L^2_k}_k, \overrightarrow{G}^{L^2_l}_l\right) \leq c\left(\left\|\overrightarrow{G}^{L^2_k}_k - \overrightarrow{G}^{L^2_l}_l\right\|\right).$$

Proof By Theorem 1.3 we know that there are positive constants $c_{vu}^1, c_{vu}^2 \in \mathbb{R}$ such that $\left\|L_1^{A_{vu}^+}(v,u)\right\| \leq c_1^{vu}\|L_1(v,u)\|$ and $\left\|L_2^{A_{vu}^+}(v,u)\right\| \leq c_2^{vu}\|L_2(v,u)\|$ for $\forall(v,u) \in \overrightarrow{G}$ if the end-operator A_{vu}^+ is linear continuous. Without loss of generality, let

$$c^{\max}\left(\overrightarrow{G}^{L^2}\right) = \max\left\{c_1^{vu}, c_2^{vu} | v, u \in V\left(\overrightarrow{G}\right)\right\}$$

and $\overrightarrow{H} = \overrightarrow{G}_k \bigcup \overrightarrow{G}_l$. We are easily know that

$$\begin{aligned}
d\left(\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2}\right) &= \sum_{v\in V\left(\overrightarrow{H}\right)} \left\|L_{kl1}^-(v)\right\| = \sum_{v\in V\left(\overrightarrow{H}\right)} \sum_{u\in N_{\overrightarrow{H}}(v)} \left\|{L_{kl1}^-}^{A_{vu}^+}(v,u)\right\| \\
&\leq c^{\max} \sum_{v\in V\left(\overrightarrow{H}\right)} \sum_{u\in N_{\overrightarrow{H}}(v)} \left\|L_{kl1}^-(v,u)\right\| \\
&= 2c^{\max}\left(\overrightarrow{H}^{L_{kl1}^-}\right) \sum_{(v,u)\in E\left(\overrightarrow{H}\right)} \left\|L_{kl1}^-(v,u)\right\| \\
&= 2c^{\max}\left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)^{L_{kl}^-} \left\|\left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)^{L_{kl}^-}\right\| \\
&= 2c^{\max}\left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)^{L_{kl}^-} \left\|\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right\|
\end{aligned}$$

by the assumption. We therefore get that

$$d\left(\overrightarrow{G}_k^{L_k^2}, \overrightarrow{G}_l^{L_l^2}\right) \leq c\left(\left\|\overrightarrow{G}_k^{L_k^2} - \overrightarrow{G}_l^{L_l^2}\right\|\right)$$

with $c = 2c^{\max}\left(\overrightarrow{G}_k \bigcup \overrightarrow{G}_l\right)^{L_{kl}^-}$. This completes the proof. □

A linear operator $\mathbf{T} : \mathscr{G}_{\mathscr{B}_1}^{\pm} \rightarrow \mathscr{G}_{\mathscr{B}_2}^{\pm}$ is a *functional* if $\mathscr{G}_{\mathscr{B}_2}^{\pm} = \mathbb{R}$ or $\mathbb{C}$, and there is a fundamental question on functionals should be answered, i.e., *are there really linear continuous functionals on harmonic flow spaces* $\mathscr{G}_{\mathscr{B}}^{\pm}$? Certainly, its answer is affirmative by results following.

Definition 2.19 *A functional* $p : \mathscr{G}_{\mathscr{B}_1}^{\pm} \rightarrow \mathbb{R}$ *is sublinear if* $p\left(\overrightarrow{G}^{L^2} + \overrightarrow{G'}^{L'^2}\right) \leq p\left(\overrightarrow{G}^{L^2}\right) + p\left(\overrightarrow{G'}^{L'^2}\right)$ *and* $p\left(\alpha\overrightarrow{G}^{L^2}\right) = \alpha p\left(\overrightarrow{G}^{L^2}\right)$ *for* $\overrightarrow{G}^{L^2}, \overrightarrow{G'}^{L'^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$ *and* $\alpha \geq 0$.

We can similarly extend the Hahn-Banach theorem, i.e., the existence of functionals in a Banach space to the harmonic flow space $\mathscr{G}_{\mathscr{B}}^{\pm}$ following.

Theorem 2.20(Hahn-Banach) *Let* $\mathscr{H}_{\mathscr{B}}^{\pm}$ *be a harmonic flow subspace of* $\mathscr{G}_{\mathscr{B}}^{\pm}$ *and let* $F : \mathscr{H}_{\mathscr{B}}^{\pm} \rightarrow \mathbb{C}$ *be a linear continuous functional on* $\mathscr{H}_{\mathscr{B}}^{\pm}$. *Then, there is a linear continuous functional* $\widetilde{F} : \mathscr{G}_{\mathscr{B}}^{\pm} \rightarrow \mathbb{C}$ *hold with*

(1) $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ *if* $\overrightarrow{G}^{L^2} \in \mathscr{H}_{\mathscr{B}}^{\pm}$;

(2) $\left\|\widetilde{F}\right\| = \|F\|$.

Proof The proof is consisting of claims following.

Claim 1. If there is a linear functional $F:\mathscr{H}^{\pm}_{\mathscr{B}}\rightarrow\mathbb{R}$ and a sublinear functional $p:\mathscr{G}^{\pm}_{\mathscr{B}}\rightarrow\mathbb{R}$ with $F\left(\overrightarrow{G}^{L^2}\right)\leq p\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}}$, then there exists a linear functional $\widetilde{F}:\mathscr{G}^{\pm}_{\mathscr{B}}\rightarrow\mathbb{R}$ such that $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)=F\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2}\in\mathscr{H}^{\pm}_{\mathscr{B}}$ and $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\leq p\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2}\in\mathscr{G}^{\pm}_{\mathscr{B}}$.

Let $\overrightarrow{G}_0^{L_0^2}\in\mathscr{G}^{\pm}_{\mathscr{B}}\setminus\mathscr{H}^{\pm}_{\mathscr{B}}$ and $\mathscr{H}^{\pm}_{1\mathscr{B}}=\left\{\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\middle|\,\alpha\in\mathbb{R},\overrightarrow{G}^{L^2}\in\mathscr{H}^{\pm}_{\mathscr{B}}\right\}$, a linear space spanned by $\overrightarrow{G}_0^{L_0^2}$ and $\mathscr{H}^{\pm}_{\mathscr{B}}$. For $\forall\overrightarrow{G}^{L^2},\overrightarrow{G'}^{L'^2}\in\mathscr{H}^{\pm}_{\mathscr{B}}$ calculation shows that

$$\begin{aligned}F\left(\overrightarrow{G}^{L^2}\right)-F\left(\overrightarrow{G'}^{L'^2}\right) &= F\left(\overrightarrow{G}^{L^2}-\overrightarrow{G'}^{L'^2}\right)\leq p\left(\overrightarrow{G}^{L^2}-\overrightarrow{G'}^{L'^2}\right)\\ &\leq p\left(\overrightarrow{G}^{L^2}+\overrightarrow{G}_0^{L_0^2}\right)+p\left(-\overrightarrow{G'}^{L'^2}-\overrightarrow{G}_0^{L_0^2}\right),\end{aligned}$$

i.e.,

$$-p\left(-\overrightarrow{G'}^{L'^2}-\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G'}^{L'^2}\right)\leq p\left(\overrightarrow{G}^{L^2}+\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G}^{L^2}\right).$$

Notice that $\overrightarrow{G}^{L^2}$, $\overrightarrow{G'}^{L'^2}$ are arbitrarily selected in $\mathscr{H}^{\pm}_{\mathscr{B}}$. There are must be

$$\sup_{\overrightarrow{G'}^{L'^2}\in\mathscr{H}^{\pm}_{\mathscr{B}}}\left\{-p\left(-\overrightarrow{G'}^{L'^2}-\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G'}^{L'^2}\right)\right\}\leq\inf_{\overrightarrow{G'}^{L'^2}\in\mathscr{H}^{\pm}_{\mathscr{B}}}\left\{\left(\overrightarrow{G}^{L^2}+\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G}^{L^2}\right)\right\},$$

which enables one to choose a number c hold with

$$\sup_{\overrightarrow{G'}^{L'^2}\in\mathscr{H}^{\pm}_{\mathscr{B}}}\left\{-p\left(-\overrightarrow{G'}^{L'^2}-\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G'}^{L'^2}\right)\right\}\leq c\leq\inf_{\overrightarrow{G'}^{L'^2}\in\mathscr{H}^{\pm}_{\mathscr{B}}}\left\{\left(\overrightarrow{G}^{L^2}+\overrightarrow{G}_0^{L_0^2}\right)-F\left(\overrightarrow{G}^{L^2}\right)\right\}$$

and define a functional F' by $F'\left(\overrightarrow{G^*}^{L^{*2}}\right)=\alpha c+F\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G^*}^{L^{*2}}\in\mathscr{H}^{\pm}_{1\mathscr{B}}$.

Clearly, F' is indeed a linear functional on $\mathscr{H}^{\pm}_{1\mathscr{B}}$ because $\mathscr{H}^{\pm}_{1\mathscr{B}}$ is linear spanned by $\overrightarrow{G}_0^{L_0^2}$ and $\mathscr{H}^{\pm}_{\mathscr{B}}$. We prove

$$F'\left(\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\right)\leq p\left(\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\right),\tag{2.11}$$

and without of loss of generality, assume $\alpha\neq 0$ because the assertion is obvious if $\alpha=0$.

Now if $\alpha>0$, by

$$c\leq p\left(\overrightarrow{G}_0^{L_0^2}+\frac{\overrightarrow{G}^{L^2}}{\alpha}\right)-F\left(\frac{\overrightarrow{G}^{L^2}}{\alpha}\right)$$

we are easily know that

$$F'\left(\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\right)\leq p\left(\alpha\overrightarrow{G}_0^{L_0^2}+\overrightarrow{G}^{L^2}\right),$$

i.e., (2.11) is true, and if $\alpha<0$, by

$$c\geq -p\left(-\overrightarrow{G}_0^{L_0^2}-\frac{\overrightarrow{G}^{L^2}}{\alpha}\right)-F\left(\frac{\overrightarrow{G}^{L^2}}{\alpha}\right)$$

we can know that (2.11) hold also. Whence, F' is a linear extension of F by $\overrightarrow{G}_0^{L_0^2}$. All such extensions of F' are denoted by $\mathscr{H}(F)$, i.e., $F'\left(\overrightarrow{G}^{L^2}\right)=F\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2}\in\mathscr{H}^{\pm}_{\mathscr{B}}$ and all extensions F' of F further with $F'\left(\overrightarrow{G}^{L^2}\right)\leq p\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2}\in\mathscr{H}(F)$ are denoted by $\widetilde{\mathscr{H}}(F)$.

We define an order $\prec$ in $\widetilde{\mathscr{H}}(F)$ by:

For $F_1, F_2 \in \widetilde{\mathscr{H}}(F)$, if $\mathscr{H}(F_1) \subset \mathscr{H}(F_2)$ and $F_1\left(\overrightarrow{G}^{L^2}\right) = F_2\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2} \in \mathscr{H}(F_1)$, then F_1 is precedent of F_2, denoted by $F_1 \prec F_2$.

Then $\left(\widetilde{\mathscr{H}}(F); \prec\right)$ is a partial order set.

Let $\mathscr{M}(F) \subset \widetilde{\mathscr{F}}(F)$ be with $(\mathscr{M}(F); \prec)$ an order subset and let

$$\mathscr{D}(F) = \bigcup_{F \in \mathscr{M}(F)} \mathscr{H}(F).$$

Notice that for $\overrightarrow{G}^{L^2} \in \mathscr{D}(F)$ there must be a $\mathscr{H}(F)$ such that $\overrightarrow{G}^{L^2} \in \mathscr{H}(F)$. By this fact, we can define a linear functional $\widehat{F}$ on $\mathscr{D}(F)$ by $\widehat{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2} \in \mathscr{H}(F)$. Since $\mathscr{M}(F)$ is an order set we know such a $\widetilde{F}$ is a uniquely linear functional with $\widehat{F}\left(\overrightarrow{G}^{L^2}\right) \leq p\left(\overrightarrow{G}^{L^2}\right)$ on $\mathscr{H}(F)$. Thus, $\widehat{F} \in \mathscr{D}(F)$ and it is an upper bound of $\mathscr{M}(F)$.

By Zorn's Lemma, there is a maximal element $\widetilde{F}$ in $\widetilde{\mathscr{H}}(F)$ with $\mathscr{H}\left(\widetilde{F}\right) = \mathscr{G}_{\mathscr{B}}^{\pm}$. Otherwise, let $\overrightarrow{G}_0^{L_0^2} \in \mathscr{G}_{\mathscr{B}}^{\pm} \setminus \mathscr{H}\left(\widetilde{F}\right)$, then we can extend $\widetilde{F}$ to a linear space spanned by $\mathscr{H}\left(\widetilde{F}\right)$ with $\overrightarrow{G}_0^{L_0^2}$, contradicts to the maximality of $\widetilde{F}$. We therefore know that $\mathscr{H}\left(\widetilde{F}\right) = \mathscr{G}_{\mathscr{B}}^{\pm}$.

