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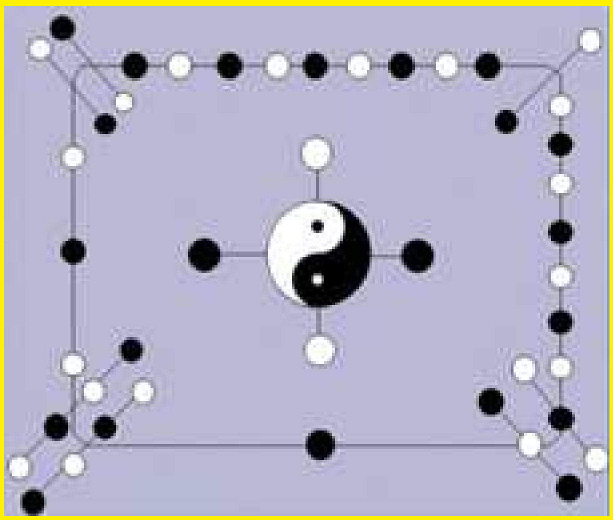

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The problems that exist in the world today cannot be solved by the level of thinking that created them.

By Albert Einstein, an American physicist.

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Some Fixed Point Results for Contractive Type Conditions in Cone b-Metric Spaces and Applications

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Abstract: In this paper, we establish some fixed point results for contractive type conditions in the framework of complete cone b-metric spaces and give some applications of our results. The results presented in this paper generalize, extend and unify several well-known comparable results in the existing literature.

Key Words: Fixed point, contractive type condition, cone b-metric space, cone.

AMS(2010): 47H10, 54H25.

§1. Introduction

Fixed point theory plays a basic role in applications of many branches of mathematics. Finding fixed point of contractive mappings becomes the center of strong research activity. Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle [2] in 1922.

In [3], Bakhtin introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see [4, 5, 11] and references therein). In recent investigation, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [14, 15, 17, 20]). The authors define an ordering by using a cone, which naturally induces a partial ordering in Banach spaces.

In 2007, Huang and Zhang [14] introduced the concept of cone metric spaces as a generalization of metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [1, 16, 20, 23] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space. In 2008, Rezapour and Hamlbarani [20] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space.

Recently, Hussain and Shah [15] introduced the concept of cone b-metric space as a general-

[1]Received July 23, 2017, Accepted February 8, 2018.

ization of b-metric space and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space. In this paper, we give some examples in cone b-metric spaces, then obtain some fixed theorems for contractive type conditions in the setting of cone b-metric spaces.

Definition 1.1([14]) *Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:)*

(c_1) P is closed, nonempty and $P \neq \{0\}$;
(c_2) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;
(c_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P. If $P^0 \neq \emptyset$ then P is called a solid cone (see [22]).

There exist two kinds of cones- normal (with the normal constant K) and non-normal ones ([12]).

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ imply } \|x\| \leq K\|y\|, \tag{1.1}$$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{1.2}$$

The least positive number K satisfying (1.1) is called the normal constant of P.

Example 1.2([22]) Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \leq x_n \leq y_n$, and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (1.2) that P is a non-normal cone.

Definition 1.3([14,24]) *Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to E$ satisfies:*

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space [14].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 1.4([14]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d\colon X \times X \to E$

defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space with normal cone P where $K = 1$.

Example 1.5([19]) Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{for all } n\}$, (X,ρ) a metric space, and $d\colon X \times X \to E$ defined by $d(x,y) = \{\rho(x,y)/2^n\}_{n\geq 1}$. Then (X,d) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 1.6([15]) *Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to E$ is said to be cone b-metric if and only if, for all x, y, $z \in X$, the following conditions are satisfied:*

(b_1) $0 \leq d(x,y)$ *with* $x \neq y$ *and* $d(x,y) = 0 \Leftrightarrow x = y$;

(b_2) $d(x,y) = d(y,x)$;

(b_3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a cone b-metric space.

Remark 1.7 The class of cone b-metric spaces is larger than the class of cone metric space since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

We give some examples, which show that introducing a cone b-metric space instead of a cone metric space is meaningful since there exist cone b-metric space which are not cone metric space.

Example 1.8([13]) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0, y \geq 0\} \subset E$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x,y) = (|x-y|^p, \alpha|x-y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X,d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

Example 1.9([13]) Let $X = \ell^p$ with $0 < p < 1$, where $\ell^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d\colon X \times X \to \mathbb{R}_+$ defined by $d(x,y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}}$, where $x = \{x_n\}$, $y = \{y_n\} \in \ell^p$. Then (X,d) is a cone b-metric space with the coefficient $s = 2^{1/p} > 1$, but not a cone metric space.

Example 1.10([13]) Let $X = \{1,2,3,4\}$, $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0, y \geq 0\}$. Define $d\colon X \times X \to E$ by

$$d(x,y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}) & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then (X,d) is a cone b-metric space with the coefficient $s = \frac{6}{5} > 1$. But it is not a cone metric space since the triangle inequality is not satisfied,

$$d(1,2) > d(1,4) + d(4,2), \quad d(3,4) > d(3,1) + d(1,4).$$

Definition 1.11([15]) *Let (X,d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then*

• $\{x_n\}$ *is a Cauchy sequence whenever, if for every* $c \in E$ *with* $0 \ll c$*, then there is a natural number* N *such that for all* $n, m \geq N$, $d(x_n, x_m) \ll c$*;*

• $\{x_n\}$ *converges to* x *whenever, for every* $c \in E$ *with* $0 \ll c$*, then there is a natural number* N *such that for all* $n \geq N$, $d(x_n, x) \ll c$*. We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

• (X, d) *is a complete cone b-metric space if every Cauchy sequence is convergent.*

In the following (X, d) will stands for a cone b-metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and $\leq$ is partial ordering in E with respect to P.

Definition 1.12([10]) *Let* (X, d) *be a metric space. A self mapping* $T\colon X \to X$ *is called quasi contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq h \max\Big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big\}$$

for all $x, y \in X$ *and* $h \in (0, 1)$ *is a constant.*

Definition 1.13([10]) *Let* (X, d) *be a metric space. A self mapping* $T\colon X \to X$ *is called Ciric quasi-contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq h \max\Big\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\Big\}$$

for all $x, y \in X$ *and* $h \in (0, 1)$ *is a constant.*

The following lemmas are often used (in particular when dealing with cone metric spaces in which the cone need not be normal).

Lemma 1.14([17]) *Let* P *be a cone and* $\{a_n\}$ *be a sequence in* E*. If* $c \in int\, P$ *and* $0 \leq a_n \to 0$ *as* $n \to \infty$*, then there exists* N *such that for all* $n > N$*, we have* $a_n \ll c$*.*

Lemma 1.15([17]) *Let* x, y, $z \in E$*, if* $x \leq y$ *and* $y \ll z$*, then* $x \ll z$*.*

Lemma 1.16([15]) *Let* P *be a cone and* $0 \leq u \ll c$ *for each* $c \in int\, P$*, then* $u = 0$*.*

Lemma 1.17([8]) *Let* P *be a cone, if* $u \in P$ *and* $u \leq k\, u$ *for some* $0 \leq k < 1$*, then* $u = 0$*.*

Lemma 1.18([17]) *Let* P *be a cone and* $a \leq b + c$ *for each* $c \in int\, P$*, then* $a \leq b$*.*

§2. Main Results

In this section we shall prove some fixed point theorems of contractive type conditions in the framework of cone b-metric spaces.

Theorem 2.1 *Let* (X, d) *be a complete cone b-metric space with the coefficient* $s \geq 1$*. Suppose* *that the mapping $T\colon X \to X$ satisfies the contractive type condition:*

$$\begin{aligned} d(Tx,Ty) &\leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) \\ &\quad + \mu[d(x,Ty) + d(y,Tx)] \end{aligned} \tag{2.1}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \mu \geq 0$ are constants such that $s\alpha + \beta + s\gamma + (s^2+s)\mu < 1$. Then T has a unique fixed point in X.

Proof Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.1), we have

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha d(x_n,x_{n-1}) + \beta d(x_n,Tx_n) + \gamma d(x_{n-1},Tx_{n-1}) \\ &\quad + \mu[d(x_n,Tx_{n-1}) + d(x_{n-1},Tx_n)] \\ &= \alpha d(x_n,x_{n-1}) + \beta d(x_n,x_{n+1}) + \gamma d(x_{n-1},x_n) \\ &\quad + \mu[d(x_n,x_n) + d(x_{n-1},x_{n+1})] \\ &= \alpha d(x_n,x_{n-1}) + \beta d(x_n,x_{n+1}) + \gamma d(x_{n-1},x_n) \\ &\quad + \mu d(x_{n-1},x_{n+1}) \\ &\leq \alpha d(x_n,x_{n-1}) + \beta d(x_n,x_{n+1}) + \gamma d(x_{n-1},x_n) \\ &\quad + s\mu[d(x_{n-1},x_n) + d(x_n,x_{n+1})] \\ &= (\alpha + \gamma + s\mu) d(x_n,x_{n-1}) \\ &\quad + (\beta + s\mu) d(x_n,x_{n+1}). \end{aligned} \tag{2.2}$$

This implies that

$$\begin{aligned} d(x_{n+1},x_n) &\leq \Big(\frac{\alpha+\gamma+s\mu}{1-\beta-s\mu}\Big) d(x_n,x_{n-1}) \\ &= \lambda\, d(x_n,x_{n-1}) \end{aligned} \tag{2.3}$$

where

$$\lambda = \Big(\frac{\alpha+\gamma+s\mu}{1-\beta-s\mu}\Big).$$

As $s\alpha + \beta + s\gamma + (s^2+s)\mu < 1$, it is clear that $\lambda < 1/s$.

Similarly, we obtain

$$d(x_{n-1},x_n) \leq \lambda\, d(x_{n-2},x_{n-1}). \tag{2.4}$$

Using (2.4) in (2.3), we get

$$d(x_{n+1},x_n) \leq \lambda^2\, d(x_{n-1},x_{n-2}). \tag{2.5}$$

Continuing this process, we obtain

$$d(x_{n+1},x_n) \leq \lambda^n\, d(x_1,x_0). \tag{2.6}$$

Let $m \geq 1$, $p \geq 1$, we have

$$\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&\quad + \cdots + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq s\lambda^m d(x_1, x_0) + s^2\lambda^{m+1} d(x_1, x_0) + s^3\lambda^{m+2} d(x_1, x_0) \\
&\quad + \cdots + s^p k^{m+p-1} d(x_1, x_0) \\
&= s\lambda^m [1 + s\lambda + s^2\lambda^2 + s^3\lambda^3 + \cdots + (s\lambda)^{p-1}] d(x_1, x_0) \\
&\leq \Big[\frac{s\lambda^m}{1 - s\lambda}\Big] d(x_1, x_0).
\end{aligned}$$

Let $0 \ll \varepsilon$ be given. Notice that $\Big[\frac{s\lambda^m}{1-s\lambda}\Big] d(x_1, x_0) \to 0$ as $m \to \infty$ for any p since $0 < s\lambda < 1$. Making full use of Lemma 1.14, we find $m_0 \in \mathbb{N}$ such that

$$\Big[\frac{s\lambda^m}{1 - s\lambda}\Big] d(x_1, x_0) \ll \varepsilon$$

for each $m > m_0$. Thus

$$d(x_m, x_{m+p}) \leq \Big[\frac{s\lambda^m}{1 - s\lambda}\Big] d(x_1, x_0) \ll \varepsilon$$

for all $m \geq 1$, $p \geq 1$. So, by Lemma 1.15, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b-metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Take $n_0 \in \mathbb{N}$ such that $d(x_n, u) \ll \frac{\varepsilon(1-s(\beta+\mu))}{s(\alpha+\mu+1)}$ for all $n > n_0$. Hence,

$$\begin{aligned}
d(Tu, u) &\leq s[d(Tu, Tx_n) + d(Tx_n, u)] \\
&= sd(Tu, Tx_n) + sd(Tx_n, u) \\
&\leq s\Big\{\alpha d(u, x_n) + \beta d(u, Tu) + \gamma d(x_n, Tx_n) \\
&\quad + \mu[d(u, Tx_n) + d(x_n, Tu)]\Big\} + sd(x_{n+1}, u) \\
&= s\Big\{\alpha d(u, x_n) + \beta d(u, Tu) + \gamma d(x_n, x_{n+1}) \\
&\quad + \mu[d(u, x_{n+1}) + d(x_n, Tu)]\Big\} + sd(x_{n+1}, u) \\
&= s(\alpha + \mu + 1) d(x_n, u) + s(\beta + \mu) d(Tu, u).
\end{aligned} \tag{2.7}$$

This implies that

$$d(Tu, u) \leq \Big(\frac{s(\alpha + \mu + 1)}{1 - s(\beta + \mu)}\Big) \ll \varepsilon,$$

for each $n > n_0$. Then, by Lemma 1.16, we deduce that $d(Tu, u) = 0$, that is, $Tu = u$. Thus u is a fixed point of T.

Now, we show that the fixed point is unique. If there is another fixed point u^* of T such that $Tu^* = u^*$, then from (2.1), we have

$$\begin{aligned} d(u,u^*) &= d(Tu,Tu^*) \\ &\leq \alpha d(u,u^*) + \beta d(u,Tu) + \gamma d(u^*,Tu^*) \\ &\quad + \mu[d(u,Tu^*) + d(u^*,Tu)] \\ &\leq \alpha d(u,u^*) + \beta d(u,u) + \gamma d(u^*,u^*) \\ &\quad + \mu[d(u,u^*) + d(u^*,u)] \\ &= (\alpha + 2\mu) d(u,u^*) \\ &\leq (s\alpha + \beta + s\gamma + (s^2+s)\mu) d(u,u^*). \end{aligned}$$

By Lemma 1.17, we have $u = u^*$. This completes the proof. □

Remark 2.2 Theorem 2.1 extends Theorem 2.1 of Huang and Xu in [13] to the case of weaker contractive condition considered in this paper.

From Theorem 2.1, we obtain the following result as corollaries.

Corollary 2.3 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the contractive condition:*

$$d(Tx,Ty) \leq \alpha d(x,y)$$

for all $x,y \in X$, where $\alpha \in \left[0, \frac{1}{s}\right)$ is a constant. Then T has a unique fixed point in X.

Proof The proof of Corollary 2.3 is immediately follows from Theorem 2.1 by taking $\beta = \gamma = \mu = 0$. This completes the proof. □

Corollary 2.4 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the contractive condition:*

$$d(Tx,Ty) \leq \beta[d(x,Tx) + d(y,Ty)]$$

for all $x,y \in X$, where $\beta \in \left[0, \frac{1}{1+s}\right)$ is a constant. Then T has a unique fixed point in X.

Proof The proof of Corollary 2.4 is immediately follows from Theorem 2.1 by taking $\alpha = \mu = 0$ and $\beta = \gamma$. This completes the proof. □

Corollary 2.5 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the contractive condition:*

$$d(Tx,Ty) \leq \mu[d(x,Ty) + d(y,Tx)]$$

for all $x,y \in X$, where $\mu \in \left[0, \frac{1}{s+s^2}\right)$ is a constant. Then T has a unique fixed point in X.

Proof The proof of Corollary 2.5 is immediately follows from Theorem 2.1 by taking

$\alpha = \beta = \gamma = 0$. This completes the proof. □

Remark 2.6 Corollaries 2.3, 2.4 and 2.5 extend Theorem 1, 3 and 4 of Huang and Zhang [14] to the case of cone b-metric space without normal constant considered in this paper.

Remark 2.7 Corollary 2.3 also extends the well known Banach contraction principle [2] to that in the setting of cone b-metric spaces.

Remark 2.8 Corollary 2.4 also extends the Kannan contraction [18] to that in the setting of cone b-metric spaces.

Remark 2.9 Corollary 2.5 also extends the Chatterjea contraction [7] to that in the setting of cone b-metric spaces.

Remark 2.10 Theorem 2.1 also extends several results from the existing literature to the case of weaker contractive condition considered in this paper in the setting of cone b-metric spaces.

Theorem 2.11 *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T: X \to X$ satisfies the contractive type condition:*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Ty) + \gamma d(y, Tx) \tag{2.8}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma \geq 0$ are constants such that $s\alpha + s(1+s)\gamma < 1$. Then T has a unique fixed point in X.

Proof Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.8), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha d(x_n, x_{n-1}) + \beta d(x_n, Tx_{n-1}) + \gamma d(x_{n-1}, Tx_n) \\ &= \alpha d(x_n, x_{n-1}) + \beta d(x_n, x_n) + \gamma d(x_{n-1}, x_{n+1}) \\ &= \alpha d(x_n, x_{n-1}) + \gamma d(x_{n-1}, x_{n+1}) \\ &\leq \alpha d(x_n, x_{n-1}) + s\gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &= (\alpha + s\gamma) d(x_n, x_{n-1}) + s\gamma d(x_n, x_{n+1}). \end{aligned} \tag{2.9}$$

This implies that

$$d(x_{n+1}, x_n) \leq \Big(\frac{\alpha + s\gamma}{1 - s\gamma}\Big) d(x_n, x_{n-1}) = \rho\, d(x_n, x_{n-1}), \tag{2.11}$$

where

$$\rho = \Big(\frac{\alpha + s\gamma}{1 - s\gamma}\Big).$$

As $s\alpha + s(s+1)\gamma < 1$, it is clear that $\rho < 1/s$.

Similarly, we obtain

$$d(x_{n-1}, x_n) \leq \rho\, d(x_{n-2}, x_{n-1}). \tag{2.11}$$

Using (2.11) in (2.10), we get

$$d(x_{n+1}, x_n) \leq \rho^2\, d(x_{n-1}, x_{n-2}). \tag{2.12}$$

Continuing this process, we obtain

$$d(x_{n+1}, x_n) \leq \rho^n\, d(x_1, x_0). \tag{2.13}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned}
d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\
&= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\
&\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\
&= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\
&\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\
&\leq s\rho^n d(x_1, x_0) + s^2 \rho^{n+1} d(x_1, x_0) + s^3 \rho^{n+2} d(x_1, x_0) \\
&\quad + \cdots + s^m \rho^{n+m-1} d(x_1, x_0) \\
&= s\rho^n [1 + s\rho + s^2\rho^2 + s^3\rho^3 + \cdots + (s\rho)^{m-1}] d(x_1, x_0) \\
&\leq \left[\frac{s\rho^n}{1 - s\rho}\right] d(x_1, x_0).
\end{aligned}$$

Let $0 \ll \varepsilon_1$ be given. Notice that $\left[\frac{s\rho^n}{1-s\rho}\right] d(x_1, x_0) \to 0$ as $n \to \infty$ since $0 < s\rho < 1$. Making full use of Lemma 1.14, we find $n_0 \in \mathbb{N}$ such that

$$\left[\frac{s\rho^n}{1 - s\rho}\right] d(x_1, x_0) \ll \varepsilon_1$$

for each $n > n_0$. Thus

$$d(x_n, x_m) \leq \left[\frac{s\rho^n}{1 - s\rho}\right] d(x_1, x_0) \ll \varepsilon_1$$

for all $n, m \geq 1$. So, by Lemma 1.15, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b-metric space, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. Take $n_1 \in \mathbb{N}$ such that $d(x_n, v) \ll \frac{\varepsilon_1(1-s\gamma)}{s(\alpha+1)}$ for all $n > n_1$. Hence,

$$\begin{aligned}
d(Tv, v) &\leq s[d(Tv, Tx_n) + d(Tx_n, v)] \\
&= sd(Tv, Tx_n) + sd(Tx_n, v) \\
&\leq s[\alpha d(v, x_n) + \beta d(v, Tx_n) + \gamma d(x_n, Tv)] + sd(x_{n+1}, v) \\
&= s[\alpha d(v, x_n) + \beta d(v, x_{n+1}) + \gamma d(x_n, Tv)] + sd(x_{n+1}, v) \\
&= s(\alpha + 1) d(v, x_n) + s\gamma d(Tv, v).
\end{aligned} \tag{2.14}$$

This implies that

$$d(Tv, v) \leq \Big(\frac{s(\alpha+1)}{1-s\gamma}\Big) d(x_n, v) \ll \varepsilon_1,$$

for each $n > n_1$. Then, by Lemma 1.16, we deduce that $d(Tv, v) = 0$, that is, $Tv = v$. Thus v is a fixed point of T.

Now, we show that the fixed point is unique. If there is another fixed point v^* of T such that $Tv^* = v^*$, then from (2.8), we have

$$\begin{array}{rcl} d(v, v^*) & = & d(Tv, Tv^*) \\ & \leq & \alpha d(v, v^*) + \beta d(v, Tv^*) + \gamma d(v^*, Tv) \\ & = & \alpha d(v, v^*) + \beta d(v, v^*) + \gamma d(v^*, v) \\ & = & (\alpha+\beta+\gamma) d(v, v^*) \\ & \leq & (s\alpha + s(1+s)\gamma) d(v, v^*). \end{array}$$

By Lemma 1.17, we have $v = v^*$. This completes the proof. □

Theorem 2.12 *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the following contractive condition: there exists*

$$u(x, y) \in \Big\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2s}, \frac{d(x, Ty) + d(y, Tx)}{2s}\Big\}$$

such that

$$d(Tx, Ty) \leq k\, u(x, y), \tag{2.15}$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant with $ks < 1$. Then T has a unique fixed point in X.

Proof Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.15), we have

$$\begin{array}{rcl} d(x_{n+1}, x_n) & = & d(Tx_n, Tx_{n-1}) \\ & \leq & k\, u(x_n, x_{n-1}) \leq \cdots \leq k^n\, u(x_1, x_0). \end{array} \tag{2.16}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{array}{rcl} d(x_n, x_m) & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ & = & sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ & \leq & sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ & = & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\ & \leq & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ & & + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \end{array}$$

$$\begin{aligned} &\leq sk^n u(x_1,x_0) + s^2k^{n+1}u(x_1,x_0) + s^3k^{n+2}u(x_1,x_0) \\ &\quad + \cdots + s^m k^{n+m-1}u(x_1,x_0) \\ &= sk^n[1 + sk + s^2k^2 + s^3k^3 + \cdots + (sk)^{m-1}]u(x_1,x_0) \\ &\leq \Big[\frac{sk^n}{1-sk}\Big]u(x_1,x_0). \end{aligned}$$

Let $0 \ll r$ be given. Notice that

$$\Big[\frac{sk^n}{1-sk}\Big]u(x_1,x_0) \to 0$$

as $n \to \infty$ since $0 < k < 1$. Making full use of Lemma 1.14, we find $n_0 \in \mathbb{N}$ such that

$$\Big[\frac{sk^n}{1-sk}\Big]u(x_1,x_0) \ll r$$

for each $n > n_0$. Thus

$$d(x_n,x_m) \leq \Big[\frac{sk^n}{1-sk}\Big]u(x_1,x_0) \ll r$$

for all $n, m \geq 1$. So, by Lemma 1.15, $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete cone b-metric space, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Take $n_1 \in \mathbb{N}$ such that $d(x_n,p) \ll \frac{r}{s(k+1)}$ for all $n > n_1$. Hence,

$$\begin{aligned} d(Tp,p) &\leq s[d(Tp,Tx_n) + d(Tx_n,p)] \\ &= sd(Tp,Tx_n) + sd(Tx_n,p) \\ &\leq sk\,u(p,x_n) + s\,d(x_{n+1},p) \\ &\leq sk\,d(p,x_n) + s\,d(x_n,p) \\ &= s(k+1)\,d(x_n,p). \end{aligned}$$

This implies that

$$d(Tp,p) \ll r,$$

for each $n > n_1$. Then, by Lemma 1.16, we deduce that $d(Tp,p) = 0$, that is, $Tp = p$. Thus p is a fixed point of T.

Now, we show that the fixed point is unique. If there is another fixed point q of T such that $Tq = q$, then by the given condition (2.15), we have

$$d(p,q) = d(Tp,Tq) \leq k\,u(p,q) = k\,d(p,q).$$

By Lemma 1.17, we have $p = q$. This completes the proof. □

Theorem 2.13 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the following contractive condition:*

$$d(Tx,Ty) \leq h\ \max\{d(x,y), d(x,Tx), d(y,Ty)\} \tag{2.17}$$

for all $x, y \in X$, where $h \in [0, 1)$ is a constant with $sh < 1$. Then T has a unique fixed point in X.

Proof Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.17), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq h \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\ &= h \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &\leq h\, d(x_n, x_{n-1}). \end{aligned} \tag{2.18}$$

Similarly, we obtain

$$d(x_{n-1}, x_n) \leq h\, d(x_{n-2}, x_{n-1}). \tag{2.19}$$

Using (2.19) in (2.18), we get

$$d(x_{n+1}, x_n) \leq h^2\, d(x_{n-1}, x_{n-2}). \tag{2.20}$$

Continuing this process, we obtain

$$d(x_{n+1}, x_n) \leq h^n\, d(x_1, x_0). \tag{2.21}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\ &\leq sh^n d(x_1, x_0) + s^2 h^{n+1} d(x_1, x_0) + s^3 h^{n+2} d(x_1, x_0) \\ &\quad + \cdots + s^m h^{n+m-1} d(x_1, x_0) \\ &= sh^n[1 + sh + s^2h^2 + s^3h^3 + \cdots + (sh)^{m-1}] d(x_1, x_0) \\ &\leq \Big[\frac{sh^n}{1 - sh}\Big] d(x_1, x_0). \end{aligned}$$

Let $0 \ll c$ be given. Notice that

$$\Big[\frac{sh^n}{1 - sh}\Big] d(x_1, x_0) \to 0$$

as $n \to \infty$ since $0 < h < 1$. Making full use of Lemma 1.14, we find $N_0 \in \mathbb{N}$ such that

$$\Big[\frac{sh^n}{1 - sh}\Big] d(x_1, x_0) \ll c$$

for each $n > N_0$. Thus

$$d(x_n, x_m) \leq \Big[\frac{sh^n}{1-sh}\Big] d(x_1, x_0) \ll c$$

for all $n, m \geq 1$. So, by Lemma 1.15, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b-metric space, there exists $q \in X$ such that $x_n \to q$ as $n \to \infty$. Take $N_1 \in \mathbb{N}$ such that $d(x_n, q) \ll \frac{c}{s(h+1)}$ for all $n > N_1$. Hence,

$$\begin{aligned}
d(Tq, q) &\leq s[d(Tq, Tx_n) + d(Tx_n, q)] \\
&= sd(Tq, Tx_n) + sd(Tx_n, q) \\
&\leq sh \max\{d(q, x_n), d(x_n, Tx_n), d(q, Tq)\} + s\, d(x_{n+1}, q) \\
&= sh \max\{d(q, x_n), d(x_n, x_{n+1}), d(q, Tq)\} + s\, d(x_{n+1}, q) \\
&\leq sh\, d(q, x_n) + s\, d(x_n, q) \\
&= s(h+1)\, d(x_n, q).
\end{aligned}$$

This implies that

$$d(Tq, q) \ll c,$$

for each $n > N_1$. Then, by Lemma 1.16, we deduce that $d(Tq, q) = 0$, that is, $Tq = q$. Thus q is a fixed point of T.

Now, we show that the fixed point is unique. If there is another fixed point q' of T such that $Tq' = q'$, then by the given condition (2.17), we have

$$\begin{aligned}
d(q, q') &= d(Tq, Tq') \\
&\leq h \max\{d(q, q'), d(q, Tq), d(q', Tq')\} \\
&= h \max\{d(q, q'), d(q, q), d(q', q')\} \\
&= h \max\{d(q, q'), 0, 0\} \\
&\leq h\, d(q, q')
\end{aligned}$$

By Lemma 1.17, we have $q = q'$. This completes the proof. □

Example 2.14([13]) Let $X = [0, 1]$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x \geq 0, y \geq 0\} \subset E$ and $d\colon X \times X \to E$ defined by $d(x, y) = (|x-y|^p, |x-y|^p)$ for all $x, y \in X$ where $p > 1$ is a constant. Then (X, d) is a complete cone b-metric space. Let us define $T\colon X \to X$ as $T(x) = \frac{1}{2}x - \frac{1}{4}x^2$ for all $x \in X$. Therefore,

$$\begin{aligned}
d(Tx, Ty) &= (|Tx - Ty|^p, |Tx - Ty|^p) \\
&= \Big(\Big|\frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y)\Big|^p, \Big|\frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y)\Big|^p\Big) \\
&= \Big(|x-y|^p.\Big|\frac{1}{2} - \frac{1}{4}(x+y)\Big|^p, |x-y|^p.\Big|\frac{1}{2} - \frac{1}{4}(x+y)\Big|^p\Big) \\
&\leq \frac{1}{2^p}(|x-y|^p, |x-y|^p) = \frac{1}{2^p} d(x, y).
\end{aligned}$$

Hence $0 \in X$ is the unique fixed point of T.

Other consequence of our result for the mapping involving contraction of integral type is the following.

Denote Λ the set of functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying the following hypothesis:

(h_1) φ is a Lebesgue-integrable mapping on each compact subset of $[0,\infty)$;

(h_2) for any $\varepsilon > 0$ we have $\int_0^\infty \varphi(t)dt > 0$.

Theorem 2.15 *Let (X,d) be a complete cone b-metric space (CCbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \quad \leq \quad \beta \int_0^{d(x,y)} \psi(t)dt$$

for all $x, y \in X$, where $\beta \in [0,1)$ is a constant with $s\beta < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Remark 2.16 Theorem 2.15 extends Theorem 2.1 of Branciari [6] from complete metric space to that setting of complete cone *b*-metric space considered in this paper.

§3. Applications

In this section we shall apply Theorem 2.1 to the first order differential equation.

Example 3.1 $X = C([1,3],\mathbb{R})$, $E = \mathbb{R}^2$, $\alpha > 0$ and

$$d(x,y) = \Big(\sup_{t\in[1,3]} |x(t)-y(t)|^2,\ \alpha \sup_{t\in[1,3]} |x(t)-y(t)|^2\Big)$$

for every $x, y \in X$, and $P = \big\{(u,v) \in \mathbb{R}^2 : u,\, v \geq 0\big\}$. Then (X,d) is a cone b-metric space. Define $T\colon X \to X$ by

$$T(x(t)) = 4 + \int_1^t \Big(x(u) + u^2\Big)e^{u-5}du.$$

For $x, y \in X$,

$$\begin{aligned} d(Tx,Ty) &= \Big(\sup_{t\in[1,3]} |T(x(t)) - T(y(t))|^2,\ \alpha \sup_{t\in[1,3]} |T(x(t)) - T(y(t))|^2\Big) \\ &\leq \Big(\int_1^3 |(x(u)-y(u))|^2 e^{-2}du,\ \alpha \int_1^3 |(x(u)-y(u))|^2 e^{-2}du\Big) \\ &= 2e^{-2}d(x,y) \\ &\leq 2e^{-1}d(x,y). \end{aligned}$$

Thus for $\alpha = \frac{2}{e} < 1$, $\beta = \gamma = \mu = 0$, all conditions of Theorem 2.1 are satisfied and so T

has a unique fixed point, which is the unique solution of the integral equation:

$$x(t) = 4 + \int_1^t \Big(x(u) + u^2\Big)e^{u-5}du,$$

or the differential equation:

$$x'(t) = \Big(x(t) + t^2\Big)e^{t-5},\ t \in [1,3],\ x(1) = 4.$$

Hence, the use of Theorem 2.1 is a delightful way of showing the existence and uniqueness of solutions for the following class of integral equations:

$$q + \int_p^t K(x(u), u)du = x(t) \in C\big([p,q],\, \mathbb{R}^n\big).$$

Now, we shall apply Corollary 2.3 to the first order periodic boundary problem

$$\begin{cases} \dfrac{dx}{dt} = F(t, x(t)), \\ x(0) = \mu, \end{cases} \tag{3.1}$$

where $F\colon [-h,h] \times [\mu-\theta, \mu+\theta]$ is a continuous function.

Example 3.2([13]) Consider the boundary problem (3.1) with the continuous function F and suppose $F(x,y)$ satisfies the local Lipschitz condition, i.e., if $|x| \le h$, $y_1, y_2 \in [\mu-\theta, \mu+\theta]$, it induces

$$|F(x,y_1) - F(x,y_2)| \le L\,|y_1 - y_2|.$$

Set $M = \max_{[-h,h]\times[\mu-\theta,\mu+\theta]} |F(x,y)|$ such that $h^2 < \min\{\theta/M^2, 1/L^2\}$, then there exists a unique solution of (3.1).

Proof Let $X = E = C([-h,h])$ and $P = \{u \in E : u \ge 0\}$. Put $d\colon X \times X \to E$ as $d(x,y) = f(t)\max_{-h\le t\le h} |x(t) - y(t)|^2$ with $f\colon [-h,h] \to \mathbb{R}$ such that $f(t) = e^t$. It is clear that (X,d) is a complete cone b-metric space.

Note that (3.1) is equivalent to the integral equation

$$x(t) = \mu + \int_0^t F(u, x(u))du.$$

Define a mapping $T\colon C([-h,h]) \to \mathbb{R}$ by $x(t) = \mu + \int_0^t F(u,x(u))du$. If

$$x(t), y(t) \in B(\mu, f\,\theta) = \{\varphi(t) \in C([-h,h]) : d(\mu,\varphi) \le f\,\theta\},$$

then from

$$\begin{aligned} d(Tx,Ty) &= f(t)\max_{-h\le t\le h}\Big|\int_0^t F(u,x(u))du-\int_0^t F(u,y(u))du\Big|^2 \\ &= f(t)\max_{-h\le t\le h}\Big|\int_0^t [F(u,x(u))-F(u,y(u))]du\Big|^2 \\ &\le h^2f(t)\max_{-h\le t\le h}|F(u,x(u))-F(u,y(u))|^2 \\ &\le h^2L^2f(t)\max_{-h\le t\le h}|x(u)-y(u)|^2=h^2L^2d(x,y), \end{aligned}$$

and

$$\begin{aligned} d(Tx,\mu) &= f(t)\max_{-h\le t\le h}\Big|\int_0^t F(u,x(u))du\Big|^2 \\ &\le h^2f\max_{-h\le t\le h}|F(u,x(u))|^2\le h^2M^2f\le f\theta, \end{aligned}$$

we speculate $T\colon B(\mu,f\theta)\to B(\mu,f\theta)$ is a contraction mapping.

