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A Note on Common Fixed Points for (ψ, α, β)-Weakly Contractive Mappings in Generalized Metric Space

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Abstract: In this paper, we establish a common fixed point theorem for mappings satisfying a (ψ, α, β)-weakly contractive condition in generalized metric space. Presented theorems extend and generalize many existing results in the literature. We prove the main results for four self mappings using any two continuous mappings.

Key Words: Fixed point theory, generalized metric space, (ψ, α, β)-weakly contractive mappings, common fixed point.

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§1. Introduction

Fixed point theory is an important part of mathematics. Moreover, its well known that the contraction mapping principle, which is introduced by S. Banach in 1922.

During the last few decades, this theorem has undergone various generalizations either by relaxing the condition on contractivity or withdrawing the requirement of completeness or sometimes even both.

In 2000, Branciari [1] obtained a very interesting generalization of metric space by changing the structure of the space. He, replaced the triangle inequality of a metric space by an inequality involving three terms instead of two called quadrilateral inequality. He, proved the Banach fixed point theorem in such space. Recently, many fixed point results have been established for this interesting space ([3],[7],[8],[9]). As such, any metric space is a generalized metric space, but the converse is not true [1].

Recently, many researchers have interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems. Choudhury and Kundu [2] established the (ψ, α, β)-weakly contraction principal to coincidence point and common fixed point results in partially ordered metric spaces. In a recent paper Isik and Turkoglu [3] proved common fixed point for (ψ, α, β)-weakly contractive mappings in generalized metric spaces for two mappings.

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The aim of this paper to prove a common fixed point for (ψ, α, β) -weakly contractive mappings in generalized metric space of four self mappings.

§2. Preliminaries

Definition 2.1([1]) *Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a generalized metric on X if the following conditions are satisfied:*

(1) $d(x, y) = 0$ *iff* $x = y$ *for all* x, y *in* X;

(2) $d(x, y) = d(y, x)$ *for all* x, y *in* X;

(3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ *for all* x, y, u, v *in* X.

The pair (X, d) is called a generalized metric space abbreviated to g.m.s .

Definition 2.2([1]) *Let (X,d) be a g.m.s and let (x_n) be a sequence in X and $x \in X$.*

(1) (x_n) *is a g.m.s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$;*

(2) x_n *is a g.m.s. Cauchy sequence if and only if for each $\in > 0$ there exists a natural number $n(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n > m > n(\epsilon)$;*

(3) (X, d) *is called a complete g.m.s if every g.m.s. Cauchy sequence is g.m.s. convergent in X;*

We denote by Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

$(\psi 1)$ ψ *is continuous and monotone non decreasing;*

$(\psi 2)$ $\psi(t) = 0$ *if and only if t=0.*

We denote by ϕ the set of function $\alpha : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

$(\alpha 1)$ α *is continuous;*

$(\alpha 2)$ $\alpha(t) = 0$ *if and only if t=0.*

We denote by Γ the set of function $\beta : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

$(\beta 1)$ β *is lower semi continuous;*

$(\beta 2)$ $\beta(t) = 0$ *if and only if t=0.*

Definition 2.3([3]) *Let A and B be mappings from a metric space (X, d) into itself. A and B are said to be weakly compatible mapping if they commute at their coincidence point i.e, $Ax = Bx$ for some x in X implies $ABx = BAx$.*

Lemma 2.1([3]) *Let a_n be a sequence of non negative real numbers. If*

$$\psi(a_{n+1}) \leq \alpha(a_n) - \beta(a_n) \tag{A}$$

for all $n \in N$, where $\psi \in \Psi$, $\alpha \in \phi$, $\beta \in \Gamma$ and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \ for \ all \ t > 0, \tag{B}$$

then the following hold:

(1) $a_{n+1} \leq a_n$ *if* $a_n > 0$*;*

(2) $a_n \to 0$ *as* $n \to \infty$*.*

Proof (1) Let, if possible $a_n < a_{n+1}$ for some $n \in N$ then using the monotone property of Ψ and from (A) we have, $\psi(a_n) \leq \psi(a_{n+1}) \leq \alpha(a_n) - \beta(a_n)$, which implies that $a_n = 0$ by (B) a contradiction with $a_n > 0$. Therefore, for all $n \in N$, $a_{n+1} \leq a_n$.