Claim 2. If $F : \mathscr{H}_{\mathscr{B}}^{\pm} \to \mathbb{C}$ is a linear continuous functional on $\mathscr{H}_{\mathscr{B}}^{\pm}$, then there is a linear continuous functional $\widetilde{F} : \mathscr{G}_{\mathscr{B}}^{\pm} \to \mathbb{C}$ hold with $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2} \in \mathscr{H}_{\mathscr{B}}^{\pm}$ and $\left\|\widetilde{F}\right\| = \|F\|$.

Let $F\left(\overrightarrow{G}^{L^2}\right) = F_1\left(\overrightarrow{G}^{L^2}\right) + iF_2\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2} \in \mathscr{H}_{\mathscr{B}}^{\pm}$, where $F_1, F_2 : \mathscr{H}_{\mathscr{B}}^{\pm} \to \mathbb{R}$ and $i^2 = 1$. Notice that

$$i\left(F_1\left(\overrightarrow{G}^{L^2}\right) + iF_2\left(\overrightarrow{G}^{L^2}\right)\right) = iF\left(\overrightarrow{G}^{L^2}\right) = F\left(i\overrightarrow{G}^{L^2}\right) = F_1\left(i\overrightarrow{G}^{L^2}\right) + iF_2\left(i\overrightarrow{G}^{L^2}\right).$$

We know that $F_1\left(i\overrightarrow{G}^{L^2}\right) = -F_2\left(\overrightarrow{G}^{L^2}\right)$. Let $p\left(\overrightarrow{G}^{L^2}\right) = \|F\|\left\|\overrightarrow{G}^{L^2}\right\|$. Then $p\left(\overrightarrow{G}^{L^2}\right)$ is a linear functional with

$$F_1\left(\overrightarrow{G}^{L^2}\right) \leq \left\|F\left(\overrightarrow{G}^{L^2}\right)\right\| \leq \|F\|\left\|\overrightarrow{G}^{L^2}\right\| = p\left(\overrightarrow{G}^{L^2}\right)$$

on $\mathscr{H}_{\mathscr{B}}^{\pm}$, i.e., F_1 is holding with conditions of Claim 1. We know that F_1 can be extended to a linear functional F_{10} on $\mathscr{G}_{\mathscr{B}}^{\pm}$ with $F_{10}\left(\overrightarrow{G}^{L^2}\right) \leq p\left(\overrightarrow{G}^{L^2}\right)$ for $\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$.

Define

$$\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F_{10}\left(\overrightarrow{G}^{L^2}\right) + iF_{20}\left(\overrightarrow{G}^{L^2}\right) = F_{10}\left(\overrightarrow{G}^{L^2}\right) - iF_{10}\left(i\overrightarrow{G}^{L^2}\right). \tag{2.12}$$

We prove $\widetilde{F}$ is a linear continuous functional satisfying conditions of Claim 2. For $\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$, calculation shows that

$$\begin{aligned}\widetilde{F}\left(i\overrightarrow{G}^{L^2}\right) &= F_{10}\left(i\overrightarrow{G}^{L^2}\right) - iF_{10}\left(-\overrightarrow{G}^{L^2}\right) = F_{10}\left(i\overrightarrow{G}^{L^2}\right) + iF_{10}\left(i\overrightarrow{G}^{L^2}\right)\\ &= i\left(F_{10}\left(\overrightarrow{G}^{L^2}\right) - iF_{10}\left(\overrightarrow{G}^{L^2}\right)\right) = i\widetilde{F}\left(\overrightarrow{G}^{L^2}\right).\end{aligned}$$

Whence, for $\forall\alpha_1 + i\alpha_2 \in \mathbb{C}$ we have

$$\begin{aligned}\widetilde{F}\left(\alpha\overrightarrow{G}^{L^2}\right) &= \widetilde{F}\left((\alpha_1 + i\alpha_2)\overrightarrow{G}^{L^2}\right) = \alpha_1 F_{10}\left(\overrightarrow{G}^{L^2}\right) + \alpha_2 F_{10}\left(i\overrightarrow{G}^{L^2}\right)\\ &= \alpha_1 F_{10}\left(\overrightarrow{G}^{L^2}\right) + i\alpha_2 F_{10}\left(\overrightarrow{G}^{L^2}\right) = \alpha\widetilde{F}\left(\overrightarrow{G}^{L^2}\right),\end{aligned}$$

i.e., $\widetilde{F}$ is a linear functional on $\mathscr{G}_{\mathscr{B}}^{\pm}$. By Claim 1 we know that $F_{10}\left(\overrightarrow{G}^{L^2}\right) = \widetilde{F}_1\left(\overrightarrow{G}^{L^2}\right)$ and $F_{20}\left(\overrightarrow{G}^{L^2}\right) = \widetilde{F}_2\left(\overrightarrow{G}^{L^2}\right)$, i.e., $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ if $\overrightarrow{G}^{L^2} \in \mathscr{H}_{\mathscr{B}}^{\pm}$.

Clearly, $\widetilde{F}$ is continuous by definition. We show that $\left\|\widetilde{F}\right\| = \|F\|$. Let $\theta = \arg\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)$. By definition, $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = \left\|\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\right\| e^{i\theta}$. Therefore,

$$\left\|\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\right\| = e^{-i\theta}\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = \widetilde{F}\left(e^{-i\theta}\overrightarrow{G}^{L^2}\right) = F_{10}\left(e^{-i\theta}\overrightarrow{G}^{L^2}\right) - iF_{10}\left(ie^{-i\theta}\overrightarrow{G}^{L^2}\right).$$

Notice that $\left\|\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\right\| \geq 0$ is a real number, we know that

$$\left\|\widetilde{F}\left(\overrightarrow{G}^{L^2}\right)\right\| = F_{10}\left(e^{-i\theta}\overrightarrow{G}^{L^2}\right) \leq p\left(e^{-i\theta}\overrightarrow{G}^{L^2}\right) = \|F\|\left\|\overrightarrow{G}^{L^2}\right\|.$$

Whence, $\left\|\widetilde{F}\right\| \leq \|F\|$. However, $\left\|\widetilde{F}\right\| \geq \|F\|$ for $\mathscr{G}_{\mathscr{B}}^{\pm} \supset \mathscr{H}_{\mathscr{B}}^{\pm}$. We get $\left\|\widetilde{F}\right\| = \|F\|$. □

Corollary 2.21 *Let $\mathscr{G}_{\mathscr{B}}^{\pm}$ be harmonic flows space with $\mathbf{O} \neq \overrightarrow{G}_0^{L_0^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$. Then, there always exists a linear continuous functional F with $\|F\| = 1$ and $\left\|F\left(\overrightarrow{G}_0^{L_0^2}\right)\right\| = \left\|\overrightarrow{G}_0^{L_0^2}\right\|$ on $\mathscr{G}_{\mathscr{B}}^{\pm}$.*

Proof Define $\mathscr{H}_{\mathscr{B}}^{\pm} = \left\{\alpha\overrightarrow{G}_0^{L_0^2}\middle|\,\alpha \in \mathbb{C}\right\}$ and a liner functional $F\left(\alpha\overrightarrow{G}_0^{L_0^2}\right) = \alpha\left\|\overrightarrow{G}_0^{L_0^2}\right\|$ on $\mathscr{H}_{\mathscr{B}}^{\pm}$. Clearly, $\left\|F\left(\overrightarrow{G}^{L^2}\right)\right\| = |\alpha|\left\|\overrightarrow{G}_0^{L_0^2}\right\| = \left\|\alpha\overrightarrow{G}_0^{L_0^2}\right\| = \left\|\overrightarrow{G}^{L^2}\right\|$ if $\overrightarrow{G}^{L^2} = \alpha\overrightarrow{G}_0^{L_0^2}$. We know that $\|F\| = 1$ on $\mathscr{H}_{\mathscr{B}}^{\pm}$ with $F\left(\overrightarrow{G}_0^{L_0^2}\right) = \overrightarrow{G}_0^{L_0^2}$. By Theorem 2.20, F can be extended to $\mathscr{G}_{\mathscr{B}}^{\pm}$. □

Corollary 2.22 *For $\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^{\pm}$, if $F\left(\overrightarrow{G}^{L^2}\right) = 0$ hold with all linear functionals F on $\mathscr{G}_{\mathscr{B}}^{\pm}$ then $\overrightarrow{G}^{L^2} = \mathbf{O}$.*

§3. Harmonic Flow Dynamics

3.1 Harmonic Flow Calculus

Let $\overrightarrow{G}^{L^2} \in \mathscr{G}_{\mathscr{B}}^2$ with $L^2 : (v,u) \to (L_1(v,u), L_2(v,u))$ for $(v,u) \in E\left(\overrightarrow{G}^{L^2}\right)$. We transform L^2 to $L^2 : (v,u) \to L_1(v,u) + iL_2(v,u)$, i.e., a complex vector and particularly, a complex number if $\mathscr{B}=\mathbb{C}$, where $i=\sqrt{-1}$, which enables one to establish calculus on harmonic flows.

Definition 3.1 *Let D be a boundary subset of $\mathbb{C}^n=\{(x_1,x_2,\cdots,x_n)|x_i\in\mathbb{C},1\leq i\leq n\}$, $\mathscr{B}=\mathbb{C}(D)$ of differentiable functions on D and all end-operators in $\mathscr{A}$ satisfying $[A,\frac{\partial}{\partial x_i}]=\mathbf{0}$ for $\forall A\in\mathscr{A}$. Define n differential operators $\partial_i:\mathscr{G}^2_{\mathscr{B}}\to\mathscr{G}^2_{\mathscr{B}}$, $1\leq i\leq n$ by*

$$\partial_i\overrightarrow{G}^{L^2}=\overrightarrow{G}^{\frac{\partial L^2}{\partial x_i}},$$

and denoted by $\frac{d\overrightarrow{G}^{L^2}}{dz}$ if $D\subset\mathbb{C}$ for $\overrightarrow{G}^{L^2}\in\mathscr{G}^2_{\mathscr{B}}$ in which the integral flow of $\overrightarrow{G}^{L^2}\in\mathscr{G}^2_{\mathscr{B}}$ along a curve $C=\{\ z(t)\ |\ \alpha\leq t\leq\beta\ \}$ of length$<+\infty$ is defined by

$$\int_C\overrightarrow{G}^{L^2}dz=\overrightarrow{G}^{\int_C L^2dz}.$$

For $\overrightarrow{G}_k^{L_k^2},\overrightarrow{G}_l^{L_l^2}\in\mathscr{G}^2_{\mathscr{B}}$ and $\lambda,\mu\in\mathbb{C}$, calculation shows that

$$\begin{aligned}\partial_i\left(\lambda\overrightarrow{G}_k^{L_k^2}+\mu\overrightarrow{G}_l^{L_l^2}\right) &= \partial_i\left(\overrightarrow{G}_k^{\lambda L_k^2}+\overrightarrow{G}_l^{\mu L_l^2}\right)\\ &= \partial_i\left(\left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{\lambda L_k^2}\bigcup\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{\mu L_l^2}\bigcup\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{\lambda L_k^2+\mu L_l^2}\right)\\ &= \left(\overrightarrow{G}_k\setminus\overrightarrow{G}_l\right)^{\lambda\frac{\partial L_k^2}{\partial x_i}}\bigcup\left(\overrightarrow{G}_l\setminus\overrightarrow{G}_k\right)^{\mu\frac{\partial L_l^2}{\partial x_i}}\bigcup\left(\overrightarrow{G}_k\bigcap\overrightarrow{G}_l\right)^{\lambda\frac{\partial L_k^2}{\partial x_i}+\mu\frac{\partial L_l^2}{\partial x_i}}\\ &= \lambda\partial_i\overrightarrow{G}_k^{L_k^2}+\mu\partial_i\overrightarrow{G}_l^{L_l^2}\end{aligned}$$

and $\partial_i\overrightarrow{G}^{L^2}\to\partial\overrightarrow{G}_0^{L_0^2}$ if $\overrightarrow{G}^{L^2}\to\overrightarrow{G}_0^{L_0^2}$, i.e., linear continuous on the boundary domain D for integers $1\leq i\leq n$. Similarly, we can also show that the integral operator $\int_C$ is linear continuous on the boundary domain D. We get the following result.