Lastly, we prove that $(B(\mu,f\theta),d)$ is complete. In fact, suppose $\{x_n\}$ is a Cauchy sequence in $B(\mu,f\theta)$. Then $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, there is $q\in X$ such that $x_n\to q$ $(n\to\infty)$, So, for each $c\in int\,P$, there exists N, whenever $n>N$, we obtain $d(x_n,q)\ll c$. Thus, it follows from

$$d(\mu,q)\le d(x_n,\mu)+d(\mu,q)\le f\theta+c$$

and Lemma 1.18 that $d(\mu,q)\le f\theta$, which means $q\in B(\mu,f\theta)$, that is, $(B(\mu,f\theta),d)$ is complete. Thus, from the above statement, all the conditions of Corollary 2.3 are satisfied. Hence T has a unique fixed point $x(t)\in B(\mu,f\theta)$ or we say that, there exists a unique solution of (3.1). $\square$

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A Note on Slant and Hemislant Submanifolds of an (ϵ)-Para Sasakian Manifold

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Abstract: In the present note we have derived some basic results pertaining to the geometry of slant and hemislant submanifolds of an (ϵ)-para Sasakian manifold. In particular, we have obtained few results on a totally umbilical slant submanifold of an (ϵ)-para Sasakian manifold. In the next section, we have obtained the integrability conditions of the involved distributions of hemislant submanifold of an (ϵ)-para Sasakian manifold. Finally we have verified the theorems by providing an example of three dimensional hemislant submanifold of (ϵ)-para Sasakian manifold.

Key Words: (ϵ)-Para Sasakian manifold, totally umbilical submanifold, slant submanifold, hemislant submanifold.

AMS(2010): 53B25, 53C15, 53C40, 53C42.

§1. Introduction

Connected almost contact metric manifold was classified by S.Tanno [13], as those automorphism group has maximum dimension. He has given following classifications:

(i) Homogeneous normal contact Riemannian manifolds with constant ϕ holomorphic sectional curvature if the sectional curvature of the plane section containing ξ, say $K(X, \xi) > 0$;

(ii) Global Riemannian product of a line (or a circle) and a Kaehlerian manifold with constant holomorphic sectional curvature, if $K(X, \xi) = 0$;

(iii) A warped product space RX_fC_n, if $K(X, \xi) < 0$.

Manifold of class (i) has Sasakian structure. The manifold of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. The manifold of class (iii) are characterized by some tensorial equations, attaining a Kenmotsu structure.

An almost paracontact structure (ϕ, ξ, η) satisfying $\phi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold was introduced by Sato [11] in 1976. After him Takahashi [14] in 1969, gave the notion of almost contact manifold equipped with an associated pseudo-Riemannian metric. Later on, motivated by these circumstances, M.M.Tripathi et.al.([15]) has drawn a relation between a semi-Riemannian metric (not necessarily Lorentzian) and an almost paracontact structure, and he named this indefinite almost paracontact metric structure an (ϵ)-

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almost paracontact structure, where the structure vector field ξ will be spacelike or timelike according as $\epsilon = 1$ or $\epsilon = -1$. Authors have discussed (ϵ)- almost paracontact manifolds and in particular (ϵ)- Sasakian manifolds in([15]).

On the other hand, the study of slant submanifolds in complex spaces was initiated by B.Y.Chen as a natural generalization of both holomorphic and totally real submanifolds in ([4]). After him, A.Lotta in 1996 extended the notion to the setting of almost contact metric manifolds [8]. Further modifications regarding semislant submanifolds were introduced by N.Papaghiuc [10]. These submanifolds are a generalized version of CR-submanifolds. After him, J.L.Cabrerizo et.al. ([2]) extended the study of semislant submanifolds of Kaehler manifold to the setting of Sasakian manifolds. The idea of hemislant submanifold was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds in [3]. Recently, B.Sahin extended the study of pseudo-slant submanifolds in Kaehler setting for their warped product. Totally umbilical proper slant submanifold of a Kaehler manifold has also been discussed in [12].

This paper contains the analysis about slant and pseudo-slant submanifolds of an (ϵ)-para Sasakian manifold. Section (1) is introductory. Section (2) gives us a view of (ϵ)-para Sasakian manifold. In section (3) we have obtained some results on a totally umbilical proper slant submanifold M of an (ϵ)-para Sasakian manifold. Finally, in section (4) we have derived some conditions for the integrability of the distributions on the hemislant submanifolds of an (ϵ)-para Sasakian manifold.

§2. Preliminaries

Let $\tilde{M}$ be an n-dimensional almost paracontact manifold [11] endowed with an almost paracontact structure (ϕ, ξ, η) consisting of a tensor field ϕ of type $(1,1)$, a structure vector field ξ and 1-form η satisfying:

$$\phi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\phi(\xi) = 0 \tag{2.3}$$

and

$$\eta \circ \phi = 0 \tag{2.4}$$

for any vector field $X, Y \in \tilde{M}$. A semi-Riemannian metric [9] on a manifold $\tilde{M}$, is a non-degenerate symmetric tensor field g of type $(0,2)$. If this metric is of index 1 then it is called Lorentzian metric ([1]). Let g be semi-Riemannian metric with index 1 in an n-dimensional almost paracontact manifold $\tilde{M}$ such that,

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.5}$$

where $\epsilon = +1$ or -1. Then $\tilde{M}$ is called an almost paracontact metric manifold associated with an (ϵ)-almost paracontact metric structure $(\phi, \xi, \eta, g, \epsilon)$([**?**]). In case, if index $(g) = 1$ then an

(ϵ)-almost paracontact metric manifold is defined as a Lorentzian paracontact manifold and if the metric is positive definite, then an (ϵ)-almost paracontact metric manifold is the usual almost paracontact metric manifold [**?**].

The condition (2.5) is equivalent to

$$g(X, \phi Y) = g(\phi X, Y) \tag{2.6}$$

equipped with

$$g(X, \xi) = \epsilon\eta(X). \tag{2.7}$$

From (2.7), it can be easily observed that

$$g(\xi, \xi) = \epsilon, \tag{2.8}$$

i.e. structure vector field ξ is never lightlike. We define

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.9}$$

and we can obtain

$$\Phi(X, \xi) = 0. \tag{2.10}$$

From (2.9), we can also calculate

$$(\nabla_X \Phi)(Y, Z) = g(\nabla_X \phi)(Y, Z) = (\nabla_X \Phi)(Z, Y). \tag{2.11}$$

An (ϵ)-almost paracontact metric manifold $\tilde{M}$ satisfying

$$2\Phi(X, Y) = (\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) \tag{2.12}$$

$\forall\ X, Y \in TM$, then $\tilde{M}$ is called an (ϵ)-paracontact metric manifold ([15]).

An (ϵ)-almost paracontact metric structure $(\phi, \xi, \eta, g, \epsilon)$ is called an (ϵ)-S-paracontact metric structure if

$$\tilde{\nabla}_X \xi = \epsilon\phi X \tag{2.13}$$

for $\forall\ X \in T\tilde{M}$. A manifold endowed with an (ϵ)-S-paracontact metric structure is called an (ϵ)-S-paracontact metric manifold. Equation (2.13) can be written as

$$\Phi(X, Y) = g(\phi X, Y) = \epsilon g(\tilde{\nabla}_X \xi, Y) = (\nabla_X \eta)(Y) \tag{2.14}$$

for $\forall\ X, Y \in TM$.

An (ϵ)-almost paracontact metric structure is called an (ϵ)-para Sasakian structure if the following relation holds

$$(\tilde{\nabla}_X \phi)(Y) = -g(\phi X, \phi Y)\xi - \epsilon\eta(Y)\phi^2 X, \tag{2.15}$$

where $\tilde{\nabla}$ is the Levi-Civita connection with respect to g on $\tilde{M}$. A manifold equipped with an

(ϵ)-para Sasakian structure is called an (ϵ)-para Sasakian manifold.

From the definition of contact CR-submanifolds of an (ϵ)-paracontact Sasakian manifold we have

Definition 2.1([7]) *An n-dimensional Riemannian submanifold M of an (ϵ)-para Sasakian manifold $\tilde{M}$ is called a contact CR-submanifold if*

(i) ξ is tangent to M;

(ii) there exists on M a differentiable distribution $D : x \longrightarrow D_x \subset T_x(M)$, such that D_x is invariant under ϕ; i.e., $\phi D_x \subset D_x$, for each $x \in M$ and the orthogonal complementary distribution $D^\perp : x \longrightarrow D_x^\perp \subset T_x{}^\perp(M)$ of the distribution D on M is totally real; i.e., $\phi D_x^\perp \subset T_x^\perp(M)$, where $T_x(M)$ and $T_x^\perp(M)$ are the tangent space and the normal space of M at x. D (resp.$D^\perp$) is the horizontal (resp. vertical) distribution. The contact CR-submanifold of an (ϵ)-para Sasakian manifold is called ξ-horizontal (resp. ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x^\perp$) for each $x \in M$.

Let TM and $T^\perp M$ be the Lie algebras of vector fields tangential to M and normal to M respectively. h and A denote the second fundamental form and the shape operator of the immersion of M into $\tilde{M}$ respectively. If ∇ is the induced connection on M, the Gauss and Weingarten formulae of M into $\tilde{M}$ are characterized by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.16}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp N \tag{2.17}$$

for any X, Y in TM and N in $T^\perp M$. $\nabla^\perp$ is the connection on normal bundle and A_N is the Weingarten endomorphism associated with N by

$$g(A_V X, Y) = g(h(X,Y), V). \tag{2.18}$$

For any $x \in M$ and $X \in T_x M$ we decompose it as

$$\phi X = TX + NX, \tag{2.19}$$

where $TX \in T_x M$ and $NX \in T_x^\perp M$.

Similarly for $V \in T_x^\perp M$ we know

$$\phi V = tV + nV, \tag{2.20}$$

where tV (resp.nV) is vertical (resp. normal) component of ϕV.

Now, for any $X, Y \in TM$, comparing the tangential and normal parts of $(\tilde{\nabla}_X \phi)Y$ by $\mathbf{P}_X Y$ and $\mathbf{Q}_X Y$ respectively. After having some brief calculation, we obtain

$$\mathbf{P}_X Y = (\tilde{\nabla}_X T)Y - A_{NY} X - th(X,Y), \tag{2.21}$$

$$\mathbf{Q}_X Y = (\tilde{\nabla}_X N)Y + h(X, TY) - nh(X,Y) \tag{2.22}$$

for any $X, Y \in TM$.

Again for any $V \in T^{\perp}M$, denoting tangential and normal parts of $(\tilde{\nabla}_X \phi)V$ by $\mathbf{P}_X V$ and $\mathbf{Q}_X V$ respectively, we have

$$\mathbf{P}_X V = (\tilde{\nabla}_X t)V - A_{nV}X + TA_V X, \tag{2.23}$$

$$\mathbf{Q}_X V = (\tilde{\nabla}_X n)V + h(tV, X) + NA_V X, \tag{2.24}$$

where the covariant derivatives of T, N, t and n are given by

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.25}$$

$$(\tilde{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{2.26}$$

$$(\tilde{\nabla}_X t)V = \nabla_X tV - t\nabla^{\perp}_X V, \tag{2.27}$$

$$(\tilde{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V \quad \forall \ \ X, Y \in TM, V \in T^{\perp}M. \tag{2.28}$$

A submanifold M of an almost contact metric manifold $\tilde{M}$ is called totally umbilical if

$$h(X, Y) = g(X, Y)H \tag{2.29}$$

for any $X, Y \in \Gamma(TM)$, where H is the mean curvature. A submanifold M is said to be totally geodesic if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is minimal if $H = 0$ on M.

§3. Slant Submanifolds

The slant submanifold of a para contact Lorentzian manifold were first defined by [5]. Hereafter, for a submanifold M of an almost contact manifold, authors in [6] assumed that the structure vector field ξ is tangential to the submanifold M, whence the tangent bundle TM can be decomposed as

$$(a) TM = D \bigoplus <\xi>,$$

where the orthogonal complementary distribution D of $<\xi>$ is known as the slant distribution on M and $<\xi>$ is the 1-dimensional distribution on M spanned by the structure vector field ξ, and they also assumed that $g(X, X) \geq 0 \ \ \forall X \ \in TM \backslash \xi$. Let M be an immersed submanifold of $\tilde{M}$. For any $x \in M$ and $X \in T_x M$, if the vectors X and ξ are linearly independent, then the angle $\theta(X) \in [0, \pi/2]$ between ϕX and $T_x M$ is well defined, if $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_x M$, then M is slant in $\tilde{M}$. The constant angle $\theta(X)$ is then called the slant angle of M in $\tilde{M}$ by [5] and which in short we denote by $Sla(M)$. If μ is ϕ-invariant of the normal bundle $T^{\perp}M$, then

$$(b) T^{\perp}M = FTM \oplus <\mu>.$$

Defining the endomorphism $P : TM \longrightarrow TM$, whose square P^2 will be denoted by Q. Then the tensor fields on M of type $(1, 1)$ determined by these endomorphism will be denoted by the same letters, respectively P and Q.

It is proved the following theorem in [6]:

For a proper slant submanifold M of an (ϵ)-para Sasakian manifold $\tilde{M}$ with slant angle θ, then

$$QX = \lambda(X - \eta(X)\xi). \tag{3.1}$$

From this theorem we can state our next theorem,

Theorem 3.1 *Let M be a submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$ such that $\xi \in TM$. Then, M is slant iff there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda(I - \eta \bigotimes \xi). \tag{3.2}$$

Furthermore, in such case, if θ is the slant angle of M, then $\lambda = cos^2\theta$. Hence for a slant manifold we have

$$g(TX, TX) = cos^2\theta(g(X,Y) - \epsilon\eta(X)\eta(Y)), \tag{3.3}$$

$$g(NX, NY) = sin^2\theta(g(X,Y) - \epsilon\eta(X)\eta(Y)) \tag{3.4}$$

for $\forall$ X, $Y \in TM$.

Proof Follows from [5]. □

Assuming M to be totally umbilical proper slant submanifold of an (ϵ)-para Sasakian manifold, we can obtain the following theorem.

Theorem 3.2 *Let M be a totally umbilical proper slant submanifold of an (ϵ)-paracontact Sasakian manifold $\tilde{M}$, then for any $X \in TM$ following conditions are equivalent:*

(i) $H \in \mu$;
(ii) $g(\nabla_{TX}\xi, X) = \epsilon[||X||^2 - \eta^2(X)]$.

Proof For any $X \in TM$ we know $h(X,TX) = g(X,TX)H = 0$. Then from (2.16) and (2.17) and the structure equation of (ϵ)-para Sasakian manifold for any vector field $X \in TM$, we calculate

$$\begin{aligned} 0 &= \phi(\nabla_X X + h(X,X)) - \nabla_X TX + A_{NX}X - \nabla^{\perp}_X NX \\ &\quad -g(\phi X, \phi X)\xi - \epsilon\eta(X)\phi^2 X. \end{aligned} \tag{3.5}$$

After using (2.19), and on comparing the tangential component we obtain

$$\begin{aligned} 0 &= ZT\nabla_X X - \nabla_X TX + th(X,X) + A_{NX}X - g(X,X)\xi \\ &\quad +2\epsilon\eta^2(X)\xi - \epsilon\eta(X)X. \end{aligned} \tag{3.6}$$

As M is totally umbilical submanifold then the term $A_{NX}X$ becomes $Xg(H, NX)$, so using this fact above equation takes the form

$$\begin{aligned} 0 &= T\nabla_X X - \nabla_X TX + th(X,X) + Xg(NX,H) + g(X,X)tH \\ &\quad + g(X,X)\xi - 2\epsilon\eta^2(X)\xi + \epsilon\eta(X)X. \end{aligned} \tag{3.7}$$

If $H \in \mu$ then from (3.6) we get

$$T\nabla_X X - \nabla_X TX = -||X||^2\xi + 2\epsilon\eta(X)[2\eta(X)\xi - X]. \tag{3.8}$$

Taking the inner product in (3.7) by ξ we obtain

$$g(\nabla_X TX, \xi) = \eta^2(X) - \epsilon||X||^2. \tag{3.9}$$

Replacing X by TX, we derive

$$g(\nabla_{TX} T^2X, \xi) = \eta^2(TX) - \epsilon||TX||^2. \tag{3.10}$$

Then from equation (3.3) and (3.4) we calculate

$$cos^2\theta g(X, \nabla_{TX}\xi) - cos^2\theta\eta(X)g(\xi, \nabla_{TX}\xi) = -cos^2\theta[\epsilon||X||^2 - \eta^2(X)]. \tag{3.11}$$

Therefore we can conclude that

$$g(X, \nabla_{TX}\xi) - \eta(X)g(\xi, \nabla_{TX}\xi) = \epsilon||X||^2 - \eta^2(X). \tag{3.12}$$

Now we know that $g(\xi,\xi) = \epsilon$. Taking the covariant derivative of this equation with respect to TX for any $X \in TM$, we obtain

$$g(\nabla_{TX}\xi, \xi) + g(\xi, \nabla_{TX}\xi) = 0, \tag{3.13}$$

which implies $g(\nabla_{TX}\xi, \xi) = 0$. Hence (3.8) becomes

$$g(X, \nabla_{TX}\xi) = \epsilon||X||^2 - \eta^2(X). \tag{3.14}$$

This proves part (ii) of the theorem. If (3.9) holds then equation (3.6) implies $H \in \mu$. This proves theorem (3.2).

Now if $\epsilon||X||^2 - \eta^2(X) = 0$, then from (3.9), we conclude

$$g(X, \nabla_{TX}\xi) = 0. \tag{3.15}$$

Replacing X by TX we have by using (3.3), we get

$$g(TX, \nabla_{T^2X}\xi) = g(\nabla_{cos^2\theta(X-\eta(X)\xi)}\xi, TX) = 0. \tag{3.16}$$

Then the above equation becomes

$$cos^2\theta g(\nabla_X\xi, TX) + cos^2\theta\eta(X)g(\nabla_\xi\xi, TX) = 0. \tag{3.17}$$

From the structure equation (2.4) we have $\nabla_\xi\xi = 0$. Thus we can write

$$cos^2\theta g(\nabla_X\xi, TX) = 0. \tag{3.18}$$

Thus from equation (3.10) we get either M is an anti-invariant submanifold or $\nabla_X\xi = 0$ i.e. ξ is a Killing vector field on M or M is trivial. If ξ is not Killing then we can take at least two linearly independent vectors X and TX to span D_θ i.e. the $dimM \geq 3$.

From above discussion we can conclude the following theorem.

Theorem 3.3 *Let M be a totally umbilical slant submanifold of an ϵ-para Sasakian manifold $\tilde{M}$ such that $\epsilon||X||^2 = \eta^2(X)$ on M then one of the following statements is true:*

(i) $H \in \Gamma(\mu)$;

(ii) M is an anti-invariant submanifold;

(iii) If M is a proper slant submanifold then $dimM \geq 3$;

(iv) M is trivial;

(v) ξ is a Killing vector field on M.

Next we prove

Theorem 3.4 *A totally umbilical proper slant submanifold M of an (ϵ)-para Sasakian manifold $\tilde{M}$ is totally geodesic if $\nabla^\perp_X H \in \Gamma(\mu)$ for any $X \in TM$.*

Proof As $\tilde{M}$ is an (ϵ)-paracontact Sasakian manifold we have

$$(\tilde{\nabla}_X\phi)Y = \tilde{\nabla}_X\phi Y - \phi\tilde{\nabla}_X Y. \tag{3.19}$$

From the fact that $\phi Y = TY + NY$ and $\tilde{M}$ is an (ϵ)-paracontact Sasakian manifold we infer

$$\begin{aligned}\tilde{\nabla}_X TY + \tilde{\nabla}_X NY &= T\nabla_X Y + N\nabla_X Y + \phi h(X,Y) - g(\phi X, \phi Y)\xi \\ &\quad -\epsilon\eta(Y)\phi^2 X.\end{aligned} \tag{3.20}$$

Using (2.25), (2.26) and (2.29) we obtain

$$\begin{aligned}\nabla_X TY + h(X,TY) - A_{NY}X &+ \nabla^\perp_X NY = T\nabla_X Y + N\nabla_X Y \\ &+g(X,Y)\phi H - g(\phi X, \phi Y)\xi - \epsilon\eta(Y)\phi^2 X.\end{aligned} \tag{3.21}$$

Taking inner product with ϕH and using the fact that $H \in \Gamma(\mu)$, from (2.5) and (2.29) we get

$$g(X,TY)g(H,\phi H)+g(\nabla^{\perp}_{X}NY,\phi H) = g(X,Y)||H||^2 - g(\phi X,\phi Y)g(\phi H,\xi)-\epsilon\eta(Y)g(\phi^2X,\phi H). \quad (3.22)$$

Now we consider

$$\tilde{\nabla}_X\phi H=\phi\tilde{\nabla}_X H+(\tilde{\nabla}_X\phi)H. \quad (3.23)$$

Using the covariant derivative of ∇

$$-A_{\phi H}X+\nabla^{\perp}_{X}\phi H = -TA_HX-NA_HX+t\nabla^{\perp}_{X}H + n\nabla^{\perp}_{X}H-g(\phi X,\phi H)\xi-\epsilon\eta(H)\phi^2X. \quad (3.24)$$

Taking inner product with NY, for any $Y\in\Gamma(TM)$ and as the submanifold considered is always tangent to ξ we obtain

$$g(\nabla^{\perp}_{X}\phi H,NY)=-g(NA_HX,NY)+g(n\nabla^{\perp}_{X}H,NY)-\epsilon\eta(H)g(\phi^2X,NY). \quad (3.25)$$

Since $n\nabla^{\perp}_{X}H\in\Gamma(\mu)$, then by (3.5) the above equation takes the form

$$g(\nabla^{\perp}_{X}\phi H,NY) = -sin^2\theta\ [g(A_HX,Y)-\epsilon\eta(A_HX)\eta(Y)] - \epsilon\eta(H)g(\phi^2X,NY). \quad (3.26)$$

Using (2.17), (2.18) and (2.29) and having some brief calculations we obtain

$$g(\tilde{\nabla}_X\phi H,NY) = -sin^2\theta[g(X,Y)-\epsilon\eta(X)\eta(Y)]\ ||H||^2 - \epsilon\eta(H)g(\phi^2X,NY). \quad (3.27)$$

The above equation can be written as

$$g(\tilde{\nabla}_X NY,\phi H) = sin^2\theta[g(X,Y)-\epsilon\eta(X)\eta(Y)]\ ||H||^2 + \epsilon\eta(H)g(\phi^2X,NY). \quad (3.28)$$

Again using the fact that $H\in\Gamma(\mu)$ and by Weingarten formula we have

$$g(\nabla^{\perp}_{X}NY,\phi H) = sin^2\theta[g(X,Y)-\epsilon\eta(X)\eta(Y)]||H||^2 + \epsilon\eta(H)g(\phi^2X,NY). \quad (3.29)$$

From (3.14) and (3.21) we get

$$sin^2\theta[g(X,Y)-\epsilon\eta(X)\eta(Y)]||H||^2+\epsilon\eta(H)g(\phi^2X,NY) = g(X,Y)||H||^2 - \epsilon\eta(Y)g(\phi^2X,\phi H). \quad (3.30)$$

The equation (3.22) has a solution if $H=0$. Hence M is totally geodesic in $\tilde{M}$. □

§4. Hemislant Submanifolds

A.Carriazo [3] introduced hemi-slant submanifolds as a special case of bislant submanifolds and he called them pseudo-slant submanifolds. This section deals with a special case of hemislant submanifolds which are totally umbilical.

Definition 4.1([5,16]) *A submanifold M of an (ϵ)-para Sasakian manifold $\tilde{M}$ is said to be a hemislant submanifold if there exist two orthogonal complementary distributions D_1 and D_2 satisfying the following properties:*

(*i*) $TM = D_1 \bigoplus D_2 \bigoplus < \xi >$;
(*ii*) D_1 *is a slant distribution with slant angle* $\theta \neq \pi/2$;
(*iii*) D_2 *is totally real i.e.,* $\phi D_2 \subseteq T^{\perp}M$.

A hemislant submanifold is called proper hemislant submanifold if $\theta \neq 0, \frac{\pi}{2}$. Further if μ is ϕ-invariant subspace of the normal bundle $T^{\perp}M$, then for pseudo-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = ND_1 \bigoplus ND_2 \bigoplus < \mu > .$$

In this section we will derive some of the integrability conditions of the involved distributions of a hemislant submanifold, which play a crucial role from a geometrical point of view.

Theorem 4.1 *Let M be a hemislant submanifold of an (ϵ)-paracontact Sasakian manifold $\tilde{M}$ then $g([X,Y],\xi) = 0$ for any $X, Y \in D_1 \oplus D_2$.*

Proof We know

$$g(X, \phi Y) = g(Y, \phi X),\ \nabla_X \xi = \epsilon \phi X. \tag{4.1}$$

Taking inner product with Y we obtain

$$g(\nabla_X \xi, Y) = \epsilon g(\phi X, Y). \tag{4.2}$$

We can write

$$g(\nabla_X Y, \xi) = -\epsilon g(\phi X, Y). \tag{4.3}$$

Interchanging X, Y we get

$$g(\nabla_Y X, \xi) = -\epsilon g(\phi Y, X). \tag{4.4}$$

Subtracting equations (4.3) and (4.4) and using (4.1) we have

$$g([X,Y], \xi) = 0. \tag{4.5}$$

This completes the proof. □

From Theorem (4.1) we can deduce the following corollaries.

Corollary 4.1 *The distribution $D_1 \oplus D_2$ on a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$ is integrable.*

Corollary 4.2 *The distribution D_1 and D_2 on a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$ is integrable.*

Proposition 4.1 *Let M be a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$, then for any Z, $W \in D_2$, the anti-invariant distribution $D_2 \oplus <\xi>$ is integrable iff*

$$A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z - \epsilon\eta(W)Z + \epsilon\eta(Z)W = 0.$$

Proof For any $Z, W \in D_2 \bigoplus \xi$ we know

$$\tilde{\nabla}_Z\phi W = (\tilde{\nabla}_Z\phi W) + \phi\tilde{\nabla}_Z W = (\tilde{\nabla}_Z\phi W) + \phi\nabla_Z W + \phi h(Z, W). \tag{4.6}$$

Using (2.16) and (2.17) we have

$$-A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W = (\tilde{\nabla}_Z\phi W) + \phi\tilde{\nabla}_Z W = (\tilde{\nabla}_Z\phi W) + \phi\nabla_Z W + \phi h(Z, W). \tag{4.6}$$

Interchanging Z and W, we obtain

$$-A_{\phi Z}W + \nabla^{\perp}_{W}\phi Z = (\tilde{\nabla}_W\phi Z) + \phi\tilde{\nabla}_W Z = (\tilde{\nabla}_W\phi Z) + \phi\nabla_W Z + \phi h(W, Z). \tag{4.8}$$

Then from (4.7) and (4.8) we calculate

$$A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z = (\tilde{\nabla}_Z\phi W) - (\tilde{\nabla}_W\phi Z) + \phi[Z, W]. \tag{4.9}$$

From (2.15) we obtain

$$A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z = \phi[Z, W] - \epsilon\eta(W)\phi^2 Z + \epsilon\eta(Z)\phi^2 W. \tag{4.10}$$

Taking inner product with ϕX, for any $X \in D_1$ we obtain

$$\begin{aligned} &g(A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z, \phi X) \\ &= g(\phi[Z, W], \phi X) - \epsilon\eta(W)g(\phi^2 Z, \phi X) + \epsilon\eta(Z)g(\phi^2 W, \phi X). \end{aligned} \tag{4.11}$$

Thus from (2.5) the above equation takes the form

$$\begin{aligned} g(\phi[Z, W], \phi X) \;\; = \;\; & g(A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z - \epsilon\eta(W)Z \\ & + \epsilon\eta(Z)W, \phi X). \end{aligned} \tag{4.12}$$

The distribution $D_2 \bigoplus <\xi>$ is integrable iff the right hand side of the above equation is zero.

Proposition 4.2 *Let M be a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$, then the anti-invariant distribution $D_1 \bigoplus <\xi>$ is integrable iff*

$$h(Y,TX)+\nabla^{\perp}_{Y}NX-h(X,TY)-\nabla^{\perp}_{X}NY\in\mu$$

for all $X,Y\in D_1 \bigoplus <\xi>$.

Proof For any $X,Y\in D_1\bigoplus\xi$, we have

$$\begin{aligned}\phi[X,Y] &= \phi[\tilde{\nabla}_Y X-\tilde{\nabla}_X Y]=\tilde{\nabla}_Y TX+\tilde{\nabla}_Y NX-\tilde{\nabla}_X TY\\ &\quad -\tilde{\nabla}_X NY-\epsilon\eta(Y)\phi^2X+\epsilon\eta(X)\phi^2Y.\end{aligned}\tag{4.13}$$

$$\begin{aligned}\phi[X,Y] &= \nabla_Y TX+h(Y,TX)-A_{NX}Y+\nabla^{\perp}_{Y}NX-\nabla_X TY-h(X,TY)\\ &\quad +A_{NY}X-\nabla^{\perp}_{X}NY-\epsilon\eta(Y)\phi^2X+\epsilon\eta(X)\phi^2Y.\end{aligned}\tag{4.14}$$

Taking the product with ϕZ, for any $Z\in D_2$, we obtain on solving

$$\begin{aligned}g(\phi[X,Y],\phi Z) &= g(h(Y,TX),\phi Z)+g(\nabla^{\perp}_{Y}NX,\phi Z)-g(h(X,TY),\phi Z)\\ &\quad -g(\nabla^{\perp}_{X}NY,\phi Z)-\epsilon\eta(Y)g(X-\eta(X)\xi,\phi Z)\\ &\quad +\epsilon\eta(Y)g(Y-\eta(Y)\xi,\phi Z).\end{aligned}\tag{4.15}$$

$$g([X,Y],Z)=g(h(Y,TX)+\nabla^{\perp}_{Y}NX-h(X,TY)-\nabla^{\perp}_{X}NY,\phi Z).\tag{4.16}$$

Thus our assertion follows from equation (4.16). □

Theorem 4.2 *Let M be a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$, then at least one of the following statements is true:*

(i) *$dim D_2=1$;*

(ii) *$H\in\mu$;*

(iii) *M is proper slant.*

Proof For any $Z,W\in TM$, we have

$$(\tilde{\nabla}_Z\phi W)+(\tilde{\nabla}_W\phi Z)=-2g(\phi Z,\phi W)\xi-\epsilon\eta(Z)\phi^2W+\epsilon\eta(W)\phi^2Z.\tag{4.17}$$

If we assume the vector fields $Z,W\in D_2$, then the above equation reduces to

$$(\tilde{\nabla}_Z\phi)W+(\tilde{\nabla}_W\phi)Z+2g(\phi Z,\phi W)\xi=0.\tag{4.18}$$

In particular if we take the above equation for one vector $Z\in D_2$, i.e

$$(\tilde{\nabla}_Z\phi)Z+g(\phi Z,\phi Z)\xi=0.\tag{4.19}$$

Again using (2.6) we have
$$(\tilde{\nabla}_Z\phi)Z + ||Z||^2\xi = 0. \tag{4.20}$$

Therefore the tangential and normal components of the above equation are $\mathbf{P}_Z Z = ||Z||^2\xi$ and $\mathbf{Q}_Z Z = 0$ respectively. From (2.21) and tangential component of (4.20) we get

$$(\tilde{\nabla}_Z T)Z = -T\nabla_Z Z = A_{NZ}Z + th(Z,Z) - ||Z||^2\xi. \tag{4.21}$$

Taking the product with $W \in D_2$, we get from (2.18)

$$g(T\nabla_Z Z, W) = g(h(Z,W), NZ) + g(th(Z,Z), W). \tag{4.22}$$

Using the fact that M is totally umbilical submanifold and for any $W \in D_2$, then the above equation takes the form

$$g(Z,W)g(H,NZ) + ||Z||^2 g(tH,W) = -g(T\nabla_Z Z, W) = 0. \tag{4.23}$$

Thus the equation (4.10) has a solution if either dim$D_2 = 1$ or $H \in \mu$ or $D_2 = 0$, i.e. M is proper slant. □

From the above conclusions we can obtain the following theorem

Theorem 4.3 *Let M be a totally umbilical hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$. Then at least one of the following statements is true:*

(i) *M is an anti-invariant submanifold;*
(ii) $g(\nabla_{TX}\xi, X) = \epsilon[||X||^2 - \eta^2(X)]$;
(iii) *M is totally geodesic submanifold;*
(iv) *dim D_2=1;*
(v) *M is a proper slant submanifold.*

Proof If $H \neq 0$ then from equation (3.19), we can conclude that the slant distribution D_1 =0 i.e. M is anti-invariant submanifold which is case (i). If $D_1 \neq 0$ and $H \in \mu$, then from theorem (3.2) we get (ii) for any $X \in TM$. Again if $H \in \mu$ then by theorem (3.4), M is totally geodesic. Lastly if $H \notin \mu$, then the equation (4.23) has a solution if either dim$D_2 = 1$ or $D_2 = 0$. Hence the theorem follows. □

Next we have the following theorem.

Theorem 4.4 *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in TM$. Then M is a pseudo-slant submanifold iff there exists a constant $\lambda \in (0,1]$ such that*

(i) D=$\{X \in TM | T^2X = -\lambda X\}$ is a distribution on M;
(ii) *For case $X \in TM$, orthogonal to D, $TX = 0$.*

Furthermore in this case $\lambda = cos^2\theta$, where θ denotes the slant angle of D.

Proof: Follows from [11]. □

Again we prove

Theorem 4.5 *Let M be a hemi-slant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$. Then $\nabla Q = 0$ iff M is an anti-invariant submanifold.*

Proof Considering the distribution $D_2 \bigoplus < \xi >$, from (4.4) we can write

$$T^2 X = \lambda(X - \eta(X)\xi). \tag{4.24}$$

Denoting the slant angle of M by θ. Then, replacing X by $\nabla_X Y$, we obtain from (4.24)

$$Q\nabla_X Y = cos^2\theta[\nabla_X Y - \eta(\nabla_X Y)\xi]. \tag{4.25}$$

for any $X, Y \in D_2 \bigoplus < \xi >$. After taking the covariant derivative of equation (4.24) we have

$$\nabla_X QY = cos^2\theta[\nabla_X Y - \eta(\nabla_X Y)\xi) - g(Y, \nabla_X \xi)\xi - \eta(Y)\nabla_X \xi]. \tag{4.26}$$

Adding equations (4.25) and (4.26) we obtain

$$\begin{aligned}(\tilde{\nabla}_X Q)Y &= cos^2\theta[\nabla_X Y - \eta(\nabla_X Y)\xi) + g(Y, \epsilon TX)\xi + \eta(Y)\epsilon TX] \\ &\quad -cos^2\theta\nabla_X Y + cos^2\theta\eta(\nabla_X Y)\xi \qquad (4.27)\end{aligned}$$

for any $X, Y \in D_2 \bigoplus < \xi >$.