(2) By (1) the sequence a_n is non-increasing, hence there is $a \geq 0$ such that $a_n \to a$ as $n \to \infty$ letting $n \to \infty$ in (A), using the lower semi continuity of β and the continuities of Ψ and α, we obtain $\psi(a) \leq \alpha(a) - \beta(a)$, which by (B) implies that $a = 0$. □

§3. Main Results

Theorem 3.1 *Let (X, d) be a Hausdorff and complete g.m.s. and let f, g, h and J be four mappings of X into itself and $f(X) \subset h(X)$, $g(X) \subset J(X)$. Without loss of generality, assume h, J are continuous, f and J, g and h both are compatible satisfying the following condition*

$$\psi(d(fx, gy) \leq \alpha(M(x, y)) - \beta(M(x, y)), \tag{1}$$

where $M(x, y) = \max\{d(Jx, hy), d(fx, Jx), d(gy, hy), d(fx, hy)\}$ for all $x, y \in X$, where $\psi \subset \Psi$, $\alpha \subset \phi$ and $\beta \subset \Gamma$ satisfying condition (B). Then, f, g, h and J have a unique common fixed point in X.

Proof Notice that $f(X) \subset h(X)$ and $g(X) \subset J(X)$. Let x_0 be any point in X. Then, there exists sequences (x_n) and (y_n) such that $y_n = fx_n = hx_{n+1}$, $y_{n+1} = gx_{n+1} = Jx_{n+2}$, $n = 0, 1, 2, 3, \cdots$. Now,

$$\psi(d(y_n, y_{n+1})) = \psi(d(fx_n, gx_{n+1})) \leq \alpha(M(x, y)) - \beta(M(x, y)), \tag{2}$$

where,

$$\begin{aligned} M(x, y) &= \max\{d(Jx_n, hx_{n+1}), d(fx_n, Jx_n), d(gx_{n+1}, hx_{n+1}), d(fx_n, hx_{n+1})\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), d(y_n, y_n)\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), 0\} \end{aligned}$$

If possible, let $d(y_n, y_{n+1}) > 0$ and $d(y_n, y_{n+1}) > d(y_{n-1}, y_n)$. Then from (2) we get that

$$\psi(d(y_n, y_{n+1})) \leq \alpha(d(y_n, y_{n+1})) - \beta(d(y_n, y_{n+1})). \tag{3}$$

By Lemma 2.1, each number in the sequence y_n is non negative and real. Hence there exists $a \geq 0$ such that $y_n \to a$ as $n \to \infty$ in (3). Using the lower semi continuity of β and the continuities of ψ and α, we obtain $\psi(a) \leq \alpha(a) - \beta(a)$. However, (B) implies $a = 0$, which is a contradiction. So $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$.

From (2) we get that

$$\psi(d(y_n, y_{n+1})) \leq \alpha(d(y_{n-1}, y_n)) - \beta(d(y_{n-1}, y_n)).$$

According to Lemma 2.1 we know that $\psi(a_{n+1}) \leq \alpha(a_n) - \beta(a_n)$ for all $n \in N$, where $\psi \subset \Psi$, $\alpha \subset \phi$ and $\beta \subset \Gamma$ and $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$. Thus, $a_{n+1} \leq a_n$ if $a_n > 0$ and $a_n \to \infty$. Therefore,

$$d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n) \text{ and } \lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{4}$$

Now, we prove that y_n is a g.m.s Cauchy sequence. If y_n is not a g.m.s Cauchy sequence, then there exists $\epsilon > 0$, for which we can find a sub sequence y_{n_k} and y_{m_k} of y_n with $m_k > n_k > k$ such that

$$d(y_{n_k}, y_{m_k}) \geq \epsilon \tag{5}$$

and corresponding to n_k, we can choose m_k in such a way that it is the smallest integer with $m_k > n_k$ satisfying (4) and

$$d(y_{n_k}, y_{m_{k-1}}) < \epsilon. \tag{6}$$

Applying (5) and (6) and the rectangular inequality, we get that

$$\epsilon \leq d(y_{n_k}, y_{m_k}) \leq d(y_{n_k}, y_{n_{k-1}}) + d(y_{n_{k-1}}, y_{m_{k-1}}) + d(y_{m_{k-1}}, y_{m_k}). \tag{7}$$

Applying (4) we get that

$$\epsilon \leq d(y_{n_{k-1}}, y_{m_{k-1}}) \tag{8}$$

and

$$d(y_{n_{k-1}}, y_{m_{k-1}}) \leq d(y_{n_{k-1}}, y_{n_k}) + d(y_{n_k}, y_{m_k}) + d(y_{m_k}, y_{m_{k-1}}).$$

By (4) and (6) we have that

$$d(y_{n_{k-1}}, y_{m_{k-1}}) \leq \epsilon. \tag{9}$$

From (8) and (9) we know that

$$d(y_{n_{k-1}}, y_{m_{k-1}}) = \epsilon. \tag{10}$$

Applying (4) and (10) in (7) we get that

$$\epsilon \leq d(y_{n_k}, y_{m_k}) \leq \epsilon, \text{ i.e., } d(y_{n_k}, y_{m_k}) = \epsilon. \tag{11}$$