Theorem 3.2 *All partial differential operators ∂_i and the integral operator $\int_C$ are linear continuous on $\mathscr{G}^2_{\mathscr{B}}$, and furthermore, on $\mathscr{G}^{\pm}_{\mathscr{B}}$ for integers $1\leq i\leq n$.*

Proof We have shown that each ∂_i is linear continuous for integers $1\leq i\leq n$. Now, we prove that $\partial_i\overrightarrow{G}^{L^2},\int_C\overrightarrow{G}^{L^2}dz\in\mathscr{G}^2_{\mathscr{B}}$ if $\overrightarrow{G}^{L^2}\in\mathscr{G}^2_{\mathscr{B}}$, i.e., hold with the continuity equations on vertices. In fact, by assumption $\overrightarrow{G}^{L^2}\in\mathscr{G}^2_{\mathscr{B}}$ and $[A,\frac{\partial}{\partial x_i}]=0$ for $\forall A\in\mathscr{A}$ there must be

$$\sum_{u\in N_G(v)}\left(L_1^{A^+_{vu}}(v,u)+iL_2^{A^+_{vu}}(v,u)\right)=L_1(v)+iL_2(v)$$

for $\forall v\in V\left(\overrightarrow{G}\right)$ by definition. Whence,

$$\begin{aligned}&\partial_i\sum_{u\in N_G(v)}\left(L_1^{A^+_{vu}}(v,u)+iL_2^{A^+_{vu}}(v,u)\right)=\sum_{u\in N_G(v)}\left(\partial_iL_1^{A^+_{vu}}(v,u)+i\partial_iL_2^{A^+_{vu}}(v,u)\right)\\ &=\sum_{u\in N_G(v)}\left((\partial_iL_1)^{A^+_{vu}}(v,u)+i\left(\partial_iL_2\right)^{A^+_{vu}}(v,u)\right)=\partial_iL_1(v)+i\partial_iL_2(v),\end{aligned}$$

i.e., $\partial_i : \mathscr{G}^2_{\mathscr{B}} \to \mathscr{G}^2_{\mathscr{B}}$ for integers $1 \le i \le n$.

Now, if $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{B}}$, then we are easily know that

$$\partial_i\left(\sum_{u\in N_G(v)}\left(L^{A^+_{vu}}(v,u) - iL^{A^+_{vu}}(v,u)\right)\right) = \partial_i L(v) - i\partial_i L(v),$$

i.e., $\partial_i : \mathscr{G}^{\pm}_{\mathscr{B}} \to \mathscr{G}^{\pm}_{\mathscr{B}}$ for integers $1 \le i \le n$.

Similarly, we can also show that the integral operators

$$\int_C : \mathscr{G}^2_{\mathscr{B}} \to \mathscr{G}^2_{\mathscr{B}} \quad \text{and} \quad \int_C : \mathscr{G}^{\pm}_{\mathscr{B}} \to \mathscr{G}^{\pm}_{\mathscr{B}}$$

are linear continuous and hold also with the continuity equation on vertices of $\overrightarrow{G}$. $\square$

Now, if $\frac{d\overrightarrow{G}^{L^2}}{dz}$ exists, then $\frac{d}{dz}L^2(v,u)$, a complex function for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ also exists. Let $L^2(v,u)(z) = M^{vu}(x,y) + iN^{vu}(x,y)$ for $(v,u) \in E\left(\overrightarrow{G}\right)$, where $z = x + iy$ and $M(x,y), N(x,y) \in \mathbb{R}^2$. Applying the Cauchy-Riemann equations in complex analysis, we are easily know that

$$\frac{dL^2}{dz} = \frac{\partial M}{\partial x} + i\frac{\partial N}{\partial x} = \frac{\partial N}{\partial y} - i\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x} - i\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} + i\frac{\partial N}{\partial x}.$$

By definition,

$$\begin{aligned}\frac{d\overrightarrow{G}^{L^2}}{dz} &= \overrightarrow{G}^{\frac{dL^2}{dz}} = \overrightarrow{G}^{\frac{\partial M}{\partial x}+i\frac{\partial N}{\partial x}} = \frac{\partial \overrightarrow{G}^M}{\partial x} + i\frac{\partial \overrightarrow{G}^N}{\partial x},\\ \frac{d\overrightarrow{G}^{L^2}}{dz} &= \overrightarrow{G}^{\frac{dL^2}{dz}} = \overrightarrow{G}^{\frac{\partial N}{\partial y}-i\frac{\partial M}{\partial y}} = \frac{\partial \overrightarrow{G}^N}{\partial y} - i\frac{\partial \overrightarrow{G}^M}{\partial y}.\end{aligned}$$

Similarly, if $\frac{d\overrightarrow{G}^{L^2}}{dz} = \overrightarrow{G'}^{L'^2}$, i.e., $d\overrightarrow{G}^{L^2} = \overrightarrow{G'}^{L'^2}dz$, then $\overrightarrow{G}^{L^2}$ is called the *primitive flow* of $\overrightarrow{G'}^{L'^2}$ and denoted by $\int \overrightarrow{G}^{L^2}dz$. Calculation shows that

$$\int_C \overrightarrow{G}^{L^2}dz = \overrightarrow{G}^{\int_C L^2dz} = \overrightarrow{G}^{\int L^2dz|_\beta - \int L^2dz|_\alpha} = \int \overrightarrow{G}^{L^2}\bigg|_\beta - \int \overrightarrow{G}^{L^2}\bigg|_\alpha$$

and particularly,

$$\int_C \overrightarrow{G}^{L^2}dz = \mathbf{O}$$

if C is the boundary curve of a simply connected domain on $\mathbb{R}^2$ and furthermore,

$$\overrightarrow{G}^{L^2}(z) = \frac{1}{2\pi i}\int_C \frac{\overrightarrow{G}^{L^2}(\zeta)}{\zeta - z}d\zeta$$

with $z \in D$ if $\overrightarrow{G}^{L^2}$ is differentiable on D and continuous on $Cl(D) = D + C$ by definition. We therefore generalize a few well-known results of complex analysis to $\mathscr{G}^2_{\mathbb{C}}$ following.

Theorem 3.3 *Let $D \subset \mathbb{C}$ be a domain with boundary curve C and $\mathscr{B} = \mathbb{C}(D)$. Then,*

(1)(C-R Equations) *A flow $\overrightarrow{G}^{L^2} \in \mathscr{G}^2_{\mathscr{B}}$ or $\mathscr{G}^{\pm}_{\mathscr{B}}$ is differentiable at $x + iy = z \in D$ if and only if*

$$\frac{\partial \overrightarrow{G}^M}{\partial x} = \frac{\partial \overrightarrow{G}^N}{\partial y} \quad \text{and} \quad \frac{\partial \overrightarrow{G}^N}{\partial x} = -\frac{\partial \overrightarrow{G}^M}{\partial y}$$

where $L^2(v,u)(z) = M^{vu}(x,y) + iN^{vu}(x,y)$, $K_1^{vu}(x,y), K_2^{vu}(x,y) \in \mathbb{R}^2$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ and both of them differentiable at (x,y);

(2)(Cauchy) *If D is simply connected on $\mathbb{R}^2$ and flow $\overrightarrow{G}^{L^2} \in \mathscr{G}^2_{\mathscr{B}}$ or $\mathscr{G}^{\pm}_{\mathscr{B}}$ is differentiable on D, then*

$$\int_{\alpha}^{\beta} \overrightarrow{G}^{L^2} dz = \int \overrightarrow{G}^{L^2}\Big|_{\beta} - \int \overrightarrow{G}^{L^2}\Big|_{\alpha},$$

where $z(\alpha)$ and $z(\beta)$ are two points on C and particularly, $\int_C \overrightarrow{G}^{L^2} dz = \mathbf{O}$;

(3)(Cauchy Integral Formula) *If $\overrightarrow{G}^{L^2} \in \mathscr{G}^2_{\mathscr{B}}$ or $\mathscr{G}^{\pm}_{\mathscr{B}}$ is differentiable on D and continuous on $Cl(D) = D + C$, then*

$$\overrightarrow{G}^{L^2}(z) = \frac{1}{2\pi i} \int_C \frac{\overrightarrow{G}^{L^2}(\zeta)}{\zeta - z} d\zeta,$$

where $z \in D$.

3.2 Harmonic Flow Dynamics

A self-adaptive system is naturally a harmonic flow over its underlying skeleton or a topological graph $\overrightarrow{G}$, particularly, an animal or a human, and all animals are in motion, internal, external or both which motives the harmonic flow dynamics, i.e., harmonic flow's status $\overrightarrow{G}^{L^2}[t]$ changes on time t for fields such as those of life or social systems. For example, the differential $\frac{d\overrightarrow{G}^{L^2}[t]}{dt}$ can be viewed as the harmonic change rate of a national economy, both in the internal and external if one models a notional economy by harmonic flow $\overrightarrow{G}^{L^2}[t]$, which is more scientific than that of the current rate of GDP, the gross domestic product of a country.

As it is well-known, the dynamic behavior of a self-adaptive system S, particularly, an animal or a human consisting of subsystems can be characterized by Lagrangians with continuity equations holds. If S is characterized by a harmonic flow $\overrightarrow{G}^{L^2}[t]$ with all subsystems by edges of $\overrightarrow{G}^{L^2}[t]$, this fact implies that Lagrangians on edges of $\overrightarrow{G}^{L^2}[t]$ hold with the continuity equation at vertices, i.e., if $L^2 : (v,u) \to L(v,u)[t] - iL(v,u)[t]$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ then $\overrightarrow{G}^{L^2}[t]$ is a harmonic flow with $L(v,u)[t] \in \mathbb{R}$ for edges $(v,u) \in E\left(\overrightarrow{G}\right)$. Whence, $\frac{d\overrightarrow{G}^{L^2}[t]}{dt}$ and $\int_{t_1}^{t_2} \overrightarrow{G}^{L^2} dt$ both are existed in $\mathscr{G}^{\pm}_{\mathscr{B}}$ by Section 3.1.

Now, if

$$\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right] : (v,u) \in E\left(\overrightarrow{G}\right) \to \mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))(v,u)\right]$$

is a differentiable functional with $[\mathscr{L}, A] = \mathbf{0}$ for $A \in \mathscr{A}$, there must be $\overrightarrow{G}^{\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]} \in \mathscr{G}^{\pm}_{\mathscr{B}}$, i.e., hold with the continuity equations on vertices of $\overrightarrow{G}$, where $\mathbf{x}=(x_1,x_2,\cdots,x_n)$. Consider the variational action

$$J\left[\overrightarrow{G}^{L^2}[t]\right]=\int_{t_1}^{t_2}\overrightarrow{G}^{\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]}dt. \qquad (3.1)$$

on a harmonic flow $\overrightarrow{G}^{L^2}[t]\in\mathscr{G}^{\pm}_{\mathscr{B}}$. By variational calculus we know that

$$\begin{aligned}\delta J\left[\overrightarrow{G}^{L^2}[t]\right] &= \delta\int_{t_1}^{t_2}\overrightarrow{G}^{\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]}dt\\ &= \overrightarrow{G}^{\delta\int_{t_1}^{t_2}\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]dt} = \overrightarrow{G}^{\int_{t_1}^{t_2}\sum\limits_{i=1}^{n}\left(\frac{\partial\mathscr{L}}{\partial x_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{x}_i}\right)\delta x_i dt}.\end{aligned}$$

According to the Hamiltonian principle there must be $\delta J\left[\overrightarrow{G}^{L^2}[t]\right]=\mathbf{O}$, i.e.,

$$\int_{t_1}^{t_2}\sum_{i=1}^{n}\left(\left(\frac{\partial\mathscr{L}}{\partial x_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{x}_i}\right)\Bigg|_{(v,u)}\right)\delta x_i dt=0 \qquad (3.2)$$

for $(v,u)\in E\left(\overrightarrow{G}\right)$. However, this can be only happened only if each coefficient of δx_i is 0 in (3.2), i.e.,

$$\left(\frac{\partial\mathscr{L}}{\partial x_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{x}_i}\right)\Bigg|_{(v,u)}=0,\qquad 1\le i\le n \qquad (3.3)$$

for $(v,u)\in E\left(\overrightarrow{G}\right)$ which results in Euler-Lagrange equations on $\overrightarrow{G}^{L^2}[t]$ following.