Here we observe that $g(Y, TX)\xi + TX\eta(Y) \neq 0$. Therefore $(\tilde{\nabla}_X Q) = 0$ iff $\theta = \frac{\pi}{2}$ holds in $D_2 \bigoplus < \xi >$. Again D_1 is anti-invariant by definition. Thus, the theorem follows. □

§5. An Example

Let us give an example of a three dimensional submanifold of (ϵ)-paracontact Sasakian manifold which is pseudo slant so as to verify the above results. Let $\mathbf{R}^3$ be a 3-dimensional Euclidean space with a rectangular coordinates (x, y, z), we put

$$\eta = dy \qquad \xi = \frac{\partial}{\partial y}. \tag{5.1}$$

We define the $(1,1)$ tensor ϕ as:

$$\phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial z} \quad \phi(\frac{\partial}{\partial z}) = \frac{\partial}{\partial x} \quad \phi(\frac{\partial}{\partial y}) = 0 \tag{5.2}$$

and we define the Riemannian metric g as

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we can easily see that (ϕ, ξ, η, g) is an (ϵ)-paracontact Sasakian manifold on $\mathbf{R}^3$.

The vector fields $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$, forms a frame of TM. We have

$$\phi e_1 = e_3, \quad \phi e_2 = 0, \quad \phi e_3 = e_1.$$

Let $D_1 =< e_2 >$, $D_2 =< e_1 >$ and $\xi =< e_3 >$. We know

$$\cos\angle(\phi X, Y) = \frac{g(\phi X, Y)}{|\phi X||Y|}.$$

Suppose $X \in D_1$ and $Y \in TM$. Then we can write $X = Ke_2$ where K is a scalar and $Y = re_1 + se_2 + te_3$ where r, s, t are scalars. Notice that

$$g(\phi X, Y) = g(\phi e_2, re_1 + se_2 + te_3) = rg(0, e_1) + sg(0, e_2) + tg(0, e_3) = 0.$$

Hence $cos\angle(\phi X, Y) = 0$ implies $\theta = \frac{\pi}{2}$ Hence the distribution D_1 is anti-invariant.

Again let us assume $U \in D_1$ and $V \in TM$. Then we can write $U = ae_1$, where a is a scalar and $V = ke_1 + le_2 + me_3$ where k, l, m are scalars. Using the formula above we get that

$$g(\phi U, V) = g(\phi(ae_1), ke_1 + le_2 + me_3) = am.$$

Hence $cos\angle(\phi U, V)$=constant. So we have obtained that the distribution D_2 is slant.

In this case, the distribution D_1 is anti-invariant while D_2 is slant. Hence the submanifold under consideration is hemislant.

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A Study on Set-Valuations of Signed Graphs

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Abstract: Let X be a non-empty ground set and $\mathcal{P}(X)$ be its power set. A set-labeling (or a set-valuation) of a graph G is an injective set-valued function $f : V(G) \to \mathcal{P}(X)$ such that the induced function $f^{\oplus} : E(G) \to \mathcal{P}(X)$ is defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$, where $f(u) \oplus f(v)$ is the symmetric difference of the sets $f(u)$ and $f(v)$. A graph which admits a set-labeling is known to be a set-labeled graph. A set-labeling f of a graph G is said to be a set-indexer of G if the associated function $f^{\oplus}$ is also injective. In this paper, we define the notion of set-valuations of signed graphs and discuss certain properties of signed graphs which admits certain types of set-valuations.

Key Words: Signed graphs, balanced signed graphs, clustering of signed graphs, set-labeled signed graphs.

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§1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [4, 8, 13] and and for the topics in signed graphs we refer to [14, 15]. Unless mentioned otherwise, all graphs considered here are simple, finite, undirected and have no isolated vertices.

1.1 An Overview of Set-Valued Graphs

Let X be a non-empty set and $\mathcal{P}(X)$ be its power set. A *set-labeling* (or a *set-valuation*) of a graph G is an injective function $f : V(G) \to \mathcal{P}(X)$ such that the induced function $f^{\oplus} : E(G) \to \mathcal{P}(X)$ is defined by $f^{\oplus}(uv) = f(u) \oplus f(v)\ \forall\ uv \in E(G)$, where $\oplus$ is the symmetric difference of two sets. A graph G which admits a set-labeling is called an *set-labeled graph* (or a set-valued graph)(see [1]).

A *set-indexer* of a graph G is an injective function $f : V(G) \to \mathcal{P}(X)$ such that the induced function $f^{\oplus} : E(G) \to \mathcal{P}(X)$ is also injective. A graph G which admits a set-indexer is called a *set-indexed graph* (see [1]).

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Several types of set-valuations of graphs have been introduced in later studies and their properties and structural characteristics of such set-valued graphs have been done extensively.

1.2 Preliminaries on Signed Graphs

An edge of a graph G having only one end vertex is known as a *half edge* of G and an edge of G without end vertices is called *loose edge* of G.

A *signed graph* (see [14, 15]), denoted by $\Sigma(G, \sigma)$, is a graph $G(V, E)$ together with a function $\sigma : E(G) \to \{+, -\}$ that assigns a sign, either $+$ or $-$, to each ordinary edge in G. The function σ is called the *signature* or *sign function* of §, which is defined on all edges except half edges and is required to be positive on free loops.

An edge e of a signed graph Σ is said to be a *positive edge* if $\sigma(e) = +$ and an edge $\sigma(e)$ of a signed graph Σ is said to be a *negative edge* if $\sigma(e) = -$. The set E^+ denotes the set of all positive edges in Σ and the set E^- denotes the set of negative edges in Σ.

A simple cycle (or path) of a signed graph Σ is said to be *balanced* (see [3, 9]) if the product of signs of its edges is $+$. A signed graph Σ is said to be a *balanced signed graph* if it contains no half edges and all of its simple cycles are balanced. It is to be noted that the number of all negative signed graph is balanced if and only if it is bipartite.

Balance or imbalance is the basic and the most important property of a signed graph. The following theorem, popularly known as *Harary's Balance Theorem*, establishes a criteria for balance in a signed graph.

Theorem 1.1([9]) *The following statements about a signed graph are equivalent.*

(*i*) *A signed graph Σ is balanced;*
(*ii*) *Σ has no half edges and there is a partition (V_1, V_2) of $V(\Sigma)$ such that $E^- = E(V_1, V_2)$;*
(*iii*) *Σ has no half edges and any two paths with the same end points have the same sign.*

Some balancing properties of certain types of signed graphs have been studied in [6, 7].

A signed graph Σ is said to be *clusterable* or *partitionable* (see [14, 15]) if its vertex set can be partitioned into subsets, called *clusters*, so that every positive edge joins the vertices within the same cluster and every negative edge joins the vertices in the different clusters. If $V(\Sigma)$ can be partitioned in to k subsets with the above mentioned conditions, then the signed graph Σ is said to be *k-clusterable*. In this paper, we study the 2-clusterability of signed graphs only.

Note that 2-clusterability always implies balance in a signed graph Σ. But, the converse need not be true. If all edges in Σ are positive edges, then Σ is balanced but not 2-clusterable.

In this paper, we introduce the notion of set-valuations of signed graphs and study the properties and characteristics of such signed graphs.

§2. Set-Labeled Signed Graphs

Motivated from the studies on set-valuations of signed digraphs in [2], and the studies on integer additive set-labeled signed graphs in [11], we define the notion of a set-labeling of a signed graph as follows.

Definition 2.1 *Let X be a non-empty set and let Σ be a signed graph, with corresponding underlying graph G and the signature σ. An injective function $f : V(\Sigma) \to \mathcal{P}(X)$ is said to be a set-labeling (or set-valuation) of Σ if f is a set-labeling of the underlying graph G and the signature of Σ is defined by $\sigma(uv) = (-1)^{|f(u)\oplus f(v)|}$. A signed graph Σ together with a set-labeling f is known as a set-labeled signed graph (or set valued signed graph) and is denoted by Σ_f.*

Definition 2.2 *A set-labeling f of a signed graph Σ is said to be a set-indexer of Σ if f is a set-indexer of the underlying graph G.*

If the context is clear, we can represent a set-valued signed graph or a set-indexed signed graph simply by Σ itself. In this section, we discuss the 2-clusterability and balance of set-valued signed graphs.

The following theorem establishes the existence of set-valuations for all signed graphs.

Theorem 2.3 *Every signed graph admits a set-labeling (and a set-indexer).*

Proof Let Σ be a signed graph whose vertex set is given by $V(\Sigma) = \{v_1, v_2, \cdots, v_n\}$. Let $X = \{1, 2, 3, \cdots, n\}$. Define a set-valued function $f : V(\Sigma) \to \mathcal{P}(X)$ such that $f(v_i) = \{i\}$, where $1 \leqslant i \leqslant n$. Clearly, f is an injective function. Then, $f^{\oplus}(v_i v_j) = \{i, j\}, \forall\ uv \in E(G)$. Note that $f^{\oplus}$ is also an injective function and hence f is a set-indexer of Σ. $\square$

We say that two sets are of *same parity* if they are simultaneously even or simultaneously odd. If two sets are not of same parity, then they are said to be the sets of *opposite parity*. The signature of an edge of a set-valued signed graph can be determined in terms of the set-labels of its end vertices, as described in the following theorem.

Theorem 2.4 *An edge e of a set-labeled signed graph is a positive edge if and only if the set-labels of its end vertices are of the same parity.*

Proof Let f be a set-labeling of a given signed graph Σ. Assume that, an edge $e = v_i v_j$ be a positive edge in Σ. Then, $|f(v_i) \oplus f(v_j)| = |f(v_i) - f(v_j)| + |f(v_j) - f(v_i)|$ is an even number. That is, $|f(v_i) - f(v_j)|$ and $|f(v_j) - f(v_i)|$ are simultaneously even or simultaneously odd. Hence, we need to consider the following cases.

Case 1. Assume that both $|f(v_i) - f(v_j)|$ and $|f(v_j) - f(v_i)|$ are even. That is, both $|f(v_i) - f(v_i) \cap f(v_j)|$ and $|f(v_j) - f(v_i) \cap f(v_j)|$ are even. Then, we have

Subcase 1.1 Let $|f(v_i)|$ be an even integer. Then, since $|f(v_i) - f(v_i) \cap f(v_j)| = |f(v_i)| - |f(v_i) \cap f(v_j)|$, we have $|f(v_i) \cap f(v_j)|$ must also be even. Hence, as $|f(v_j) - f(v_i) \cap f(v_j)| = |f(v_j)| - |f(v_i) \cap f(v_j)|$ is even, we have $|f(v_j)|$ is even.

Subcase 1.2 Let $|f(v_i)|$ be an odd integer. Then, since $|f(v_i) - f(v_i) \cap f(v_j)| = |f(v_i)| - |f(v_i) \cap f(v_j)|$ is even, we have $|f(v_i) \cap f(v_j)|$ must be odd. Hence, as $|f(v_j) - f(v_i) \cap f(v_j)| = |f(v_j)| - |f(v_i) \cap f(v_j)|$ is even, we have $|f(v_j)|$ is odd.

Case 2. Assume that both $|f(v_i) - f(v_j)|$ and $|f(v_j) - f(v_i)|$ are odd. That is, both $|f(v_i) - f(v_i) \cap f(v_j)|$ and $|f(v_j) - f(v_i) \cap f(v_j)|$ are odd. Then, we have

Subcase 2.1 Let $|f(v_i)|$ be an even integer. Then, since $|f(v_i) - f(v_i) \cap f(v_j)|$ is odd, we have $|f(v_i) \cap f(v_j)|$ must be odd. Hence, as $|f(v_j) - f(v_i) \cap f(v_j)|$ is odd, we have $|f(v_j)|$ is even.

Subcase 2.2 Let $|f(v_i)|$ be an odd integer. Then, since $|f(v_i) - f(v_i) \cap f(v_j)|$ is odd, we have $|f(v_i) \cap f(v_j)|$ must be even. Then, as $|f(v_j) - f(v_i) \cap f(v_j)| = |f(v_j)| - |f(v_i) \cap f(v_j)|$ is odd, we have $|f(v_j)|$ is odd. □

As a contrapositive of Theorem 2.4, we can prove the following theorem also.

Theorem 2.5 *An edge e of a set-labeled signed graph is a negative edge if and only if the set-labels of its end vertices are of the opposite parity.*

The following result is an immediate consequence of Theorems 2.4 and 2.5.

Corollary 2.6 *A set-valued signed graph Σ is balanced if and only if every cycle in Σ has even number of edges whose end vertices have opposite parity set-labels.*

Proof Note that the number of negative edges in any cycle of a balanced signed graph is even. Hence, the proof is immediate from Theorem 2.5. □

The following theorem discusses a necessary and sufficient condition for a set-valued signed graph to be 2-clusterable.

Theorem 2.7 *A set-valued signed graph is 2-clusterable if and only if at least two adjacent vertices in Σ have opposite parity set-labels.*

Proof First, assume that at least two adjacent vertices in the set-valued signed graph Σ have opposite parity set-labels. If $e = v_iv_j$ be an edge of Σ such that $f(v_i)$ and $f(v_j)$ are of opposite parity, then $\sigma(v_iv_j) = -$. Then, we can find (U_1, U_2) be a partition of $V(\Sigma)$ such that U_1 contains one end vertex of every negative edge and U_2 contains the other end vertex of every negative edge. Therefore, Σ is 2-clusterable.

Conversely, assume that Σ is 2-clusterable. Then, there exist two non-empty subsets U_1 and U_2 of $V(\Sigma)$ such that $U_1 \cup U_2 = V(\Sigma)$. Since Σ is a connected signed graph, at least one vertex in U_1 is adjacent some vertices in U_2 and vice versa. Let $e = v_iv_j$ be such an edge in Σ Since Σ is 2-clusterable, e is a negative edge and hence $f(v_i)$ and $f(v_j)$ are of opposite parity. This completes the proof. □

Theorem 2.8 *Let f be a set-indexer defined on a signed graph Σ whose underlined graph G is an Eulerian graph. If Σ is balanced, then*

$$\sum_{e \in E(\Sigma)} |f^{\oplus}(e)| \equiv 0 \pmod 2.$$

Proof Let the underlying graph G of Σ is Eulerian. Then, $G = \bigcup_{i=1}^{k} C_i$, where each C_i is a cycle such that C_i and C_j are edge-disjoint for $i \neq j$. Let E_i be the edge set of the cycle C_i.

Since f is a set-indexer of Σ, we have $f(E_i) \cap f(E_j) = \emptyset$, for $i \neq j$. Hence, we have

$$\sum_{e \in E(\S)} |f(e)| = \sum_{i=1}^{k} \sum_{e_i \in E_i} f(e_i). \tag{1}$$

Consider the cycle C_i. Let A_i be the set all positive edges and B_i be the set of all negative edges in the cycle C_i. Then, the set-labels of edges in A_i are of even parity and those of edges in B_i are of odd parity. That is, $|f^{\oplus}(e)| \equiv 0 \pmod 2$ for all $e \in A_i$ and hence we have

$$\sum_{e \in A_i} |f(e)| \equiv 0 \pmod 2. \tag{2}$$

Since Σ is balanced, the number of negative edges in C_i is even. Therefore, the number of elements in B_i must be even. That is, the number of edges having odd parity set-labels in C_i is even. Therefore, being a sum of even number of odd integers, we have

$$\sum_{e_i \in B_i} |f(e_i)| \equiv 0 \pmod 2. \tag{3}$$

From Equations (2) and (3), we have

$$\sum_{e \in E_i} |f(e)| = \sum_{e_i \in A_i} |f(e_i)| + \sum_{e_i \in B_i} |f(e_i)| \equiv 0 \pmod 2. \tag{4}$$

Therefore, by Equations (1) and (4), we can conclude that

$$\sum_{e \in E(\S)} |f^{\oplus}(e)| \equiv 0 \pmod 2. \qquad \square$$

From the above results, we infer the most important result on a set-valued signed graph as follows.

Theorem 2.9 *If a signed graph Σ admits a vertex set-labeling, then Σ is balanced.*

Proof Let Σ be a signed graph which admits a set-labeling. If all vertices of Σ have the same parity set-labels, then by Theorem 2.4, all edges of Σ are positive edges and hence Σ is balanced.

Next, let that Σ contains vertices with opposite parity set-labels. Let A_i be the set of all vertices with odd parity set-labels and B_i be the set of all even parity set-labels. First, assume v_i be a vertex in A_i whose adjacent vertices are in B_i. Then, v_i is one end vertex of some negative edges in Σ. If v_i is not in a cycle of Σ, then none of these negative edges will be a part in any cycle of Σ.

If v_i is an internal vertex of a cycle C, then it is adjacent to two vertices, say v_j and v_k, which are in B_i. Hence, the edges v_iv_j and v_iv_k are negative edges. If two vertices v_i and v_j are adjacent in the cycle C, then v_i is adjacent to one more vertex, say v_k and the vertex v_j is also adjacent to one more vertex v_l and in the cycle C, the edges v_iv_k and v_jv_l are negative edges and the edge v_iv_j is a positive edge. If the vertices v_i and v_j are not adjacent, then also each of them induce two negative, which may not be distinct always. However, in each case the number of negative edges will be even. This condition can be verified in all cases when any number element of A_i are the vertices of any cycle C in Σ. Hence, the umber of negative edges in any cycle of a set-labeled signed graph is even. Hence, Σ is balanced.

Hence, in this case, the number negative edges in C will always be even. □

It is interesting to check whether the converse of the above theorem is valid. In context of set-labeling of signed graphs, a necessary and sufficient condition for a signed graph Σ is to be balanced is given in the following theorem.

Theorem 2.10 *A signed graph Σ is balanced if and only if it admits a set-labeling.*

Proof The proof is an immediate consequence of Theorems 2.4 and 2.9. □

In view of Theorems 2.4 and 2.10, we have

Theorem 2.11 *Any set-labeled signed graph is balanced.*

§3. **Conclusion**

In this paper, we have discussed the characteristics and properties of the signed graphs which admit set-labeling with a focus on 2-clusterability and balance of these signed graphs. There are several open problems in this area. Some of the open problems that seem to be promising for further investigations are following.

Problem 3.1 *Discuss the k-clusterability of different types of set-labeled signed graphs for $k > 2$.*

Problem 3.2 *Discuss the balance, 2-clusterability and general k-clusterability of other types of set-labeling of signed graphs such as topological set-labeling, topogenic set-labeling, graceful set-labeling, sequential set-labeling etc.*

Problem 3.3 *Discuss the balance and 2-clusterability and general k-clusterability of different set-labeling of signed graphs, with different set operations other than the symmetric difference of sets.*

Further studies on other characteristics of signed graphs corresponding to different set-labeled graphs are also interesting and challenging. All these facts highlight the scope for further studies in this area.

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Extended Quasi Conformal Curvature Tensor on $N(k)$-Contact Metric Manifold

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Abstract: In this paper certain results on $N(k)$-contact metric manifold endowed with a extended quasi conformal curvature tensor are formulated. First, we consider ξ-extended quasi conformally flat $N(k)$-contact metric manifold. Next we describe extended quasi-conformally semi-symmetric and extended quasi conformal pseudo-symmetric $N(k)$-contact metric manifold. Finally, we study the conditions $\widetilde{C}_e(\xi, X) \cdot R = 0$ and $\widetilde{C}_e(\xi, X) \cdot S = 0$ on $N(k)$-contact metric manifold.

Key Words: $N(k)$-contact metric manifold, quasi conformal curvature tensor, extended quasi conformal curvature tensor, η-Einstein.

AMS(2010): 53C10, 53C15, 53C25.

§1. Introduction

In 1968, Yano and Sawaki [20] introduced the notion of quasi conformal curvature tensor $\tilde{C}$ on a Riemannian manifold M and is given by

$$\begin{aligned}\widetilde{C}(X,Y)Z &= aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right] \\ &- \frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right]\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{1.1}$$

for all $X,\ Y\ \in TM$, where a and b are constants and r is a scalar curvature. If $a = 1$ and $b = -\frac{1}{2n-1}$, then quasi conformal curvature tensor reduces to conformal curvature tensor.

The extended form of quasi conformal curvature tensor [8] is given by

$$\widetilde{C}_e(X,Y)Z = \widetilde{C}(X,Y)Z - \eta(X)\widetilde{C}(\xi,Y)Z - \eta(Y)\widetilde{C}(X,\xi)Z - \eta(Z)\widetilde{C}(X,Y)\xi. \tag{1.2}$$

On the other hand Tanno [19] introduced a class of contact metric manifolds for which the characteristic vector field ξ belongs to the k-nullity distribution for some real number k. Such manifolds are known as $N(k)$-contact metric manifolds. The authors Blair, Kim and Tripathi [2]

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gave the classification of $N(k)$-contact metric manifold satisfying the condition $Z(\xi, X) \cdot Z = 0$. Also quasi conformal curvature tensor on a sasakian manifold has been studied by De et al., [11]. Recently in [10], the authors study certain properties of $N(k)$-contact metric manifold endowed with a concircular curvature tensor.

Motivated by these studies the present paper is organized as follows: After giving preliminaries and basic formulas in Section 2, we study ξ-extended quasi conformally flat $N(k)$-contact metric manifolds in Section 3 and we found that the manifold is η-Einstein and also it admits a η-parallel Ricci tensor. In fact Section 4 is devoted to the study of extended quasi-conformally semi-symmetric $N(k)$-contact metric manifold and proved that the manifold is either locally isometric to $E^{n+1} \times S^n(4)$ or it is extended quasi-conformally flat. Then, in Section 5, we consider extended quasi conformal pseudo-symmetric $N(k)$-contact metric manifold and we found that the manifold reduces to η-Einstein. Finally in the last section, we discuss $N(k)$-contact metric manifolds satisfying conditions $\widetilde{C}_e(\xi, X) \cdot R = 0$ and $\widetilde{C}_e(\xi, X) \cdot S = 0$

§2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be a contact manifold if it carries a global differentiable 1-form η which satisfies the condition $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Also a contact manifold admits an almost contact structure (ϕ, ξ, η), where ϕ is $(1,1)$-tensor field, ξ is a characteristic vector field and η is a global 1-form such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \cdot \phi = 0. \tag{2.1}$$

An almost contact metric structure is said to be be normal if the induced complex structure J on the product manifold $M \times R$ is defined by,

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

is integrable, where X is tangent to M, t is the coordinate of R and f is a smooth function on $M \times R$. Let g be the Riemannian metric with almost contact structure (ϕ, ξ, η) i.e.,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

From (2.1), it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X), \tag{2.2}$$

for all $X, Y \in TM$. An almost contact metric structure is called contact metric structure if $g(X, \phi Y) = d\eta(X, Y)$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds;

$$\nabla_X \xi = -\phi X - \phi h X. \tag{2.3}$$

A normal contact metric manifold is a Sasakian manifold. An almost metric manifold is

Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X,$$

where ∇ is the Levi-Civita connection of the Riemannian metric g.

As a generalization of both $R(X,Y)\xi = 0$ and Sasakian case, the authors Blair, Koufogiorgos and Papantoniou [4] introduced the idea of (k,μ)-nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ of a contact metric manifold M is defined by

$$N(k,\mu): \;\; p \to N_p(k,\mu) = \{Z \in T_pM : R(X,Y)Z = (kI + \mu h)(g(Y,Z)X - g(X,Z)Y)\},$$

where $(k,\mu) \in R^2$. A contact metric manifold with $\xi \in N(k,\mu)$ is called a (k,μ)- contact metric manifold. If $\mu = 0$, the (k,μ)- nullity distribution reduces to k- nullity distribution [19]. The k- nullity distribution $N(k)$ of a Riemannian manifold is defined by

$$N(k): \;\; p \to N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{2.4}$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$- contact metric manifold [2]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [3]. In an $N(k)$-contact metric manifold, the following relations holds:

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.5}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{2.6}$$

$$\begin{aligned} S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ &\quad +[2nk - 2(n-1)]\eta(X)\eta(Y), \end{aligned} \tag{2.7}$$

$$S(X,\xi) = 2nk\eta(X), \quad Q\xi = 2nk\xi. \tag{2.8}$$

Also in an $N(k)$-contact metric manifold, extended quasi conformal curvature tensor satisfies the following:

$$\begin{aligned} \widetilde{C}_e(X,Y)\xi &= \left[a\left(\frac{r}{2n(2n+1)} - k\right) + 2b\left(\frac{r}{2n+1} - nk\right)\right](\eta(Y)X - \eta(X)Y) \\ &\quad - b[\eta(Y)QX - \eta(X)QY], \end{aligned} \tag{2.9}$$

$$\begin{aligned} \widetilde{C}_e(\xi,X)Y &= \left[a\left(k - \frac{r}{2n(2n+1)}\right) + 2b\left(nk - \frac{r}{2n+1}\right)\right](\eta(Y)X \\ &\quad - \eta(X)\eta(Y)\xi) + b[\eta(Y)QX - 2nk\eta(X)\eta(Y)\xi] = -\widetilde{C}_e(X,\xi)Y, \end{aligned} \tag{2.10}$$

$$\widetilde{C}_e(\xi,\xi)X = 0, \tag{2.11}$$

$$\eta(\widetilde{C}_e(X,Y)\xi) = \eta(\widetilde{C}_e(\xi,X)Y) = \eta(\widetilde{C}_e(X,\xi)Y) = \eta(\widetilde{C}_e(X,Y)Z) = 0. \tag{2.12}$$

By virtue of (1.2), let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold and using (1.1) and (2.8), we get

$$\sum_{i=1}^{2n} g(\widetilde{C}_e(e_i, Y)Z, e_i) = Lg(Y, Z) + MS(Y, Z) + N\eta(Y)\eta(W), \tag{2.13}$$

where,

$$\begin{aligned} L &= b(r - 2nk) - \left[\frac{r(2n-1)}{2n+1}\left(\frac{a}{2n} + 2b\right)\right] - \left[ka + 2nkb - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right], \\ M &= a + b(2n-3) \end{aligned}$$

and

$$N = 4nkb - (4n-3)\left[ak + 2nkb - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right] - 2b(r - 2nk).$$

Definition 2.1 *A (2n+1)-dimensional $N(k)$-contact metric manifold M is said to be η-Einstein if its Ricci tensor S is of the form*

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y),$$

for any vector fields X and Y, where α and β are constants. If β=0, then the manifold M is an Einstein manifold.

Definition 2.2 *In a $(2n+1)$-dimensional $N(k)$-contact metric manifold, if the Ricci tensor S satisfies $(\nabla_W S)(\phi X, \phi Y) = 0$, then the Ricci tensor is said to be η-parallel.*

In [1], Baikoussis and Koufogiorgos proved the following lemma.

Lemma 2.1 *Let M be an η-Einstein manifold of dimension $(2n+1)(n \geq 1)$. If ξ belongs to the k-nullity distribution, then $k = 1$ and the structure is Sasakian.*

§3. ξ-Extended Quasi Conformally Flat $N(k)$-Contact Metric Manifolds

Definition 3.1 *A $(2n+1)$-dimensional $N(k)$-contact metric manifold is said to be ξ-extended quasi conformally flat if*

$$\widetilde{C}_e(X, Y)\xi = 0 \quad \text{for all} \quad X, Y \in TM. \tag{3.1}$$

Let us consider ξ-extended quasi conformally flat $N(k)$-contact metric manifold. Then from (3.1) and (2.9), it can be easily seen that

$$\begin{aligned} 0 &= \left[a\left(\frac{r}{2n(2n+1)} - k\right) + 2b\left(\frac{r}{2n+1} - nk\right)\right](\eta(Y)X - \eta(X)Y) \\ &- b[\eta(Y)QX - \eta(X)QY]. \end{aligned} \tag{3.2}$$

Taking inner product of (3.2) with respect to W, we get

$$\begin{aligned} 0 &= \left[a\left(\frac{r}{2n(2n+1)} - k\right) + 2b\left(\frac{r}{2n+1} - nk\right)\right](\eta(Y)g(X,W) - \eta(X)g(Y,W)) \\ &- b[\eta(Y)S(X,W) - \eta(X)S(Y,W)]. \end{aligned}$$

On plugging $Y = \xi$ in above equation, gives

$$S(X,W) = Ag(X,W) + B\eta(X)\eta(W), \tag{3.3}$$

where

$$A = \left[\frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) - 2nk - \frac{ka}{b}\right] \text{ and } B = \left[4nk + \frac{ka}{b} - \frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right)\right].$$

Hence we can state the following:

Theorem 3.1 *A (2n+1)-dimensional ξ-extended quasi conformally flat $N(k)$-contact metric manifold is an η-Einstein manifold.*

Hence in view of Lemma 2.1 and above result, we can state the following result:

Corollary 3.1 *Let M be a $(2n+1)$-dimensional ξ-extended quasi conformally flat $N(k)$-contact metric manifold, then $k = 1$ and the structure is Sasakian.*

Replacing X and W by ϕX and ϕW in (3.3) and using (2.1), we obtain

$$S(\phi X, \phi W) = M' g(\phi X, \phi W). \tag{3.4}$$

Now taking the covariant derivative of (3.4) with respect to U, yields

$$(\nabla_U S)(\phi X, \phi W) = \frac{dr(U)}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) g(\phi X, \phi W).$$

If we consider $N(k)$-contact metric manifold with constant scalar curvature, then above equation becomes

$$(\nabla_U S)(\phi X, \phi W) = 0.$$

Hence this leads us to the following result:

Corollary 3.2 *A $(2n+1)$-dimensional ξ-extended quasi conformally flat $N(k)$-contact metric manifold with constant scalar curvature admits a η-parallel Ricci tensor.*

§4. Extended Quasi-Conformally Semi-Symmetric $N(k)$-Contact Metric Manifold

Let us consider an extended quasi-conformally semi-symmetric $N(k)$-contact metric manifold i.e.,

$$R(\xi, X) \cdot \widetilde{C}_e = 0.$$

Then the above condition turns into,

$$\begin{aligned} 0 &= R(\xi, X)\widetilde{C}_e(U, V)W - \widetilde{C}_e(R(\xi, X)U, V)W \\ &\quad -\widetilde{C}_e(U, R(\xi, X)V)W - \widetilde{C}_e(U, V)R(\xi, X)W. \end{aligned} \tag{4.1}$$

In view of (2.6), equation (4.1) can be written as

$$\begin{aligned} 0 &= k\Big[g(X, \widetilde{C}_e(U, V)W)\xi - \eta(\widetilde{C}_e(U, V)W)X - g(X, U)\widetilde{C}_e(\xi, V)W \\ &\quad + \eta(U)\widetilde{C}_e(X, V)W - g(X, V)\widetilde{C}_e(U, \xi)W + \eta(V)\widetilde{C}_e(U, X)W \\ &\quad - g(X, W)\widetilde{C}_e(U, V)\xi + \eta(W)\widetilde{C}_e(U, V)X\Big]. \end{aligned} \tag{4.2}$$

Which implies that either $k = 0$ or

$$\begin{aligned} &\Big[g(X, \widetilde{C}_e(U, V)W)\xi - \eta(\widetilde{C}_e(U, V)W)X - g(X, U)\widetilde{C}_e(\xi, V)W + \eta(U)\widetilde{C}_e(X, V)W \\ &- g(X, V)\widetilde{C}_e(U, \xi)W + \eta(V)\widetilde{C}_e(U, X)W - g(X, W)\widetilde{C}_e(U, V)\xi + \eta(W)\widetilde{C}_e(U, V)X\Big] = 0. \end{aligned}$$

Now taking inner product of above equation with ξ and then using (2.12), we get

$$g(X, \widetilde{C}_e(U, V)W) = 0.$$

Which implies that $\widetilde{C}_e(U, V)W = 0$. Hence we can state the following:

Theorem 4.1 *An extended quasi-conformally semi-symmetric $N(k)$-contact metric manifold is either locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$ or the manifold is extended quasi-conformally flat.*

§5. Extended Quasi Conformal Pseudo-Symmetric $N(k)$-Contact Metric Manifold

Definition 5.1 *A $(2n+1)$-dimensional $N(k)$-contact metric manifold M is said to be extended quasi conformal pseudo-symmetric if*

$$(R(X, Y) \cdot \widetilde{C}_e)(U, V)W = L_{\widetilde{C}_e}[((X \wedge Y) \cdot \widetilde{C}_e)(U, V)W], \tag{5.1}$$

holds for any vector fields X, Y, U, V, $W \in TM$, where $L_{\widetilde{C}_e}$ is function of M. The endomorphism $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{5.2}$$

Now we prove the following result:

Theorem 5.1 *Let M be a $(2n + 1)$-dimensional extended quasi conformal pseudo-symmetric*

$N(k)$-contact metric manifold. Then either $L_{\widetilde{C}_e} = k$ or the the manifold is η-Einstein.

