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Centered Triangular Mean Graphs

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Abstract: A graph $G = (V, E)$ with p vertices and q edges is said to have centered triangular mean labeling if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from S, where S is a set of non-negative integers in such a way that for each edge $e = uv$, $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ and the resulting edge labels are the first q centered triangular numbers. A graph that admits a centered triangular mean labeling is called centered triangular mean graph. In this paper, we prove that the graphs P_n, $K_{1,n}$, $B_{m,n}$, coconut tree, caterpillar $S(n_1, n_2, n_3, \ldots, n_m)$, $St(n_1, n_2, n_3, \ldots, n_m)$, $Bt(\underbrace{n, n, \ldots, n}_{\text{m times}})$ and $P_m@P_n$ are centered triangular mean graphs.

Key Words: Mean labeling, Smarndache mean m-labelling, triangular mean labeling, triangular mean graph, centered triangular mean labeling, centered triangular mean graph.

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§1. Introduction

By a graph, we mean a finite, simple and undirected one. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. For various graph theoretic notations and terminology we follow Harary [2] and for number theory Burton[1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and a detailed survey on graph labeling is available in [3]. The concept of mean labeling was introduced by Somasundaram et al.[6]. A graph $G = (V, E)$ with p vertices and q edges is called a mean graph if there is an injective function f that maps $V(G)$ to $\{0, 1, 2, \cdots, q\}$ such that each edge uv is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd and the resulting labels of the edges are distinct and are $0, 1, 2, \cdots, q$. Generally, if each edge uv is labeled with $\frac{f(u)+f(v)}{m}$ if f maps $V(G)$ to $\{0, 1, 2, \cdots, q, 2, 4, \cdots, 2q, \cdots, m, 2m, 4m \cdots, qm\}$ such that $\frac{f(u)+f(v)+m-k}{m}$ if $f(u)+f(v) \equiv k(mod m)$ the resulting labels of the edges are distinct and are $0, 1, 2, \cdots, q$, such a labeling f is called a *Smarandache mean m-labeling* and G a *Smarandache m-mean graph*, where $m \geq 2$ is an integer. Particularly, a Smarandache 2-mean

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graph is nothing else but a mean graph. Several papers have been published on mean labeling and its variations.

Seenivasan et al.[5] introduced the concept of triangular mean labeling in 2007. A triangular mean labeling of a graph $G = (V, E)$ with p vertices and q edges is an injection $f : V(G) \longrightarrow \{0, 1, 2, 3, \cdots, T_q\}$ such that for each edge $e = uv$, the edge labels are defined as $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ such that the values of the edges are the first q triangular numbers. A graph that admits a triangular mean labeling is called triangular mean graph.

Recently, the number theory has a strong impact on graph theory. A triangular number[1] is a number obtained by adding all positive numbers less than or equal to a given positive integer n. If the n^{th} triangular number[1] is denoted by T_n, then $T_n = \frac{1}{2}n(n+1)$. A centered triangular number is a centered figurative number that represents a triangle with a dot in the center and all other dots surrounding the center in successive triangular layers. If the n^{th} centered triangular number is denoted by cT_n, then $cT_n = \frac{1}{2}(3n^2 - 3n + 2)$. The first few centered triangular numbers are $1,\ 4,\ 10,\ 19,\ 31,\ 46,\ 64,\ 85, \cdots$.

The figurative representations of the first four centered triangular numbers are shown in Figure 1.

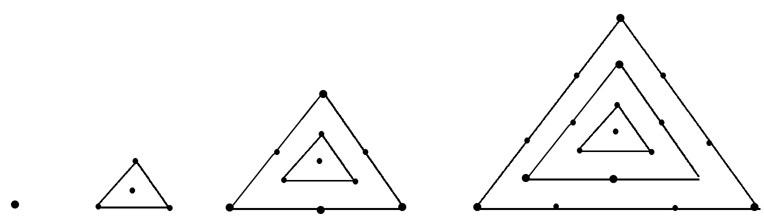

Figure 1

The notion of centered triangular sum labeling was due to Murugesan et al.[4] in 2013. Motivated by the results in [4] and [5] and using the centered triangular concept in number theory [1] we define a new labeling called centered triangular mean labeling. A graph $G = (V, E)$ with p vertices and q edges is said to have centered triangular mean labeling if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from S, where S is a set of non-negative integers such that for each edge $e = uv$, $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$, and the resulting labels of the edges are the first q centered triangular numbers. A graph that admits a centered triangular mean labeling is called centered triangular mean graph. In this paper, we prove that the graph $P_n,\ K_{1,n},\ B_{m,n}$, coconut tree, caterpillar $S(n_1, n_2,\ n_3, \cdots, n_m)$, $St(n_1, n_2,\ n_3, \cdots, n_m)$, $Bt(\underbrace{n, n, \cdots, n}_{\text{m times}})$ and $P_m@P_n$ are centered triangular mean graphs. We use the following definitions in the subsequent sequel.