Theorem 3.4 *If $L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))(v,u)$ is a Lagrangian on edge (v,u) and $\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]$: $(v,u)\to\mathscr{L}\left[L^2(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))(v,u)\right]$ is a differentiable functional on a harmonic flow $\overrightarrow{G}^{L^2}[t]$ for $(v,u)\in E\left(\overrightarrow{G}\right)$ with $[\mathscr{L},A]=\mathbf{0}$ for $A\in\mathscr{A}$, then*

$$\frac{\partial\overrightarrow{G}^{\mathscr{L}}}{\partial x_i}-\frac{d}{dt}\frac{\partial\overrightarrow{G}^{\mathscr{L}}}{\partial\dot{x}_i}=\mathbf{O},\qquad 1\le i\le n. \qquad (3.4)$$

Let the polynomial expansion of $\mathscr{L}$ be

$$\mathscr{L}\left[L^2\right]=\mathscr{L}(0)+\frac{1}{1!}\frac{d\mathscr{L}}{d(L^2)}\Bigg|_{L=0}L^2+\cdots+\frac{1}{m!}\frac{d^m\mathscr{L}}{d(L^2)^m}\Bigg|_{L=0}\left(L^2\right)^m+o\left(\left(L^2\right)^m\right) \qquad (3.5)$$

on L^2 with an approximation

$$\mathscr{L}\left[L^2\right]=\mathscr{L}(0)+\frac{1}{1!}\frac{d\mathscr{L}}{d(L^2)}\Bigg|_{L=0}L^2+\cdots+\frac{1}{m!}\frac{d^m\mathscr{L}}{d(L^2)^m}\Bigg|_{L=0}\left(L^2\right)^m$$

of m terms. Calculation shows that

$$\frac{\partial \mathscr{L}}{\partial x_i} = \left.\frac{d\mathscr{L}}{d(L^2)}\right|_{L=0} \frac{\partial L^2}{\partial x_i} + \cdots + \frac{1}{(m-1)!} \left.\frac{d^m \mathscr{L}}{d(L^2)^m}\right|_{L=0} \left(L^2\right)^{m-1} \frac{\partial L^2}{\partial x_i}$$

and

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}_i} = \left.\frac{d\mathscr{L}}{d(L^2)}\right|_{L=0} \frac{d}{dt}\frac{\partial L^2}{\partial \dot{x}_i} + \cdots + \frac{1}{(m-1)!} \left.\frac{d^m \mathscr{L}}{d(L^2)^m}\right|_{L=0} \frac{d}{dt}\left(\left(L^2\right)^{m-1} \frac{\partial L^2}{\partial \dot{x}_i}\right).$$

Whence,

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}_i} &= \left.\frac{d\mathscr{L}}{d(L^2)}\right|_{L=0} \left(\frac{\partial L^2}{\partial x_i} - \frac{d}{dt}\frac{\partial L^2}{\partial \dot{x}_i}\right) + \cdots \\ &+ \frac{1}{(m-1)!} \left.\frac{d^m \mathscr{L}}{d(L^2)^m}\right|_{L=0} \left(\left(L^2\right)^{m-1} \frac{\partial L^2}{\partial x_i} - \frac{d}{dt}\left(\left(L^2\right)^{m-1} \frac{\partial L^2}{\partial \dot{x}_i}\right)\right) = 0\end{aligned}$$

for $(v,u) \in E\left(\overrightarrow{F}\right)$ and integers $1 \leq i \leq n$. Particularly, if $\mathscr{L}$ is linear dependent on L^2, we get the following conclusion.

Corollary 3.5 *If $\mathscr{L}$ is linear dependent on L^2, then*

$$\frac{\partial \overrightarrow{G}^{L^2}}{\partial x_i} - \frac{d}{dt}\frac{\partial \overrightarrow{G}^{L^2}}{\partial \dot{x}_i} = \mathbf{O}, \quad 1 \leq i \leq n.$$

Corollary 3.5 enables one to define the *Lagrangian* of a harmonic flow $\overrightarrow{G}^{L^2}[t]$ by $\mathscr{L}\left[\overrightarrow{G}^{L^2}[t]\right]$: $(v,u) \rightarrow L(v,u) - iL(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ which is generally dependent on $L(v,u)$, $(v,u) \in E\left(\overrightarrow{G}\right)$ in Theorem 3.4. If it is independent on $L(v,u)$, $(v,u) \in E\left(\overrightarrow{G}\right)$, we get an interesting result following.

Corollary 3.6(Euler-Lagrange) *If the Lagrangian $\mathscr{L}\left[\overrightarrow{G}^{L^2}[t]\right]$ of a harmonic flow $\overrightarrow{G}^{L^2}[t]$ is independent on (v,u), i.e., all Lagrangians $L^2(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)$, $(v,u) \in E\left(\overrightarrow{G}\right)$ are synchronized, then the dynamic behavior of $\overrightarrow{G}^{L^2}[t]$ can be characterized by n equations*

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad 1 \leq i \leq n, \tag{3.6}$$

which are essentially equivalent to the Euler-Lagrange equations of bouquet $\overrightarrow{B}_1^{L^2} \in \overrightarrow{B}_{1\mathscr{B}}^{\pm}$, i.e, dynamic equations on a particle P.

For example, let

$$\mathscr{L}\left[L^2(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)\right] = \sum_{i=1}^{n} c_i \dot{x}_i^2 - \sum_{1 \leq i,j \leq n} c_{ij} x_i x_j,$$

which is independent on $(v,u) \in\in E\left(\overrightarrow{G}\right)$. Then

$$\frac{\partial \mathscr{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}_i} = 2c_{ij}x_j - 2c_i\ddot{x}_i.$$

We get a system of n differential equations

$$\begin{cases} c_1\ddot{x}_1 - \sum\limits_{j\neq 1} c_{1j}x_j = 0, \\ c_2\ddot{x}_2 - \sum\limits_{j\neq 2} c_{2j}x_j = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots , \\ c_n\ddot{x}_n - \sum\limits_{j\neq n} c_{nj}x_j = 0 \end{cases}$$

for harmonic flow $\overrightarrow{G}^{L^2}[t]$ by Corollary 3.6. The solvability of equations (3.4) is answered in results following.

Theorem 3.7 *Let $\overrightarrow{G} \in \mathscr{G}$ and $n \geq 1$ be an integer. A Cauchy problem*

$$\begin{cases} \mathcal{F}\left(\mathbf{x}, X, \frac{\partial X}{\partial x_1}, \cdots, \frac{\partial X}{\partial x_n}, \frac{\partial^2 X}{\partial x_1 \partial x_2}, \cdots\right) = 0, \\ X|_{\mathbf{x}_0} = \overrightarrow{G}^{L^0} \end{cases} \tag{3.7}$$

is solvable in a boundary domain $D \subset \mathscr{G}^{\pm}_{\mathbb{C}^n}$ if and only if the Cauchy problem

$$\begin{cases} \mathcal{F}\left(\mathbf{x}, X_{vu}, \frac{\partial X_{vu}}{\partial x_1}, \cdots, \frac{\partial X_{vu}}{\partial x_n}, \frac{\partial^2 X_{vu}}{\partial x_1 \partial x_2}, \cdots\right) = 0, \ u \in N_G(v), \\ X_{vu_1}^{A^+_{vu_1}} + X_{vu_2}^{A^+_{vu_2}} + \cdots + X_{vu_{\rho(v)}}^{A^+_{vu_{\rho(v)}}} = L(v), \ u_i \in N_G(v), \ 1 \leq i \leq \rho(v), \\ X_{vu_i}|_{\mathbf{x}_0} = L_0(v, u_i), \ 1 \leq i \leq \rho(v), \\ L_0^{A^+_{vu_1}}(v, u_1) + L_0^{A^+_{vu_2}}(v, u_2) + \cdots + L_0^{A^+_{vu_{\rho(v)}}}(v, u_{\rho(v)})) = L_0(v) \end{cases} \tag{3.8}$$

is solvable in $D|_v$ for $\forall v \in V\left(\overrightarrow{G}\right)$, where D_v, $\rho(v)$ denote respectively the domain of D constraint on the closure neighborhood $N_{\overrightarrow{G}}(v)\bigcup\{v\}$ and the valency of v in $\overrightarrow{G}$, and particularly, if the solution of

$$\mathcal{F}\left(\mathbf{x}, X_{vu}, \frac{\partial X_{vu}}{\partial x_1}, \cdots, \frac{\partial X_{vu}}{\partial x_n}, \frac{\partial^2 X_{vu}}{\partial x_1 \partial x_2}, \cdots\right) = 0 \tag{3.9}$$

is linearly dependent on the initial value $L_0(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ with $\overrightarrow{G}^{L_0^2} \in \mathscr{G}^{\pm}_{\mathbb{C}^n}$ and $[\frac{\partial}{\partial x_i}, A] = \mathbf{0}$ for $A \in \mathscr{A}$, then (3.7) is solvable on $D \subset \mathscr{G}^{\pm}_{\mathbb{C}^n}$.

Proof Clearly, if (3.7) is solvable in $D \subset \mathscr{G}^{\pm}_{\mathbb{C}^n}$, without loss of generality, let the solution be $\overrightarrow{G}^{L^2}$, then $\overrightarrow{G}^{L^2}$ holds with (3.8). Conversely, if (3.8) is solvable in D_v for $\forall v \in V\left(\overrightarrow{G}\right)$, then there are solutions X_{vu} on edges $(v,u) \in E\left(\overrightarrow{G}\right)$, hold with the continuity equations on vertices of $\overrightarrow{G}^{L^2}$.

Now, if the solution X_{vu} of

$$\mathcal{F}\left(\mathbf{x}, X_{vu}, \frac{\partial X_{vu}}{\partial x_1}, \cdots, \frac{\partial X_{vu}}{\partial x_n}, \frac{\partial^2 X_{vu}}{\partial x_1 \partial x_2}, \cdots\right) = 0$$

is linearly dependent on the initial value $L_0(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, there is a linear functional H such that $X = H\left(\mathbf{x}, L_0(v,u)\right)$ holds with (3.9).

Notice that $\overrightarrow{G}^{L_0^2} \in \mathscr{G}^{\pm}_{\mathbb{C}^n}$ and $[\frac{\partial}{\partial x_i}, A] = \mathbf{0}$ for $A \in \mathscr{A}$ by assumption. We know that

$$\sum_{u \in N_{G^0}(v)} {L^2}_0^{A^+_{vu}}(v,u) = L_0^2(v)$$

for $v \in V\left(\overrightarrow{G}\right)$ by definition. Whence,

$$H\left(L_0(v)\right) = H\left(\sum_{u \in N_{G^0}(v)} {L^2}_0^{A^+_{vu}}(v,u)\right) = \sum_{u \in N_{G^0}(v)} \left(H\left(L_0^2(v,u)\right)\right)^{A^+_{vu}}$$

i.e., hold with the continuity equation at vertex v for $v \in V\left(\overrightarrow{G}\right)$. Therefore, if we define $L^2 : v \to H\left(L_0(v)\right)$ for $v \in V\left(\overrightarrow{G}\right)$ and $L^2 : (v,u) \to H\left(\mathbf{x}, L_0(v,u)\right)$ for $(v,u) \in E\left(\overrightarrow{G}\right)$, we get a harmonic flow $\overrightarrow{G}^{L^2} \in \mathscr{G}^{\pm}_{\mathscr{C}^n}$ which holds with (3.7). □

Theorem 3.7 enables one to extend solutions of differential equations in a domain $D \subset \mathbb{C}^n$ to $\mathscr{G}^{\pm}_{\mathbb{C}^n}$ if the solution is linearly dependent on initial values. For example, we have know the solution of the heat equation

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

is linearly dependent on the initial values $X(\mathbf{x}, t_0) = \varphi(\mathbf{x})$ in $\mathbb{R}^n \times \mathbb{R}$ if $\varphi(\mathbf{x})$ is continuous and bounded in $\mathbb{R}^n$, where c is a non-zero constant in. We get the conclusion following.

Corollary 3.8 *Let $\overrightarrow{G} \in \mathscr{G}$ with $\overrightarrow{G}^{L_0^2} \in \mathscr{G}^{\pm}_{\mathbb{R}^n \times \mathbb{R}}$ and $n \geq 1$ be an integer. If $[\frac{\partial}{\partial x_i}, A] = \mathbf{0}$ for $A \in \mathscr{A}$, then the Cauchy problem*

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

with $X|_{t=t_0} = \overrightarrow{G}^{L_0^2} \in \mathscr{G}^{\pm}_{\mathscr{V}}$ is solvable on a domain $D \subset \mathscr{G}^{\pm}_{\mathbb{R} \times \mathbb{R}}$ if $L_0(v,u)$ is continuous and bounded in $\mathbb{R}^n$ for $(v,u) \in E\left(\overrightarrow{G}\right)$.

§4. Balance Recovery

A flow $\overrightarrow{G}^{L^2}$ maybe not continuity. Even it is, it maybe not harmonic. *How to transform a non-continuity or non-harmonic flow to a continuity or harmonic flow, i.e., balance recovery?* We consider this problem in the following.

Definition 4.1 *Let $\overrightarrow{G}^{L^2}$ be a flow with $L^2(v,u)=(L_1(v,u),L_2(v,u))$, $L_1(v,u),L_2(v,u)\in\mathscr{B}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ and $v\in V\left(\overrightarrow{G}\right)$. Define an action operations O on v, i.e., input or output an additional flow A at vertex v with $O(v)=(A,A)$, where $A\in\mathscr{B}$ such as those shown in Fig.*8 *following.*

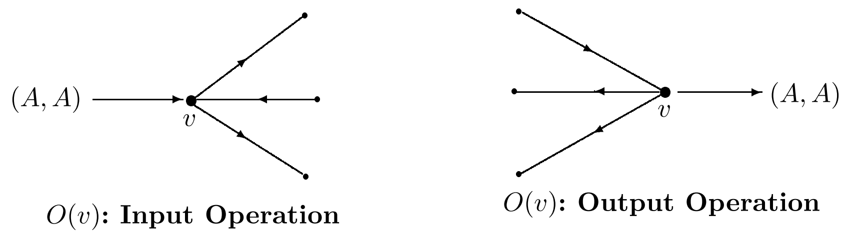

Fig.8

Observation 4.2 If a continuity flow $\overrightarrow{G}^L$ is influenced externally by $A_1,A_2,\cdots,A_s$ respectively on vertices $v_i\in V\left(\overrightarrow{G}\right)$, $1\leq i\leq s$ which results in an imbalanced flow $\overrightarrow{G}^{L'}$, we can offset inputs $-A_1,-A_2,\cdots,-A_s$ on vertices $v_i\in V\left(\overrightarrow{G}\right)$, $1\leq i\leq s$ and obtain the continuity flow $\overrightarrow{G}^L$ immediately.