Now,

$$\psi(d(y_{n_k}, y_{m_k})) = \psi(d(fx_{n_k}, gx_{m_k})) \leq \alpha(M(x, y)) - \beta(M(x, y)),$$

where,

$$\begin{aligned} M(x,y) &= \max\{d(Jx_{n_k}, hx_{m_k}), d(fx_{n_k}, Jx_{n_k}), d(gx_{m_k}, hx_{m_k}), d(fx_{n_k}, hx_{m_k})\} \\ &= \max\{d(y_{n_k}, y_{m_k}), d(y_{n_k}, y_{n_{k-1}}), d(y_{m_k}, y_{m_k}), d(y_{n_k}, y_{m_k})\} \\ &= \max\{\epsilon, 0, 0, \epsilon\} \ \text{ (by (11) and (4))}. \end{aligned}$$

we therefore get that $\psi(\epsilon) \leq \alpha(\epsilon) - \beta(\epsilon)$ and $\epsilon = 0$ by lemma 2.1, which is a contradiction as we assume that $\epsilon > 0$. Then it follows that y_n is a g.m.s Cauchy sequence and hence y_n is convergent in the complete g.m.s space (X, d). Let $lim_{n\to\infty} y_n = z$, i.e.,

$$\lim_{n\to\infty} fx_n = lim_{n\to\infty} hx_{n+1} = lim_{n\to\infty} gx_{n+1} = lim_{n\to\infty} Jx_{n+2} = z. \tag{12}$$

Notice that J is a continuous function, we know that

$$\psi(d(fJx_n, gx_{n+1})) \leq \alpha(M(x,y)) - \beta(M(x,y)), \tag{13}$$

where,

$$M(x,y) = \max\{d(JJx_n, hx_{n+1}), d(Jfx_n, JJx_n), d(gx_{n+1}, hx_{n+1}), d(Jfx_n, hx_{n+1})\}$$

Let $n \to \infty$ in the above. We get that

$$\begin{aligned} M(x,y) &= \max\{d(Jz, z), d(Jz, Jz), d(z, z), d(Jz, z)\} \\ &= \max\{d(Jz, z), 0, 0, d(Jz, z)\} \\ &= d(Jz, z). \end{aligned}$$

Let $n \to \infty$ on both sides in (12). We know that $\psi(d(Jz, z)) \leq \alpha(d(Jz, z)) - \beta(d(Jz, z))$. By Lemma 2.1 we have $d(Jz, z) = 0$, i.e.,

$$Jz = z. \tag{14}$$

Now,

$$\psi(d(fz, gx_{n+1})) \leq \alpha(M(x,y)) - \beta(M(x,y)), \tag{15}$$

where,

$$M(x,y) = \max\{d(Jz, hx_{n+1}), d(fz, Jz), d(gx_{n+1}, hx_{n+1}), d(fz, hx_{n+1})\}$$

by (1) and (14). Now, taking $n \to \infty$ on above we get that

$$\begin{aligned} M(x,y) &= \max\{d(z, z), d(z, z), d(z, z), d(fz, z)\} \\ &= \max\{0, 0, 0, d(fz, z)\} = d(fz, z). \end{aligned}$$

Similarly, let $n \to \infty$ in (14) on both sides we get that

$$\psi(d(fz, z)) \leq \alpha(d(fz, z)) - \beta(d(fz, z)).$$

Applying Lemma 2.1 we get that $d(fz, z) = 0$, i.e.,

$$fz = z. \tag{16}$$

So, from (14) and (16) we know that

$$Jz = fz = z. \tag{17}$$

Similarly as h is continuous in X we can prove

$$hz = gz = z. \tag{18}$$

From (17) and (18) we get that

$$Jz = fz = hz = gz = z. \tag{19}$$

So z is a common fixed point of f, g, h and J.

Now, we prove that z is unique. If $w(\neq z)$ is another fixed point. Notice that

$$\psi(d(z, w) = \psi(d(fz, gw)) \leq \alpha(M(x, y)) - \beta M(x, y)), \tag{20}$$

where,

$$\begin{aligned} M(x, y) &= \max\{d(Jz, hw), d(fz, Jz), d(gw, hw), d(fz, hw)\} \\ &= \max\{d(z, w), d(z, z), d(w, w), d(z, w)\} = max(d(z, w), 0, 0, d(z, w)), \end{aligned}$$

which enables us to get that $\psi(d(z, w) \leq \alpha(d(z, w)) - \beta(d(z, w))$ from (20). By Lemma 2.1 we get $d(z, w) = 0$, i.e., $z = w$. Thus the fixed point z is unique. □

§4. Conclusion

The main result is an extension of the result [3] to the set of generalized metric space. This paper is also a generalization of many existing results in this literature.

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