Definition 1.1 *The bistar $B_{m,n}$ is a graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2.*

Definition 1.2 *A caterpillar is a tree with a path P_m: $v_1, v_2, \cdots, v_m$, called spine with leaves(pendant vertices) known as feet attached to the vertices of the spine by edges known as legs. If every spine vertex v_i is attached with n_i number of leaves then the caterpillar is* *denoted by* $S(n_1, n_2, \cdots, n_m)$.

Definition 1.3 *The shrub* $St(n_1, n_2, \cdots, n_m)$ *is a graph obtained by connecting a vertex* v_0 *to central vertex of each of* m *number of stars.*

Definition 1.4 *The banana tree* $Bt(n_1, n_2, \cdots, n_m)$ *is a graph obtained by connecting a vertex* v_0 *to one leaf of each of* m *number of stars.*

Definition 1.5 *The graph* $P_m@P_n$ *is obtained from* P_m *and* m *copies of* P_n *by identifying one pendant vertex of the* i^{th} *copy of* P_n *with* i^{th} *vertex of* P_m *where* P_m *is a path of length of* $m-1$.

§2. Centered Triangular Mean Graphs

Theorem 2.1 *The path* $P_n (n \geqslant 1)$ *is a centered triangular mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n. Define $f : V(P_n) \longrightarrow S$ as follows:

$f(v_1) = 0,$

$f(v_j) = 2(cT_{j-1} - cT_{j-2} + cT_{j-3} - \cdots + (-1)^j cT_1)$ for $2 \leqslant j \leqslant n.$

Let $e_i = v_i v_{i+1} (1 \leqslant i \leqslant n-1)$ be the edges of P_n. For each vertex label f, the induced edge label f^* is defined to be $f^*(e_i) = cT_i$ for $1 \leqslant i \leqslant n-1$. Then f is a centered triangular mean labeling. Hence P_n is a centered triangular mean graph. □

The centered triangular mean labeling of P_5 is given in Figure 2.

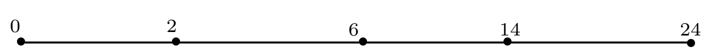

Figure 2

Theorem 2.2 *The star graph* $K_{1,n} (n \geq 1)$ *admits centered triangular mean labeling.*

Proof Let v be the apex vertex and let $v_1, v_2, \cdots, v_n$ be the pendant vertices of the star $K_{1,n}$. Define $f : V(K_{1,n}) \longrightarrow S$ to be $f(v) = 0, \quad f(v_j) = 2cT_j$ for $1 \leq j \leq n$.

For each vertex label f, the induced edge label f^* is defined to be $f^*(vv_j) = cT_j$ for $1 \leq j \leq n$. Then the induced edge labels are the centered triangular numbers $cT_1, \ cT_2, \ cT_3, \cdots, cT_n$. Hence $K_{1,n}$ is a centered triangular mean graph. □

The centered triangular mean labeling of $K_{1,8}$ is shown in Figure 3.

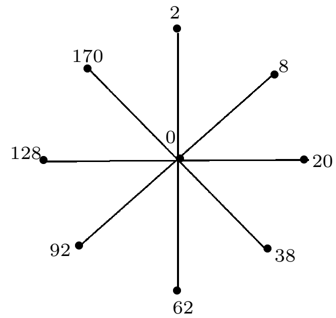

Figure 3

Theorem 2.3 *The bistar $B_{m,n}(m \geqslant 1,\ n \geqslant 1)$is a centered triangular mean graph.*

Proof Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and $E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Define $f : V(B_{m,n}) \longrightarrow S$ as follows:

$f(u) = 0,\ f(v) = 2,$
$f(u_i) = 2cT_{i+1}$ for $1 \leq i \leq m,$
$f(v_j) = 2(cT_{m+j+1} - 1)$ for $1 \leq j \leq n.$

For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(uv) = cT_1,$
$f^*(uu_i) = cT_{i+1}$ for $1 \leqslant i \leqslant m,$
$f^*(vv_j) = cT_{m+j+1}$ for $1 \leqslant j \leqslant n.$

Then the induced edge labels are the first $m + n + 1$ centered triangular numbers. Hence $B_{m,n}$ is a centered triangular mean graph. □

The centered triangular labeling of $B_{4,5}$ is given in Figure 4.