Denoted by $\varpi\left(\overrightarrow{G}^L\right)$ the number of acted vertices by O and $o\left(\overrightarrow{G}^L\right)$ the number of conservation vertices in $\overrightarrow{G}^L$. Then Observation 4.2 implies the following result.

Proposition 4.3 $\varpi\left(\overrightarrow{G}^L\right)+o\left(\overrightarrow{G}^L\right)=\left|\overrightarrow{G}\right|$.

A continuity flow $\overrightarrow{G}^{L^2}$ maybe not a harmonic flow even for the circuit $\overrightarrow{C}$ such as those shown in Fig.9,

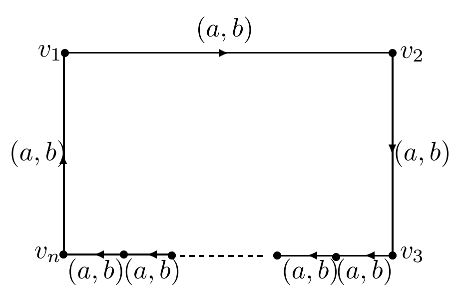

Fig.9

where $a\neq -b$, which naturally brings about the following problem.

Problem 4.4 *Can a flow $\overrightarrow{G}^{L^2}$ (continuity or not) be transformed to a harmonic flow, i.e., balanced at everywhere by input and output operations O on vertices of $\overrightarrow{G}$? And generally, if a flow $\overrightarrow{G}^{L^2}[t]$ evolves on the time t, can it be transformed to a harmonic flow by action O within interval $[t_1,t_2]$ of times?*

The answer of this problem is affirmative, which in fact consists of the foundation of traditional Chinese medicine theory.

Theorem 4.5 *For a flow $\overrightarrow{G}^{L^2}$ (continuity or not) over a Banach space $\mathscr{B}$ with $\left[A, \frac{d}{dt}\right] = \mathbf{0}$ and $\left[A, \int_t^0 \right] = \mathbf{0}$ for $A \in \mathscr{A}$, there are input or output operations O on vertices of $\overrightarrow{G}$ which transforms $\overrightarrow{G}^{L^2}$ to a harmonic flow, i.e., balanced on everywhere, and generally, if $\overrightarrow{G}^{L^2}[t]$ is continuous on time t holding with a harmonic flow $\overrightarrow{G}^{L^2}[0]$, then there are input or output operations O on vertices of $\overrightarrow{G}$ which transforms $\overrightarrow{G}^{L^2}[t]$ to a harmonic flow on time t.*

Proof Notice that $L^2: \ (v,u) \to (L_1(v,u), L_2(v,u))$ with $L_1(v,u), L_2(v,u) \in \mathscr{B}$. First, it is clear that a non-continuity flow $\overrightarrow{G}^{L^2}$ can be transferred to a continuity flow. Without loos of generality, let $v \in V\left(\overrightarrow{G}\right)$ be such a vertex with

$$\sum_{u \in V\left(\overrightarrow{G}\right)} L_1^{A_{vu}}(v,u) \neq \sum_{w \in V\left(\overrightarrow{G}\right)} L_1^{A_{wv}}(w,v), \quad \text{or} \quad \sum_{u \in V\left(\overrightarrow{G}\right)} L_2^{A_{vu}}(v,u) \neq \sum_{w \in V\left(\overrightarrow{G}\right)} L_2^{A_{wv}}(w,v).$$

Then, we can let O act on v by input $O(v) = (S_1, S_2)$ with

$$\begin{aligned} S_1 &= \sum_{u \in V\left(\overrightarrow{G}\right)} L_1^{A_{vu}}(v,u) - \sum_{w \in V\left(\overrightarrow{G}\right)} L_1^{A_{wv}}(w,v), \\ S_2 &= \sum_{u \in V\left(\overrightarrow{G}\right)} L_2^{A_{vu}}(v,u) - \sum_{w \in V\left(\overrightarrow{G}\right)} L_2^{A_{wv}}(w,v). \end{aligned}$$

Clearly, v becomes a conservation vertex after such an action. Notice that such action can be acted on all non-conservation vertices in $\overrightarrow{G}^{L^2}$ and get a continuity flow $\overrightarrow{G}^{L^2}$ finally.

Second, there exists a labeled graph $\overrightarrow{G}^{L'}$ on $\overrightarrow{G}$ such that $\overrightarrow{G}^{L^2} + \overrightarrow{G}^{L'}$ is a harmonic flow. Define a labeling L' on $\overrightarrow{G}$ by

$$L': \ (v,u) \to \begin{cases} \left(-\dfrac{L_1(v,u)+L_2(v,u)}{2}, \ -\dfrac{L_1(v,u)+L_2(v,u)}{2}\right) & \text{if } u \in N^(v), \\ (\mathbf{0}, \mathbf{0}) & \text{otherwise.} \end{cases}$$

Then, calculation shows that

$$\begin{aligned} L^2 + L': \ (v,u) \quad &\to \quad \left(L_1(v,u) - \frac{L_1(v,u)+L_2(v,u)}{2}, \ L_2(v,u) - \frac{L_1(v,u)+L_2(v,u)}{2}\right) \\ &= \left(\frac{L_1(v,u)-L_2(v,u)}{2}, \ -\frac{L_1(v,u)-L_2(v,u)}{2}\right), \end{aligned}$$

i.e., the flows on the edge (v,u) are in balance which implies that $\overrightarrow{G}^{L^2} + \overrightarrow{G}^{L'}$ is harmonic.

For $\forall v \in V\left(\overrightarrow{G}\right)$ let

$$V_v = \sum_{w \in N^-(v)} \frac{L_1^{A_{wv}}(w,v) + L_2^{A_{wv}}(w,v)}{2} - \sum_{u \in N^+(v)} \frac{L_1^{A_{vu}}(v,u) + L_2^{A_{vu}}(v,u)}{2}$$

and define $O(v)=(V_v,\ V_v)$ on the vertex v with allocation

$$\left(-\frac{L_1(v,u)+L_2(v,u)}{2},\ -\frac{L_1(v,u)+L_2(v,u)}{2}\right),$$
$$\left(\frac{L_1(w,v)+L_2(w,v)}{2},\ \frac{L_1(w,v)+L_2(w,v)}{2}\right)$$

respective on edges (v,u) for $u\in N^-(v)$ and (w,v) for $w\in N^+(v)$. Clearly, O transforms $\overrightarrow{G}^{L^2}$ to a flow with

$$\left(\frac{L_1(v,u)-L_2(v,u)}{2},\ -\frac{L_1(v,u)-L_2(v,u)}{2}\right)$$

on edges (v,u) for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. Whence, we get a harmonic flow.

Now, if $\overrightarrow{G}^{L^2}[t]$ is continuous on time t holding with a harmonic flow $\overrightarrow{G}^{L^2}[0]$, we consider the differential flow

$$\frac{d}{dt}\left(\overrightarrow{G}^{L^2}[t]\right)=\overrightarrow{G}^{\frac{d}{dt}\left(L^2[t]\right)}$$

and apply the method of the previous. For $\forall v\in V\left(\overrightarrow{G}\right)$ we define

$$\begin{aligned}V_v' &= \sum_{w\in N^-(v)}\frac{\frac{d}{dt}\left(L_1^{A_{wv}}(w,v)[t]\right)+\frac{d}{dt}\left(L_2^{A_{wv}}(w,v)[t]\right)}{2}\\ &\quad-\sum_{u\in N^+(v)}\frac{\frac{d}{dt}\left(L_1^{A_{vu}}(v,u)[t]\right)+\frac{d}{dt}\left(L_2^{A_{vu}}(v,u)[t]\right)}{2},\end{aligned}$$

i.e., V_v' in $\frac{d}{dt}\left(\overrightarrow{G}^{L^2}[t]\right)$ and let $O'(v)=(V_v',\ V_v')$ on vertex v with allocation

$$\left(-\frac{\frac{d}{dt}\left(L_1(v,u)[t]\right)+\frac{d}{dt}\left(L_2(v,u)[t]\right)}{2},\ -\frac{\frac{d}{dt}\left(L_1(v,u)[t]\right)+\frac{d}{dt}\left(L_2(v,u)[t]\right)}{2}\right),$$
$$\left(\frac{\frac{d}{dt}\left(L_1(w,v)[t]\right)+\frac{d}{dt}\left(L_2(w,v)[t]\right)}{2},\ \frac{\frac{d}{dt}\left(L_1(w,v)[t]\right)+\frac{d}{dt}\left(L_2(w,v)[t]\right)}{2}\right)$$

respective on edges (v,u) for $u\in N^-(v)$ and (w,v) for $w\in N^+(v)$ in this case. Clearly, $O':v\in V\left(\overrightarrow{G}\right)\to O'(v)$ transforms $\frac{d}{dt}\left(\overrightarrow{G}^{L^2}[t]\right)$ to a flow with

$$\left(\frac{\frac{d}{dt}\left(L_1(v,u)[t]\right)-\frac{d}{dt}\left(L_2(v,u)[t]\right)}{2},\ -\frac{\frac{d}{dt}\left(L_1(v,u)[t]\right)-\frac{d}{dt}\left(L_2(v,u)[t]\right)}{2}\right)$$

on edges (v,u) for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. Now, considering the integral flow

$$\int_0^t\frac{d}{dt}\left(\overrightarrow{G}^{L^2}[t]\right)dt=\overrightarrow{G}^{\int_0^t\frac{d}{dt}\left(L^2[t]\right)dt},$$

we immediately get a harmonic flow on time t by that $\overrightarrow{G}^{L^2}[0]$ is a such one. □

Theorem 4.5 implies that for any continuity flow $\overrightarrow{G}^{L^2}[t]$ with

$$\left[A, \frac{d}{dt}\right] = \mathbf{0} \quad \text{and} \quad \left[A, \int_t^0\right] = \mathbf{0}$$

for $A \in \mathscr{A}$ there are always input or output operations O on vertices $v_1, v_2, \cdots, v_s$ of $\overrightarrow{G}$ which transforms a $\overrightarrow{G}^{L^2}[t]$ to a harmonic flow if $\overrightarrow{G}^{L^2}[0]$ is a harmonic flow, and these vertices do not dependent on $\overrightarrow{G}^{L^2}[t]$ is variable or not.

Definition 4.6 *A vertex $v \in V\left(\overrightarrow{G}^{L^2}[t]\right)$ is zero-acted on time t if $O(v) = \mathbf{0}$.*

Theorem 4.7 *Let $\overrightarrow{G}^{L^2}[t]$ be a continuity flow on time t. Then, all vertices of $\overrightarrow{G}^{L^2}[t]$ are zero-acted.*

Proof By the proof of Theorem 4.5, a vertex $v \in V\left(\overrightarrow{G}^{L^2}[t]\right)$ is zero-acted, i.e., $O(v) = \mathbf{0}$ if and only if

$$\sum_{w\in N^-(v)} \frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2} = \sum_{u\in N^+(v)} \frac{L_1^{A_{vu}}(v,u)[t] + L_2^{A_{vu}}(v,u)[t]}{2}.$$

Notice that

$$\sum_{w\in N^-(v)} L_1^{A_{wv}}(w,v)[t] = \sum_{u\in N^+(v)} L_1^{A_{vu}}(v,u)[t], \tag{4.1}$$

$$\sum_{w\in N^-(v)} L_2^{A_{wv}}(w,v)[t] = \sum_{u\in N^+(v)} L_2^{A_{vu}}(v,u)[t] \tag{4.2}$$

by definition. We naturally know that

$$\sum_{w\in N^-(v)} \frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2} = \sum_{u\in N^+(v)} \frac{L_1^{A_{vu}}(v,u)[t] + L_2^{A_{vu}}(v,u)[t]}{2}.$$

Adding (4.1) with (4.2) and divided the sum by 2, we get the result. □

Notice that $O(v) = \mathbf{0}$ does not implies there are not needed action O on $v \in V\left(\overrightarrow{G}^{L^2}[t]\right)$ for transforming $\overrightarrow{G}^{L^2}[t]$ to a harmonic flow, for instance the continuity flow $\overrightarrow{C}_n^{L^2}$ shown in Fig.9. But, *how to hold on a zero-action O*? In fact, we can not realize O just one input or output action on v. In this case, O is decomposed into 2 actions, i.e., $O = O_1 + O_2$ with

$$O_1(v) = -\sum_{w\in N^-(v)} \frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2}, \quad O_2(v) = \sum_{u\in N^+(v)} \frac{L_1^{A_{vu}}(v,u)[t] + L_2^{A_{vu}}(v,u)[t]}{2}$$

and each O_1 or O_2 action on v allocates respectively

$$\left(-\frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2}, -\frac{L_1^{A_{wv}}(w,v)[t] + L_2^{A_{wv}}(w,v)[t]}{2}\right)$$

on edges $(w,v),\ w\in N^-(v)$ and

$$\left(\frac{L_1^{A_{vu}}(v,u)[t]+L_2^{A_{vu}}(v,u)[t]}{2},\frac{L_1^{A_{vu}}(v,u)[t]+L_2^{A_{vu}}(v,u)[t]}{2}\right)$$

on edges $(v,u),\ u\in N^+(v)$, such as those shown in Fig.10.