Proof Let us consider a $(2n+1)$-dimensional extended quasi conformal pseudo symmetric $N(k)$-contact metric manifold. Taking $Y = \xi$ in (5.1), we get

$$(R(X,\xi)\cdot \widetilde{C}_e)(U,V)W = L_{\widetilde{C}_e}[((X\wedge\xi)\cdot\widetilde{C}_e)(U,V)W]. \tag{5.3}$$

By virtue of (5.2) and (2.12), right hand side of above equation becomes

$$\begin{aligned}&L_{\widetilde{C}_e}[-g(X,\widetilde{C}_e(U,V)W)\xi - \eta(U)\widetilde{C}_e(X.V)W + g(X,U)\widetilde{C}_e(\xi,V)W - \eta(V)\widetilde{C}_e(U,X)W\\ &+g(X,V)\widetilde{C}_e(U,\xi)W - \eta(W)\widetilde{C}_e(U,V)X + g(X,W)\widetilde{C}_e(U,V)\xi].\end{aligned} \tag{5.4}$$

In view of (2.5), left hand side of (5.3) gives

$$\begin{aligned}&k[-g(X,\widetilde{C}_e(U,V)W)\xi - \eta(U)\widetilde{C}_e(X.V)W + g(X,U)\widetilde{C}_e(\xi,V)W - \eta(V)\widetilde{C}_e(U,X)W\\ &+g(X,V)\widetilde{C}_e(U,\xi)W - \eta(W)\widetilde{C}_e(U,V)X + g(X,W)\widetilde{C}_e(U,V)\xi].\end{aligned} \tag{5.5}$$

By considering (5.5) and (5.4) in (5.3) with $V = \xi$, we get

$$\begin{aligned}&(L_{\widetilde{C}_e} - k)[-g(X,\widetilde{C}_e(U,\xi)W)\xi - \eta(U)\widetilde{C}_e(X.\xi)W + g(X,U)\widetilde{C}_e(\xi,\xi)W\\ &-\eta(\xi)\widetilde{C}_e(U,X)W + g(X,\xi)\widetilde{C}_e(U,\xi)W - \eta(W)\widetilde{C}_e(U,\xi)X + g(X,W)\widetilde{C}_e(U,\xi)\xi] = 0.\end{aligned} \tag{5.6}$$

By using (2.9)-(2.11) in (5.6), we have either $(L_{\widetilde{C}_e} - k) = 0$ or

$$\begin{aligned}\widetilde{C}_e(U,X)W &= \left[a\left(k - \frac{r}{2n(2n+1)}\right) + 2b\left(nk - \frac{r}{2n+1}\right)\right]\{\eta(W)g(X,U)\xi\\ &- 2\eta(U)\eta(X)\eta(W)\xi + \eta(U)\eta(W)X + g(X,W)\eta(U)\xi - g(X,W)U\}\\ &+ b[\eta(W)S(X,U)\xi - 4nk\eta(U)\eta(X)\eta(W)\xi + \eta(U)\eta(W)QX\\ &+ 2nkg(X,W)\eta(U)\xi - g(X,W)QU].\end{aligned} \tag{5.7}$$

On contracting (5.7) with respect to U and then using (2.13), we have

$$S(X,W) = A'g(X,W) + B'\eta(X)\eta(W),$$

where,

$$\begin{aligned}A' &= \frac{1}{a+b(2n-3)}\left[(2-2n)ka + (6-2n)2nkb - \frac{r(3-4n)}{2n+1}\left(\frac{a}{2n} + 2b\right) - 2rb\right],\\ B' &= \frac{1}{a+b(2n-3)}\left[(4n-3)\left(ka + 2nkb - \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right) + 2br - 12nkb\right].\end{aligned}$$

Thus M is a η-Einstein manifold. □

§6. $N(k)$-Contact Metric Manifold Satisfying $\widetilde{C}_e(\xi,X)\cdot R=0$ and $\widetilde{C}_e(\xi,X)\cdot S=0$

First we consider an $N(k)$-contact metric manifold satisfying $\widetilde{C}_e(\xi,X)\cdot R=0$. Now it follows from above condition that

$$0=\widetilde{C}_e(\xi,X)R(U,V)Y-R(\widetilde{C}_eU,V)Y-R(U,\widetilde{C}_e(\xi,X)V)Y-R(U,V)\widetilde{C}_e(\xi,X)Y. \tag{6.1}$$

By virtue of (2.10) in (6.1), gives

$$\begin{aligned}0 &= \left[a\left(k-\frac{r}{2n(2n+1)}\right)+2b\left(nk-\frac{r}{2n+1}\right)\right]\{\eta(R(U,V)Y)[X-\eta(X)\xi] \\ &- \eta(U)[R(X,V)Y-\eta(X)R(\xi,V)Y]-\eta(V)[R(U,X)Y-\eta(X)R(U,\xi)Y] \\ &- \eta(Y)[R(U,V)X-\eta(X)R(U,V)\xi]\}+b\{\eta(R(U,V)Y)[QX-2nk\eta(X)\xi] \\ &- \eta(U)[R(QX,V)Y-2nk\eta(X)R(\xi,V)Y]-\eta(V)[R(U,QX)Y \\ &- 2nk\eta(X)R(U,\xi)Y]-\eta(Y)[R(U,V)QX-2nk\eta(X)R(U,V)\xi]\}.\end{aligned} \tag{6.2}$$

Considering $U=\xi$ in (6.2), gives

$$\begin{aligned}0 &= \left[a\left(k-\frac{r}{2n(2n+1)}\right)+2b\left(nk-\frac{r}{2n+1}\right)\right]\{\eta(R(\xi,V)Y)[X-\eta(X)\xi] \\ &- [R(X,V)Y-\eta(X)R(\xi,V)Y]-\eta(V)[R(\xi,X)Y-\eta(X)R(\xi,\xi)Y] \\ &- \eta(Y)[R(\xi,V)X-\eta(X)R(\xi,V)\xi]\}+b\{\eta(R(\xi,V)Y)[QX-2nk\eta(X)\xi] \\ &- [R(QX,V)Y-2nk\eta(X)R(\xi,V)Y]-\eta(V)[R(\xi,QX)Y \\ &- 2nk\eta(X)R(\xi,\xi)Y]-\eta(Y)[R(\xi,V)QX-2nk\eta(X)R(\xi,V)\xi]\}.\end{aligned} \tag{6.3}$$

Taking inner product of (6.3) with respect to ξ and then by virtue of (2.5) and (2.6), we obtain

$$0=k\eta(Y)\left[S(V,X)-A''g(V,X)-B''\eta(V)\eta(X)\right], \tag{6.4}$$

where $A''=\left[\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)-2nk-\frac{ka}{b}\right]$ and $B''=\left[4nk+\frac{ka}{b}-\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)\right]$.

Since for an $N(k)$-contact metric manifolds $\eta(Y)\neq 0$, then (6.4) yields either $k=0$ or

$$S(V,X)=A''g(V,X)+B''\eta(V)\eta(X).$$

This leads us to the following:

Theorem 6.1 *Let M be a $(2n+1)$-dimensional $N(k)$-contact metric manifold satisfying the condition $\widetilde{C}_e(\xi,X)\cdot R=0$. Then M reduces to η-Einstein manifold or it is locally isometric to $E^{n+1}\times S^n(4)$ for $n>1$ and flat for $n=1$.*

Next we prove the following result:

Theorem 6.2 *Let M be an $(2n+1)$-dimensional $N(k)$-contact metric manifold satisfying $\widetilde{C}_e(\xi,X)\cdot S=0$. Then the Ricci tensor S is given by the equation (6.7).—*

Proof Let us consider an $N(k)$-contact metric manifold satisfying the condition $\widetilde{C}_e(\xi, X) \cdot S = 0$. Then it can be easily seen that

$$S(\widetilde{C}_e(\xi, X)Y, W) + S(Y, \widetilde{C}_e(\xi, X)W) = 0. \tag{6.5}$$

By virtue of (2.10), it follows from above equation that

$$\begin{aligned} 0 = &\left[a\left(k - \frac{r}{2n(2n+1)}\right) + 2b\left(nk - \frac{r}{2n+1}\right)\right]\{\eta(Y)[S(X, W) - \eta(X)S(\xi, W)] \\ &+ \eta(W)[S(Y, X) - \eta(X)S(Y, \xi)]\} + b\{\eta(Y)[S(QX, W) - 2nk\eta(X)S(\xi, W)] \\ &+ \eta(W)[S(Y, QX) - 2nk\eta(X)S(Y, \xi)]\}. \end{aligned} \tag{6.6}$$

On plugging $Y = \xi$ in (6.6) and making necessary calculation, we have

$$S(QX, W) = MS(X, Y) + N\eta(X)\eta(W), \tag{6.7}$$

where,

$$\begin{aligned} M &= \left[\frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right) - \frac{ak}{b} - 2nk\right], \\ N &= \left[\frac{2nk^2 a}{b} + 8n^2k^2 - \frac{2nkr}{b(2n+1)}\left(\frac{a}{2n} + 2b\right)\right]. \end{aligned}$$

Hence the proof. □

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The Examination of Eigenvalues and Eigenfunctions of the Sturm-Liouville Fuzzy Problem According to Boundary Conditions

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Abstract: In this paper, the eigenvalues and the eigenfunctions of the homogeneous fuzzy Sturm-Liouville problem are examined under the three different situations according to the boundary conditions. This examination is studied under the approach of Hukuhara differentiability. The examples are given for this situations.

Key Words: Fuzzy boundary value problems, second-order fuzzy differential equations, Hukuhara differentiability, eigenvalue, eigenfunction.

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§1. Introduction

Approaches to fuzzy boundary value problems can be of two types. The first approach assumes that the derivative in the boundary value problem can be considered as a derivative of fuzzy function. This derivative can be Hukuhara derivative or a derivative in generalized sense [1,2,11,12]. The second approach is based on generating the fuzzy solution from crisp solution. In particular case, this approach can be of three ways. The first one uses the extension principle [3]. The second way uses the concept of differential inclusion [10]. In the third way the fuzzy problem is considered to be a set of crisp problem [6,7].

In this paper, the eigenvalues and the eigenfunctions of the homogeneous fuzzy Sturm-Liouville problem are examined under the approach of Hukuhara differentiability. It is seen that applied method for this examination is different according to given boundary conditions. Therefore, the eigenvalues and the eigenfunctions of the problem are examined under the three different situation and the examples are given for this situations.

§2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

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Definition 2.1([11]) *A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ satisfying the following properties:*

(i) u is normal;

(ii) u is convex fuzzy set;

(iii) u is upper semi-continuous on $\mathbb{R}$;

(iv) $cl\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2.2([12]) *Let $u \in \mathbb{R}_F$. The α-level set of u, denoted , $[u]^\alpha$, $0 < \alpha \leq 1$, is*

$$[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq 0\}.$$

If $\alpha = 0$, the support of u is defined

$$[u]^0 = cl\{x \in \mathbb{R} \mid u(x) > 0\}.$$

The notation, denotes explicitly the α-level set of u. The notation, $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ denotes explicitly the α-level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

The following remark shows when $[\underline{u}_\alpha, \overline{u}_\alpha]$ is a valid α-level set.

Remark 2.3([11]) The sufficient and necessary conditions for $[\underline{u}_\alpha, \overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

(i) $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

(ii) $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

(iii) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 2.4([13]) *If A is a symmetric triangular numbers with supports $[\underline{a}, \overline{a}]$, the $\alpha-$level sets of $[A]^\alpha$ is $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$.*

Definition 2.5([12]) *For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$, where means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.*

The metric structure is given by the Hausdorff distance

$$D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}, \quad ([10])$$

by

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha|\}.$$

Definition 2.6([13]) *Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the Hukuhara difference of fuzzy numbers u and v, and it is denoted by $w = u \ominus v$.*

Definition 2.7([11]) *Let $f : [a,b] \to \mathbb{R}_F$ and $t_0 \in [a,b]$. We say that f is Hukuhara differential at t_0, if there exists an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(t_0+h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ and the limits*

$$\lim_{h \to 0^+} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0-h)}{h} = f'(t_0).$$

Here the limits are taken in the metric space $(\mathbb{R}_F, D)$.

Theorem 2.8([5]) *Let $f : I \to \mathbb{R}_F$ be a function and denote $[f(t)]^\alpha = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right]$, for each $\alpha \in [0,1]$. If f is Hukuhara differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f'(t)\right]^\alpha = \left[\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)\right]$.*

Definition 2.9([9]) *Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be a function. f^+ and f^- are not the negative function defined as*

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) < 0 \end{cases}, \quad f^-(x) = \begin{cases} -f(x), & f(x) \leq 0 \\ 0, & f(x) > 0 \end{cases}$$

The function f^+ and f^- are called the positive piece and negative piece of f, respectively.

§3. The Eigenvalues and the Eigenfunctions of the Sturm-Liouville Fuzzy Problem According to the Boundary Conditions

Let

$$\begin{aligned} Ly &= p(x)y'' + q(x)y, \ p'(x) = 0, \\ A, B, C, D &\geq 0, \ A^2 + B^2 \neq 0 \text{ and } C^2 + D^2 \neq 0. \end{aligned}$$

(I) Consider the eigenvalues of the fuzzy boundary value problem

$$Ly + \lambda y = 0, \ x \in [a,b] \tag{3.1}$$

$$Ay(a) + By'(a) = 0, \tag{3.2}$$

$$Cy(b) + Dy'(b) = 0. \tag{3.3}$$

Let be functions $\underline{\phi}_\alpha$, $\underline{\psi}_\alpha$, $\overline{\phi}_\alpha$, $\overline{\psi}_\alpha$ the solution of the fuzzy boundary value problem (3.1)-(3.3). The eigenvalues of the fuzzy boundary value problem (3.1)-(3.3) if and only if are consist of the zeros of functions $\underline{W}_\alpha(\lambda)$ and $\overline{W}_\alpha(\lambda)$, where ([8])

$$\underline{W}_\alpha(\lambda) = W\left(\underline{\phi}_\alpha, \underline{\psi}_\alpha\right)(x,\lambda) = \underline{\phi}_\alpha(x,\lambda)\underline{\psi}'_\alpha(x,\lambda) - \underline{\psi}_\alpha(x,\lambda)\underline{\phi}'_\alpha(x,\lambda),$$

$$\overline{W}_{\alpha}(\lambda) = W\left(\overline{\phi}_{\alpha}, \overline{\psi}_{\alpha}\right)(x,\lambda) = \overline{\phi}_{\alpha}(x,\lambda)\,\overline{\psi}'_{\alpha}(x,\lambda) - \overline{\psi}_{\alpha}(x,\lambda)\,\overline{\phi}'_{\alpha}(x,\lambda).$$

Example 3.1 Consider the fuzzy Sturm-Liouville problem

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(1) + y'(1) = 0. \tag{3.4}$$

Let be $\lambda = k^2$, $k > 0$,

$$\phi(x,\lambda) = \sin(kx)$$

be the solution of the classical differential equation $y'' + \lambda y = 0$ satisfying the condition $y(0) = 0$ and

$$\psi(x,\lambda) = \left(\cos(k) + \frac{\sin(k)}{k}\right)\cos(kx) + \left(\sin(k) - \frac{\cos(k)}{k}\right)\sin(kx)$$

be the solution satisfying the condition $y(1) = 1$, $y'(1) = -1$. Then,

$$[\phi(x,\lambda)]^{\alpha} = \left[\underline{\phi}_{\alpha}(x,\lambda), \overline{\phi}_{\alpha}(x,\lambda)\right] = [\alpha, 2-\alpha]\sin(kx) \tag{3.5}$$

is the solution of the fuzzy differential equation $y'' + \lambda y = 0$ satisfying the condition $y(0) = 0$ and

$$[\psi(x,\lambda)]^{\alpha} = \left[\underline{\psi}_{\alpha}(x,\lambda), \overline{\psi}_{\alpha}(x,\lambda)\right] = [\alpha, 2-\alpha]\,\psi(x,\lambda) \tag{3.6}$$

is the solution satisfying the condition $y(1) = 1$, $y'(1) = -1$. Since the eigenvalues of the fuzzy Sturm-Liouville problem (3.4) are zeros the functions $\underline{W}_{\alpha}(\lambda)$ and $\overline{W}_{\alpha}(\lambda)$, $\underline{W}_{\alpha}(\lambda)$ is obtained as

$$\underline{W}_{\alpha}(\lambda) = \alpha^2\{(-k\cos(k) - \sin(k))\sin^2(kx) + (k\sin(k) - \cos(k))\cos(kx)\sin(kx) +$$

$$+(-k\cos(k) - \sin(k))\cos^2(kx) - (k\sin(k) - \cos(k))\cos(kx)\sin(kx)\}$$

$$\underline{W}_{\alpha}(\lambda) = -\alpha^2(k\cos(k) + \sin(k))$$

and similarly $\overline{W}_{\alpha}(\lambda)$ is obtained as

$$\overline{W}_{\alpha}(\lambda) = -(2-\alpha)^2(k\cos(k) + \sin(k)).$$

From here, yields

$$\underline{W}_{\alpha}(\lambda) = 0 \Rightarrow k\cos(k) + \sin(k) = 0,$$

$$\overline{W}_{\alpha}(\lambda) = 0 \Rightarrow k\cos(k) + \sin(k) = 0.$$

Computing the values k satisfying the equation $k\cos(k) + \sin(k) = 0$, we have

$$k_1 = 2.028757838,\ k_2 = 4.913180439,\ k_3 = 7.978665712,\ k_4 = 11.08553841, \cdots$$

We show that this values are k_n, $n = 1, 2, \cdots$ Substituing this values in (3.5),(3.6), we obtain

$$[\phi_n(x)]^{\alpha} = \left[\underline{\phi_n}_{\alpha}(x), \overline{\phi_n}_{\alpha}(x)\right] = [\alpha, 2-\alpha]\sin(k_n x),$$

$$[\psi_n(x)]^\alpha = \left[\underline{\psi_n}_\alpha(x), \overline{\psi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\left(\sin(k_n) - \frac{\cos(k_n)}{k_n}\right)\sin(k_n x).$$

As

$$[\alpha, 2-\alpha](\sin(k_n x))^+$$

and

$$[\alpha, 2-\alpha]\left(\left(\sin(k_n) - \frac{\cos(k_n)}{k_n}\right)\sin(k_n x)\right)^+, \ [\phi_n(x)]^\alpha$$

and $[\psi_n(x)]^\alpha$ are a valid $\alpha-$level set. Let be $k_n x \in [(n-1)\pi, n\pi]$, $n = 1, 2, \cdots$

(i) If n is single, $\sin(k_n x) \geq 0$. Then $[\phi_n(x)]^\alpha$ is a valid $\alpha-$level set.

(ii) If n is double, $\sin(k_n x) \leq 0$. Also, since $x \in [0,1]$, $k_n \in [(n-1)\pi, n\pi]$, and according to Fig.1, $\sin(k_n) - \frac{\cos(k_n)}{k_n} < 0$ for n is double. Then $[\psi_n(x)]^\alpha$ is a valid $\alpha-$level set.

Consequently, $k_n x \in [(n-1)\pi, n\pi]$, $n = 1, 2, \cdots$

(i) If n is single, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{1n}(x)]^\alpha = [\alpha, 2-\alpha]\sin(k_n x),$$

(ii) If n is double, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{2n}(x)]^\alpha = [\alpha, 2-\alpha]\left(\sin(k_n) - \frac{\cos(k_n)}{k_n}\right)\sin(k_n x),$$

(iii) If $\alpha = 1$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = \sin(k_n x).$$

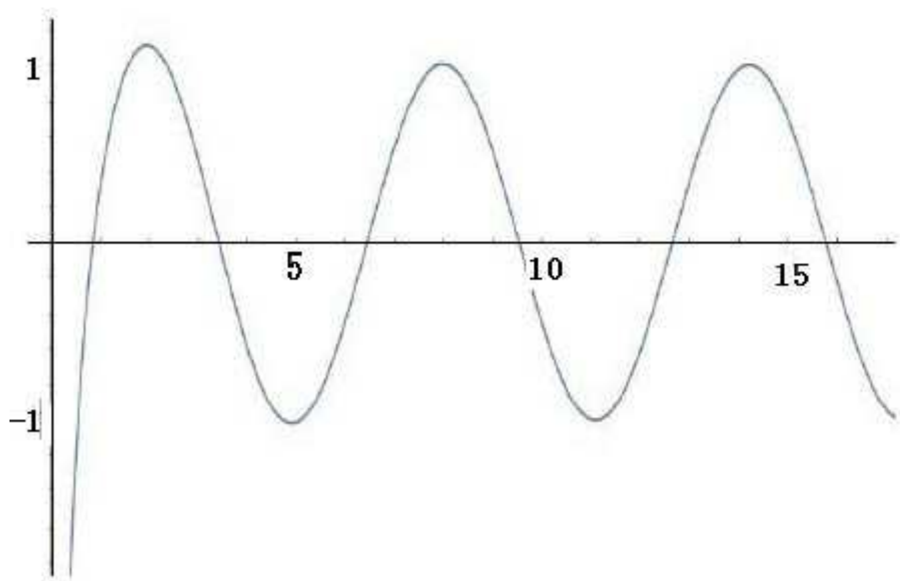

Fig.1 The graphic of the function $f(k) = \sin(k) - \frac{\cos(k)}{k}$

(II) Consider the eigenvalues of the fuzzy boundary value problem,

$$Ly + \lambda y = 0,\ x \in (a, b) \tag{3.7}$$

$$Ay(a) = By^{'}(a), \tag{3.8}$$

$$Cy(b) = Dy^{'}(b). \tag{3.9}$$

$$[\phi(x,\lambda)]^{\alpha} = \left[\underline{\phi}_{\alpha}(x,\lambda), \overline{\phi}_{\alpha}(x,\lambda)\right]$$

is the solution of the fuzzy differential equation (3.7) satisfying the conditions $y(a) = B$, $y^{'}(a) = A$ and

$$[\psi(x,\lambda)]^{\alpha} = \left[\underline{\psi}_{\alpha}(x,\lambda), \overline{\psi}_{\alpha}(x,\lambda)\right]$$

is the solution satisfying the conditions

$$y(b) = D,\ y^{'}(b) = C.$$

Hence, the method which is applied for the fuzzy boundary value problem (3.1)-(3.3) is valid for the problem (3.7)-(3.9).

Example 3.2 Consider the fuzzy Sturm-Liouville problem

$$y^{''} + \lambda y = 0,\ y(0) = 0,\ y^{'}(1) = y(1). \tag{3.10}$$

Let $\lambda = k^2$, $k > 0$. Then,

$$[\phi(x,\lambda)]^{\alpha} = \left[\underline{\phi}_{\alpha}(x,\lambda), \overline{\phi}_{\alpha}(x,\lambda)\right] = [\alpha, 2-\alpha]\sin(kx) \tag{3.11}$$

is the solution of the fuzzy differential equation $y^{''} + \lambda y = 0$ satisfying the condition $y(0) = 0$ and

$$[\psi(x,\lambda)]^{\alpha} = \left[\underline{\psi}_{\alpha}(x,\lambda), \overline{\psi}_{\alpha}(x,\lambda)\right] = [\alpha, 2-\alpha]\,\psi(x,\lambda) \tag{3.12}$$

is the solution satisfying the condition $y^{'}(1) = y(1)$, where

$$\psi(x,\lambda) = \left(\cos(k) - \frac{\sin(k)}{k}\right)\cos(kx) + \left(\sin(k) + \frac{\cos(k)}{k}\right)\sin(kx).$$

Since the eigenvalues of the fuzzy Sturm-Liouville problem (3.10) are zeros the functions $\underline{W}_{\alpha}(\lambda)$ and $\overline{W}_{\alpha}(\lambda)$, we obtained

$$\underline{W}_{\alpha}(\lambda) = -\alpha^2(k\cos(k) - \sin(k)),\ \overline{W}_{\alpha}(\lambda) = -(2-\alpha)^2(k\cos(k) - \sin(k)).$$

Computing the values k satisfying the equation $k\cos(k) - \sin(k) = 0$, we have

$$k_1 = 4.493409458,\ k_1 = 7.725251837,\ k_3 = 10.90412166,\ k_4 = 14.06619391, ...$$

We show that this values are k_n, $n = 1, 2, \cdots$. Substituting this values in (3.11), (3.12), we obtain

$$[\phi_n(x)]^\alpha = \left[\underline{\phi_n}_\alpha(x), \overline{\phi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\sin(k_n x),$$

$$[\psi_n(x)]^\alpha = \left[\underline{\psi_n}_\alpha(x), \overline{\psi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\left(\sin(k_n) + \frac{\cos(k_n)}{k_n}\right)\sin(k_n x).$$

Let $k_n x \in [(n-1)\pi, n\pi]$, $n = 1, 2, \cdots$

(i) If n is single, $\sin(k_n x) \geq 0$. Then $[\phi_n(x)]^\alpha$ is a valid $\alpha-$level set.

(ii) If n is double, $\sin(k_n x) \leq 0$. Also, since $x \in [0,1]$, $k_n \in [(n-1)\pi, n\pi]$ and according to Fig.2, $\sin(k_n) + \frac{\cos(k_n)}{k_n} < 0$ for n is double. Then $[\psi_n(x)]^\alpha$ is a valid $\alpha-$level set.

Consequently, $k_n x \in [(n-1)\pi, n\pi]$, $n = 1, 2, \cdots$

(i) If n is single, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{1n}(x)]^\alpha = [\alpha, 2-\alpha]\sin(k_n x),$$

(ii) If n is double, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{2n}(x)]^\alpha = [\alpha, 2-\alpha]\left(\sin(k_n) + \frac{\cos(k_n)}{k_n}\right)\sin(k_n x),$$

(iii) If $\alpha = 1$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = \sin(k_n x).$$

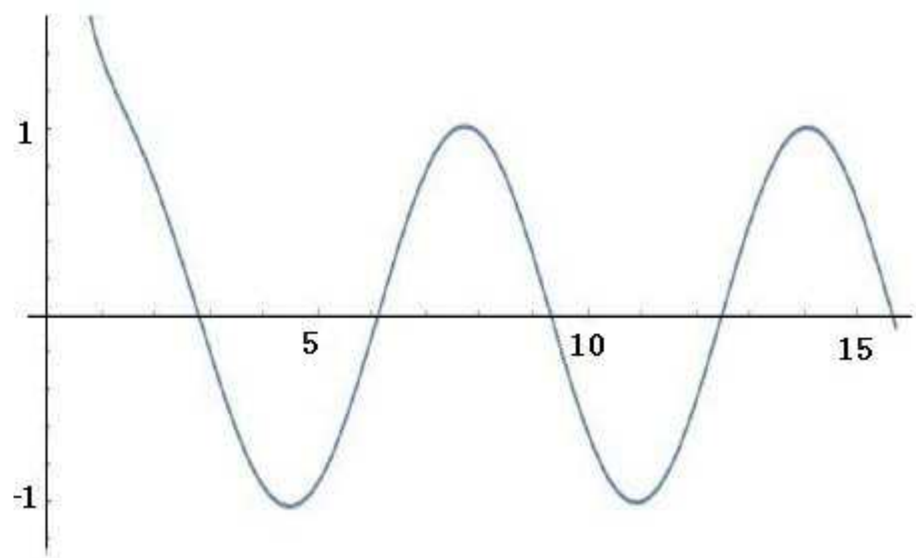

Fig.2 The graphic of the function $f(k) = \sin(k) + \frac{\cos(k)}{k}$

(III) Consider the eigenvalues of the fuzzy boundary value problem

$$Ly + \lambda y = 0, \ x \in (a,b), \tag{3.13}$$

$$-Ay(a) + By'(a) = 0, \tag{3.14}$$

$$-Cy(b) + Dy^{'}(b) = 0. \tag{3.15}$$

Let be $[y]^{\alpha} = \left[\underline{y}_{\alpha}, \overline{y}_{\alpha}\right]$ the general solution of the fuzzy differential equation (3.13). From the boundary condition (3.14)

$$-A\left[\underline{y}_{\alpha}(a,\lambda), \overline{y}_{\alpha}(a,\lambda)\right] + B\left[\underline{y}_{\alpha}(a,\lambda), \overline{y}_{\alpha}(a,\lambda)\right]^{'} = 0.$$

Using the Hukuhara differentiability, the fuzzy arithmetic and -$[y]^{\alpha} = \left[-\overline{y}_{\alpha}, -\underline{y}_{\alpha}\right]$, we obtained

$$\left[-A\overline{y}_{\alpha}(a,\lambda) + B\underline{y}_{\alpha}^{'}(a,\lambda), -A\underline{y}_{\alpha}(a,\lambda) + B\overline{y}_{\alpha}^{'}(a,\lambda)\right] = 0.$$

From here, the equations

$$-A\overline{y}_{\alpha}(a,\lambda) + B\underline{y}_{\alpha}^{'}(a,\lambda) = 0 \tag{3.16}$$

$$-A\underline{y}_{\alpha}(a,\lambda) + B\overline{y}_{\alpha}^{'}(a,\lambda) = 0 \tag{3.17}$$

are obtained. So we can not decompose the lower solution and upper solution. Therefore we can not find the function $\underline{\phi}_{\alpha}(x,\lambda)$ satisfying the condition (3.16) and the function $\overline{\phi}_{\alpha}(x,\lambda)$ satisfying the condition (3.17) of the fuzzy differential equation (3.13). Consequently, there is not the function

$$[\phi(x,\lambda)]^{\alpha} = \left[\underline{\phi}_{\alpha}(x,\lambda), \overline{\phi}_{\alpha}(x,\lambda)\right]$$

satisfying the condition (3.14) of the fuzzy differential equation (3.13). Similarly, there is not the function

$$[\psi(x,\lambda)]^{\alpha} = \left[\underline{\psi}_{\alpha}(x,\lambda), \overline{\psi}_{\alpha}(x,\lambda)\right]$$

satisfying the condition (3.15) of the fuzzy differential equation (3.13).

Therefore, the method which is applied for the fuzzy boundary value problem (3.1)-(3.3) is not valid for the problem (3.13)-(3.15).

Example 3.3 Consider the fuzzy Sturm-Liouville problem

$$y^{''} + \lambda y = 0,\ y(0) = 0,\ y^{'}(1) - y(1) = 0. \tag{3.18}$$

Let be $[y]^{\alpha} = \left[\underline{y}_{\alpha}, \overline{y}_{\alpha}\right]$ and $\lambda = k^2,\ k > 0$. Then, the lower and upper solutions of the fuzzy diferential equation in (3.18) are

$$\underline{y}_{\alpha}(x,\lambda) = c_1(\alpha)\cos(kx) + c_2(\alpha)\sin(kx), \tag{3.19}$$

$$\overline{y}_{\alpha}(x,\lambda) = c_3(\alpha)\cos(kx) + c_4(\alpha)\sin(kx). \tag{3.20}$$

From the boundary conditon $y(0) = 0$

$$\underline{y}_{\alpha}(0,\lambda) = c_1(\alpha) = 0,\ \overline{y}_{\alpha}(0,\lambda) = c_3(\alpha) = 0.$$

From the boundary conditon $y'(1) - y(1) = 0$,

$$\underline{y}'_\alpha(1,\lambda) - \overline{y}_\alpha(1,\lambda) = 0,\ \overline{y}'_\alpha(1,\lambda) - \underline{y}_\alpha(1,\lambda) = 0$$

and from here, we obtain the system of equations

$$\underline{y}'_\alpha(1,\lambda) - \overline{y}_\alpha(1,\lambda) = kc_2(\alpha)\cos(k) - c_4(\alpha)\sin(k) = 0,$$

$$\overline{y}'_\alpha(1,\lambda) - \underline{y}_\alpha(1,\lambda) = kc_4(\alpha)\cos(k) - c_2(\alpha)\sin(k) = 0.$$

If

$$\begin{vmatrix} k\cos(k) & -\sin(k) \\ -\sin(k) & k\cos(k) \end{vmatrix} = k^2\cos^2(k) - \sin^2(k) = 0,$$

there is the non-zero solution of the system of equation. Computing the values k satisfying this equation, we have

$$k_1 = 2.028757838,\ k_2 = 4.493409458,\ k_3 = 4.913180439,\ k_4 = 7.725251837, \cdots$$

We show that this values are k_n, $n = 1, 2, \cdots$ Substituing this values in (3.19), (3.20), we obtain

$$\underline{y}_{n\alpha}(x) = c_2(\alpha)\sin(k_n x),\ \overline{y}_{n\alpha}(x) = c_4(\alpha)\sin(k_n x),$$

$$[y_n(x)]^\alpha = \left[\underline{y}_{n\alpha}(x), \overline{y}_{n\alpha}(x)\right].$$

As

$$\frac{\partial \underline{y}_{n\alpha}(x)}{\partial\alpha} \geq 0,\ \frac{\partial \overline{y}_{n\alpha}(x)}{\partial\alpha} \leq 0 \text{ and } \underline{y}_{n\alpha}(x) \leq \overline{y}_{n\alpha}(x),$$

$[y_n(x)]^\alpha$ is valid $\alpha-$level set. Then $k_n x \in [m\pi, (m+1)\pi]$, $m = 0, 1, \cdots$

(i) If m is double, since $\sin(k_n x) \geq 0$, it must be $\frac{\partial c_2(\alpha)}{\partial\alpha} \geq 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} \leq 0$ and $c_2(\alpha) \leq c_4(\alpha)$,

(ii) If m is single, since $\sin(k_n x) \leq 0$, it must be $\frac{\partial c_2(\alpha)}{\partial\alpha} \leq 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} \geq 0$ and $c_2(\alpha) \geq c_4(\alpha)$.

Consequently, $k_n x \in [m\pi, (m+1)\pi]$, $m = 0, 1, \cdots$

(i) If m is double, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{1n}(x)]^\alpha = [c_2(\alpha)\sin(k_n x),\ c_4(\alpha)\sin(k_n x)],$$

for $c_2(\alpha)$ and $c_4(\alpha)$ satisfying the inequalities $\frac{\partial c_2(\alpha)}{\partial\alpha} > 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} < 0$ and $c_2(\alpha) < c_4(\alpha)$.

(ii) If m is single, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{2n}(x)]^\alpha = [c_2(\alpha)\sin(k_n x),\ c_4(\alpha)\sin(k_n x)],$$

for $c_2(\alpha)$ and $c_4(\alpha)$ satisfying the inequalities $\frac{\partial c_2(\alpha)}{\partial\alpha} < 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} > 0$ and $c_2(\alpha) > c_4(\alpha)$.

(iii) If $\frac{\partial c_2(\alpha)}{\partial\alpha} = 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} = 0$ and $c_2(\alpha) = c_2 = c_4 = c_4(\alpha)$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = \sin(k_n x).$$

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Nonholonomic Frames for Finsler Space with Deformed Special (α, β) Metric

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Abstract: The purpose of present paper to determine the Finsler spaces due to deformation of special Finsler (α, β) metrics. Consequently, we obtain the nonholonomic frame with the help of Riemannian metric $\alpha^2 = a_{ij}(x)y^i y^j$, one form metric $\beta = b_i(x)y^i$ and Douglas metric $L(\alpha, \beta) = \left(\alpha + \frac{\beta^2}{\alpha}\right)$ such as forms

$$\begin{aligned} \text{I. } L(\alpha,\beta) &= \alpha\beta, \\ \text{II. } L(\alpha,\beta) &= \left(\alpha + \frac{\beta^2}{\alpha}\right)\beta = \alpha\beta + \frac{\beta^3}{\alpha}. \end{aligned}$$

The first metric of the above deformation is obtained by the product of Riemannian metric and one form and second one is the product of Douglas metric and 1-form metric.

Key Words: Finsler space, (α, β)-metrics, Riemannian metric, one form metric, Douglas metric, GL-metric, nonholonomic Finsler frame.

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§1. Introduction

P.R. Holland [1], [2] studies about the nonholonomic frame on space time which was based on the consideration of a charged particle moving in an external electromagnetic field in the year 1982. In the year 1987, R.S. Ingarden [3] was the first person, to point out that the Lorentz force law can be written in above case as geodesic equation on a Finsler space called Randers space. Further in 1995, R.G. Beil [5], [6] have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries.

In the present paper we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M. In this case we have considered the fundamental tensor field might be the deformation of two different special Finsler spaces from the (α, β)- metrics. First we consider a nonholonomic frame for a Finsler space with (α, β)- metrics such as first product of Riemannian metric and 1-form metric and second is the product of Douglas metric and 1-form metric. Further we obtain

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a corresponding frame for each of these two Finsler deformations. This is an extension work of Ioan Bucataru and Radu Miron [10], Tripathi [14, 16] and Narasimhamurthy [15].