Figure 4

Theorem 2.4 *The coconut tree $T(n, m)$, obtained by identifying the central vertex of the star $K_{1,m}$ with a pendant vertex of a path P_n, is a centered triangular mean graph.*

Proof Let $u_0, u_1, u_2, \cdots, u_n$ be the vertices of a path, having path length $n(n \geqslant 1)$ and $v_1,\ v_2, \cdots, v_m$ be the pendant vertices being adjacent with u_0. Define $f : V(T(n, m)) \longrightarrow S$ as follows:

$f(u_0) = 0,$
$f(u_1) = 2cT_{m+1},$
$f(v_i) = 2cT_i$ for $1 \leqslant i \leqslant m,$

$f(u_j) = 2(cT_{m+j} - cT_{m+j-1} + cT_{m+j-2} - \ldots + (-1)^{j+1}cT_{m+1})$ for $2 \leqslant j \leqslant n$.

For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(u_0v_i) = cT_i$ for $1 \leqslant i \leqslant m$,
$f^*(u_0u_1) = cT_{m+1}$,
$f^*(u_ju_{j+1}) = cT_{m+j+1}$ for $1 \leqslant j \leqslant n-1$.

Then the induced edge labels are the first $m+n$ centered triangular numbers. Hence, coconut tree admits centered triangular mean labeling. □

The centered triangular mean labeling of $T(3,7)$ is given in Figure 5.

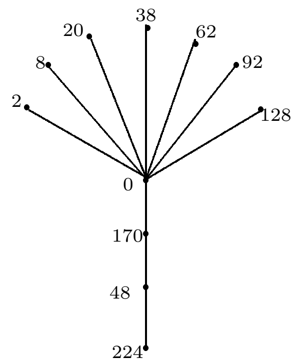

Figure 5

Theorem 2.5 *The caterpillar $S(n_1, n_2, \ldots, n_m)$ is a centered triangular mean graph.*

Proof Let $v_1, v_2, \cdots, v_m$ be the vertices of the path P_m and $v_i^j(1 \leqslant i \leqslant n_j,$ $1 \leqslant j \leqslant m)$ be the pendant vertices joined with $v_j(1 \leqslant j \leqslant m)$ by an edge. Then

$$V(S(n_1, n_2, \cdots, n_m)) = \{v_j, v_i^j : 1 \leqslant i \leqslant n_j \ 1 \leqslant j \leqslant m\}$$
$$E(S(n_1, n_2, \cdots, n_m)) = \{v_tv_{t+1}, v_jv_i^j : 1 \leqslant t \leqslant m-1,\ 1 \leqslant i \leqslant n_j,\ 1 \leqslant j \leqslant m\}.$$

We define $f : V(S(n_1, n_2, \ldots, n_m)) \longrightarrow S$ as follows:

$f(v_1) = 0$, $f(v_j) = 2(cT_{j-1} - cT_{j-2} + cT_{j-3} - \cdots + (-1)^j cT_1)$ for $2 \leqslant j \leqslant m$,
$f(v_i^1) = 2cT_{m-1+i}$ for $1 \leqslant i \leqslant n_1$,
$f(v_i^j) = 2cT_{m-1+n_1+n_2+\cdots+n_{j-1}+i} + (-1)^{j-1}2(cT_1 - cT_2 + cT_3 - \cdots + (-1)^j cT_{j-1})$ for $1 \leqslant i \leqslant n_j$ and $2 \leqslant j \leqslant m$.

For the vertex label f, the induced edge label f^* is defined as follows:

$f^*(v_jv_{j+1}) = cT_j$ for $1 \leqslant j \leqslant m-1$,
$f^*(e_i^1) = cT_{m-1+i}$ for $1 \leqslant i \leqslant n_1$,
$f^*(e_i^j) = cT_{m-1+n_1+n_2+\cdots+n_{j-1}+i}$ for $1 \leqslant j \leqslant n_j$ and $2 \leqslant j \leqslant m$.

Then the edge labels are the centered triangular numbers

$$cT_1, cT_2, \cdots, cT_{m-1}, cT_m, \cdots, cT_{m-1+n_1+n_2,\cdots+n_m}$$

and also the vertex labels are different. Hence $S(n_1, n_2, \cdots, n_m)$ is a centered triangular mean graph. □

The centered triangular mean labeling of $S(3,5,4,6)$ is shown in Figure 6.