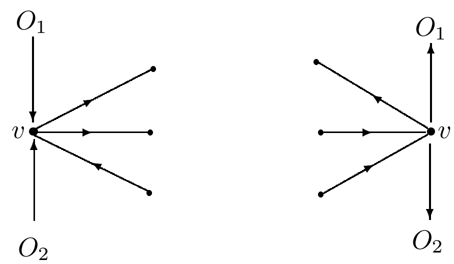

Fig.10

However, a calculation immediately enables one getting the numbers $\varpi\left(\overrightarrow{P}_n^{L^2}\right)$ and $\varpi\left(\overrightarrow{C}_n^{L^2}\right)$.

Theorem 4.8 *If $\overrightarrow{P}_n^{L^2}[t]$ and $\overrightarrow{C}_n^{L^2}[t]$ are continuity flows with $\left[A,\frac{d}{dt}\right]=\mathbf{0}$, $\left[A,\int_t^0\right]=\mathbf{0}$ for $A\in\mathscr{A}$, then $\varpi\left(\overrightarrow{P}_n^{L^2}[t]\right)=1$ and $\varpi\left(\overrightarrow{C}_n^{L^2}[t]\right)=1$ if $\overrightarrow{P}_n^{L^2}[0]$ and $\overrightarrow{C}_n^{L^2}[0]$ are harmonic.*

Proof This fact is an immediate conclusion by calculation. Assuming flows of $\overrightarrow{P}_n^{L^2}[t]$ shown in Fig.11 in time t with $a_i\neq -b_i$ for integers $1\leq i\leq n$,

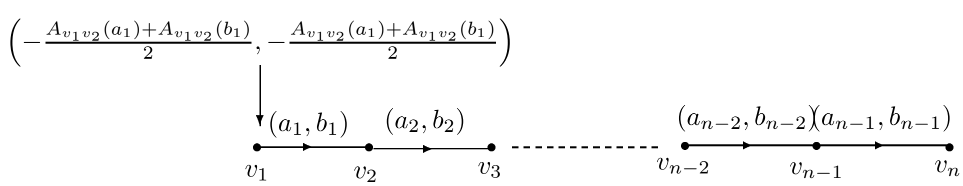

Fig.11

we can calculate flows on its edges by an input action

$$O(v_1)=\left(-\frac{A_{v_1v_2}(a_1)+A_{v_1v_2}(b_1)}{2},-\frac{A_{v_1v_2}(a_1)+A_{v_1v_2}(b_1)}{2}\right)$$

on vertex v_1. In fact, the flow on edge (v_1,v_2) is

$$\left(a_1-\frac{a_1+b_1}{2},b_1-\frac{a_1+b_1}{2}\right)=\left(\frac{a_1-b_1}{2},-\frac{a_1-b_1}{2}\right).$$

Then, we can determine flows on edges $(v_2,v_3),\cdots,(v_{n-2},v_{n-1}),(v_{n-1},v_n)$ by the conservation laws on vertices $v_2,v_3,\cdots,v_{n-1}$. For example, let the flows on edge (v_2,v_3), $\cdots$, (v_{n-2},v_{n_1}),

(v_{n-1}, v_n) be (a_2', b_2'), $\cdots$, (a_{n-2}', b_{n-2}'), (a_{n-1}', b_{n-1}'), respectively. Then, we must know that

$$A_{v_2v_1}\left(\frac{a_1-b_1}{2}\right) = A_{v_2v_3}\left(a_2'\right) \quad \text{and} \quad A_{v_2v_1}\left(-\frac{a_1-b_1}{2}\right) = A_{v_2v_3}\left(b_2'\right),$$

by the conservation law on vertex v_2, i.e., $A_{v_2v_3}\left(a_2'\right) = -A_{v_2v_3}\left(b_2'\right)$ or $A_{v_2v_3}\left(a_2'+b_2'\right) = \mathbf{0}$. There must be $a_2' = -b_2'$ by the linearity assumption on end-operators in $\mathscr{A}$. Continuing this process on vertices $v_3, \cdots, v_{n-1}$, we finally get $a_3' = -b_3', \cdots, a_{n-2}' = -b_{n-2}'$ and $a_{n-1}' = -b_{n-1}'$, i.e., a harmonic flow on $\overrightarrow{P}_n$. Whence, $\varpi\left(\overrightarrow{P}_n^{L^2}[t]\right) = 1$.

Similarly, we know that $\varpi\left(\overrightarrow{C}_n^{L^2}[t]\right) = 1$ by a zero-action O on any vertex of $\overrightarrow{C}_n^{L^2}$. This completes the proof. $\square$

The proof of Theorem 4.8 enables one to know that a vertex v with of $\rho(v) = 2$ is not needed in the calculation of $\varpi\left(\overrightarrow{G}^{L^2}[t]\right)$. We introduce a conception following.

Definition 4.9 *If $\overrightarrow{G}$ is a graph, the topological neighborhood $N^p(v)$ on vertex $v \in V\left(\overrightarrow{G}\right)$ is defined to be all vertices in $\overrightarrow{G}$ connecting v with an induced path of $\overrightarrow{G}$. If $p(v,u)$ is such a path for $u \in N^p(v)$, the induced subgraph $\langle\ (\{v\} \bigcup V(p(u,v))) \setminus \{u\} \mid u \in N^p(v)\}\rangle$, denoted by $[v]_p$ is called the claw graph on v in $\overrightarrow{G}$.*

Clearly, if all $p(v,u) = \overrightarrow{P}_2$ in $\overrightarrow{G}$, $N^p(v) = N(v)$, and if $\overrightarrow{G}^{L^2}$ is a continuity flow with all induced paths connecting vertex $v \in \left(\overrightarrow{G}^{L^2}\right)$ balanced, then $\left(\overrightarrow{G} \setminus \{[v]_P\}\right)^{L^2}$ is also a continuity flow by the proof of Theorem 4.8. This fact enables one to get $\varpi\left(\overrightarrow{G}^{L^2}[t]\right)$ following.

Theorem 4.10 *If $\overrightarrow{G}^{L^2}[t]$ is a continuity flow on a connected graph $\overrightarrow{G}$ of order≥ 3, then there are vertices $v^1, v^2, \cdots, v^{k_0} \in V\left(\overrightarrow{G}[t]\right)$ such that*

$$\overrightarrow{G} \setminus \left\{\left[v^1\right]_p, \left[v^2\right]_p, \cdots, \left[v^{k_0}\right]_p\right\} = \left(\bigcup_{i=1}^{k_1} \overrightarrow{C}_{n_i}\right) \bigcup \left(\bigcup_{j=1}^{k_2} \overrightarrow{P}_{n_j}\right) \bigcup \left(\bigcup_{l=1}^{k_3} \overrightarrow{P}_1\right), \tag{4.3}$$

where $n_i \geq 3, n_j \geq 2, k_i \geq 0$ for integers $0 \leq i \leq 2$, $k_0 + \sum\limits_{i=1}^{k_1} n_i + \sum\limits_{j=1}^{k_2} n_j + k_3 = \left|\overrightarrow{G}\right|$ and $\varpi\left(\overrightarrow{G}^{L_2}[t]\right) \leq k_0 + k_1 + k_2$, which implies that

$$\varpi\left(\overrightarrow{G}^{L_2}[t]\right) = \min\left\{k_0 + k_1 + k_2 \middle|\ \textit{all triad } k_0, k_1, k_2 \textit{ holds with equality } (4.3)\right\}$$

if $\overrightarrow{G}^{L^2}[0]$ is harmonic.

Proof Clearly, there are only 2 continuity graphs $\overrightarrow{C}_3$ and $\overrightarrow{P}_3$ if $\left|\overrightarrow{G}\right| = 3$ and $\varpi\left(\overrightarrow{G}^{L^2}[t]\right) = 1$ by Theorem 4.8. If $\left|\overrightarrow{G}\right| = 4$, for $\forall v \in V\left(\overrightarrow{G}\right)$, we know that $\overrightarrow{G} \setminus \{[v]_p\} = \overrightarrow{P}_3$, $\overrightarrow{C}_3$, the disjoint union of $\overrightarrow{P}_2$ with $\overrightarrow{P}_1$ or 3 isolated vertices and $\varpi\left(\overrightarrow{G}^{L^2}[t]\right) = 1$ or 2, i.e., the result is true for integers $\left|\overrightarrow{G}\right| \leq 4$.

Suppose the result is true for all graphs $\overrightarrow{G}$ with $\left|\overrightarrow{G}\right| \leq k$. We prove it is also true in the case of $\left|\overrightarrow{G}\right| \leq k+1$. The proof is divided into 2 cases following.

Case 1. $\overrightarrow{G}$ is 2-connected.

In this case, for $\forall v \in V\left(\overrightarrow{G}\right)$, $\overrightarrow{G}\setminus\{[v]_P\}$ is connected. By the induction assumption, there are vertices $v^1, v^2, \cdots, v^{k_0}$ in $\overrightarrow{G}$ such that

$$\left(\overrightarrow{G}\setminus\{[v]_p\}\right)\setminus\left\{\left[v^1\right]_p,\left[v^2\right]_p,\cdots,\left[v^{k_0}\right]_p\right\}=\left(\bigcup_{i=1}^{k_1}\overrightarrow{C}_{n_i}\right)\bigcup\left(\bigcup_{j=1}^{k_2}\overrightarrow{P}_{n_j}\right)\bigcup\left(\bigcup_{l=1}^{k_3}\overrightarrow{P}_1\right),$$

i.e.,

$$\overrightarrow{G}\setminus\left\{[v]_p,\left[v^1\right]_p,\left[v^2\right]_p,\cdots,\left[v^{k_0}\right]_p\right\}=\left(\bigcup_{i=1}^{k_1}\overrightarrow{C}_{n_i}\right)\bigcup\left(\bigcup_{j=1}^{k_2}\overrightarrow{P}_{n_j}\right)\bigcup\left(\bigcup_{l=1}^{k_3}\overrightarrow{P}_1\right).$$

Applying Theorem 4.8, we immediately know that $\varpi\left(\overrightarrow{G}^{L_2}[t]\right) \leq (k_0+1)+k_1+k_2$ by the initial condition and the result follows.

Case 2. $\overrightarrow{G}$ is 1-connected.

In this case, there is a cut vertex $v \in V\left(\overrightarrow{G}\right)$ such that $\overrightarrow{G}\setminus\{[v]_P\}$ is a disjoint union of s connected blocks $\overrightarrow{B}_1, \overrightarrow{B}_2, \cdots, \overrightarrow{B}_s$, $s \geq 2$. It is obvious that $\left|\overrightarrow{B}_i\right| \leq k$. Without of loss of generality, assume $\left|\overrightarrow{B}_i\right| \geq 3$. Then there are vertices $v^{i_1}, v^{i_2}, \cdots, v^{i_{k_0}}$ in $\overrightarrow{B}_i$ such that

$$\overrightarrow{B}_i\setminus\left\{\left[v^{i_1}\right]_p,\left[v^{i_2}\right]_p,\cdots,\left[v^{i_{k_0}}\right]_p\right\}=\left(\bigcup_{i=1}^{k_1\left(\overrightarrow{B}_i\right)}\overrightarrow{C}_{n_i}\right)\bigcup\left(\bigcup_{j=1}^{k_2\left(\overrightarrow{B}_i\right)}\overrightarrow{P}_{n_j}\right)\bigcup\left(\bigcup_{l=1}^{k_3\left(\overrightarrow{B}_i\right)}\overrightarrow{P}_1\right),$$

i.e., $\varpi\left(\overrightarrow{B}_i^{L^2}[t]\right) \leq i_{k_0}\left(\overrightarrow{B}_i\right)+k_1\left(\overrightarrow{B}_i\right)+k_2\left(\overrightarrow{B}_i\right)$ by the induction assumption for integers $1 \leq i \leq s$. Whence,