§2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with(α,β)-metrics [11].

Definition 2.1 *A Finsler space $F^n = \{M, F(x,y)\}$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by*

$$F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}, \tag{2.1}$$

where $\alpha^2(x,y) = a_{ij}(x)y^iy^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

The first Finsler spaces with (α,β)–metrics were introduced by the physicist G. Randers in 1940, are called Randers spaces [4]. Recently, R.G. Beil suggested a more general case by considering, $a_{ij}(x)$ the components of a Riemannian metric on the base manifold M, $a\,(x,y) > 0$ and $b\,(x,y) \geq 0$ Two functions on TM and $B\,(x,y) = B_i\,(x,y)\,(dx^i)$ a vertical 1-form on TM. Then

$$g_{ij}(x,y) = a(x,y)a_{ij}(x) + b(x,y)B_i(x,y)B_j(x,y).$$

Now a days the above generalized Lagrange metric is known as the Beil metric. The metric tensor g_{ij} is also known as a Beil deformation of the Riemannian metric a_{ij}. It has been studied and applied by R. Miron and R.K. Tavakol in General Relativity for $a(x,y) = \exp(2\sigma(x,y))$ and b = 0. The case a(x, y) = 1 with various choices of b and B_i was introduced and studied by R.G. Beil for constructing a new unified field theory [6]. Further Bucataru [12] considered the class of Lagrange spaces with (α,β)-metric and obtained some new and interesting results in the year 2002.

A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P. R. Holland. If we do not ask for the function L to be homogeneous of order two with respect to the (α,β) variables, then we have a Lagrange space with (α,β)-metric. Next we defined some different Finsler space with (α,β)-metrics.

Further consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space(M,F). Taking into account the homogeneity of α and F we have the following formulae:

$$\begin{aligned} p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j}; \quad p_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i}; \\ l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^i}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = P_i + b_i, \\ l^i = \frac{1}{L}p^i; l^il_i = p^ip_i = 1; l^ip_i = \frac{\alpha}{L}; p^il_i = \frac{L}{\alpha}; \end{aligned} \tag{2.2}$$

$$b_iP^i = \frac{\beta}{\alpha}; b_il^i = \frac{\beta}{L}$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by [13],

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_iP_j + P_ib_j - \frac{\beta}{\alpha}p_ip_j = \frac{L}{\alpha}(a_{ij} - p_ip_j) + l_il_j. \tag{2.3}$$

Theorem 2.1([10]) *For a Finsler space (M,F) consider the metric with the entries:*

$$Y^i_j = \sqrt{\frac{\alpha}{L}}(\delta^i_j - l^il_j + \sqrt{\frac{\alpha}{L}}p^ip_j) \tag{2.4}$$

defined on TM. Then $Y_j = Y^i_j(\frac{\partial}{\partial y^i}), \quad j \in 1,2,3,\cdots,n$ *is a nonholonomic frame.*

Theorem 2.2([7]) *With respect to frame the holonomic components of the Finsler metric tensor* $a_{\alpha\beta}$ *is the Randers metric* g_{ij}*, i.e,*

$$g_{ij} = Y^\alpha_i Y^\beta_j a_{\alpha\beta}. \tag{2.5}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $y_i = a_{ij}y^j, \beta^i = a^{ij}b_j$, and so on. For a Finsler space with (α,β)–metric $F^2(x,y) = L\{\alpha(x,y),\beta(x,y)\}$ we have the Finsler invariants [13].

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial\alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial\beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial\alpha\partial\beta}; \rho_{-2} = \frac{1}{2\alpha^2}(\frac{\partial^2 L}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial\alpha}), \tag{2.6}$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with (α,β)-metric we have

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{2.7}$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)–metric is given by [13]

$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\{b_i(x)y_j + b_j(x)y_i\} + \rho_{-2}y_iy_j. \tag{2.8}$$

From (2.8) we can see that $g_{ij}$is the result of two Finsler deformations

$$I. \qquad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j),$$

$$II. \quad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho^2_{-1})b_ib_j. \tag{2.9}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (2.9) is according to the theorem (7.9.1) in [10], given by

$$X^i_j = \sqrt{\rho}\delta^i_j - \frac{1}{\beta^2}\{\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}}\}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{2.10}$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by

$$h_{ij} = X_i^k X_j^l a_{kl}. \tag{2.11}$$

Again the frame that corresponds to the II_{nd} deformation (2.9) given by

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}\right)}\}b^i b_j, \tag{2.12}$$

where $C^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \tag{2.13}$$

Theorem 2.3([10]) *Let $F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}$ be the metric function of a Finsler space with (α,β) metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (2.10) and (2.12) respectively.

§3. Nonholonomic Frames for Finsler Space with Deformed (α, β) Metric

In this section we consider two cases of nonholonomic Finlser frames with special (α,β)-metrics, such an I^{st} Finsler frame product of Riemannian metric, one form metric and II^{nd} Finsler frame product of Douglas metric and 1-form metric.

3.1 Nonholonomic Frame for $L = \alpha\beta$

In the first case, for a Finsler space with the fundamental function $L = \alpha\beta$ the Finsler invariants (2.6) are given by

$$\begin{aligned} \rho_1 &= \frac{\beta}{2\alpha}, \quad \rho_0 = 0, \\ \rho_{-1} &= \frac{1}{2\alpha}, \quad \rho_{-2} = \neg\frac{\beta}{2\alpha^3}, \\ B^2 &= \frac{1}{4\alpha^4}(\alpha^2 b^2 - \beta^2). \end{aligned} \tag{3.1}$$

Using (3.1) in (2.10) we have

$$X_j^i = \sqrt{\frac{\beta}{2\alpha}}\delta_j^i - \frac{1}{4\alpha^2\beta^2}\left[\sqrt{\frac{\beta}{2\alpha}} + \sqrt{\frac{\beta - 4\alpha^4\beta}{2\alpha}}\right](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j). \tag{3.2}$$

Again using (3.1) in (2.12) we have

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{2\beta C^2}{\alpha}}\}b^i b_j, \tag{3.3}$$

where $C^2 = \frac{\beta}{2\alpha}b^2 - \frac{1}{2\alpha^3\beta}(\alpha^2 b^2 - \beta^2)^2$.

Theorem 3.1 *Let $L = \alpha\beta$ be the metric function of a Finsler space with (α, β) metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is nonholonomic Finsler Frame with X_k^i and Y_j^k are given by (3.2) and (3.3) respectively.

3.2 Nonholonomic Frame for $L = (\alpha + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \frac{\beta^3}{\alpha}$

In the second case, for a Finsler space with the fundamental function $L = (\alpha + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \frac{\beta^3}{\alpha}$ the Finsler invariants (2.6) are given by

$$\begin{aligned}
\rho_1 &= \frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}, \quad \rho_0 = \frac{3\beta}{\alpha}, \\
\rho_{-1} &= \frac{\alpha^2 - 3\beta^2}{2\alpha^3}, \quad \rho_{-2} = \frac{3\beta^3 - \alpha^2\beta}{2\alpha^5}, \\
B^2 &= \frac{(\alpha^2 - 3\beta)^2(\alpha^2 b^2 - \beta^2)}{4\alpha^8}.
\end{aligned} \tag{3.4}$$

Using (3.4) in (2.10) we have

$$\begin{aligned}
X_j^i &= \sqrt{\frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}}\delta_j^i - \frac{(\alpha^2 - 3\beta^2)^2}{4\alpha^6\beta^2}[\sqrt{\frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}} \\
&+ \sqrt{\frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3} + \frac{2\alpha^5\beta}{(3\beta^2 - \alpha^2)}}](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j).
\end{aligned} \tag{3.5}$$

Again using (3.4) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{\alpha\beta C^2}{\alpha^2 - 3\beta^2}}\}b^i b_j, \tag{3.6}$$

where $C^2 = \frac{\beta(\alpha^2 - \beta^2)}{2\alpha^3}b^2 + \frac{(3\beta^2 - \alpha^2)}{2\alpha^3\beta}(\alpha^2 b^2 - \beta^2)^2$.

Theorem 3.2 *Let $L = (\alpha + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \frac{\beta^3}{\alpha}$ be the metric function of a Finsler space with (α,β) metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is nonholonomic Finsler Frame with X_k^i and Y_j^k are given by (3.5) and (3.6) respectively.

§4. Conclusions

Nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli and Bucataru ([7], [8]) has been determined such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [9]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with (α,β)-metric, it appears a natural question: does how many Finsler space with(α,β)-metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with (α, β)-metrics.

In this work, we consider the Douglas metric, Riemannian metric and 1-form metric we determine the nonholonomic Finsler frames. But, in Finsler geometry, there are many(α, β)-metrics in future work we can determine the frames for them also.

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Homology of a Type of Octahedron

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Abstract: In this paper we find out the homology of a type of octahedron with six vertices, twelve edges and eight faces and have shown that it is analogous with the homology of a chain complex $0 \longrightarrow \mathbf{R}^6 \longrightarrow \mathbf{R}^{12} \to \mathbf{R}^8 \to 0$ and also find out the singular homology and the Euler characteristic of this type of octahedron which is equal to $\sum\limits_{n=0}^{\infty} dim_R(H_n(S))$, where S is a octahedron.

Key Words: Homology module, singular homology, homotopy.

AMS(2010): 13D05, 18G55, 18G35, 55U15, 18G20.

§1. Introduction

Homology classes were first defined rigorously by Henri Poincaré in his seminal paper "Analysis situs" in 1895 referring to the work of Riemann,Betti and von Dyck. The homology group was further developed by Emmy Noether [1] and, independently, by Leopold Vietoris and Walther Mayer [2] in the period 1925-28.

In mathematics (especially algebraic topology and abstract algebra), homology is a certain general procedure to associate a sequence of abelian groups or modules with a given mathematical object such as a topological space or a group. So, in algebraic topology, singular homology refers to the study of a certain set of algebraic invariants of a topological space X, the so-called homology groups $H_n(X)$. Intuitively spoken, singular homology counts, for each dimension n, the n-dimensional holes of a space.

The abstract algebra invariants such as ring,field were used to make concept of homology more rigorous and these developments give rise to mathematical branches such as homological algebra and K-Theory.

Homological algebra is a tool used to prove nonconstructive existence theorems in algebra (and in algebraic topology). It also provides obstructions to carrying out various kinds of constructions; when the obstructions are zero, the construction is possible. Finally, it is detailed enough so that actual calculations may be performed in important cases.

Let f and g be matrices whose product is zero. If $g.v = 0$ for some column vector v, say, of length n, we can not always write $v = f.u$ for some row vector u. This failure is measured by the defect

$$d = n - rank(f) - rank(g).$$

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Let R be a ring and let U, V and W be the modules over R. In modern language, f and g represent linear maps

$$U \xrightarrow{\quad f \quad} V \xrightarrow{\quad g \quad} W$$

with $gf = 0$, and d is the dimension of the homology module

$$H = \ker(g)/f(U)$$

Given an R-module homomorphism $f : A \longrightarrow B$, one is immediately led to study the kernel $\ker f$ and image $\mathrm{im} f$. Given another map $g : B \longrightarrow C$, we can form the sequence

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C, \tag{1}$$

where A, B and C are the modules over R. We say that such a sequence is exact (at B) if $\ker(g) = \mathrm{im}(f)$. This implies in particular that the composite $gf : A \longrightarrow C$ is zero and finally brings our attention to sequence (1) such that $gf = 0$.

The word polyhedron has slightly different meanings in geometry and algebraic geometry. In elementary geometry, a polyhedron is simply a three-dimensional solid which consists of a collection of polygons, usually joined at their edges. In [4], S. Dey et al. studied homology of a type of heptahedron. Here we consider polyhedron octahedron with eight faces, six vertices and twelve edges.

In this paper, first we find out that homology of a type of octahedron is analogous to the homology of a chain complex, $0 \longrightarrow \mathbf{R}^7 \longrightarrow \mathbf{R}^{12} \longrightarrow \mathbf{R}^7 \longrightarrow 0$ and we also find out the matrices of this complex. Next we show computationally, $H_2(S) \cong H_0(S) \cong \mathbf{R}$ and $H_1(S) = 0$ and the Euler characteristic of this type of octahedron which is equal to $\sum_{n=0}^{\infty} dim_R(H_n(S))$.

§2. Homology of a Octahedron

We can obtain a chain complex from a geometric object. We refer to the Weibel's book [3] for some details of the construction. We illustrate it with a octahedron S in Fig.1 following.

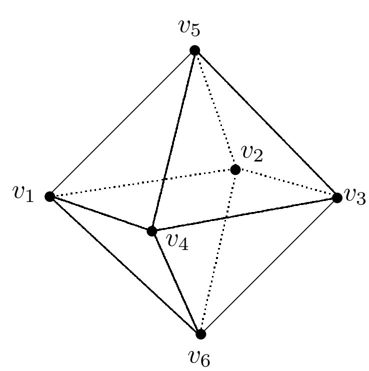

Fig.1

Level the vertex set of S as $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and then the twelve edges e_{12}, e_{23}, e_{34}, $e_{41}, e_{15}, e_{25}, e_{35}, e_{45}, e_{16}, e_{26}, e_{36}, e_{46}$, where $e_{ij}{=}e_{ji}$ for $i, j = 1, 2, 3, 4, 5, 6$ can be ordered as

$$\begin{aligned} E \;=\; & \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_1, v_5\}, \{v_2, v_5\}, \\ & \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_6\}, \{v_2, v_6\}, \{v_3, v_6\}, \{v_4, v_6\}\} \end{aligned}$$

and seven faces $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$ can be ordered as

$$\begin{aligned} F \;=\; & \{\{v_1, v_2, v_5\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_5\}, \\ & \{v_4, v_1, v_5\}, \{v_1, v_2, v_6\}, \{v_2, v_3, v_6\}, \{v_4, v_1, v_6\}\}. \end{aligned}$$

Let R be a ring and let $C_i(S)$ be the free R-module on the set V, E, F, respectively. Define maps ∂_0, ∂_1, ∂_2: $F \to E$ by removing the first, second, and third vertices, respectively except first face for each map. For the first face we define each map in such a way so that we can construct the homology. So, ∂_0, ∂_1, ∂_2 are given by

$$\begin{aligned} \partial_0 : f_1 &\longrightarrow e_{25} & \partial_1 : f_1 &\longrightarrow e_{15} & \partial_2 : f_1 &\longrightarrow e_{12} \\ f_2 &\longrightarrow e_{35} & f_2 &\longrightarrow e_{25} & f_2 &\longrightarrow e_{23} \\ f_3 &\longrightarrow e_{45} & f_3 &\longrightarrow e_{35} & f_3 &\longrightarrow e_{34} \\ f_4 &\longrightarrow e_{15} & f_4 &\longrightarrow e_{45} & f_4 &\longrightarrow e_{41} \\ f_5 &\longrightarrow e_{26} & f_5 &\longrightarrow e_{16} & f_5 &\longrightarrow e_{12} \\ f_6 &\longrightarrow e_{36} & f_6 &\longrightarrow e_{26} & f_6 &\longrightarrow e_{23} \\ f_7 &\longrightarrow e_{46} & f_7 &\longrightarrow e_{36} & f_7 &\longrightarrow e_{34} \\ f_7 &\longrightarrow e_{16} & f_7 &\longrightarrow e_{46} & f_7 &\longrightarrow e_{41} \end{aligned}$$

The set maps ∂_i yield $k+1$ module maps $C_k \longrightarrow C_{k-1}$, which we also call ∂_i, their alternating sum $d_i = \Sigma(-1)^i \partial_i$ is the map $C_k \longrightarrow C_{k-1}$, where $(0 \leq i \leq k \leq n)$ in the chain complex C. We can then define the map

$$d_2 = \partial_0 - \partial_1 + \partial_2 : C_2 \longrightarrow C_1,$$

which is given by

$$\begin{aligned} f_1 &\longrightarrow e_{25} - e_{15} + e_{12} \\ f_2 &\longrightarrow e_{35} - e_{25} + e_{23} \\ f_3 &\longrightarrow e_{45} - e_{35} + e_{34} \\ f_4 &\longrightarrow e_{15} - e_{45} + e_{41} \\ f_5 &\longrightarrow e_{26} - e_{16} + e_{12} \\ f_6 &\longrightarrow e_{36} - e_{26} + e_{23} \end{aligned}$$

$$
\begin{aligned}
f_7 &\longrightarrow e_{46} - e_{36} + e_{34} \\
f_8 &\longrightarrow e_{16} - e_{46} + e_{41}
\end{aligned}
$$

We can define maps $\partial_0, \partial_1 : E \longrightarrow V$ by removing the first, second vertices, respectively. Therefore we have

$$
\begin{aligned}
\partial_0 : e_{12} &\longrightarrow v_2 & \qquad \partial_1 : e_{12} &\longrightarrow v_1 \\
e_{23} &\longrightarrow v_3 & e_{23} &\longrightarrow v_2 \\
e_{34} &\longrightarrow v_4 & e_{34} &\longrightarrow v_3 \\
e_{41} &\longrightarrow v_1 & e_{41} &\longrightarrow v_4 \\
e_{15} &\longrightarrow v_5 & e_{15} &\longrightarrow v_1 \\
e_{25} &\longrightarrow v_5 & e_{25} &\longrightarrow v_2 \\
e_{35} &\longrightarrow v_5 & e_{35} &\longrightarrow v_3 \\
e_{45} &\longrightarrow v_5 & e_{45} &\longrightarrow v_4 \\
e_{16} &\longrightarrow v_6 & e_{16} &\longrightarrow v_1 \\
e_{26} &\longrightarrow v_6 & e_{26} &\longrightarrow v_2 \\
e_{36} &\longrightarrow v_6 & e_{36} &\longrightarrow v_3 \\
e_{46} &\longrightarrow v_6 & e_{46} &\longrightarrow v_4
\end{aligned}
$$

We can define map $d_1 = \partial_0 - \partial_1$ from C_1 *to* C_0, and it is given by

$$
\begin{aligned}
e_{12} &\longrightarrow v_2 - v_1 \\
e_{23} &\longrightarrow v_3 - v_2 \\
e_{34} &\longrightarrow v_4 - v_3 \\
e_{41} &\longrightarrow v_1 - v_4 \\
e_{15} &\longrightarrow v_5 - v_1 \\
e_{25} &\longrightarrow v_5 - v_2 \\
e_{35} &\longrightarrow v_5 - v_3 \\
e_{45} &\longrightarrow v_5 - v_4 \\
e_{16} &\longrightarrow v_6 - v_1 \\
e_{26} &\longrightarrow v_6 - v_2 \\
e_{36} &\longrightarrow v_6 - v_3 \\
e_{46} &\longrightarrow v_6 - v_4
\end{aligned}
$$

By viewing $C_0 = \mathbf{R}^6$, $C_1 = \mathbf{R}^{12}$, and $C_2 = \mathbf{R}^8$, the maps d_1 and d_2 are given by the following matrices

$$d_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$d_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Because $d_1 d_2$ is easily computed to be zero matrix, the sequence

$$\cdots C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

is a complex. We compute the homology $V_n(S)$ of this complex with the help of Matlab. By Finding the column space of d_1, we find $\mathrm{im}(d_1)$. This space has a basis consisting of the vectors

$$\{(-1,1,0,0,0,0),(0,-1,1,0,0,0),(0,0,-1,1,0,0),(-1,0,0,0,1,0),(-1,0,0,0,0,1)\}.$$

We note that by adding $(0,0,0,0,0,1)$ that we get a basis for $\mathbf{R}^6$. Therefore

$$C_0/\mathrm{im}(d_1) \cong \mathbf{R}.$$

Thus

$$V_0(S) = \mathbf{R}.$$

Now, $\ker(d_1)$ has a basis

$$\left\{\begin{array}{cc} (1,1,1,1,1,1,1), & (1,0,1,1,0,1,1), \\ (1,0,0,1,0,0,1), & (1,0,0,0,0,0,0), \\ (0,-1,-1,-1,0,0,0), & (0,1,0,0,0,0,0,0), \\ (0,0,1,0,0,0,0), & (0,0,0,1,0,0,0), \\ (0,0,0,0,-1,-1,-1), & (0,0,0,0,1,0,0), \\ (0,0,0,0,0,1,0), & (0,0,0,0,0,0,1) \end{array}\right\}$$

Again, $\mathrm{im}(d_2)$ has a basis

$$\left\{\begin{array}{cc} \{(1,0,0,0,0,0,0,0), & (0,1,0,0,0,0,0), \\ (0,0,1,0,0,0,0), & (0,0,0,1,0,0,0), \\ (0,0,0,0,1,0,0), & (0,0,0,0,0,1,0), \\ (0,0,0,0,0,0,1), & (0,0,0,0,-1,-1,-1) \\ (-1,0,0,1,-1,0,0), & (1,-1,0,0,0,-1,0), \\ (0,1,-1,0,0,0,-1), & (0,0,1,-1,1,1,1)\} \end{array}\right\}$$

If u_i are the basis vectors of $\ker(d_1)$,then the following vectors of $\mathrm{im}(d_2)$ can be constructed in the following way:

$$\begin{array}{lll} (-1,0,0,1,-1,0,0) & \text{is} & u_9 - u_4 + u_{12} + u_{10} + u_{11}, \\ (1,-1,0,0,0,-1,0) & \text{is} & u_4 - u_6 - u_{11}, \\ (0,1,-1,0,0,0,-1) & \text{is} & u_6 - u_7 - u_{12} \qquad\qquad \text{and} \\ (0,0,1,-1,1,1,1) & \text{is} & u_7 - u_8 + u_{10} + u_{12} + u_{11}. \end{array}$$

The rest of the elements of $\mathrm{im}(d_2)$ can be found in $\ker(d_1)$.

Thus we see that $\mathrm{im}(d_2) = \ker(d_1)$. Therefore, $V_1(S) = 0$. Finally, $\ker(d_2)$ has a basis of one element $\{(-1,-1,-1,-1,1,1,1,1)\}$. So, $V_2(S) = \ker d_2 = \mathbf{R}$. To summarize, the singular homology $V_n(S)$ of the Octahedron is

$$\begin{aligned} V_0(S) &= V_2(S) = \mathbf{R}, \\ V_1(S) &= 0, \\ V_n(S) &= 0 \ \text{ if } \ n \geq 3. \end{aligned}$$

The *Euler characteristic* is a fundamental invariant for the classification of surfaces, so it is particulary useful that it can be calculated with homological algebra. The Euler characteristic of such surface H is $v - e + f$, where v is the number of vertices, e is the number of edges and f is the number of faces. Now, the Euler characteristic of octahedron is 2, which is equal to $\sum\limits_{n=0}^{\infty} dim_R(V_n(S))$. This is the same as the Euler Characteristic of a sphere as a octahedron is homeomorphic to a sphere, so it is homotopic to a sphere.

§3. Conclusion

One can find the homology of other polyhedron like prism, decahedron etc.

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Even Modular Edge Irregularity Strength of Graphs

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Abstract: A new graph characteristic, even modular edge irregularity strength of graphs is introduced. Estimation on this parameter is obtained and the precise values of this parameter are obtained for some families of graphs.

Key Words: Irregular labeling, modular irregular labeling, even modular edge irregular labeling, vertex k-labeling, irregularity strength, modular irregularity strength, even modular edge irregularity strength, Smarandachely p-modular edge irregularity strength.

AMS(2010): 05C78.

§1. Introduction

Let $G = (V, E)$ be a simple graph, having at most one isolated vertex and no component of order 2. A map that carries vertex set (edge set or both) as domain to the positive integers $\{1, 2, \cdots, k\}$ is called *vertex k-labeling* (*edge k-labeling or total k-labeling*). Well-known parameter irregularity strength of a graph introduced by Chartrand et al. [6]. A simple graph G is called irregular if there exists an edge k-labeling $\lambda : E(G) \to \{1, 2, \cdots, k\}$ such that the weight of a vertex v under the labeling defined by $w_\lambda(v) = \sum \lambda(uv)$, are pairwise distinct. The minimum value of k, for which G is irregular, called irregularity strength of G denoted by $s(G)$.

The parameter irregularity strength of a graph is attracted by numerous authors. Aigner and Triesh [1] proved that $\mathrm{s}(G) \leqslant n-1$ if G is a connected graph of order n, and $\mathrm{s}(G) \leqslant n+1$ otherwise. Nierhoff [15] refined their method and showed that $\mathrm{s}(G) \leqslant n-1$ for all graphs with finite irregularity strength, except for K_3. This bound is tight e.g. for stars. In particular Faudree and Lehel [8] showed that if G is d-regular ($d \geqslant 2$), then $\left\lceil \frac{n+d-1}{d} \right\rceil \leqslant \mathrm{s}(G) \leqslant \left\lceil \frac{n}{2} \right\rceil + 9$, and they conjectured that $\mathrm{s}(G) \leqslant \left\lceil \frac{n}{d} \right\rceil + c$ for some constant c. Przybylo in [16] proved that $\mathrm{s(G)} \leqslant 16\frac{\mathrm{n}}{\mathrm{d}} + 6$. Kalkowski, Karonski and Pfender [12] showed that $\mathrm{s}(G) \leqslant 6\frac{n}{\delta} + 6$, where δ is the minimum degree of graph G. Currently Majerski and Przybylo [13] proved that $\mathrm{s}(G) \leqslant (4 + o(1))\frac{n}{\delta} + 4$ for graphs with minimum degree $\delta \geqslant \sqrt{n} \ln n$. Other interesting results on the irregularity strength can be found in [3, 4, 5, 7, 9]. For recent survey of graph labeling refer the paper [10].

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Ali Ahmad et al.[2] introduced edge irregularity strength of a graph as follows: Consider a simple graph G together with a vertex k-labeling $\chi : V(G) \to \{1, 2, \cdots, k\}$. The weight of an edge xy in G, denoted by $wt(xy) = \chi(x) + \chi(y)$. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges e and f there is $wt(e) \neq wt(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$. The lower bound of $es(G)$ was given by the following inequality

$$es(G) \geqslant max\{\left\lceil \frac{E(G)+1}{2} \right\rceil, \Delta\}$$

where Δ is the maximum degree of graph G. Ibrahim Tarawneh et al. [11], determined the exact value of edge irregularity strength of corona graphs of path P_n with P_2, P_n with K_1 and P_n with S_m.

Martin Bača et al. [14] introduced modular irregularity strength of a graph. An edge labeling $\psi : E(G) \to \{1, 2, \cdots, k\}$ is called *modular irregular k-labeling* if there exists a bijective weight function $\sigma : V(G) \to Z_n$ defined by $\sigma(x) = \sum \psi(xy)$ called *modular weight* of the vertex x, where Z_n is the group of integers modulo n and the sum is over all vertices y adjacent to x. They defined the *modular irregularity strength* of a graph G, denoted by ms(G), as the minimum k for which G has a modular irregular k-labeling.

Motivated by the edge irregularity strength of graphs we introduce a new parameter, an even modular edge irregularity strength of graph, a modular version of edge irregularity strength.

Let $G = (V, E)$ be a (n, m)-graph together with a vertex k-labeling $\rho : V(G) \to \{1, 2, \cdots, k\}$. Define a set of edge weight $W = \{wt(uv) : wt(uv) = \rho(u) + \rho(v), \forall\ uv \in E\}$. Vertex labeling ρ is called *even modular edge irregular labeling* if there exists a bijective map $\sigma : W \to M$ defined for each edge weight $wt(uv)$ there corresponds an element $x \in M$ such that $wt(uv) \equiv x \pmod{2m}$, where $M = \{0, 2, 4, \cdots, 2(m-1)\}$. We define the *even modular edge irregularity strength* of a graph G, denoted by emes(G), as the minimum k for which G has an even modular edge irregular labeling. If there doesn't exist an even modular edge irregular labeling for G, we define emes$(G) = \infty$. Generally, if $M = \{0, p, 2p, \cdots, (m-1)p\}$ for a prime number p, such a modular edge irregular labeling is called a *Smarandache p-modular edge irregular labeling* and the minimum k for which G has a Smarandachely p-modular edge irregular labeling is denoted by emes$^p(G)$. Clearly, emes$^2(G) =$ emes(G).

The main aim of this paper is to show a lower bound of the even modular edge irregularity strength and determine the precise values of this parameter for some families of graphs.

§2. Main Results

Following theorem gives the lower bound of even modular edge irregularity strength of a graph.

Theorem 2.1 *Let G be a (n,m)-graph. Then $emes(G) \geq m$.*

Proof Let G be a (n,m)-graph together with an even modular edge irregular labeling

$\rho : V(G) \to \{1, 2, \cdots, k\}$. Consider the even edge weights of G, there should be an edge e such that $wt(e) \equiv 0 \pmod{2m}$. Since the weight of e must be at least $2m$, $emes(G) \geq m$. □

Lemma 2.1 *Let $(d_1, d_2, \cdots, d_n)$ be the degree sequence of a graph G and let $(l_1, l_2, \ldots, l_n)$ be the corresponding vertex labels of an even modular edge irregular labeling of G. Then the sum of all the edge weights denoted as S is equal to the sum of the product of degree with its corresponding labels, that is,*

$$S = \sum_{e \in E} wt(e) = \sum_{i=1}^{n} d_i l_i.$$

Lemma 2.2 *In any even modular edge irregular labeling of C_n, labels of all vertices are of same parity.*

Proof By definition, weight of an edge is sum of the labels of its end vertices. To obtain an even edge weight, both the labels must be either odd or even, and hence all the vertex labels of C_n are of same parity. □

Theorem 2.2 *Let C_n be a cycle of order $n \geq 3$. Then*

$$emes(C_n) = \begin{cases} n+1, & \text{if } n \equiv 0 \pmod 4, \\ n, & \text{if } n \equiv 1 \pmod 4, \\ n+2, & \text{if } n \equiv 3 \pmod 4, \\ \infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof Let $V(C_n) = \{v_i : i = 1, 2, \cdots, n\}$ be the vertex set and let $E(C_n) = \{e_i = v_i v_{i+1} : i = 1, 2, \cdots, n\}$ be the edge set of the cycle C_n. Define the vertex labeling $\rho : V \to \{1, 2, \cdots, n+2\}$ as follows:

$$\rho(v_i) = 2i - 1, \ 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$$

If $n \equiv 0, 1 \pmod 4$, then, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$\rho(v_{n+1-i}) = \begin{cases} 2i - 1, & \text{i is odd} \\ 2i + 1, & \text{i is even} \end{cases}$$

If $n \equiv 3 \pmod 4$, then for $2 \leq i \leq \lceil \frac{n}{2} \rceil$,

$$\rho(v_{n+2-i}) = \begin{cases} 2i - 1, & \text{i is odd} \\ 2i + 1, & \text{i is even} \end{cases}$$

We can easily check that the above labeling ρ, is an even modular edge irregular labeling of C_n. Thus,

$$emes(C_n) \leq \begin{cases} n+1, & \text{if } n \equiv 0 \pmod 4 \\ n, & \text{if } n \equiv 1 \pmod 4 \\ n+2, & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Now let us find the lower bound of $emes(C_n)$ as follows:

Case 1. Suppose $n \equiv 0 \pmod 4$. Consider the set of even edge weights $W(C_n) = \{2, 4, 6, \cdots, 2n\}$. To obtain the weight 2 for an edge, we must assign label 1 to both of its end vertices, and hence all the vertices of C_n must receive odd labels by Lemma 2.2. Since the heaviest weight is $2n$, $emes(C_n) \geq n+1$.

Case 2. Suppose $n \equiv 1 \pmod 4$. By Theorem 2.1, $emes(C_n) \geq n$.

Case 3. Suppose $n \equiv 3 \pmod 4$. Assume that the cycle C_n has the set of even edge weights $W(C_n) = \{2, 4, \cdots, 2n\}$, then $\frac{S}{2}$ is even, where S is the sum of the weights. Since the least weight is 2, by Lemma 2.2 all the vertex labels of C_n must be odd and hence $\sum_{i=1}^{n} l_i$ is odd, which is a contradiction to $\sum_{i=1}^{n} l_i = \frac{S}{2}$ by Lemma 2.1.

Assume that C_n has the set of even edge weights $W(C_n) = \{4, 6, \cdots, 2n+2\}$. Now the sum of the labels,

$$\sum_{i=1}^{n} l_i = \frac{S}{2} = \frac{(n+1)(n+2)}{2} - 1$$

is odd and hence each label must odd. Heaviest weight $2n+2$ can be obtained by assigning the label at least $n+2$. Thus, $emes(C_n) \geq n+2$.

Case 4. Suppose $n \equiv 2 \pmod 4$. If the cycle C_n has an even modular edge irregular labeling, then the sum of the edge weights $S \equiv 2 \pmod 4$, and hence $\frac{S}{2}$ is odd. When $n \equiv 2 \pmod 4$, sum of the labels $\sum_{i=1}^{n} l_i$ is even, which is a contradiction to $\sum_{i=1}^{n} l_i = \frac{S}{2}$. Thus, $emes(C_n) = \infty$, if $n \equiv 2 \pmod 4$. □

Theorem 2.3 *Let P_n be a path of order $n \geq 2$. Then*

$$emes(P_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

Proof Let $V(P_n) = \{v_i : i = 1, 2, \cdots, n\}$ be the vertex set and let $E(P_n) = \{e_i = v_i v_{i+1} : i = 1, 2, \cdots, n\}$ be the edge set of the path P_n.

Define the vertex n-labeling $\theta : V \to \{1, 2, \cdots, n\}$ as follows:

For $1 \leq i \leq n$,

$$\theta(v_i) = \begin{cases} i, & \text{i is odd} \\ i-1, & \text{i is even.} \end{cases}$$

Clearly, θ is an even modular edge irregular labeling of P_n. Thus, the upper bound of

$emes(P_n)$ can be obtained as follows:

$$emes(P_n) \leq \begin{cases} n, & \text{if n is odd} \\ n-1, & \text{if n is even.} \end{cases}$$

Let us find the lower bound of $emes(P_n)$.

Case 1. Assume that n is odd. Consider the optimal even edge weight $W(P_n) = \{2, 4, \cdots, 2(n-1)\}$. Since the least weight is 2, all the vertices of P_n must receive odd labels. To obtain the heaviest weight $2(n-1)$, we must assign vertex label at least n. Thus, $emes(P_n) \geq n$.

Case 2. Assume that n is even. In this case, the lower bound can be obtain directly from Theorem 2.1. □

Theorem 2.4 *Let $K_{1,n}$ be a star graph of order $n+1, n \geq 1$. Then $emes(K_{1,n}) = 2n-1$.*

Proof Let $V(K_{1,n}) = \{x, v_i : i = 1, 2, \cdots, n\}$ be the vertex set and let $E(K_{1,n}) = \{e_i = xv_i : i = 1, 2, \cdots, n\}$ be the edge set of the path $K_{1,n}$.