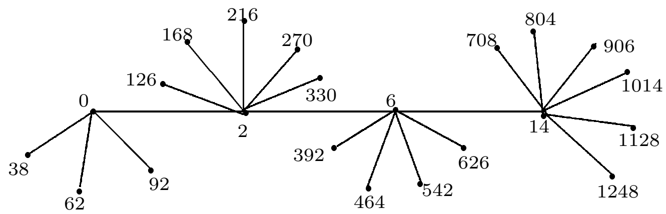

Figure 6

Theorem 2.6 *The shrub $St(n_1, n_2, \ldots, n_m)$ is a centered triangular mean graph.*

Proof let $v_0, v_j, u_i^j (1 \leqslant j \leqslant m,\ 1 \leqslant i \leqslant n_j)$ be the vertices of $St(n_1, n_2, \cdots, n_m)$. Then $E(St(n_1, n_2, \cdots, n_m)) = \{v_0v_j,\ v_ju_i^j$ for $1 \leqslant i \leqslant n_j$ and $1 \leqslant j \leqslant m\}$. Define $f : V(St(n_1, n_2, \cdots, n_m)) \longrightarrow S$ as follows:

$f(v_0) = 0$,
$f(v_j) = 2cT_j$ for $1 \leqslant j \leqslant m$,
$f(u_i^j) = 2(cT_{m+n_1+n_2+\cdots+n_{j-1}+i} - cT_j)$ for $1 \leqslant i \leqslant n_j$ and $1 \leqslant j \leqslant m$.

Let $e_i^j = v_ju_i^j$ for $1 \leqslant i \leqslant n_j$ and $1 \leqslant j \leqslant m$. For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(v_0v_j) = cT_j$ for $1 \leqslant j \leqslant m$,
$f^*(e_i^j) = cT_{m+n_1+n_2+\cdots+n_{j-1}+i}$ for $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant n_j$.

Then f is a centered triangular mean labeling of $St(n_1, n_2, \cdots, n_m)$. □

The centered triangular mean labeling of $St(4,5,4)$ is shown in Figure 7.

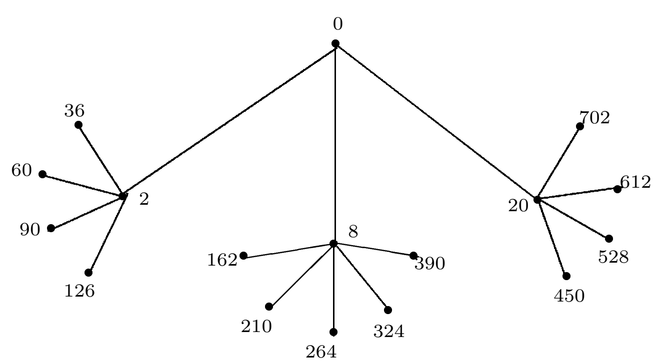

Figure 7

Theorem 2.7 *The banana tree $Bt(\underbrace{n, n, \ldots, n}_{m\ times})$ is a centered triangular mean graph.*

Proof Let $v_0, v_j, u_i^j (1 \leqslant j \leqslant m,\ 1 \leqslant i \leqslant n)$ be the vertices of $Bt(\underbrace{n, n, \cdots, n}_{\text{m times}})$. Then

$E(Bt(\underbrace{n,n,\ldots,n}_{\text{m times}})) = \{v_0u_1^j, v_iu_i^j$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m\}$. Define $f : V(Bt(\underbrace{n,n,\ldots,n}_{\text{m times}})) \longrightarrow S$ as follows:

$$f(v_0) = 0, \quad f(v_j) = 2(cT_{m+j} - cT_j) \quad \text{for } 1 \leqslant j \leqslant m,$$
$$f(u_1^j) = 2cT_j \text{ for } 1 \leqslant j \leqslant m,$$
$$f(u_i^j) = 2(cT_{2m+(j-1)(n-1)+i-1} - cT_{m+j} + cT_j) \text{ for } 2 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant m.$$

Let $e_i^j = v_ju_i^j$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. For each vertex label f, the induced edge label f^* is defined as follows:

$$f^*(v_0u_1^j) = cT_j \text{ for } 1 \leqslant j \leqslant m,$$
$$f^*(v_ju_1^j) = cT_{m+j} \text{ for } 1 \leqslant j \leqslant m,$$
$$f^*(e_i^j) = cT_{2m+(j-1)(n-1)+(i-1)} \text{ for } 1 \leqslant j \leqslant m \text{ and } 2 \leqslant i \leqslant n.$$

Therefore f is a centered triangular mean labeling of $Bt(\underbrace{n,n,\cdots,n}_{\text{m times}})$. □

The centered triangular mean labeling of $Bt(\underbrace{3,3,\cdots,3}_{\text{8 times}})$ is shown in Figure 8.