$$\begin{aligned}
&\left(\overrightarrow{G}\setminus\{[v]_p\}\right)\setminus\left(\bigcup_{i=1}^{s}\left\{\left[v^{i_1}\right]_p,\left[v^{i_2}\right]_p,\cdots,\left[v^{i_{k_0}}\right]_p\right\}\right)\\
&=\left(\bigcup_{i=1}^{s}\overrightarrow{B}_i\right)\setminus\left(\bigcup_{i=1}^{s}\left\{\left[v^{i_1}\right]_p,\left[v^{i_2}\right]_p,\cdots,\left[v^{i_{k_0}}\right]_p\right\}\right)\\
&=\bigcup_{i=1}^{s}\left(\overrightarrow{B}_i\setminus\left\{\left[v^{i_1}\right]_p,\left[v^{i_2}\right]_p,\cdots,\left[v^{i_{k_0}}\right]_p\right\}\right)\\
&=\bigcup_{i=1}^{s}\left(\left(\bigcup_{i=1}^{k_1(B_i)}\overrightarrow{C}_{n_i}\right)\bigcup\left(\bigcup_{j=1}^{k_2(B_i)}\overrightarrow{P}_{n_j}\right)\bigcup\left(\bigcup_{j=1}^{k_3(B_i)}\overrightarrow{P}_1\right)\right)\\
&=\left(\bigcup_{i=1}^{k_1}\overrightarrow{C}_{n_i}\right)\bigcup\left(\bigcup_{j=1}^{k_2}\overrightarrow{P}_{n_j}\right)\bigcup\left(\bigcup_{l=1}^{k_3}\overrightarrow{P}_1\right),
\end{aligned}$$

where $k_1 = \sum\limits_{i=1}^{s} k_1\left(\overrightarrow{B}_i\right)$, $k_2 = \sum\limits_{i=1}^{s} k_2\left(\overrightarrow{B}_i\right)$ and $k_3 = \sum\limits_{i=1}^{s} k_3\left(\overrightarrow{B}_i\right)$, i.e.,

$$\overrightarrow{G} \setminus \left(\{[v]_p\} \bigcup \left(\bigcup_{i=1}^{s} \left\{\left[v^{i_1}\right]_p, \left[v^{i_2}\right]_p, \cdots, \left[v^{i_{k_0}}\right]_p\right\}\right)\right)$$
$$= \left(\bigcup_{i=1}^{k_1} \overrightarrow{C}_{n_i}\right) \bigcup \left(\bigcup_{j=1}^{k_2} \overrightarrow{P}_{n_j}\right) \bigcup \left(\bigcup_{j=1}^{k_3} \overrightarrow{P}_1\right),$$

which implies that $\varpi\left(\overrightarrow{G}^{L^2}[t]\right) \leq (k_0+1) + k_1 + k_2$ by the induction assumption, Theorem 4.8 and initial condition with $k_0 = \sum\limits_{i=1}^{s} i_{k_0}\left(\overrightarrow{B}_i\right)$.

Combining Cases 1 and 2, we know $\varpi\left(\overrightarrow{G}^{L^2}[t]\right) \leq k_0 + k_1 + k_2$ is true for all graphs $\overrightarrow{G}$ and follows with

$$\varpi\left(\overrightarrow{G}^{L^2}[t]\right) = \min\left\{k_0 + k_1 + k_2 \mid \text{all triad } k_0, k_1, k_2 \text{ holds with equality (4.3)}\right\}.$$

This completes the proof. □

We immediately get a corollary on the number of $\varpi\left(\overrightarrow{G}^{L_2}[t]\right)$ by Theorem 4.9.

Corollary 4.11 *Let $\overrightarrow{G}$ be one of graphs $\overrightarrow{S}_{1,n}$, $\overrightarrow{W}_{1,n}$, $T^k(\overrightarrow{S}_{1,n})$, $T^{k,l}(\overrightarrow{W}_{1,n})$, $\overrightarrow{K}_n$, $\overrightarrow{K}_{n,m}$, $\overrightarrow{K}_{n_1,n_2,\cdots,n_s}$, $s \geq 2$, $\overrightarrow{P}_n \times \overrightarrow{P}_2$ and $\overrightarrow{C}_n \times \overrightarrow{P}_2$, where $T^k(\overrightarrow{S}_{1,n})$ is the topological subdivision of $\overrightarrow{S}_{1,n}$ by subdividing k times, $T^{k,l}(\overrightarrow{W}_{1,n})$ is the graph obtained by respectively subdividing k times on spoke edges, l times on wheel edges in $\overrightarrow{W}_{1,n}$. Then, $\varpi\left(\overrightarrow{G}^{L_2}[t]\right)$ is shown in Table 1 if $\overrightarrow{G}^{L^2}$ satisfies $\left[A, \frac{d}{dt}\right] = \mathbf{0}$, $\left[A, \int_t^0\right] = \mathbf{0}$ for $A \in \mathscr{A}$ and $\overrightarrow{G}^{L^2}[0]$ is harmonic.*

$\overrightarrow{G}$	$\overrightarrow{G}^{L^2}$	$n,\ k$	$\varpi\left(\overrightarrow{G}^{L_2}[t]\right)$
$\overrightarrow{S}_{1,n}$	$\overrightarrow{S}_{1,n}^{L^2}$	$n \geq 2$	1
$T^k\left(\overrightarrow{S}_{1,n}\right)$	$T^k\left(\overrightarrow{S}_{1,n}\right)^{L^2}$	$n \geq 3,\ k \geq 2$	1
$\overrightarrow{W}_{1,n}$	$\overrightarrow{W}_{1,n}^{L^2}$	$n \geq 3$	2
$T^{k,l}\left(\overrightarrow{W}_{1,n}\right)$	$T^{k,l}\left(\overrightarrow{W}_{1,n}\right)^{L^2}$	$n \geq 3, k \leq 2$	2
$\overrightarrow{K}_n$	$\overrightarrow{K}_n^{L^2}$	$n \geq 3$	$n-2$
$\overrightarrow{K}_{n,m}$	$\overrightarrow{K}_{n,m}^{L^2}$	$m \geq n, n \neq 2$	n
$\overrightarrow{K}_{n,m}$	$\overrightarrow{K}_{n,m}^{L^2}$	$m = n = 2$	1
$\overrightarrow{K}_{2,\cdots,2(s\ times)}$	$\overrightarrow{K}_{2,\cdots,2(s\ times)}^{L^2}$	$s \geq 3$	$2s-3$
$\overrightarrow{K}_{n_1,n_2,\cdots,n_s}$	$\overrightarrow{K}_{n_1,n_2,\cdots,n_s}^{L^2}$	$3 \leq n_1 \leq \cdots \leq n_s, s \geq 3$	$n_1 + n_2 + \cdots + n_{s-1}$

Table 1

For a tree $\overrightarrow{T}$, there is only one path $u - v$ connecting 2 vertices u, v in $\overrightarrow{T}$. We can get an efficient way for getting the number $\varpi\left(\overrightarrow{T}^{L^2}[t]\right)$. Let $V_3\left(\overrightarrow{T}\right)$ be all vertices v of valency≥ 3

connecting with a leaf by a path in $\overrightarrow{T}$. Clearly, $T \setminus V_3\left(\overrightarrow{T}\right)$ is still a tree, and we can recursively define

$$\begin{aligned}
&V_3^0\left(\overrightarrow{T}\right) = V_3\left(\overrightarrow{T}\right),\\
&V_3^1\left(\overrightarrow{T}\right) = V_3(\overrightarrow{T} \setminus V_3^0\left(\overrightarrow{T}\right)),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,\\
&V_3^m\left(\overrightarrow{T}\right) = V_3\left(\overrightarrow{T} \setminus \left(\bigcup_{i=0}^{m} V_3^i\left(\overrightarrow{T}\right)\right)\right),
\end{aligned}$$

where m is the minimum number such that $\overrightarrow{T} \setminus \left(\bigcup\limits_{i=0}^{m} V_3^i\left(\overrightarrow{T}\right)\right)$ is a path P or an empty set. Notice that if $\overrightarrow{T} \setminus \left(\bigcup\limits_{i=0}^{m} V_3^i\left(\overrightarrow{T}\right)\right)$ is not empty, there must be a vertex $v \in V_3^{m-1}$ adjacent to an internal vertex of P, $|P| \geq 3$. Otherwise, $\overrightarrow{T} \setminus \left(\bigcup\limits_{i=0}^{m} V_3^i\left(\overrightarrow{T}\right)\right)$ must be empty by definition, a contradiction. Denoted by $V_{\geq 3} = \bigcup\limits_{i=0}^{m} V_3^i\left(\overrightarrow{T}\right)$, $|V_{\geq 3}| = n_3$. By Theorem 4.10, we get a result on the number of $\varpi\left(\overrightarrow{T}^{L^2}\right)$ following.

Corollary 4.12 *Let $\overrightarrow{T}$ be a tree with vertices of valency≥ 3. Then*

$$\varpi\left(\overrightarrow{T}^{L^2}[t]\right) = n_3 + n_\delta$$

if $\overrightarrow{T}^{L^2}$ satisfies $\left[A, \frac{d}{dt}\right] = \mathbf{0}$, $\left[A, \int_t^0\right] = \mathbf{0}$ for $A \in \mathscr{A}$ and $\overrightarrow{T}^{L^2}[0]$ is harmonic, where

$$n_\delta = \begin{cases} 0, & if\ T \setminus V_{\geq 3} = \emptyset;\\ 1, & otherwise.\end{cases}$$

Proof Notice that $\overrightarrow{T} \setminus V_{\geq 3}$ is an empty set or a path, and all vertices in $V_{\geq 3}$ should be acted by input or output operations O. Otherwise, there must be vertices of valency≥ 3 in $\overrightarrow{T} \setminus \left(\bigcup\limits_{i=0}^{m} V_3^i\left(\overrightarrow{T}\right)\right)$, contradicts to the assumption on number m. Whence, we get that

$$\varpi\left(\overrightarrow{T}^{L^2}[t]\right) = n_3 + n_\delta. \qquad \square$$

Let $\overrightarrow{C}_n = v_1 v_2 \cdots v_n v_1$ be a circuit and let $\overrightarrow{P}_m = u_1^i u_2^i \cdots u_m$ be a path disjoint with $\overrightarrow{C}_n$. Define a graph $\overrightarrow{C}_n \odot \overrightarrow{P}_m$ by identifying v_1 with u_1, and if $\overrightarrow{P}^i_{m_i} = u_1^i u_2^i \cdots u_{m_i}$, $1 \leq i \leq s$ are s distinct paths and disjoint with $\overrightarrow{C}_n$, define

$$\overrightarrow{C}_n \bigodot_{i=1}^{s} \overrightarrow{P}^i_{m_i} = \left(\cdots\left(\left(\overrightarrow{C}_n \bigodot \overrightarrow{P}^1_{m_1}\right) \bigodot \overrightarrow{P}^2_{m_2}\right) \cdots \bigodot \overrightarrow{P}^s_{m_s}\right),$$

i.e., identifying $\overrightarrow{C}_n$ with paths $\overrightarrow{P}^i_{m_1}$ one by one, each with different identified vertices on $\overrightarrow{C}_n$.

Similar to Corollary 4.12 we get the following result.

Corollary 4.13 *For integers $n \geq 3, m_i \geq 2$ and $s \leq n$, there are*

$$\varpi\left(\left(\overrightarrow{C}_n \bigodot_{i=1}^{s} \overrightarrow{P}^i_{m_i}\right)^{L^2}[t]\right) = \begin{cases} s & if\ d(v_i, v_{i+1}) > 1,\ i(mod\ s); \\ \left\lceil \frac{s}{2} \right\rceil & if\ d(v_i, v_{i+1}) = 1,\ i(mod\ s) \end{cases}$$

if $\left(\overrightarrow{C}_n \bigodot_{i=1}^{s} \overrightarrow{P}^i_{m_i}\right)^{L^2}$ *satisfies* $\left[A, \frac{d}{dt}\right] = \mathbf{0}$, $\left[A, \int_t^0\right] = \mathbf{0}$ *for* $A \in \mathscr{A}$ *and* $\left(\overrightarrow{C}_n \bigodot_{i=1}^{s} \overrightarrow{P}^i_{m_i}\right)^{L^2}[0]$ *is harmonic.*

Corollary 4.14 $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) = n-1$, $\varpi\left(\left(\overrightarrow{C}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) = n$ *if* $\left(\overrightarrow{C}_n \bigodot_{i=1}^{s} \overrightarrow{P}^i_{m_i}\right)^{L^2}$ *satisfies* $\left[A, \frac{d}{dt}\right] = \mathbf{0}$, $\left[A, \int_t^0\right] = \mathbf{0}$ *for* $A \in \mathscr{A}$ *and* $\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[0]$, $\left(\overrightarrow{C}_n \times \overrightarrow{P}_2\right)^{L^2}[0]$ *both are harmonic.*

Proof Clearly, there are $2(n-2)$ vertices of valency 3, and 4 vertices of valency 2 in $\overrightarrow{P}_n \times \overrightarrow{P}_2$. Notice that there are no vertices with valency≥ 3 in graph $\overrightarrow{G} \setminus \{v^1, v^2, \cdots, v^{k_0}\}$, i.e., any vertex of valency≥ 3 should be acted itself by an action O or adjacent to acted vertices in Theorem 4.10. Whence, there are $n-2$ vertices should be acted by an input or output operation O at least. But, if there are just $n-2$ such acted vertices, there are must be a vertex of valency 3 or a circuit in the resulted subgraph by deleted these $n-2$ vertices, i.e., there are an additional vertex should be acted also. By Theorem 4.10,

$$\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) \geq n-2+1 = n-1.$$

Now, let $\overrightarrow{P}_n = v_1v_2\cdots v_n$ and $\overrightarrow{P}_2 = u_1u_2$.