Define the vertex labeling $\lambda_1 : V \to \{1, 2, \cdots, 2n-1\}$ as follows:

$\lambda_1(x) = 1,$

$\lambda_1(v_i) = 2i - 1, 1 \leq i \leq n.$

From the above even modular edge irregular labeling λ_1, upper bound of $emes(K_{1,n})$ is obtained as follows, $emes(K_{1,n}) \leq 2n - 1$.

Consider the optimal even edge weights $W(K_{1,n}) = \{2, 4, \ldots, 2n\}$. Since the least weight is 2, the vertex x must be label with 1. To obtain the heaviest weight $2n$, we must assign label at least $2n-1$ to other end vertex. Thus, $emes(K_{1,n}) \geq 2n - 1$. Hence the theorem. □

Theorem 2.5 *Let $K_{2,n}$ be the complete bipartite graph of order $n+2, n \geq 2$. Then $emes(K_{2,n}) = 2n + 1$.*

Proof Let $V(K_{2,n}) = \{x, y, v_i : i = 1, 2, \cdots, n\}$ be the vertex set and let $E(K_{2,n}) = \{xv_i, yv_i : i = 1, 2, \cdots, n\}$ be the edge set of the complete bipartite graph $K_{2,n}$.

Define the vertex labeling $\lambda_2 : V \to \{1, 2, \cdots, 2n+1\}$ as follows:

$\lambda_2(x) = 1, \quad \lambda_2(y) = 2n + 1$

$\lambda_2(v_i) = 2i - 1, 1 \leq i \leq n.$

From the above even modular edge irregular labeling λ_2, upper bound of $emes(K_{2,n})$ is obtained as follows: $emes(K_{2,n}) \leq 2n + 1$.

Consider the even edge weights of $K_{2,n}$ as $2, 4, \cdots, 4n$. Since the least edge weight is 2, all the vertices must receive odd labels. Therefore, we must assign label at least $2n+1$, to obtain the heaviest weight $4n$. Hence $emes(K_{2,n}) \geq 2n + 1$. □

A rectangular graph R_n, $n \geq 2$, is a graph obtained from the path P_{n+1} by replacing each edge of the path by a rectangle C_4. Let

$$V(R_n)=\{v_i : i=1,2,\cdots,2n\}\bigcup\{u_j : j=1,2,\cdots,n+1\}$$

be the vertex set and let

$$\begin{aligned}E(R_n) &= \{v_{2i-1}v_{2i} : i=1,2,\cdots,n\}\bigcup\{u_iu_{i+1} : i=1,2,\cdots,n\}\\ &\quad\bigcup\{v_{2i-1}u_i : i=1,2,\cdots,n\}\bigcup\{v_{2i-2}u_i : i=2,3,\cdots,n+1\}\end{aligned}$$

be the edge set of the the rectangular graph R_n. The following theorem gives the precise value of even modular edge irregularity strength of rectangular graph.

Theorem 2.6 *Let R_n be a rectangular graph of order $3n+1$, $n\geq 2$. Then $emes(R_n)=4n+1$.*

Proof Define the vertex labeling $\alpha : V\to\{1,2,\cdots,4n+1\}$ as follows:

$\alpha(v_i)=2i-1, 1\leq i\leq 2n,$
$\alpha(u_i)=4i-3, 1\leq i\leq n+1.$

Upper bound $emes(R_n)\leq 4n+1$ can be obtained from the above labeling α.

Consider the even edge weights of R_n as $2,4,\cdots,8n$. Since the least weight is 2, all the vertex labels must be odd. Therefore, we must assign label at least $4n+1$, to obtain the heaviest weight $8n$. Hence, $emes(R_n)\geq 4n+1$. □

Theorem 2.7 *Let tP_4, $t\geq 1$, denote the disjoint union of t copies of path P_4. Then $emes(tP_4)=3t$.*

Proof Let $V(tP_4)=\{u_{ij} : 1\leq i\leq t, 1\leq j\leq 4\}$ be the vertex set and let $E(tP_4)=\{u_{i1}u_{i2},\ u_{i2}u_{i3},\ u_{i3}u_{i4} : 1\leq i\leq t\}$ be the edge set of tP_4. Define the vertex labeling $\beta : V\to\{1,2,\cdots,3t\}$ as follows:

$\beta(u_{i1})=\beta(u_{i2})=3i-2,\ 1\leq i\leq t,$
$\beta(u_{i3})=\beta(u_{i4})=3i,\ 1\leq i\leq t.$

Clearly, β is an even modular edge irregular labeling of tP_4 and hence $emes(tP_4)\leq 3t$. The lower bound $emes(tP_4)\geq 3t$ can be obtained directly from Theorem 2.1. Hence, we get that $emes(tP_4)=3t$. □

Theorem 2.8 *Let tC_3, $t\geq 2$, denote the disjoint union of t copies of cycle C_3. Then $emes(tC_3)=3t+2$.*

Proof Let $V(tC_3)=\{v_{ij} : 1\leq i\leq t, 1\leq j\leq 3\}$ be the vertex set and let $E(tC_3)=\{v_{i1}v_{i2}, v_{i2}v_{i3}, v_{i1}v_{i3} : 1\leq i\leq t\}$ be the edge set of tC_3. Define the vertex labeling $\theta : V\to\{1,2,\cdots,3t+2\}$ as follows:

$$\theta(v_{i1})=\begin{cases}1, & i=1\\ 3t, & 2\leq i\leq t,\end{cases}\qquad \theta(v_{i2})=3i+2, 1\leq i\leq t,\qquad\text{and}\qquad \theta(v_{i3})=\begin{cases}3, & i=1\\ 3i+1, & 2\leq i\leq t,\end{cases}$$

Clearly, θ is an even modular edge irregular labeling of tP_4 and hence $emes(tC_3) \leq 3t+2$.

Consider the optimal edge weights of tC_3 as $4, 6, 8, \cdots, 6t+2$. Since any two adjacent vertices of tC_3 can not receive the same labels, we must assign label at least $3t+2$ to get the heaviest label $6t+2$. Hence, $emes(tC_3) \geq 3t+2$. □

Ladder graph $L_n = K_2 \times P_n, n \geq 3$ is formed by taking two isomorphic copies of P_n and joining the corresponding vertices by an edge. Let $V = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set and let

$$E = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_iv_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_iv_i : 1 \leq i \leq n\}$$

be the edge set of L_n. The following theorem gives the precise value of even modular edge irregularity strength of ladder graph.

Theorem 2.9 *Let $L_n = K_2 \times P_n, n \geq 3$ be the ladder graph. Then*

$$emes(L_n) = \begin{cases} 3n-2, & \textit{if n is odd,} \\ 3n-1, & \textit{if n is even.} \end{cases}$$

Proof Defined the vertex labeling $\phi : V \to \{1, 2, \cdots, 3n-1\}$ as follows:

$$\phi(u_i) = \begin{cases} 3i-2, & \text{if i is odd} \\ 3i-3, & \text{if i is even} \end{cases} \quad 1 \leq i \leq n,$$

$$\phi(v_i) = \begin{cases} 3i-2, & \text{if i is odd} \\ 3i-1, & \text{if i is even.} \end{cases} \quad 1 \leq i \leq n.$$

Clearly, ϕ is an even modular edge irregular labeling of L_n and hence

$$emes(L_n) \leq \begin{cases} 3n-2, & \text{if n is odd,} \\ 3n-1, & \text{if n is even.} \end{cases}$$

Lower bound $emes(L_n) \geq 3n-2$, can be obtained directly from Theorem 2.1, when n is odd.

Suppose n is even. Consider optimal edge weights of L_n as $2, 4, \cdots, 6n-4$. Since L_n has a span cycle, all the vertices of L_n must receive the labels of same parity. Furthermore, to obtain the edge weight 2, the corresponding end vertices must be label 1, and hence all the labels must be odd. Thus $emes(L_n) \geq 3n-1$. Hence the theorem. □

§3. Conclusion

In this paper we introduced a new graph parameter, the even modular edge irregularity strength, $emes(G)$, as a modular version of edge irregularity strength. We determined the exact value of even modular edge irregularity strength of some families of graphs and a lower bound of *emes* is obtained. However, the determination of upper bound is still open.

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Finslerian Hypersurfaces and Quartic Change of Finsler Metric

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Abstract: In the present paper we have studied the Finslerian hypersurfaces and quartic change of a Finsler metric. The relationship with Finslerian hypersurface and the other which is finslerian hypersurface given by quartic change have been obtained. We have also proved that quartic change makes three type of hyper surfaces invarient under certain condition. These three type of hyper surfaces are hyperplanes of first, second, and third kind.

Key Words: Finsler metric, Finslerian hyperspaces, quartic change, hyperplanes of first, second, and third kind.

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§1. Introduction

Let (M^n, L) be an n- dimensional Finsler space on a differential Manifold M^n, equipped with fundamental function $L(x,y)$. In 1984 C. Shibata [11] introduced the transformation of Finsler metric:

$$L^*(x,y) = f(L,\beta), \tag{1.1}$$

where $\beta = b_i y^i$, $b_i(x)$ are the components of a covariant vector in (M^n, L) and f is positively homogeneous function of degree one in L and β. The change of metric is called a β- change. A particular β- change of a Finsler metric function is a quartic change of metric function is defined as

$$L^* = (L^4 + \beta^4)^{1/4}. \tag{1.2}$$

If $L(x,y)$ reduces to the metric function of Riemann space then $L^*(x,y)$ reduces to the metric function of space generated by quartic metric. Due to this reason this transformation (1.2) has been called the quartic change of Finsler metric.

On the other hand, in 1985, M. Matsumoto investigated the theory of Finslerian hypersurface [4]. He has defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds.These names were given by Rapesak [8]. Kikuchi [3] gave other name following haimovichi [1]. In the year 2005, Prasad and Tripathi [7] studied the Finslerian Hy-

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persurfaces and Kropina change of a Finsler metric and obtained different results in his paper. Again, in the year 2005, Prasad, Chaubey and Patel [6] studied the Finslerian Hypersurfaces and Matsumoto change of a Finsler metric and obtained different results.

In the present paper, using the field of linear frame ([2, 3, 5]) we shall consider Finslerian hypersurfaces given by a quartic change of a Finsler metric.Our purpose is to give some relations between the original Finslerian hypersurface and the other which is Finselrian hypersurface given by quartic change. We also show that a quartic change makes three types of hypersurfaces invariant under certain condition.

§2. **Preliminaries**

Let M^n be an n-dimensional smooth manifold and $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with the fundamental function $L(x, y)$ on M^n. The metric tensor $g_{ij}(x, y)$ and Cartan's C-tensor $C_{ijk}(x, y)$ are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

respectively and we can introduce the Cartan's Connection
$C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ in F^n.

A hypersurface M^{n-1} of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where u^α are Gaussian coordinates on M^{n-1} and Greek indices vary from 1 to $n-1$. Here we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \partial x^i/\partial u^\alpha$ is of rank $n-1$. The following notations are also employed: $B^i_{\alpha\beta} = \partial^2 x^i/\partial u^\alpha \partial u^\beta$, $B^i_{o\beta} = v^\alpha B^i_{\alpha\beta}$. If the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, we may then write $y^i = B^i_\alpha(u)v^\alpha$ i.e. v^α is thought of as the supporting element of M^{n-1} at the point (u^α). Since the function $\bar{L}(u, v) = L(x(u), y(u, v))$ gives to a Finsler metric of M^{n-1}, we get a $n-1$ -dimensional Finsler space $F^{n-1} = (M^{n-1}, \bar{L}(u, v))$.

At each point (u^α) of F^{n-1}, the unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}B^i_\alpha N^j = 0, \quad g_{ij}N^iN^j = 1. \tag{2.1}$$

If (B^α_i, N_i) is the inverse matrix of (B^i_α, N^i), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0, \quad N^iN_i = 1 \textit{ and } B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j.$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \quad N_i = g_{ij}N^j. \tag{2.2}$$

For the induced Cartan's connection $IC\Gamma = (F^\alpha_{\beta\gamma}, N^\alpha_\beta, C^\alpha_{\beta\gamma})$ on F^{n-1}, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature vector H_α are respectively given by [8]

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta) + M_\alpha H_\beta, \quad H_\beta = N_i(B^i_{0\beta} + N^i_j B^j_\beta), \tag{2.3}$$

where

$$M_\alpha = C_{ijk}B^i_\alpha N^j N^k. \tag{2.4}$$

Contracting $H_{\alpha\beta}$ by v^α, we immediately get $H_{0\beta} = H_{\alpha\beta}v^\alpha = H_\beta$. Furthermore the second fundamental v-tensor $M_{\alpha\beta}$ is given by [10]

$$M_{\alpha\beta} = C_{ijk}B^i_\alpha B^j_\beta N^k. \tag{2.5}$$

§3. Quartic Changed Finsler Space

Let $F^n = (M^n, L)$ be a given Finsler space and let $b_i(x)dx^i$ be a one form on M^n. We shall define on M^n a function $L^*(x,y)(> 0)$ by the equation (1.2) where we put $\beta(x,y) = b_i(x)y^i$. To find the metric tensor g^*_{ij}, the angular metric tensor h^*_{ij}, the Cartan's C-tensor C^*_{ijk} of $F^{*n} = (M^n, L^*)$ we use the following results.

$$\partial\beta/\partial y^i = b_i, \quad \partial L/\partial y^i = l_i \quad \partial l_j/\partial y^i = L^{-1}h_{ij}, \tag{3.1}$$

where h_{ij} are components of angular metric tensor of F^n given by

$$h_{ij} = g_{ij} - l_i l_j = L\partial^2 L/\partial y^i \partial y^j.$$

The successive differentiation of (1.2) with respect to y^i and y^j give

$$l^*_i = \frac{L^3 l_i + \beta^3 b_i}{(L^4+\beta^4)^{3/4}} \tag{3.2}$$

$$\begin{aligned} h^*_{ij} \quad = \quad & \frac{1}{(L^4+\beta^4)^{3/2}}[L^2(L^4+\beta^4)h_{ij} + 3\beta^4 L^2 l_i l_j \\ & +3L^4\beta^2 b_i b_j - 3L^3\beta^3(l_i b_j + l_j b_i)] \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we get the following relation between metric tensors of F^n and F^{*n}

$$\begin{aligned} g^*_{ij} \quad = \quad & \frac{1}{(L^4+\beta^4)^{3/2}}[L^2(L^4+\beta^4)g_{ij} + 2\beta^4 L^2 l_i l_j \\ & +\beta^2(3L^4+\beta^4)\beta^2 b_i b_j - 2L^3\beta^3(l_i b_j + l_j b_i)]. \end{aligned} \tag{3.4}$$

Differentiating (3.4) with respect to y^k and using (3.1) we get the following relation between the Cartan's C-tensor of F^n and F^{*n}

$$C^*_{ijk} = \left(\frac{L^2}{\sqrt{L^4+\beta^4}}\right)C_{ijk} - \left(\frac{L^2\beta^3}{(L^4+\beta^4)^{3/2}}\right)(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) \\ -3\left(\frac{(\beta^4-L^4)}{(L^4+\beta^4)^{5/2}}\right)m_i m_j m_k, \tag{3.5}$$

where $m_i = b_i - (\beta/L)l_i$. It is to be noted that

$$m_i l^i = 0,\ m_i b^i = b^2 - \beta^2/L^2,\ h_{ij}l^j = 0, h_{ij}m^j = h_{ij}b^i = m_i, \tag{3.6}$$

where $m^i = g^{ij}m_j = b^i - (\beta/L)l^i$.

§4. Hypersurface Given by a Quartic Change

Consider a Finslerian hypersurface $F^{n-1} = (M^{n-1}, \bar{L}(u,v))$ of the F^n and another Finslerian hypersurface $F^{*n-1} = (M^{n-1}, \bar{L}^*(u,v))$ of the F^{*n} given by the quartic change. Let N^i be the unit normal vector at each point of F^{n-1} and (B^α_i, N_i) be the inverse matrix of (B^i_α, N^i). The functions B^i_α may be considered as components of $n-1$ linearly independent tangent vectors of F^{n-1} and they are invariant under quartic change. Thus we shall show that a unit normal vector $N^{*i}(u,v)$ of F^{*n-1} is uniquely determined by

$$g^*_{ij}B^i_\alpha N^{*j} = 0, \quad g^*_{ij}N^{*i}N^{*j} = 1. \tag{4.1}$$

Contrtacting (3.4) by N^iN^j and paying attention to (2.1) and $l_iN^i = 0$, we have

$$g^*_{ij}N^iN^j = \frac{L^2(L^4+\beta^4) + \beta^2(3L^4+\beta^4)(b_iN^i)^2}{(L^4+\beta^4)^{3/2}}.$$

Therefore we obtain

$$g^*_{ij}\left(\pm\frac{(L^4+\beta^4)^{3/4}N^i}{\sqrt{L^2(L^4+\beta^4)+\tau(b_iN^i)^2}}\right)\left(\pm\frac{(L^4+\beta^4)^{3/4}N^j}{\sqrt{L^2(L^4+\beta^4)+\tau(b_iN^i)^2}}\right) = 1.$$

where

$$\tau = \beta^2(3L^4+\beta^4).$$

Hence we can put

$$N^{*i} = \frac{(L^4+\beta^4)^{3/4}N^i}{\sqrt{L^2(L^4+\beta^4)+\tau(b_iN^i)^2}}, \tag{4.2}$$

where we have choosen the positive sign in order to fix an orientation.

Using (2.1), (3.4), (4.1) and (4.2) we obtain from the first condition of (4.1),

$$[-\frac{2L^3}{(L^4+\beta^4)^{3/2}}l_iB^i_\alpha+\frac{\beta^2(3L^4+\beta^4)}{(L^4+\beta^4)^{3/2}}l_iB^i_\alpha]\frac{(L^4+\beta^4)^{3/4}b_jN^j}{\sqrt{L^2(L^4+\beta^4)+\tau(b_iN^i)^2}}=0$$

If

$$-2L^3\beta l_iB^i_\alpha+(3L^4+\beta^4)b_iB^i_\alpha=0$$

then contracting it by v^α and using $y^i=B^i_\alpha v^\alpha$ we get $L=0$ which is a contradiction with assumption that $L>0$. Hence $b_iN^i=0$. Therefore (4.2) is rewritten as

$$N^{*i}=\frac{(L^4+\beta^4)^{1/4}N^i}{L}. \tag{4.3}$$

Summary the above, we obtain

Proposition 4.1 *For a field of linear frame $(B^i_1,B^i_2,\cdots,B^i_{n-1},N^i)$ of F^n there exists a field of linear frame $(B^i_1,B^i_2,\cdots,B^i_{n-1},N^{*i}=\frac{(L^4+\beta^4)^{1/4}N^i}{L})$ such that (4.1) is satisfied along F^{*n-1} and then b_i is tangential to both the hypersurface F^{n-1} and F^{*n-1}.*

The quantities $B^{*\alpha}_i$ are uniquely defined along F^{n-1} by

$$B^{*\alpha}_i=g^{*\alpha\beta}g^*_{ij}B^j_\beta,$$

where $(g^{*\alpha\beta})$ is the inverse matrix of $(g^*_{\alpha\beta})$. Let $(B^{*\alpha}_i,N^*_i)$ be the inverse of (B^i_α,N^i), then we have $B^i_\alpha B^{*\beta}_i=\delta^\beta_\alpha, B^i_\alpha N^*_i=0, N^{*i}N^*_i=1$ and furthermore $B^i_\alpha B^{*\alpha}_j+N^{*i}N^*_j=\delta^i_j$. We also get $N^*_i=g^*_{ij}N^{*j}$ which in view of $(3.4),(2.2)$ and (4.3) gives

$$N^*_i=\frac{L}{(L^4+\beta^4)^{1/4}}N_i. \tag{4.4}$$

We denote the Cartan's connection of F^n and F^{*n} by $(F^i_{jk},N^i_j,C^i_{jk})$ and $(F^{*i}_{jk},N^{*i}_j,C^{*i}_{jk})$ respectively and put $D^i_{jk}=F^{*i}_{jk}-F^i_{jk}$ which will be called difference tensor. We choose the vector field b_i in F^n such that

$$D^i_{jk}=A_{jk}b^i-B_{jk}l^i, \tag{4.5}$$

where A_{jk} and B_{jk} are components of a symmetric covariant tensor of second order. Since $N_ib^i=0$ and $N_il^i=0$, from (4.5) we get

$$N_iD^i_{jk}=0,\quad N_iF^{*i}_{jk}=N_iF^i_{jk},\quad \text{and}\quad N_iD^i_{0k}=0. \tag{4.6}$$

Therefore from (3.3) and (4.3) we get

$$H^*_\alpha=\frac{L}{(L^4+\beta^4)^{1/4}}H_\alpha. \tag{4.7}$$

If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of enveloping space F^n, then F^{n-1} is called a hyperplane of the first kind [?]. A hyperplane of the first kind is characterized by $H_\alpha = 0$. Hence from (4.7), we have

Theorem 4.1 *If $b_i(x)$ be a vector field in F^n satisfying* (4.5)*, then a hypersurface F^{n-1} is a hyperplane of the first kind if and only if the hypersurface F^{*n-1} is a hyperplane of the first kind.*

Next contracting (3.5) by $B^\alpha_i N^{*j} N^{*k}$ and paying attention to (4.4), $m_i N^i = 0, h_{jk} N^j N^k = 1$ and $h_{ij} B^i_\alpha N^j = 0$ we get

$$M^*_\alpha = M_\alpha - \frac{\beta^3}{(L^4+\beta^4)^{3/2}} m_i B^i_\alpha. \tag{4.8}$$

From (2.3),(4.4) (4.5), (4.6), (4.7) and (4.8), we have

$$H^*_{\alpha\beta} = \frac{L}{(L^4+\beta^4)^{1/4}} (H_{\alpha\beta} - \frac{\beta^3}{(L^4+\beta^4)^{3/2}} m_i B^i_\alpha H_\beta). \tag{4.9}$$

If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also h-path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the second kind [?]. A hyperplane of the second kind is characterized by $H_{\alpha\beta} = 0$. Since $H_{\alpha\beta} = 0$ implies that $H_\alpha = 0$, from (4.7) and (4.9) we have the following

Theorem 4.2 *If $b_i(x)$ be a vector field in F^n satisfying* (4.5)*, then a hypersurface F^{n-1} is a hyperplane of the second kind if and only if the hypersurface F^{*n-1} is a hyperplane of the second kind.*

Finally contrcting (3.5) by $B^i_\alpha B^j_\beta N^{*k}$ and paying attention to (4.3) we have

$$M^*_{\alpha\beta} = \frac{L}{(L^4+\beta^4)^{1/4}} M_{\alpha\beta}. \tag{4.10}$$

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, the F^{n-1} is called a hyperplane of the third kind [8]. A hyperplane of the third kind is characterized by $H_{\alpha\beta} = 0, M_{\alpha\beta} = 0$. From (4.7), (4.9) and (4.10) we have

Theorem 4.3 *If $b_i(x)$ be a vector field in F^n satisfying* (4.5)*, then a hypersurface F^{n-1} is a hyperplane of the third kind if and only if the hypersurface F^{*n-1} is a hyperplane of the third kind.*

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Adjacency Matrices of Some Directional Paths and Stars

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Abstract: Graph labeling where vertices and edges are assigned values subject to certain conditions have been motivated by various applied mathematical fields. Most of the problems in graph labeling are discussed on undirected graphs. Bloom G.S. and Hsu D.F. defined the labeling on directed graphs. In this paper, we discuss the adjacency matrices of graceful digraphs such as unidirectional paths,alternating paths,many orientations of directed star and a class of directed bistar. We also discuss the adjacency matrices of unidirectional paths and alternating paths if they are odd digraceful.

Key Words: Digraceful labeling, digraceful graph, Smarandachely H-digraceful graph, adjacency matrix, unidirectional paths, alternating paths.

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§1. Introduction

Most of the applications in Engineering and Science have undirected graphs are in the problems.The graph labeling problems are widely focus on undirected graphs.There exists situations in which directed graphs are playing a key role in some problems. There we using graph labeling in directed graphs. In 1980,Bloom G.S. and Hsu D.F.([3],[4],[5])extend the concept of graceful labeling of undirected graphs to digraphs.They investigate the graceful labeling problems of digraphs. Graceful digraphs are related in a variety ways to other areas of Mathematics.

The underlying graph UG(D) of a digraph D is obtained from D by removing the direction of each arc in D. Here we consider UG(D) is connected and has no self loops or multiple edges. In otherwords the digraph D is simply connected. The vertex set and edge set of a simple connected digraph are denoted by V(D) and E(D) where $E(D) = \overrightarrow{uv}, uv\epsilon V(D)$. For an arc $\overrightarrow{uv}$ the first vertex u is its tail and second vertex v is its head.

For all terminology and notations in graph theory,we follow Harary[1] and for all terminology regarding graceful labeling,we follow [2]. A connected graph with p vertices and q edges is called graceful if it is possible to label the vertices of x with pairwise distinct integers f(x) in $\{0, 1, 2, 3, \cdots, q\}$ so that each edge, xy, is labeled $|f(x) - f(y)|$, the resulting edge labels are pairwise distinct (and thus from the entire set $\{1, 2, 3, \cdots, q\}$). A connected graph with p vertices and q edges is called odd graceful if it is possible to label the vertices of x with pairwise

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distinct integers f(x) in $\{0, 1, 2, 3, \cdots, 2q-1\}$ so that each edge, xy, is labeled $|f(x) - f(y)|$,the resulting edge labels are pairwise distinct (and thus from the entire set $\{1, 3, 5, \cdots, 2q-1\}$).

Definition 1.1([4]) *A digraph D with p vertices and q edges is said to be digraceful if there exists an injection $f : V(D) \to \{0, 1, 2, \cdots, q\}$ such that the induced function $f' : E(D) \to \{1, 2, 3, \cdots, q\}$ which is denoted by $f'(u, v) = [f(v) - f(u)](modq + 1)$ for every directed edge (u, v) is a bijection, where $[v](modn)$ denotes the least positive integer of v modulo n. If the edge values are all distinct then the labeling is called a digraceful labeling of a digraph. A digraph is called a graceful digraph if it has a digraceful labeling.*

Generally, a *Smarandachely H-digraceful graph* is such a digraceful graph G that each subgraph $H' \prec G$ is a graceful digraph if $H' \cong H$. Clearly, a Smarandachely G-digraceful graph is nothing else but a digraceful graph. Let f be a labeling of a digraph D from V(D) to $\{0, 1, 2, \cdots, q\}$ such that for each arc $\overrightarrow{uv} \epsilon D, f(\overrightarrow{uv}) = f(u) - f(v)$ if f(u)> f(v),otherwise $f(\overrightarrow{uv}) = q + 1 + f(u) - f(v)$.We call f a digraceful labeling of D if the arc label set $\{f(\overrightarrow{uv}) : \overrightarrow{uv} \epsilon E(D)\} = \{1, 2, 3, \cdots, q\}$.Therefore D is called a graceful digraph.

Definition 1.2 *A digraph D with p vertices and q edges is said to be odd digraceful if there exists an injection $f : V(D) \to \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f' : E(D) \to \{1, 2, 3, \cdots, 2q-1\}$ which is denoted by $f'(u, v) = [f(v) - f(u)](mod2q)$ for every directed edge (u, v) is a bijection,where [v](mod n) denotes the least positive integer of v modulo n.If the edge values are all distinct then the labeling is called a odd digraceful labeling of a digraph.A digraph is called a odd graceful digraph if it has a odd digraceful labeling.*

Let f be a labeling of a digraph D from V(D) to $\{0, 1, 2, \cdots, 2q-1\}$ such that for each arc $\overrightarrow{uv} \epsilon D, f(\overrightarrow{uv}) = f(u) - f(v)$ if f(u)> f(v),otherwise $f(\overrightarrow{uv}) = q + 1 + f(u) - f(v)$.We call f a odd digraceful labeling of D if the arc label set $\{f(\overrightarrow{uv}) : \overrightarrow{uv} \epsilon E(D)\} = \{1, 2, 3, \cdots, q\}$.
Therefore D is called a odd graceful digraph.

In next two sections, we develop a generalized adjacency matrix of unidirectional paths and alternating paths based on graceful labeling and odd graceful labeling.

§2. Adjacency Matrices of Unidirectional Paths

Definition 2.1 *Let D be a graph with $V(D) = \{v_1, v_2, v_3, \cdots, v_p\}$.Then the matrix $A_D = [x_{ij}]$ defined by*

$$x_{ij} = \begin{cases} +1, & \textit{if node } v_i \textit{ is connected to } v_j \textit{ and directed from } v_i \textit{ to } v_j \\ -1, & \textit{if node } v_j \textit{ is connected to } v_i \textit{ and directed from } v_i \textit{ to } v_j \\ 0, & \textit{otherwise} \end{cases} \tag{0.1}$$

is called the adjacency matrix of D.

Definition 2.2 *Let D be a digraph with q edges and a labeling*

$f : V(D) \to \{0, 1, 2, \cdots, q\}$. Then the $(q+1) \times (q+1)$ matrix $M_D = [x_{ij}]$ defined by

$$x_{ij} = \begin{cases} +1, & \text{if } xy \in E(D) \text{ for } f(x) = i \text{ and } f(y) = j \text{ directed from } x \text{ to } y \\ -1, & \text{if } yx \in E(D) \text{ for } f(y) = i \text{ and } f(x) = j \text{ directed from } x \text{ to } y \\ 0, & \text{otherwise} \end{cases} \tag{0.2}$$

is called the generalized adjacency matrix of D induced by the labeling f.

The generalized adjacency matrix of the digraph is skew symmetric.The generalized adjacency matrix allows all zeros for rows and columns corresponding to missing labels, while in the adjacency matrix such rows corresponding to vertices of degree zero.If two graphs have the same adjacency matrix,then they are isomorphic, but if two graphs have the same generalized adjacency matrix they may not be isomorphic.

Definition 2.3 *Let A be a $p \times p$ matrix.Then the k^{th} diagonal line of A is the collection of entries $D_k = \{a_{ij}/j - i = k\}$,counting multiplicity.*

It is clear that $0 \leq |k| \leq p - 1$. When $M = M_D$ is a generalized adjacency matrix,M is symmetric and $D_k = D_{-k}$. The entry ± 1 corresponding to an edge between vertices v_i and v_j lies on the $(j - i)^{th}$ diagonal line.The main diagonal line is D_k for $k = 0$. The odd diagonal lines are D_k for $k = \pm 1, \pm 3, \pm 5, \cdots, \pm 2q - 1$ and the even diagonal line are D_k for $k = \pm 2, \pm 4, \pm 6, \cdots, \pm 2q - 2$.

Definition 2.4([6]) *A finite group (G,.) of order n is said to be sequenceable if its elements can be arranged in a sequence $a_0 = e, a_1, a_2, \cdots, a_{n-1}$ in such a way that the partial products $b_0 = a_0, b_1 = a_0a_1, b_2 = a_0a_1a_2, \cdots, b_{n-1} = a_0a_1a_2 \cdots a_{n-1}$ are all distinct.*

Definition 2.7 *If both the indegree and outdegree of all the internal vertices of a directed path are one,then it is called unidirectional path and is denoted by $\overrightarrow{P_n}$.*

Theorem 2.1([4]) *$\overrightarrow{P_n}$ on n vertices is graceful if and only if z_n is sequenceable.*

Theorem 2.2 *Let D be a labeled unidirectional path with q edges and let $[M_D]$ be the generalized adjacency matrix for D. Then D is digraceful if and only if $[M_D]$ has exactly one entry ± 1 in each diagonal lines,except the main diagonal line of zeros.*

Proof It is to be noted that the matrix $[M_D]$ is a staircase shaped matrix and skew symmetric.The main diagonal line is D_k for $k = 0$.

Suppose that $[M_D]$ has exactly one entry ± 1 in each diagonal lines except the main diagonal. Suppose to the contrary that the labeling of D that induces $[M_D]$ is not digraceful.Then there are distinct edges v_rv_s and v_tv_u with edge labels $s - r = t - u = k > 0$. This implies that the sum of all the elements in a_{ij} if $j - i = \pm k$ is either 2 or 0.In the upper triangular matrix the sum of a_{ij} is 0 and in lower triangular matrix the sum of a_{ij} is 2 contradicting the assumption that $[M_D]$ has exactly one entry ± 1 in each diagonal D_k. Therefore labeling of D is graceful.

Let f be an digraceful labeling on D and consider $[M_D]$. Then, for all $k = \pm1, \pm3, \pm5, \cdots, \pm q$, there is exactly one nonzero entry a_{ij} =±1 for $j - i = k$, contributing to D_k(k$\neq 0$) since each edge has a unique label. Then $[M_D]$ has exactly one entry one in each diagonal line except the main diagonal line. This completes the proof. □

Theorem 2.3 *Unidirectional path $\overrightarrow{P_n}$ on n vertices is odd digraceful if n is even.*

Theorem 2.4 *Let D be a labeled unidirectional path with q edges and let $[M_D]$ be the generalized adjacency matrix for D . Then D is odd digraceful if and only if $[M_D]$ has exactly one entry* ±1 *in each odd diagonal lines and all the entries are 0 in the even diagonal lines including the main diagonal line of zeros.*

Proof It is to be noted that the matrix $[M_D]$ is a staircase shaped matrix and skew symmetric.The main diagonal line is D_k for $k = 0$. The odd diagonal lines are D_k for $k = \pm1, \pm3, \pm5, \cdots, \pm2q-1$ and the even diagonal line are D_k for $k = \pm2, \pm4, \pm6, \cdots, \pm2q-2$.

Suppose that $[M_D]$ has exactly one entry 1 in each odd diagonal lines and all the diagonal entries are zero in the even diagonal lines including the main diagonal. Suppose to the contrary that the labeling of G that induces $[M_D]$ is not odd dIgraceful Then there are distinct edges v_rv_s and v_tv_u with edge labels $s - r = t - u = k > 0$. This implies that there exists atleast one even diagonal which has a non zero entry and at least one odd diagonal which has all entries are zero contradicting the assumption that $[M_D]$ has exactly one entry 1 in each odd diagonal D_k for $k = \pm1, \pm3, \pm5, \cdots, \pm2q-1$. Therefore labeling of is odd graceful.