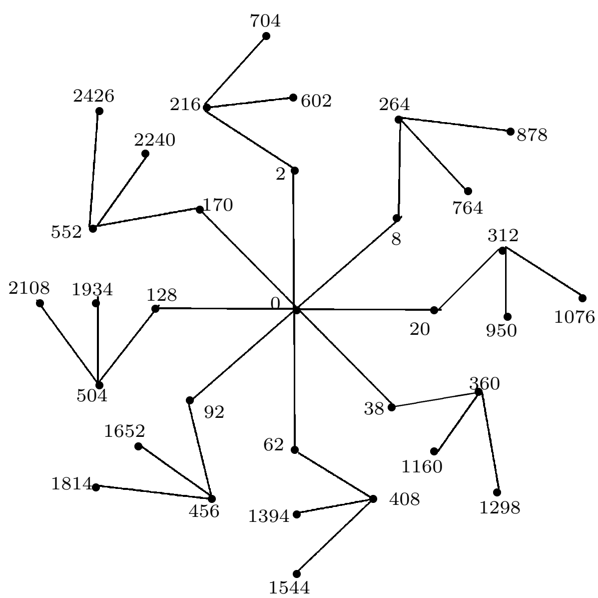

Figure 8

Theorem 2.8 *The graph $P_m@P_n$ is a centered triangular mean graph.*

proof Let $\{v_j, u_j^i,\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$ be the vertices of $P_n@P_m$ with $v_j = u_j^1$, $(1 \leqslant j \leqslant m)$. Then $E(P_n@P_m) = \{\ v_jv_{j+1}, u_j^iu_j^{i+1}\ :\ 1 \leqslant j \leqslant m-1,\ 1 \leqslant i \leqslant n-1\}$. Define $f : V(P_n@P_m) \longrightarrow S$ as follows:

$$f(u_1^1) = 0, \quad f(u_j^1) = 2(cT_{j-1} - cT_{j-2} + cT_{j-3} - \ldots + (-1)^j cT_1) \text{ for } 2 \leqslant j \leqslant m,$$

$f(u_1^2) = 2cT_m, \quad f(u_j^2) = 2(cT_{m+j-1} - cT_{j-1} + cT_{j-2} - \ldots + (-1)^{j-1}cT_1)$ for $2 \leqslant j \leqslant m$,

$$\begin{aligned} f(u_j^i) &= 2(cT_{(i-1)m+j-1} - cT_{(i-2)m+j-1} + cT_{(i-3)m+j-1} - \cdots \\ &\quad +(-1)^{i-1}cT_{m+j-1}) + (-1)^{i-1}2(cT_{j-1} - cT_{j-2} + cT_{j-3} - \ldots + (-1)^j cT_1) \end{aligned}$$

for $1 \leqslant j \leqslant m,\ 3 \leqslant i \leqslant n$. For each vertex label f, the induced edge label f^* is defined as follows:

$f^*(v_jv_{j+1}) = cT_j$ for $1 \leqslant j \leqslant m-1$,
$f^*(u_j^1u_j^2) = cT_{m+j-1}$ for $1 \leqslant j \leqslant m$,
$f^*(u_j^iu_j^{i+1}) = cT_{mi+j-1}$ for $1 \leqslant j \leqslant m$ and $2 \leqslant i \leqslant n-1$.

Therefore f is a centered triangular labeling of $P_n@P_m$. □

The centered triangular mean labeling of $P_4@P_4$ is shown in Figure 9.

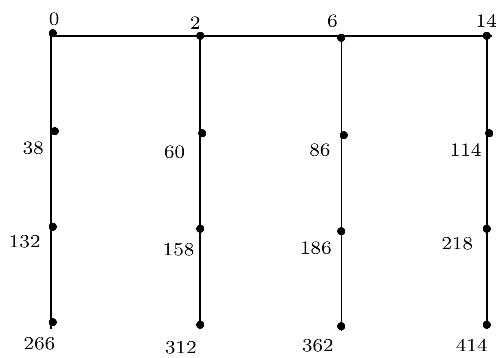

Figure 9

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