Case 1. If $n \equiv 0(\text{mod } 2)$, let O act on vertices $(v_2, u_1), (v_3, u_2), (v_4, u_1), \cdots, (v_{n-2}, u_1)$ of $\overrightarrow{P}_n \times \overrightarrow{P}_2$, then $\overrightarrow{P}_n \times \overrightarrow{P}_2 \setminus \left\{[(v_2, u_1)]_p, [(v_3, u_2)]_p, [(v_4, u_1)]_p, \cdots, [(v_{n-1}, u_2)]_p\right\}$ is an empty set. Whence, $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) \leq n-1$ by Theorem 4.8. Therefore, $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) = n-1$ in this case.

Case 2. If $n \equiv 1(\text{mod } 2)$, let O act on vertices $(v_2, u_1), (v_3, u_2), (v_4, u_1), \cdots, (v_{n-2}, u_1)$ of $\overrightarrow{P}_n \times \overrightarrow{P}_2$, then $\overrightarrow{P}_n \times \overrightarrow{P}_2 \setminus \left\{[(v_2, u_1)]_p, [(v_3, u_2)]_p, [(v_4, u_1)]_p, \cdots, [(v_{n-2}, u_1)]_p\right\}$ is $\overrightarrow{C}_4$, i.e., $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) \leq n-2+1 = n-1$ by Theorem 4.8. We get $\varpi\left(\left(\overrightarrow{P}_n \times \overrightarrow{P}_2\right)^{L^2}[t]\right) = n-1$ in this case.

Similarly, let $\overrightarrow{C}_n = v_1v_2\cdots v_nv_1$ and $\overrightarrow{P}_2 = u_1u_2$. Then, there are $2n$ vertices of valency 3 in $\overrightarrow{C}_n \times \overrightarrow{P}_2$. The action O should be acted on n vertices of $\overrightarrow{C}_n \times \overrightarrow{P}_2$ at least because if O acts on vertices less than n, then there must be an integer $i, 1 \leq i \leq n$ such that (v_i, u_1), (v_i, u_2) both in the resulted graph $\overrightarrow{G}'$ by deleted claw graphs on these acted vertices in $\overrightarrow{C}_n \times \overrightarrow{P}_2$, i.e., O must acts on (v_{i-1}, u_1), (v_{i+1}, u_1) and (v_{i-1}, u_2), (v_{i+1}, u_2) $(mod\ n)$. Otherwise, one of the valence of (v_i,u_1), (v_i,u_2) must be 3 in $\overrightarrow{G}'$, i.e., there are additional vertices in $\overrightarrow{G}'$ should be acted also by Theorem 4.10. However, if O must acts on (v_{i-1},u_1), (v_{i+1},u_1) and (v_{i-1},u_2), (v_{i+1},u_2) but not on (v_i,u_1), (v_i,u_2) for $(mod\ n)$, there are must be $2\lceil\frac{n}{2}\rceil \geq n$ acted vertices, i.e., $\varpi\left(\left(\overrightarrow{C}_n\times\overrightarrow{P}_2\right)^{L^2}[t]\right)\geq n$.

Let O act on vertices $(v_1,u_1),(v_2,u_1),\cdots,\ (v_{n-1},u_1),(v_n,u_2)$ of $\overrightarrow{C}_n\times\overrightarrow{P}_2$. Then, the resulted graph $\overrightarrow{C}_n\times\overrightarrow{P}_2\setminus\left\{[(v_1,u_1)]_p,[(v_2,u_1)]_p,\cdots,[(v_{n-1},u_1)]_p,[(v_n,u_2)]_p\right\}$ is a path P_{n-1}. Similar to the proof of Theorem 4.8 we know such a path is already harmonic after the final action O, i.e., $\varpi\left(\left(\overrightarrow{C}_n\times\overrightarrow{P}_2\right)^{L^2}[t]\right)\leq n$. We therefore get $\varpi\left(\left(\overrightarrow{C}_n\times\overrightarrow{P}_2\right)^{L^2}[t]\right)=n$. □

§5 Harmonic Flow Model with Healing in Chinese Medicine

Today, we all known that a human body maybe invaded by wind, cold, hot, humidity, dry and fire in the nature which maybe result in that a human gets sick. As we have introduced in the first section there are 12 meridians, i.e., LU, LI, ST, SP, HT, SI, BL, KI, PC, SI, GB and LR meridians on a human body such as those shown in Fig.12 by the Standard China National Standard (GB 12346-90), whose combined with du meridian (DU) and ren meridian (RN) on anterior or posterior thoracic vertebrae consist of the 14 main meridians of a human body.

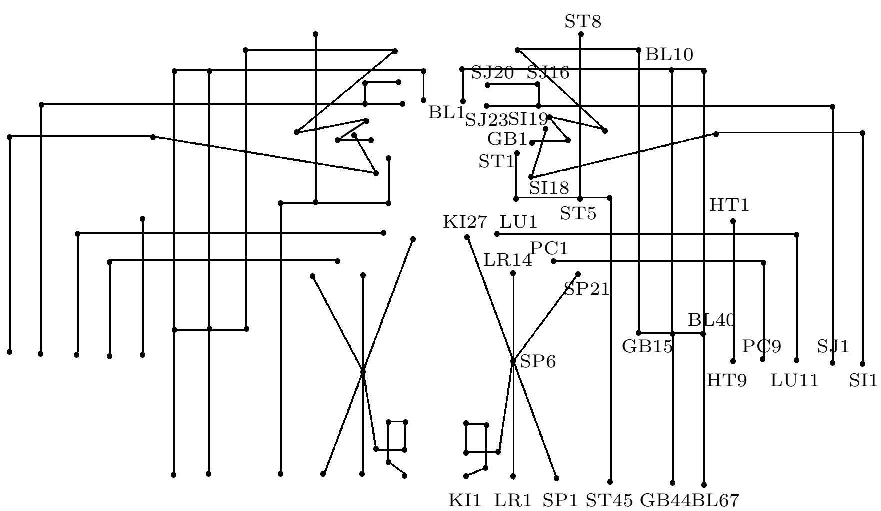

Fig.12 12 Meridians of Human Body

All of these 12 meridians are left-right symmetric with RN, DU meridians on the central axis of a human body and run with the whole life of a human, i.e., *a human is sicked if and only if there exist acupoints on meridians of body which are imbalanced.* For the recovery of a patient, the traditional Chinese doctor applies acupuncture needles inserting in acupoints on meridians for a while, then pulling out for constraint the ruler of reducing the excess with supply the insufficient by quickly or slowly and the staying time for recovery of the $\{Y^-,Y^+\}$ balance. However, it is surprised the western doctor that there are no more need of medicines unless acupuncture needles on acupoints of body for the healing of an illness such as those shown in Fig.13.

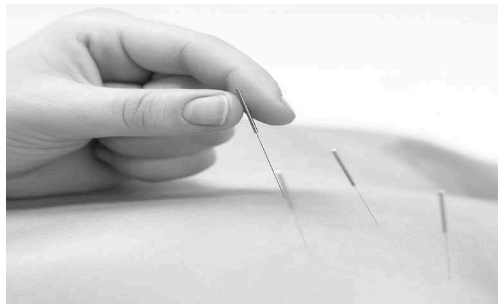

Fig.13

The traditional Chinese medicine treatment theory is the recovery of the $\{Y^-, Y^+\}$ balance on a human body which is essentially equivalent a human body to nothing else but a harmonic flow $\overrightarrow{G}^{L^2}$ over a non-connected graph $\overrightarrow{G}$ as follows:

(1) Paths: LU, LI, SP, HT, SI, KI, PC and LR meridians;

(2) Trees: GB, ST and SJ meridians;

(3) $C_n \bigodot P_{m_1} \bigodot P_{m_2}$: BL meridian.

Certainly, the 14 meridians do not run separately but conjointly in the human body. But *how do they run*? There are 2 viewpoints on this question at least in traditional Chinese medicine:

View 1.(Inner Canon of Emperor, [29]) Hand Yin meridians: *from the chest to the hand*, Hand Yang meridians: *from the hand to the head* and Foot Yang meridians: *from the head to the foot*, Foot Yin meridians: *from the foot to the chest* such as those shown in Fig.14,

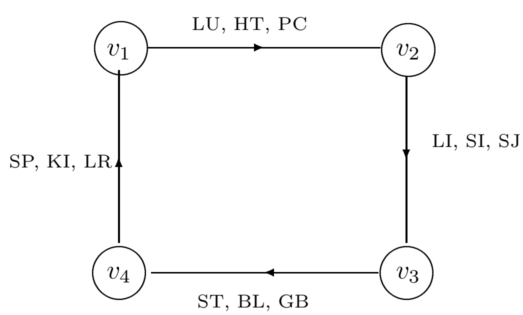

Fig.14

where v_1 =chest, v_2 =hand, v_3 =head and v_4 =foot. Furthermore, a Taoist priest Mr.Zhu spoke his seeing inside in one of danada breathing that all these meridians are with Yin and Yang in pair, i.e., $\{LU, LI\}$, $\{ST, SP\}$, $\{HT, SI\}$, $\{BL, KI\}$, $\{PC, SJ\}$, $\{GB, LR\}$, but the RN meridian and DU meridian are run respectively themselves in 2 cycles ([30]), coincident with the requirement of Chinese Qigong for getting through the RN meridian with the DU meridian.

View 2.(National Standard of 14 Meridian Pictures) The 12 meridians run from the chest to the hand, then to the head, then to the foot and then to the chest, connected respectively the first of later meridian with the end of the former meridian such as those shown in Fig.15,

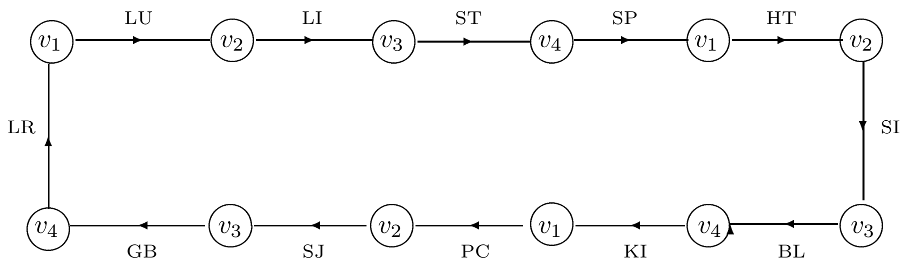

Fig.15

where v_1 =chest, v_2 =hand, v_3 =head and v_4 =foot. Therefore, there are only 2 connected graphs $\overrightarrow{G}_1, \overrightarrow{G}_2$ by View 1 and 1 connected graph $\overrightarrow{G}_3$ by View 2 such as those shown in Fig.16.

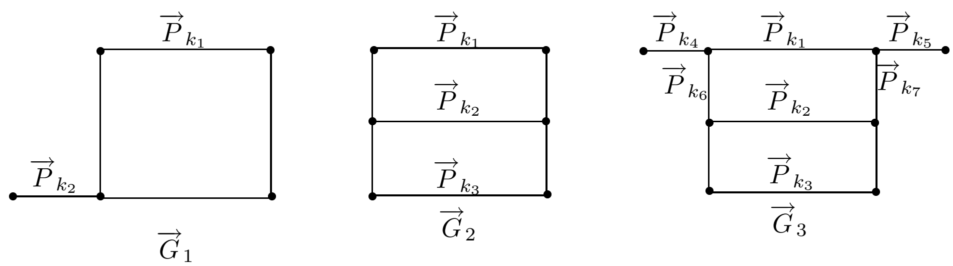

Fig.16

where $\overrightarrow{P}_k$ denotes the path of length $k-1$ with a direction and integers $k_i \geq 4$ for $1 \leq i \leq 7$.

Notice that there is always a harmonic flow $\overrightarrow{C}_n^{L^2}[0]$ for a healthy human and $\varpi\left(\overrightarrow{C}_n^{L^2}[t]\right) = 1$, $\varpi\left(\overrightarrow{P}_n^{L^2}[t]\right) = 1$ by Theorem 4.8.

Applying Theorem 4.10, we know that $\varpi\left(\overrightarrow{G}_1^{L_1^2}[t]\right) \leq 2$, $\varpi\left(\overrightarrow{G}_2^{L_2^2}[t]\right) \leq 4$ and $\varpi\left(\overrightarrow{G}_2^{L_2^2}[t]\right) \leq 7$. Thus, we acupuncture 1 needle if the imbalance appears on LU, LI, SP, HT, SI, KI, PC or LR meridian and 2 needles on original acupoints of a human body if the imbalance appears on GB, ST or SJ meridian at the early of illness, but if it is on the BL meridian or it develops serious, i.e., the imbalance is on meridians more than 3 there are needed simultaneously 3 needles on acupoints for a while at least for a patient recovery.

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