Let f be an odd graceful labeling on G and consider $[M_D]$.Then for all $k = \pm1, \pm3, \pm5, \cdots$, $\pm2q-1$, there is exactly one nonzero entry a_{ij} =±1 such that $j - i = k$,contributing to D_k since each edge has a unique label.Then $[M_D]$ has exactly one entry one in each odd diagonal line. Also in odd graceful labeling there is no even number edge labeling,so the diagonal line D_k are 0 for $k = \pm2, \pm4, \pm6, \cdots, \pm2q-2$. This completes the proof. □

§3. Adjacency Matrices of Alternating Paths

Now we consider the adjacency matrix in digraceful labeling of alternating path $\overrightarrow{AP_p}$. An alternating path $\overrightarrow{AP_p}$ with p vertices is an oriented path in which any two consecutive arcs have opposite directions.Let $v_1, v_2, v_3, \cdots, v_p$ be the vertices of $\overrightarrow{AP_p}$ and the arcs of $\overrightarrow{AP_p}$ are $\overrightarrow{v_1v_2}, \overrightarrow{v_3v_2}, \overrightarrow{v_3v_4}, \cdots, \overrightarrow{v_{p-1}v_p}$ when p is even or $\overrightarrow{v_1v_2}, \overrightarrow{v_3v_2}, \overrightarrow{v_3v_4}, \cdots, \overrightarrow{v_pv_{p-1}}$ when p is odd. An alternating path $\overrightarrow{AP_p}$ is digraceful based on the following labeling
$f : V(\overrightarrow{AP_p}) \to \{0, 1, 2, 3, \cdots, p-1\}$ defined by

$f(v_1) = 0$;
$f(v_{2i}) = p - i$ for $i = 1, 2, 3 \cdots, \left\lfloor \frac{p}{2} \right\rfloor$;
$f(v_{2i+1}) = i$ for $i = 1, 2, 3, \cdots, \left\lfloor \frac{p-1}{2} \right\rfloor$.

We have the following theorem.

Theorem 3.1 *Let $D = \overrightarrow{AP_p}$ be a labeled alternating path with q edges and let $[M_D]$ be the* *generalized adjacency matrix for D. Then $[M_D]$ has exactly one entry +1 in each diagonals of the upper triangular matrix and exactly one entry -1 in each diagonals of the lower triangular matrix if and only if D is digraceful.*

Proof It is to be noted that the matrix is skew symmetric. First assume that $[M_D]$ has exactly one entry +1 in each diagonals of upper triangular matrix and exactly one entry -1 in each diagonals of lower triangular matrix.Suppose to the contrary that the D is not digraceful.Then there are distinct edge labels $h - g = f - q = k > 0$. This implies that the sum of all the elements in a_{ij} is 0 if j-i = $\pm k$, contradicting the assumption.So D is digraceful.

Let f be a digraceful labeling on D and consider $[M_D]$.consider the arc $\overrightarrow{v_iv_j}$ on D.Since D is digraceful the labeling on the vertex v_j is greater than labeling on v_i. Then for all diagonals D_k for k = $\pm 1, \pm 2, \pm 3, \cdots, \pm q$ there is exactly one nonzero entry $a_{ij} = 1$ for $j > i$ and $a_{ij} = -1$ for $j < i$. □

Now we consider the adjacency matrix in odd digraceful labeling of alternating path $\overrightarrow{AP_p}$. An alternating path $\overrightarrow{AP_p}$ is odd digraceful based on the following labeling $f : V(\overrightarrow{AP_p}) \rightarrow \{0, 1, 2, 3, \cdots, 2p - 3\}$ defined by

$f(v_1) = 0;$
$f(v_{2i}) = 2p - 2i - 1, fori = 1, 2, 3, \cdots, \lfloor \frac{p}{2} \rfloor;$
$f(v_{2i+1}) = 2i, fori = 1, 2, 3, \cdots, \lfloor \frac{p-1}{2} \rfloor.$

Corollary 3.1 *Let $D = \overrightarrow{AP_p}$ be a labeled alternating path with q edges and let $[M_D]$ be the generalized adjacency matrix for D. Then $[M_D]$ has exactly one entry +1 in each odd diagonals of the upper triangular matrix and exactly one entry -1 in each odd diagonals of the lower triangular matrix if and only if D is odd digraceful.Also all the entries are zero for even diagonal lines.*

§4. Adjacency Matrices of Many Orientations of Directed Star

In this section we consider the adjacency matrices of many orientations of directed star.Let $K_{1,m}$ be an orientations of a star $K_{1,m}$ on $(m + 1)$ vertices.In ([7]) Bing Yao, Ming Yao and Hui Cheng proved the following conditions on gracefulness of directed star.

(1) A directed star $\overrightarrow{K_{1,m}}$ is digraceful if m is odd;

(2) A directed star $\overrightarrow{K_{1,m}}$ is digraceful if m is even and one of out-degree and in-degree of the center w of $\overrightarrow{K_{1,m}}$ must be even.

We have the following theorem based on adjacency matrices on graceful directed stars.

Theorem 4.1 *The directed star $\overrightarrow{K_{1,m}}$ is digraceful if and only if the adjacency matrices are any one of the following two forms:*

(1) *There exists exactly one entry 1 or -1 in each diagonal lines except the main diagonal;*

(2) *There exists both ± 1 in the diagonals $D_{\pm 1}$ and the remaining diagonal lines have exactly one entry ± 1. The main and last diagonal lines have all entries are zeros.*

Proof The adjacency matrix is skew symmetric.The directed star have no self loops,the entries in the main diagonal line are all zeros. First suppose that the directed star $\overrightarrow{K_{1,m}}$ is digraceful. The center vertex in the directed star is connected to all the other vertices.The labeling on the center vertex has either zero or any one of the numbers from 1 to m-1.If the labeling on the center vertex is zero,then it is connected to all the m labeled vertices.since each edge have a unique label,in each edge of the adjacency matrix there must exists an entry ± 1.Suppose the labeling on the center vertex is other than zero.The labeling on the center vertex is any one of the numbers from 1 to $m-1$.If r is the labeling on the center vertex,then there exists at least two vertices connected to the center vertex have labelings $r-1$ and $r+1$.The diagonal lines in the adjacency matrix corresponding to these vertices are $D_{\pm 1}$ and they have both ± 1. Also there exists a labeling from center vertex to all other remaining vertices.So the remaining diagonal lines have exactly one entry ± 1.Since there is no connection between the labeling zero and m,the last diagonal is zero.Conversly suppose that the adjacency matrix satisfies any one of the given conditions.The directed star $\overrightarrow{K_{1,m}}$ is digraceful followed by Theorem 2.1. □

§5. Adjacency Matrices of a Class of Directed Bistar

A ditree H with diameter three is called a directed bistar. In ([7]), we have the following description about a class of directed bistars $T(s,t)$ Dibistar(I): The vertex set and arc set of a directed bistar T(s,t) are defined as $V(T(s,t)) = u_i, u, v, v_j : i\varepsilon[1, s-1], j\varepsilon[1,t]$ and $A(T(s,t)) = \overrightarrow{uu_i}, \overrightarrow{uv}, \overrightarrow{vv_j} : i\varepsilon[1, s-1], j\varepsilon[1,t]$,respectively,where u is the root of T(s,t).Clearly,the in-degrees $d_T^-(s,t)(u) = 0$ and $d_T^-(s,t)(v) = 1$ and the out degrees $d_T^+(s,t)(u) = s$ and $d_T^+(s,t)(v) = t$.In ([7]) it is to be proved that every directed bistar T(2l+1,2k-2l) defined by Dibistar(I) is digraceful for integers $k > l \geq 0$.

Theorem 5.1 *The directed bistar T(2l+1,2k-2l) defined by dibistar(I) is digraceful if and only if the adjacency matrices have at least* ± 1 *in each diagonal lines except 4l diagonal lines in which*

(1) *2l diagonal entries contains both 1 and -1;*

(2) *2l diagonal entries are all zero except the main diagonal.*

Proof If $l = 0$, the directed bistar T(1,2k)is a directed star $K_{1,2k+1}$. The result follows from theorem(6). Let v_1 and v_2 be the centers of the directed bistar $T(2l+1, 2k-2l)$. Suppose the center v_1 is connected to $2l$ vertices and the center v_2 is connected to 2k-2l+1 vertices.Since the bistar is digraceful,each edge has a unique labeling.Let the labeling on v_2 is zero which is not connected to 2l vertices.The labeling contributed a non zero entry ± 1 in each diagonal lines of the adjacency matrix except the $2l$ diagonal lines and the main diagonal. Also the labeling on v_1 is any integer from 1 to $2k+1$ and it contributed a non zero entry ± 1 in $2l$ diagonal lines of the adjacency matrix. So the $2l$ diagonal entries contains both 1 and -1. The converse follows from Theorem 2.1. □

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Minimum Equitable Dominating Randić Energy of a Graph

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Abstract: Let G be a graph with vertex set $V(G)$, edge set $E(G)$ and d_i is the degree of its i-th vertex v_i, then the Randić matrix $R(G)$ of G is the square matrix of order n, whose (i, j)-entry is equal to $\frac{1}{\sqrt{d_i d_j}}$ if the i-th vertex v_i and j-th vertex v_j of G are adjacent, and zero otherwise. The Randic energy [3] $RE(G)$ of the graph G is defined as the sum of the absolute values of the eigenvalues of the Randić matrix $R(G)$. A subset ED of $V(G)$ is called an equitable dominating set [11], if for every $v_i \in V(G) - ED$ there exists a vertex $v_j \in ED$ such that $v_iv_j \in E(G)$ and $|d_i(v_i) - d_j(v_j)| \leq 1$. In the contrast, such a dominating set ED is Smarandachely if $|d_i(v_i) - d_j(v_j)| \geq 1$. Recently, Adiga, et.al. introduced, the minimum covering energy $Ec(G)$ of a graph [1] and S. Burcu Bozkurt, et.al. introduced, Randić Matrix and Randić Energy of a graph [3]. Motivated by these papers, Minimum equitable dominating Randić energy of a graph $RE_{ED}(G)$ of some graphs are worked out and bounds on $RE_{ED}(G)$ are obtained.

Key Words: Randić matrix and its energy, minimum equitable dominating set and minimum equitable dominating Randić energy of a graph.

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§1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix $A(G)$ of the graph G is a square matrix, whose (i, j)-entry is equal to 1 if the vertices v_i and v_j are adjacent, otherwise zero [12]. Since $A(G)$ is symmetric, its eigenvalues are all real. Denote them by $\lambda_1, \lambda_2, \ldots, \lambda_n$, and as a whole, they are called the spectrum of G and denoted by Spec(G). The energy of graph [12] G is

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The literature on energy of a graph and its bounds can refer [4,8,9,10,12]. The Randić matrix $R(G) = (r_{ij})$ of G is the square matrix of order n, where

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$$(r_{ij}) = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent vertices in } G; \\ 0, & \text{otherwise.} \end{cases}$$

The Randic energy [3] $RE(G)$ of the graph G is defined as the sum of the absolute values of the eigenvalues of the Randić matrix $R(G)$. Let $\rho_1, \rho_2, \cdots, \rho_n$ be the eigenvalues of the Randić matrix $R(G)$. Since $R(G)$ is symmetric matrix, these eigenvalues are real numbers and their sum is zero. Randić energy [3] can be defined as

$$RE(G) = \textstyle\sum_{i=1}^{n} |\rho_i|$$

For details of Randić energy and its bounds, can refer [2, 3, 5, 6, 7].

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and ED is minimum equitable dominating set of G. Minimum equitable dominating Randić matrix of G is $n \times n$ matrix $R_{ED}(G) = (r_{ij})$, where

$$(r_{ij}) = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent vertices in } G; \\ 1, & \text{if } i = j \text{ and } v_i \in ED; \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $R_{ED}(G)$ is denoted by $det(\rho I - R_{ED}(G)) = |\rho I - R_{ED}(G)|$. Since $R_{ED}(G)$ is symmetric, its eigenvalues are real numbers. If the distinct eigenvalues of $R_{ED}(G)$ are $\rho_1 > \rho_2 > \cdots > \rho_r$ with their multiplicities are $m_1, m_2, \cdots, m_r$ then spectrum of $R_{ED}(G)$ is denoted by

$$SpecR_{ED}(G) = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

The minimum equitable dominating Randić energy of G is defined as

$$RE_{ED}(G) = \sum_{i=1}^{n} |\rho_i|.$$

Example 1.1 Let W_5 be a wheel graph, with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, and let its minimum equitable dominating set be $ED = \{v_1\}$. Then minimum equitable dominating Randić matrix $R_{ED}(W_5)$ is

$$R_{ED}(W_5) = \begin{bmatrix} 1 & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{12}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{\sqrt{12}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{12}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

$$SpecR_{ED}(W_5) = \begin{pmatrix} -0.6666 & 0 & 0.2324 & 1.4342 \\ 1 & 2 & 1 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of W_5 is $RE_{ED}(W_5) = 2.3332$.

§2. Bounds for the Minimum Equitable Dominating Randić Energy of a Graph

Lemma 2.1 *If $\rho_1, \rho_2, \cdots, \rho_n$ are the eigenvalues of $R_{ED}(G)$. Then*

$$\sum_{i=1}^{n} \rho_i = |ED|$$

and

$$\sum_{i=1}^{n} \rho_i^2 = |ED| + 2\sum_{i<j} \frac{1}{d_i d_j},$$

where ED is minimum equitable dominating set.

Proof (i) The sum of eigenvalues of $R_{ED}(G)$ is

$$\sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} r_{ii} = |ED|.$$

(ii) Consider, the sum of squares of $\rho_1, \rho_2, \ldots, \rho_n$ is,

$$\begin{aligned}
\sum_{i=1}^{n} \rho_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\ r_{ji} = \sum_{i=1}^{n} (r_{ii})^2 + \sum_{i\neq j} r_{ij}\ r_{ji} \\
&= \sum_{i=1}^{n} (r_{ii})^2 + 2\sum_{i<j} (r_{ij})^2 \\
\sum_{i=1}^{n} \rho_i^2 &= |ED| + 2\sum_{i<j} \frac{1}{d_i d_j}.
\end{aligned}$$

□

Upper and lower bounds for $RE_{ED}(G)$ is similar proof to McClelland's inequalities [10], are given below

Theorem 2.2(Upper Bound) *Let G be a graph with ED is minimum equitable dominating set. Then*

$$RE_{ED}(G) \ \le \ \sqrt{n\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right]}.$$

Proof Let $\rho_1, \rho_2, \cdots, \rho_n$ be the eigenvalues of $R_{ED}(G)$. By Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right), \tag{1}$$

where a and b are any real numbers.

If $a_i = 1$, $b_i = |\rho_i|$ in (1), we get

$$\left(\sum_{i=1}^{n} |\rho_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1^2\right)\left(\sum_{i=1}^{n} |\rho_i|^2\right), \quad [RE_{ED}(G)]^2 \leq n\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right]$$

by Lemma 2.1,

$$RE_{ED}(G) \leq \sqrt{n\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right]}.$$

Theorem 2.3(Lower Bound) *Let G be a graph with $|ED|$ is minimum equitable dominating set and d_i is degree of v_i. Then*

$$RE_{ED}(G) \geq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} + n(n-1)D^{\frac{2}{n}}},$$

where $D = \prod_{i=1}^{n} |\rho_i|$.

Proof Consider

$$[RE_{ED}(G)]^2 = \left[\sum_{i=1}^{n} |\rho_i|\right]^2 = \sum_{i=1}^{n} |\rho_i|^2 + \sum_{i \neq j} |\rho_i|\ |\rho_j|. \tag{2}$$

By using arithmetic and geometric mean inequality, we have

$$\begin{aligned}
\frac{1}{n(n-1)}\sum_{i\neq j} |\rho_i|\ |\rho_j| &\geq \left(\prod_{i\neq j} |\rho_i|\ |\rho_j|\right)^{\frac{1}{n(n-1)}} \\
\sum_{i\neq j} |\rho_i|\ |\rho_j| &\geq n(n-1)\left(\prod_{i=1}^{n} |\rho_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
\sum_{i\neq j} |\rho_i|\ |\rho_j| &\geq n(n-1)\left(\prod_{i=1}^{n} |\rho_i|\right)^{2/n}.
\end{aligned} \tag{3}$$

Now using (3) in (2), we get

$$[RE_{ED}(G)]^2 \geq \sum_{i=1}^{n} |\rho_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\rho_i|\right)^{2/n},$$

$$[RE_{ED}(G)]^2 \geq |ED| + 2\sum_{i<j}\frac{1}{d_i d_j} + n(n-1)D^{\frac{2}{n}}, \quad \text{where} \quad D = \prod_{i=1}^{n} |\rho_i|,$$

$$RE_{ED}(G) \geq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} + n(n-1)D^{\frac{2}{n}}}.$$

§3. Bounds for Largest Eigenvalue of $R_{ED}(G)$ and its Energy

Proposition 3.1 *Let G be a graph and $\rho_1(G) = max_{1\leq i\leq n}\{|\rho_i|\}$ be the largest eigenvalue of $R_{ED}(G)$. Then*

$$\sqrt{\frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right]} \leq \rho_1(G) \leq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}.$$

Proof Consider,

$$\begin{aligned}\rho_1^2(G) &= max_{1\leq i\leq n}\ \{|\rho_i|^2\} \leq \sum_{i=1}^{n}|\rho_i|^2 = |ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\\ \rho_1(G) &\leq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}\end{aligned}$$

Next,

$$\begin{aligned}n\ \rho_1^2(G) &\geq \sum_{i=1}^{n}\rho_i^2 = |ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\\ \rho_1^2(G) &\geq \frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right]\\ \rho_1(G) &\geq \sqrt{\frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right]}\end{aligned}$$

Therefore,

$$\sqrt{\frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right]} \leq \rho_1(G) \leq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}.$$

Proposition 3.2 *If G is a graph and $n \leq |ED| + 2\sum_{i<j}\frac{1}{d_i d_j}$, then*

$$RE_{ED}(G) \leq \frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n} + \sqrt{(n-1)\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - \left(\frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}\right)^2\right]}.$$

Proof We know that,

$$\sum_{i=1}^{n}\rho_i^2 = |ED| + 2\sum_{i<j}\frac{1}{d_i d_j}, \quad \sum_{i=2}^{n}\rho_i^2 = |ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - \rho_1^2 \tag{4}$$

By Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=2}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=2}^{n} a_i^2\right)\left(\sum_{i=2}^{n} b_i^2\right).$$

If $a_i = 1$ and $b_i = |\rho_i|$, we have

$$\begin{aligned}
\left(\sum_{i=2}^{n} |\rho_i|\right)^2 &\leq (n-1)\sum_{i=2}^{n} |\rho_i|^2 \\
[RE_{ED}(G) - |\rho_1|]^2 &\leq (n-1)\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - \rho_1^2\right] \\
RE_{ED}(G) &\leq \rho_1 + \sqrt{(n-1)\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - \rho_1^2\right]} \qquad (5)
\end{aligned}$$

Consider the function,

$$F(x) = x + \sqrt{(n-1)\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - x^2\right]}.$$

Then,

$$F'(x) = 1 - \frac{x\sqrt{(n-1)}}{\sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - x^2}}.$$

Here $F(x)$ is decreasing in

$$\left(\frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right], |ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right).$$

We know that $F'(x) \leq 0$,

$$1 - \frac{x\sqrt{(n-1)}}{\sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - x^2}} \leq 0.$$

We have

$$x \geq \sqrt{\frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}}.$$

Since,

$$n \leq |ED| + 2\sum_{i<j}\frac{1}{d_i d_j} \quad \text{and} \quad \frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n} \leq \rho_1,$$

we have

$$\sqrt{\frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}} \leq \frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n} \leq \rho_1 \leq |ED| + 2\sum_{i<j}\frac{1}{d_i d_j}.$$

Then, equation (5) become

$$RE_{ED}(G) \leq \frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n} + \sqrt{(n-1)\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - \left(\frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}\right)^2\right]}.$$

□

§4. Minimum Equitable Dominating Randić Energy of Some Graphs

Theorem 4.1 *If K_n is complete graph with n vertices, then minimum equitable dominating Randić energy of K_n is*

$$RE_{ED}(K_n) = \frac{3n-5}{n-1}.$$

Proof Let K_n be the complete graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and minimum equitable dominating set is $ED = \{v_1\}$, we have characteristic polynomial of $R_{ED}(K_n)$ is

$$|\rho I - R_{ED}(K_n)| = \begin{vmatrix} \rho - 1 & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \rho & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \rho & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \rho & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \rho & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \rho \end{vmatrix}_{n\times n}.$$

$R'_k = R_k - R_2,\ k = 3, 4, \cdots, n-1, n$. Then, we get $\left(\rho + \frac{1}{n-1}\right)$ common from R_3 to R_n and we have

$$|\rho I - R_{ED}(K_n)| = \left(\rho + \frac{1}{n-1}\right)^{n-2} \begin{vmatrix} \rho - 1 & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \rho & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 1 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}_{n\times n}.$$

$C_2' = C_2 + C_3 + \cdots + C_n$, we get the characteristic polynomial

$$\begin{aligned} |\rho I - R_{ED}(K_n)| &= \left(\rho + \frac{1}{n-1}\right)^{n-2}\left[\rho^2 - \left(\frac{2n-3}{n-1}\right)\rho + \left(\frac{n-3}{n-1}\right)\right], \\ SpecR_{ED}(K_n) &= \begin{pmatrix} \frac{-1}{n-1} & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} \\ n-2 & 1 & 1 \end{pmatrix}. \end{aligned}$$

The minimum equitable dominating Randić energy of K_n is

$$RE_{ED}(K_n) = \frac{3n-5}{n-1}. \qquad \square$$

Theorem 4.2 *If $S_n, (n \geq 4)$ is star graph with n vertices, then minimum equitable dominating Randić energy of S_n is*

$$RE_{ED}(S_n) = n.$$

Proof Let $S_n, (n \geq 4)$ be the star graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and minimum equitable dominating set is $ED = \{v_1, v_2, \cdots, v_n\}$, we have characteristic polynomial of $R_{ED}(S_n)$ is

$$|\rho I - R_{ED}(S_n)| = \begin{vmatrix} \rho-1 & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \cdots & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} \\ \frac{-1}{\sqrt{n-1}} & \rho-1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{\sqrt{n-1}} & 0 & \rho-1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & \cdots & \rho-1 & 0 & 0 \\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & \cdots & 0 & \rho-1 & 0 \\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & \cdots & 0 & 0 & \rho-1 \end{vmatrix}_{n\times n}.$$

$R_k' = R_k - R_2,\ k = 3, 4, \cdots, n$. Then taking $(\rho - 1)$ common from R_3 to R_n, we get

$$|\rho I - R_{ED}(S_n)| = (\rho-1)^{n-2}\begin{vmatrix} \rho-1 & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \cdots & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} \\ \frac{-1}{\sqrt{n-1}} & \rho-1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 1 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}_{n\times n}.$$

$C_2' = C_2 + C_3 + \cdots + C_n$. Then, the characteristic polynomial

$$|\rho I - R_{ED}(S_n)| = \rho(\rho-1)^{n-2}(\rho-2),$$

$$SpecR_{ED}(S_n) = \begin{pmatrix} 0 & 1 & 2 \\ \\ 1 & n-2 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of S_n is $RE_{ED}(S_n) = n$. □

Theorem 4.3 *If $K_{m,n}$, where $m < n$ and $|m-n| \geq 2$ is complete bipartite graph with $m+n$ vertices, then minimum equitable dominating Randić energy of $K_{m,n}$ is $RE_{ED}(K_{m,n}) = m+n$.*

Proof Let $K_{m,n}$, where $m < n$ and $|m-n| \geq 2$ be the complete bipartite graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \cdots, u_n\}$ and minimum equitable dominating set is $ED = V(G)$, we have characteristic polynomial of $R_{ED}(K_{m,n})$ is

$$|\rho I - R_{ED}(K_{m,n})| = \begin{vmatrix} \rho-1 & 0 & \ldots & 0 & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ 0 & \rho-1 & \ldots & 0 & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho-1 & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ 0 & 0 & \cdots & 0 & \rho-1 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \rho-1 & 0 & \cdots & 0 & 0 \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & 0 & \rho-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & 0 & 0 & \cdots & \rho-1 & 0 \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & 0 & 0 & \cdots & 0 & \rho-1 \end{vmatrix}.$$

$R'_k = R_k - R_m,\ k = 1, 2, 3, \cdots, m-1$ and $R'_d = R_d - R_{m+1},\ d = m+2, m+3, \cdots, m+n$. Then, taking $(\rho - 1)$ common from R_1 to R_{m-1} and R_{m+2} to R_{m+n}, we get

$$(\rho-1)^{m+n-2} \begin{vmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \rho-1 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \rho-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

$C'_{m+1} = C_{m+1} + C_{m+2} + \cdots + C_{m+n}$,

$$(\rho-1)^{m+n-2}\begin{vmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \rho-1 & \frac{-n}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \rho-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

The characteristic polynomial $|\rho I - R_{ED}(K_{m,n})| = \rho\,(\rho-1)^{m+n-2}\,(\rho-2)$,

$$SpecR_{ED}(K_{m,n}) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of $K_{m,n}$ is $RE_{ED}(K_{m,n}) = m+n$. $\square$

Theorem 4.4 *If $K_{n\times 2}$, $(n\geq 3)$ is cocktail party graph with $2n$ vertices, then minimum equitable dominating Randić energy of $K_{n\times 2}$ is $\frac{4n-6}{n-1}$.*

Proof Let $K_{n\times 2}$, $(n\geq 3)$ be the cocktail party graph with vertex set $V(G)=\{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and minimum equitable dominating set is $ED=\{v_1, u_1\}$, we have characteristic polynomial of $R_{ED}(K_{n\times 2})$ is

$$|\rho I - R_{ED}(K_{n\times 2})| = \begin{vmatrix} \lambda-1 & 0 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & \lambda-1 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & 0 & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \lambda & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & 0 \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & 0 & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \lambda & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \lambda & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & 0 & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \lambda \end{vmatrix}.$$

$R'_k = R_k - R_3,\ k = 4, 5, \cdots, n$ and $R'_{n+k} = R_{n+k} - R_{k+1},\ k = 2, 3, \cdots, n$, we get

$$|\rho I - R_{ED}(K_{n\times 2})| = \begin{vmatrix} \lambda - 1 & 0 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & \lambda - 1 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & 0 & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & 0 & \frac{-1}{2n-2} - \lambda & \lambda + \frac{1}{2n-2} & \cdots & 0 & \frac{-1}{2n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \frac{-1}{2n-2} - \lambda & 0 & \cdots & \lambda + \frac{1}{2n-2} & \frac{-1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} \\ 0 & 0 & -\lambda & 0 & \cdots & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & 0 & \cdots & 0 & \lambda \end{vmatrix}.$$

$C'_3 = C_3 + C_4 + \cdots + C_n + C_{n+1} + \cdots + C_{2n}$ and $C'_k = C_k + C_{n+(k-1)},\ k = 4, 5, \cdots, n+1$. We get

$$|\rho I - R_{ED}(K_{n\times 2})| = \begin{vmatrix} \lambda - 1 & 0 & -1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & \lambda - 1 & -1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda - \frac{(2n-4)}{(2n-2)} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & 0 & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & 0 & 0 & \lambda + \frac{1}{n-1} & \cdots & 0 & \frac{-1}{2n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda + \frac{1}{n-1} & \frac{-1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}.$$

The characteristic polynomial

$$\begin{aligned} |\rho I - R_{ED}(K_{n\times 2})| &= \rho^{n-1}(\rho - 1)\left(\rho + \frac{1}{n-1}\right)^{n-2}\left[\rho^2 - \left(\frac{2n-3}{n-1}\right)\rho + \left(\frac{n-3}{n-1}\right)\right], \\ SpecR_{ED}(K_{n\times 2}) &= \begin{pmatrix} \frac{-1}{n-1} & 0 & 1 & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} \\ n-2 & n-1 & 1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

The minimum equitable dominating Randić energy of $K_{n\times 2}$ is

$$RE_{ED}(K_{n\times 2}) = \frac{4n-6}{n-1},$$

where $n \geq 3$. □

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Linear Cyclic Snakes as Super Vertex Mean Graphs

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Abstract: A super vertex mean labeling f of a (p,q) - graph $G(V,E)$ is defined as an injection from E to the set $\{1,2,3,\cdots,p+q\}$ that induces for each vertex v the label defined by the rule $f^v(v) = \text{Round}\left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge label and the induced vertex labels is $\{1,2,3,\cdots,p+q\}$. All the cycles, C_n, $n \geq 3$ and $n \neq 4$ are super vertex mean graphs. Our attempt in this paper is to show that all the linear cyclic snakes, including kC_4, are also super vertex mean graphs, even though C_4 is not an SVM graph. We also define the term Super Vertex Mean number of graphs.

Key Words: Super vertex mean label, Smarandachely super H-vertex mean labeling, linear cyclic snakes, SVM number.

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§1. Introduction

The graphs considered here will be finite, undirected and simple. The symbols V(G) and E(G) will denote the vertex set and edge set of a graph G respectively. p and q denote the number of vertices and edges of G respectively. A graph of order p and size q is often called a (p,q) - graph.

A labeling of a graph G is an assignment of labels either to the vertices or edges. There are varieties of vertex as well as edge labeling that are already in the literature. Mean labeling was introduced by Somasundaram and Ponraj [14]. Super mean labeling was introduced by D.Ramya et al.[10]. Some results on mean labeling and super mean labeling are given in [5, 6, 9, 10, 11, 13, 14] and [15]. Lourdusamy and Seenivasan [5] introduced vertex mean labeling as an edge analogue of mean labeling.

Continuing on the same line and inspired by the above mentioned concepts, Lourdusamy et al. [7] brought in a new extension of mean labeling, called Super vertex mean labeling of graphs.

§2. Super Vertex Mean Labeling

Definition 2.1 *A super vertex mean labeling f of a (p,q) - graph $G(V,E)$ is defined as an*

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injection from E to the set $\{1,2,3,\cdots,p+q\}$ that induces for each vertex v the label defined by the rule $f^v(v) = Round\left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge labels and the induced vertex labels is $\{1,2,3,\cdots,p+q\}$.

Furthermore, a super vertex mean labeling f on G is Smarandachely super H-vertex mean labeling if the induced vertex labels on vertex in H' is $\{1,2,3,\cdots,p(H')+q(H')\}$ for each subgraph $H' \cong H$ in G, where $p(H')$ and $q(H')$ are respectively the order and the size of H'.

A graph that accepts super vertex mean labeling is called a super vertex mean (hereafter, SVM) graph. The following results have already been proved in [7] and [8]. We use them for our further study on the super vertex mean behavior of linear cyclic snakes. Before entering into the results, we define the term cyclic snakes.

Definition 2.2 *A kC_n-snake has been defined as a connected graph in which all the blocks are isomorphic to the cycle C_n and the block-cut point graph is a path P, where P is the path of minimum length that contains all the cut vertices of a kC_n-snake. Barrientos [13] has proved that any kC_n-snake is represented by a string $s_1, s_2, s_3, \cdots, s_{k-2}$ of integers of length $k-2$, where the i^{th} integer, s_i on the string is the distance between i^{th} and $i+1^{th}$ cut vertices along the path, P, from one extreme and is taken from $S_n = \{1,2,3,\cdots,\lfloor\frac{n}{2}\rfloor\}$. A kC_n-snake is said to be linear if each integer s_i of its string is equal to $\lfloor\frac{n}{2}\rfloor$.*

Remark 2.3 The strings obtained from both the extremes are assumed to be the same. A linear cyclic snake, kC_n is obtained from k copies of C_n by identifying the vertex $v_{i,r+1}$ in the i^{th} copy of C_n at a vertex $v_{i+1,1}$ in the $(i+1)^{th}$ copy of C_n, where $1 \le i \le k-1$ and $n = 2r$ or $n = 2r+1$, depending upon whether n is even or odd respectively.

Note 2.4 that $v_{i,r+1} = v_{i+1,1}$ for $1 \le i \le k-1$, and we consider this vertex as $v_{i+1,1}$ throughout this paper.

§3. Known Results

We have known results listed in the following.

(1) *All the cycles except C_4 are SVM graphs*([8]);

(2) *All odd cycles, C_n can be SVM labeled as many as $\lfloor\frac{n}{2}\rfloor$ ways and every even cycle, C_n, except C_4 can have $\lfloor\frac{n}{2}\rfloor - 1$ types of SVM labelings*([8]);

(3) *A linear triangular snake, kC_3 with k blocks is an SVM graph*([7]).

§4. Linear Cyclic Snakes of Higher Orders

We proceed to prove that other linear cyclic snakes too are super vertex mean graphs.

Theorem 4.1 *Linear quadrilateral snakes, kC_4 with $k \ge 2$ blocks are SVM graphs.*

Proof Let kC_4 be a linear quadrilateral snake with p vertices and q edges. Then $p = 3k+1$ and $q = 4k$. Define $f : E(kC_4) \to \{1, 2, 3, \cdots, 7k+1\}$ to be $f(u_i u_{i+1}) = 7i$ if $1 \le i \le k-1$ and $7k+1$ $i = k$. When $1 \le i \le k-1$ and $k \ge 2$,

$$f(e_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j = 1 \\ 7i-5, & \text{if } 2 \le i \le k-1, \text{ and } j = 1 \\ 7i-1, & \text{if } 1 \le i \le k-1, \text{ and } j = 2 \\ 7i, & \text{if } 1 \le i \le k-1, \text{ and } j = 3 \\ 7i-4, & \text{if } 1 \le i \le k-1, \text{ and } j = 4. \end{cases}$$

When $i = k$, $k \ge 2$ and k is even,

$$f(e_{i,j}) = \begin{cases} 7k-6, & \text{if } j = 1, \\ 7k-3, & \text{if } j = 2, \\ 7k-1, & \text{if } j = 3, \\ 7k+1, & \text{if } j = 4. \end{cases}$$

When $i = k$, $k \ge 3$ and k is odd,

$$f(e_{i,j}) = \begin{cases} 7k-5, & \text{if } j = 1, \\ 7k-2, & \text{if } j = 2, \\ 7k+1, & \text{if } j = 3, \\ 7k-4, & \text{if } j = 4. \end{cases}$$

It can be easily verified that f is injective. The induced vertex labels are as follows:

When $1 \le i \le k-1$ and $k \ge 2$,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 7i-6, & \text{if } 2 \le i \le k-1, \text{ and } j = 1, \\ 7i-3, & \text{if } 1 \le i \le k-1, \text{ and } j = 2, \\ 7i-2, & \text{if } 1 \le i \le k-1, \text{ and } j = 4. \end{cases}$$

When $i = k$, $k \ge 2$, and k is even,

$$f^v(v_{i,j}) = \begin{cases} 7k-5, & \text{if } j = 1, \\ 7k-4, & \text{if } j = 2, \\ 7k-2, & \text{if } j = 3, \\ 7k, & \text{if } j = 4. \end{cases}$$

When $i = k$, $k \geq 3$, and k is odd,

$$f^v(v_{i,j}) = \begin{cases} 7k-6, & \text{if } j = 1, \\ 7k-3, & \text{if } j = 2, \\ 7k, & \text{if } j = 3, \\ 7k-1, & \text{if } j = 4. \end{cases}$$

It can be easily verified that the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 7k+1\}$ in two following cases:

Case 1. When k is even,

$$\begin{aligned} f(E) &= \{1, 6, 7, 3, 9, 13, 14, 10, 16, 20, 21, 27, \cdots, 7k-12, \\ &\quad 7k-8, 7k-7, 7k-11, 7k-6, 7k-3, 7k-1, 7k+1\} \end{aligned}$$

and,

$$f^v(V) = \{2, 4, 8, 5, 11, 15, 12, 18, 22, 19, \cdots, 7k-10, 7k-5, 7k-9, 7k-4, 7k-2, 7k\}.$$

Therefore,

$$\begin{aligned} f(E) \cup f^v(V) &= \{1, 2, 3, 4, \cdots, 7k-12, 7k-11, 7k-10, 7k-9, 7k-8, \\ &\quad 7k-7, 7k-6, 7k-5, 17k-4, 7k-3, 7k-2, 7k-2, 7k, 7k+1\}. \end{aligned}$$

Case 2. When k is odd,

$$\begin{aligned} f(E) &= \{1, 6, 7, 3, 9, 13, \cdots, 7k-12, 7k-8, \\ &\quad 7k-7, 7k-11, 7k-5, 7k-2, 7k+1, 7k-4\} \end{aligned}$$

and,

$$f^v(V) = \{2, 4, 8, 5, \cdots, 7k-10, 7k-6, 7k-9, 7k-3, 7k, 7k-1\}.$$

Therefore,

$$\begin{aligned} f(E) \cup f^v(V) &= \{1, 2, 3, 4, \cdots, 7k-12, 7k-11, 7k-10, 7k-9, 7k-8, \\ &\quad 7k-7, 7k-6, 7k-5, 17k-4, 7k-3, 7k-2, 7k-2, 7k, 7k+1\}. \end{aligned}$$

Thus it has been proved that the labeling $f : E(kC_4) \to \{1, 2, 3, \cdots, 7k+1\}$ is a super vertex mean labeling. □

Example 4.2 The super vertex-mean labelings of two linear quadrilateral snakes with 4 and 3 blocks are shown in figures 1 and 2 respectively.

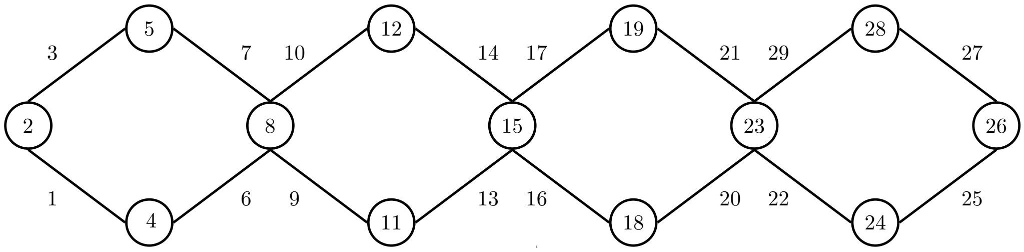

Figure 1 A Super vertex-mean labeling of a linear quadrilateral snake with 4 blocks

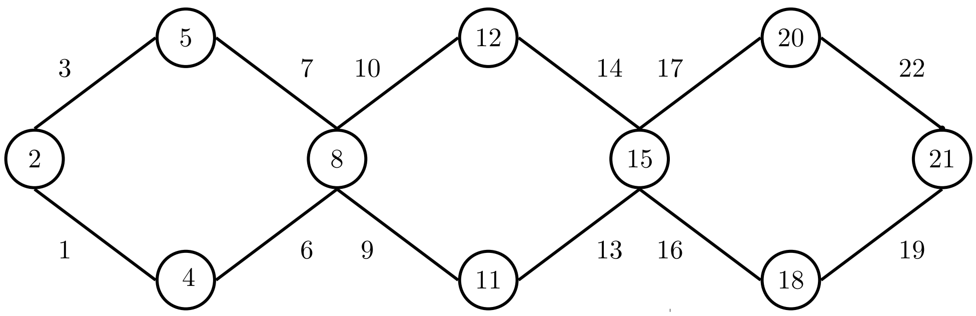

Figure 2 A super vertex-mean labeling of a linear quadrilateral snake with 3 blocks

Theorem 4.3 *All linear pentagonal snakes, kC_5 with $k, k \geq 2$ blocks are SVM graphs.*

Proof Let kC_5 be a linear pentagonal snake with $k, k \geq 2$ blocks of C_5. Here $p = 4k+1$, $q = 5k$ and $p + q = 9k + 1$. Define $f : E(kC_5) \to \{1, 2, 3, \cdots, 9k+1\}$ as follows:

When $i = 1$,

$$f(e_{i,j}) = \begin{cases} 5, & \text{if } j = 1, \\ 2j+4, & \text{if } 2 \leq j \leq 3, \\ 1, & \text{if } j = 4, \\ 3, & \text{if } j = 5. \end{cases}$$

And When $2 \leq i \leq k$,

$$f(e_{i,j}) = \begin{cases} 9i-9, & \text{if } j = 1, \\ 9i+2j-5, & \text{if } 2 \leq j \leq 3, \\ 9i+3j-18, & \text{if } 4 \leq j \leq 5. \end{cases}$$

It can be easily verified that f is injective. Then, the induced vertex labels are as follows:

When $i = 1$,

$$f^v(v_{i,j}) = \begin{cases} 4, & \text{if } j = 1, \\ 7, & \text{if } j = 2, \\ 6, & \text{if } j = 4, \\ 2, & \text{if } j = 5. \end{cases}$$

When $2 \leq i \leq k$,

$$f^v(v_{i,j}) = \begin{cases} 9i + 2j - 9, & \text{if } 1 \leq j \leq 2 \\ 9i - 2j + 6, & \text{if } 4 \leq j \leq 5 \\ 9k, & \text{if } i = k, \text{ and } j = 3. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 9k+1\}$. Therefore, all linear pentagonal snakes kC_5 are super vertex mean graphs. □

Example 4.4 A graph given in Figure 3 is an SVM labeling of a linear pentagonal snake with 3 blocks.

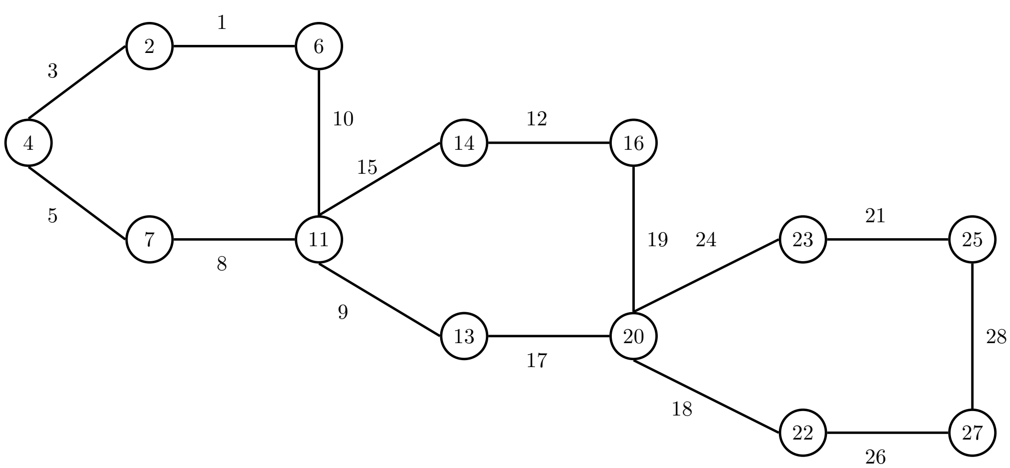

Figure 3 An SVM labeling of a linear pentagonal snake, $3C_5$.

Theorem 4.5 *All linear hexagonal snakes, $kC_6, k \geq 2$ are super vertex mean graphs.*

Proof Let kC_6 be a hexagonal snake with $k, k \geq 2$ blocks of C_6. $p = 5k + 1$ and $q = 6k$ and $p + q = 11k + 1$. Define $f : E(G_n) \to \{1, 2, 3, \cdots, 11k + 1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 4, \\ 7, & \text{if } i = 1, \text{ and } j = 5, \\ 1, & \text{if } i = 1, \text{ and } j = 6, \\ 11i - 4j, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 2, \\ 11i + 3j - 11, & \text{if } 2 \leq i \leq k, \text{ and } 3 \leq j \leq 4, \\ 11i - 8j + 37, & \text{if } 2 \leq i \leq k, \text{ and } 5 \leq j \leq 6. \end{cases}$$

It can be easily verified that f is injective. Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 11i + 3j - 12, & \text{if } 1 \le i \le k, \text{ and } 1 \le j \le 2, \\ 8, & \text{if } i = 1, \text{ and } j = 3, \\ 11i - 5, & \text{if } 2 \le i \le k, \text{ and } j = 3, \\ 11k, & \text{if } i = k, \text{ and } j = 4, \\ 11i - 1, & \text{if } 1 \le i \le k, \text{ and } j = 5, \\ 11i - 7, & \text{if } 1 \le i \le k, \text{ and } j = 6. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 11k + 1\}$.

Therefore, linear hexagonal snakes, all kC_6 with k blocks of C_6 are super vertex mean graphs. □

Example 4.6 A graph given in Figure 4 is an SVM labeling of a linear hexagonal snake with 3 blocks.

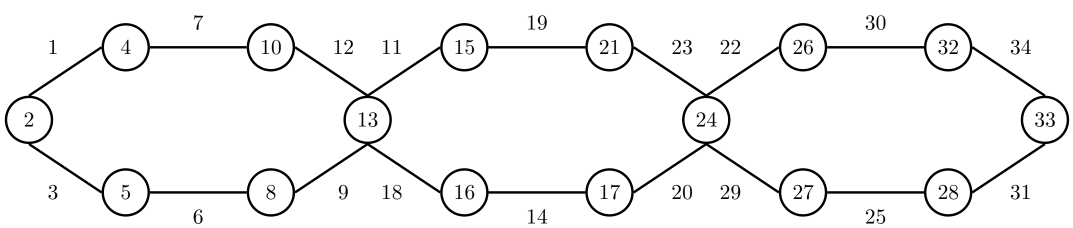

Figure 4 An SVM labeling of a linear hexagonal snake, $3C_6$.

Theorem 4.7 *All linear heptagonal snakes, $kC_7, k \ge 2$ are super vertex mean graphs.*

Proof Let kC_7 be a linear heptagonal snake with $k, k \ge 2$ blocks of C_7. $p = 6k + 1$ and $q = 7k$ and $p + q = 13k + 1$. Define $f : E(G_n) \to \{1, 2, 3, \cdots, 13k + 1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 4j - 1, & \text{if } i = 1, \text{ and } 1 \le j \le 3, \\ 30 - 4j, & \text{if } i = 1, \text{ and } 4 \le j \le 6, \\ 1, & \text{if } i = 1, \text{ and } j = 7, \\ 13i + 2j - 7, & \text{if } 2 \le i \le k, \text{ and } 1 \le j \le 4, \\ 13i - 13, & \text{if } 2 \le i \le k, \text{ and } j = 5, \\ 13i + 2j - 21, & \text{if } 2 \le i \le k, \text{ and } 6 \le j \le 7. \end{cases}$$

It can be easily verified that f is injective. Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 4j - 3, & \text{if } i = 1, \text{ and } 2 \leq j \leq 3, \\ 32 - 4j, & \text{if } i = 1, \text{ and } 5 \leq j \leq 7, \\ 13i - 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ 13i + 2j - 8, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq 3, \\ 13i - 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 5, \\ 13i + 3j - 29, & \text{if } 2 \leq i \leq k, \text{ and } 6 \leq j \leq 7, \\ 13k, & \text{if } i = k, \text{ and } j = 4. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 13k + 1\}$.

Therefore, all linear heptagonal snakes, kC_7 with k blocks of C_7 are super vertex mean graphs. □

Example 4.8 A graph given in Figure 5 is an SVM labeling of a linear heptagonal snake, $3C_7$.

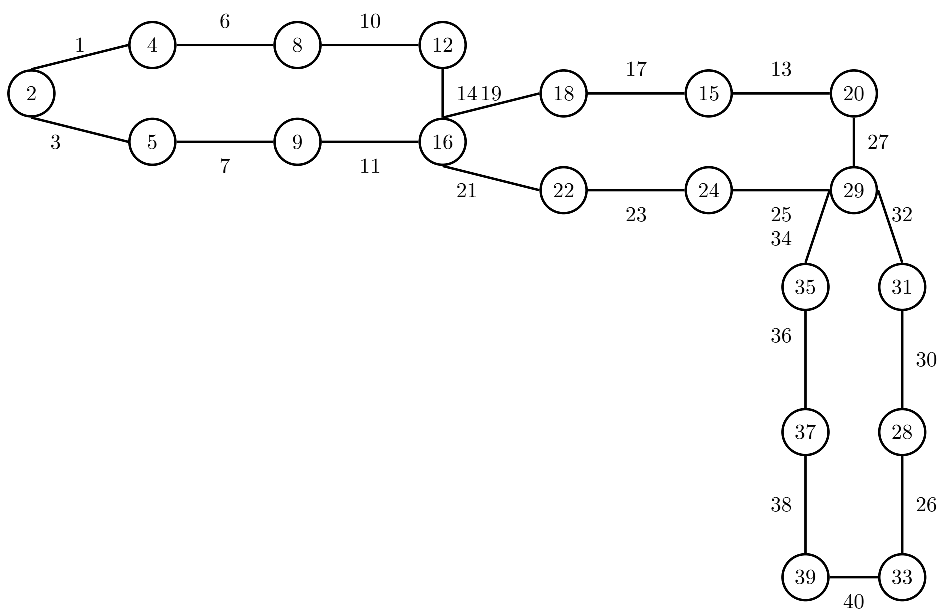

Figure 4 A Super Vertex Mean Labeling of $3C_7$ linear snake.

Theorem 4.9 *Let kC_n be a linear cyclic snake with $k, k \geq 2$ blocks of $C_n, n \geq 8$ and $n \equiv 0(mod\ 2)$. Then kC_n is a super vertex mean graph.*

Proof Let kC_n be a linear cyclic snake with $k, k \geq 2$ blocks of $C_n, n \geq 8$ and $n \equiv 0(mod\ 2)$. Let $n = 2r, r \geq 4$. Now, $p = (n-1)k + 1$ and $q = nk$ and $p + q = (2n-1)k + 1$. Define

$f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k+1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \le j \le 3, \\ 4j-3, & \text{if } i = 1, \text{ and } 4 \le j \le r, \\ 4n-4j+4, & \text{if } i = 1, \text{ and } r+1 \le j \le n-2, \\ 7, & \text{if } i = 1, \text{ and } j = n-1, \\ 1, & \text{if } i = 1, \text{ and } j = n, \\ (2n-1)i-2n+8, & \text{if } 2 \le i \le k, \text{ and } j = 1, \\ (2n-1)i-2n+4, & \text{if } 2 \le i \le k, \text{ and } j = 2, \\ (2n-1)i-2n+10, & \text{if } 2 \le i \le k, \text{ and } j = 3, \\ (2n-1)i-2n+4j-2, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (2n-1)i+2n-4j+5, & \text{if } 2 \le i \le k, \text{ and } r+1 \le j \le n-1, \\ (2n-1)(i-1), & \text{if } i = k, \text{ and } j = n. \end{cases}$$

And, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j-1, & \text{if } i = 1, \text{ and } 2 \le j \le 4, \\ 4j-5, & \text{if } i = 1, \text{ and } 5 \le j \le r, \\ 4n-4j+6, & \text{if } i = 1, \text{ and } r+2 \le j \le n-1, \\ 4, & \text{if } i = 1, \text{ and } j = n, \\ (2n-1)i-2n+3, & \text{if } 2 \le i \le k, \text{ and } j = 1, \\ (2n-1)i-2n+6, & \text{if } 2 \le i \le k, \text{ and } j = 2, \\ (2n-1)i-2n+7, & \text{if } 2 \le i \le k, \text{ and } j = 3, \\ (2n-1)i-2n+4j-4, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (2n-1)i+2n-4j+7, & \text{if } 2 \le i \le k, \text{ and } r+2 \le j \le n-1, \\ (2n-1)i-2n+5, & \text{if } 2 \le i \le k, \text{ and } j = n, \\ (2n-1)k, & \text{if } i = k, \text{ and } j = r+1. \end{cases}$$

We prove the theorem by mathematical induction on r, where $n = 2r, r \ge 4$. The above edge labeling function $f(e)$ and the induced vertex labeling function $f^v(v)$ are expressed in terms of r as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \le j \le 3, \\ 4j-3, & \text{if } i = 1, \text{ and } 4 \le j \le r, \\ 8r-4j+4, & \text{if } i = 1, \text{ and } r+1 \le j \le 2r-2, \\ 7, & \text{if } i = 1, \text{ and } j = 2r-1, \\ 1, & \text{if } i = 1, \text{ and } j = 2r \end{cases}$$

and

$$f(e_{i,j}) = \begin{cases} (4r-1)i-4r+8, & \text{if } 2 \leq i \leq k, \text{ and } j=1, \\ (4r-1)i-4r+4, & \text{if } 2 \leq i \leq k, \text{ and } j=2, \\ (4r-1)i-4r+10, & \text{if } 2 \leq i \leq k, \text{ and } j=3, \\ (4r-1)i-4r+4j-2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (4r-1)i+4r-4j+5, & \text{if } 2 \leq i \leq k, \text{ and } r+1 \leq j \leq 2r-1, \\ (4r-1)(i-1), & \text{if } i=k, \text{ and } j=2r. \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i=1, \text{ and } j=1, \\ 3j-1, & \text{if } i=1, \text{ and } 2 \leq j \leq 4, \\ 4j-5, & \text{if } i=1, \text{ and } 5 \leq j \leq r, \\ 8r-4j+6, & \text{if } i=1, \text{ and } r+2 \leq j \leq 2r-1, \\ 4, & \text{if } i=1, \text{ and } j=2r, \\ (4r-1)i-4r+3, & \text{if } 2 \leq i \leq k, \text{ and } j=1, \\ (4r-1)i-4r+6, & \text{if } 2 \leq i \leq k, \text{ and } j=2, \\ (4r-1)i-4r+7, & \text{if } 2 \leq i \leq k, \text{ and } j=3, \\ (4r-1)i-4r+4j-4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (4r-1)i+4r-4j+7, & \text{if } 2 \leq i \leq k, \text{ and } r+2 \leq j \leq 2r-1, \\ (4r-1)i-4r+5, & \text{if } 2 \leq i \leq k, \text{ and } j=2r, \\ (4r-1)k, & \text{if } i=k, \text{ and } j=r+1. \end{cases}$$

We prove that the theorem is true when $r = 4$, $n = 8$.

When $r = 4$ the linear cyclic snake is a linear octagonal snake with $k, k \geq 2$ cycles of C_8.

Now, $p = 7k+1$ and $q = 8k$ and $p+q = 15k+1$. Define $f : E(kC_n) \to \{1, 2, 3, \cdots, 15k+1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i=1, \text{ and } 1 \leq j \leq 3, \\ 13, & \text{if } i=1, \text{ and } j=4, \\ 36-4j, & \text{if } i=1, \text{ and } 5 \leq j \leq 6, \\ 7, & \text{if } i=1, \text{ and } j=7, \\ 1, & \text{if } i=1, \text{ and } j=8, \\ 15i-8, & \text{if } 2 \leq i \leq k, \text{ and } j=1, \\ 15i-12, & \text{if } 2 \leq i \leq k, \text{ and } j=2, \\ 15i-6, & \text{if } 2 \leq i \leq k, \text{ and } j=3, \\ 15i-2, & \text{if } 2 \leq i \leq k, \text{ and } j=4, \\ 15i-4j+21, & \text{if } 2 \leq i \leq k, \text{ and } 5 \leq j \leq 7, \\ 15i-15, & \text{if } i=k, \text{ and } j=8. \end{cases}$$

It can be easily verified that f is injective. The induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i=1, \text{ and } j=1, \\ 3j-1, & \text{if } i=1, \text{ and } 2 \le j \le 4, \\ 38-4j, & \text{if } i=1, \text{ and } 6 \le j \le 7, \\ 4, & \text{if } i=1, \text{ and } j=8, \\ 15i-13, & \text{if } 2 \le i \le k, \text{ and } j=1, \\ 15i-10, & \text{if } 2 \le i \le k, \text{ and } j=2, \\ 15i-9, & \text{if } 2 \le i \le k, \text{ and } j=3, \\ 15i-4, & \text{if } 2 \le i \le k, \text{ and } j=4, \\ 15i-4j+23, & \text{if } 2 \le i \le k, \text{ and } 6 \le j \le 7, \\ 15i-11, & \text{if } 2 \le i \le k, \text{ and } j=8, \\ 15k, & \text{if } i=k, \text{ and } j=5. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 15k+1\}$ as follows:

$$\begin{cases} f(E) & = \{3,6,9,13,16,12,7,1\} \bigcup \\ & \{22,18,24,28,31,27,23,15\} \bigcup \cdots, \\ & \{15k-8, 15k-12, 15k-6, 15k-2, 15k+1, \\ & 15k-3, 15k-7, 15k-15\}. \\ f^v(V) & = \{2,5,8,11,14,10,4\} \bigcup \\ & \{17,20,21,26,29,25,19\} \bigcup \cdots, \\ & \{15k-13, 15k-10, 15k-9, 15k-4, 15k-1, \\ & 15k-5, 15k-11, 15k\}. \\ f(E) \bigcup f^v(V) & = \{1,3,6,7,9,12,13,16\} \bigcup \\ & \{2,4,5,8,10,11,14\} \cup \\ & \{15,18,22,23,24,27,28,31\} \bigcup \\ & \{17,19,20,21,25,26,29\} \bigcup \cdots, \\ & \{15k-15, 15k-12, 15k-8, 15k-7, 15k-6, \\ & 15k-3, 15k-2, 15k+1\} \bigcup \\ & \{15k-13, 15k-11, 15k-10, 15k-9, 15k-5, \\ & 15k-4, 15k-1, 15k\}. \\ & = \{1,2,3,4,\cdots,29,30,31,\cdots,15k-2,15k-1,15k,15k+1\}. \end{cases}$$

Thus the theorem is true when $r = 4$.

Now we assume that the theorem is true for $r-1, r \ge 5$ (i.e., for $n-2, n \ge 10$). In this case, $p = (n-3)k+1 = (2r-3)k+1$ and $q = (n-2)k = (2r-2)k$ and $p+q = (2n-5)k+1 = (4r-5)k+1$.

The induction hypothesis is that the edge labeling,

$$f : E(kC_{2r-2}) \to \{1, 2, 3, \cdots, (4r-5)k+1\}$$

defined as follows, is a super vertex mean labeling, where $r \geq 5, n \geq 10$, $n \equiv 0(mod\ 2)$ and $k \geq 2$.

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j-3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r-1, \\ 8r-4j-4, & \text{if } i = 1, \text{ and } r \leq j \leq 2r-4, \\ 7, & \text{if } i = 1, \text{ and } j = 2r-3, \\ 1, & \text{if } i = 1, \text{ and } j = 2r-2, \\ (4r-5)i-4r+12, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r-5)i-4r+8, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r-5)i-4r+14, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r-5)i-4r+4j+2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r-1, \\ (4r-5)i+4r-4j+1, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq 2r-3, \\ (4r-5)(i-1), & \text{if } i = k, \text{ and } j = 2r-2. \end{cases}$$

and the induced vertex labeling is,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j-1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 4j-5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r-1, \\ 8r-4j-2, & \text{if } i = 1, \text{ and } r+1 \leq j \leq 2r-3, \\ 4, & \text{if } i = 1, \text{ and } j = 2r-2, \\ (4r-5)i-4r+7, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r-5)i-4r+10, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r-5)i-4r+11, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r-5)i-4r+4j, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r-1, \\ (4r-5)i+4r-4j+3, & \text{if } 2 \leq i \leq k, \text{ and } r+1 \leq j \leq 2r-3, \\ (4r-5)i-4r+9, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r-2, \\ (4r-5)k, & \text{if } i = k, \text{ and } j = r. \end{cases}$$

Now we prove that the result is true for any r. If we replace r with $r+1$ in the above mapping we get,

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j-3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\ 8r-4j+4, & \text{if } i = 1, \text{ and } r+1 \leq j \leq 2r-2 \end{cases}$$

and

$$f(e_{i,j}) = \begin{cases} 7, & \text{if } i = 1, \text{ and } j = 2r-1, \\ 1, & \text{if } i = 1, \text{ and } j = 2r, \\ (4r-1)i - 4r + 8, & \text{if } 2 \le i \le k, \text{ and } j = 1, \\ (4r-1)i - 4r + 4, & \text{if } 2 \le i \le k, \text{ and } j = 2, \\ (4r-1)i - 4r + 10, & \text{if } 2 \le i \le k, \text{ and } j = 3, \\ (4r-1)i - 4r + 4j - 2, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (4r-1)i + 4r - 4j + 5, & \text{if } 2 \le i \le k, \text{ and } r+1 \le j \le 2r-1, \\ (4r-1)(i-1), & \text{if } i = k, \text{ and } j = 2r. \end{cases}$$

and,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \le j \le 4, \\ 4j - 5, & \text{if } i = 1, \text{ and } 5 \le j \le r, \\ 8r - 4j + 6, & \text{if } i = 1, \text{ and } r+2 \le j \le 2r-1, \\ 4, & \text{if } i = 1, \text{ and } j = 2r, \\ (4r-1)i - 4r + 3, & \text{if } 2 \le i \le k, \text{ and } j = 1, \\ (4r-1)i - 4r + 6, & \text{if } 2 \le i \le k, \text{ and } j = 2, \\ (4r-1)i - 4r + 7, & \text{if } 2 \le i \le k, \text{ and } j = 3, \\ (4r-1)i - 4r + 4j - 4, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (4r-1)i + 4r - 4j + 7, & \text{if } 2 \le i \le k, \text{ and } r+2 \le j \le 2r-1, \\ (4r-1)i - 4r + 5, & \text{if } 2 \le i \le k, \text{ and } j = 2r, \\ (4r-1)k, & \text{if } i = k, \text{ and } j = r+1. \end{cases}$$

This is equivalent to the original labeling in terms of n, which is given in the beginning of the proof, and it is clear that $f(E) \cup f^v(V) = \{1, 2, 3, \cdots, (4r-1)k-1, (4r-1)k, (4r-1)k+1\}$. Thus the theorem is proved by mathematical induction. □

Example 4.10 A graph given in Figure 6 is an SVM labeling of a linear cyclic snake $2C_{12}$.

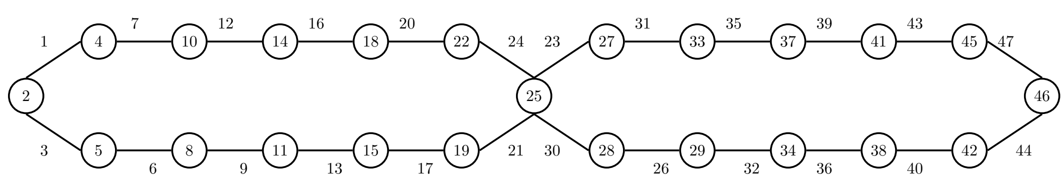

Figure 5 A super vertex mean labeling of a linear cyclic snake $2C_{12}$.

Theorem 4.11 *Let kC_n be a linear cyclic snake with $k, k \ge 2$ blocks of $C_n, n \ge 9$ and $n \equiv 1(mod\ 4)$. Then kC_n is a super vertex mean graph.*

Proof Let kC_n be a cyclic snake with $k, k \ge 2$ blocks of $C_n, n \ge 9$ and $n \equiv 1(mod\ 4)$. Let

$n = 2r + 1, r \geq 4$, and $r = 2s, s \geq 2$ so that $n = 4s + 1$. Now $p = (n-1)k + 1$ and $q = nk$ and $p + q = (2n-1)k + 1$. Define $f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k + 1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} n, & \text{if } i = 1, \text{ and } j = 1, \\ 2j + n - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq r + 1, \\ 2j - n - 2, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq n, \\ (2n-1)i - n - 1, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (2n-1)i - n + 2j - 2, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq r - 3, \\ (2n-1)i - n + 2j - 1, & \text{if } 2 \leq i \leq k, \text{ and } r - 2 \leq j \leq r - 1, \\ (2n-1)i - n + 2j, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq r + 1, \\ (2n-1)i - 8, & \text{if } 2 \leq i \leq k, n \neq 9 \text{ and } j = r + 2, \\ (2n-1)i - 7, & \text{if } 2 \leq i \leq k, n = 9 \text{ and } j = r + 2, \\ (2n-1)i - 2n + 1, & \text{if } 2 \leq i \leq k, \text{ and } j = r + 3, \\ (2n-1)i - 3n + 2j - 3, & \text{if } 2 \leq i \leq k, \text{ and } r + 4 \leq j \leq n + 1 - s, \\ (2n-1)i - 3n + 2j - 2, & \text{if } 2 \leq i \leq k, \text{ and } n + 2 - s \leq j \leq n - 1, \\ (2n-1)i - n, & \text{if } 2 \leq i \leq k, \text{ and } j = n. \end{cases}$$

and, the induced vertex labels are as follows:

When $i = 1$, and $n \geq 9$,

$$f^v(v_{i,j}) = \begin{cases} n - 1, & \text{if } j = 1, \\ 2j + n - 2, & \text{if } 2 \leq j \leq r, \\ n + 1, & \text{if } j = r + 2, \\ 2j - 16, & \text{if } r + 3 \leq j \leq n. \end{cases}$$

and when $2 \leq i \leq k$, and $n = 9$,

$$f^v(v_{i,j}) = \begin{cases} 17i - 13, & \text{if } j = 1, \\ 17i + 3j - 14, & \text{if } 2 \leq j \leq 4, \\ 17i, & \text{if } j = 5 \text{ and } i = k, \\ 17i - 3, & \text{if } j = 6, \\ 22, & \text{if } j = 7, \\ 19, & \text{if } j = 8, \\ 23, & \text{if } j = 9. \end{cases}$$

and when $2 \leq i \leq k$, and $n \geq 13$,

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i - 3r - 1, & \text{if } j = 1, \\ (2n-1)i - n + 2j - 3, & \text{if } 2 \leq j \leq r - 3 \end{cases}$$

and

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i - n + 2j - 2, & \text{if } r-2 \leq j \leq r-1, \\ (2n-1)i - 2, & \text{if } j = r, \\ (2n-1)i, & \text{if } i = k, \text{ and } j = r+1, \\ (2n-1)i - 3, & \text{if } j = r+2, \\ (2n-1)i - n - 3, & \text{if } j = r+3, \\ (2n-1)i - 3n + 2j - 4, & \text{if } r+4 \leq j \leq n+1-s, \\ (2n-1)i - 3n + 2j - 3, & \text{if } n+2-s \leq j \leq n-1, \\ (2n-1)i - n - 2, & \text{if } j = n. \end{cases}$$

We can easily prove the theorem by the technique of mathematical induction on s. The remaining of the proof is left as an exercise. □

Theorem 4.13 *Let kC_n be a linear cyclic snake with $k, k \geq 2$ blocks of $C_n, n \geq 11$ and $n \equiv 3(mod\ 4)$. Then kC_n is a super vertex mean graph.*

Proof Let kC_n be a linear cyclic snake with $k, k \geq 2$ blocks of $C_n, n \geq 11$ and $n \equiv 3(mod\ 4)$. Let $n = 2r+1, r \geq 5$, and $r = 2s+1, s \geq 2$ so that $n = 4s+3$. Now, $p = (n-1)k+1$ and $q = nk$ and $p+q = (2n-1)k+1$. Define $f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k+1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 4j - 1, & \text{if } i = 1, \text{ and } 1 \leq j \leq r, \\ 4n - 4j + 2, & \text{if } i = 1, \text{ and } r+1 \leq j \leq 2r, \\ 1, & \text{if } i = 1, \text{ and } j = n, \\ (2n-1)i - 2n + 2j + 7, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 3, \\ (2n-1)i - 2n + 4j + 1, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r-1, \\ (2n-1)i - 3r + 3j - 2, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq r+1, \\ (2n-1)i + 2n - 4j + 2, & \text{if } 2 \leq i \leq k, \text{ and } r+2 \leq j \leq 2r-2, \\ (2n-1)i - 2n + 1, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r-1, \\ (2n-1)i - 4n + 2j + 7, & \text{if } 2 \leq i \leq k, \text{ and } 2r \leq j \leq n. \end{cases}$$

and, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 4j - 3, & \text{if } i = 1, \text{ and } 2 \leq j \leq r, \\ 4n - 4j + 4, & \text{if } i = 1, \text{ and } r+2 \leq j \leq n, \\ (2n-1)i - 2n + 4, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (2n-1)i - 2n + 2j + 6, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq 3 \\ (2n-1)i - 2n + 4j - 1, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (2n-1)i - 1, & \text{if } 2 \leq i \leq k, \text{ and } j = r+2 \end{cases}$$

and

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i + 2n - 4j + 4, & \text{if } 2 \leq i \leq k, \text{ and } r+3 \leq j \leq 2r-2, \\ (2n-1)i - 2n + 8, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r-1, \\ (2n-1)i - 2n + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r, \\ (2n-1)i - 2n + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r+1, \\ (2n-1)k, & \text{if } i = k, \text{ and } j = r+1. \end{cases}$$

We can easily prove that the above labeling is an SVM labeling of kC_n, where $k \geq 2$ blocks of $C_n, n \geq 11$ and $n \equiv 3(mod\ 4)$, by using the technique of mathematical induction on s, where $n = 4s + 3$. Thus the theorem can easily be proved. □

§5. Super Vertex Mean Number

The concept of super vertex mean number arises from the earlier concepts such as, mean number, super mean number etc. M.Somasundaram and R.Ponraj have introduced the term mean number of a graph [16] and they have found the mean number of many standard graphs. Later on, A.Nagarajan et.al. introduced the concept of super mean number of a graph [9] and proved the existence of it for any graph by finding out the limit values of it. Encouraged by their works we introduce this new concept which we like to name as super vertex mean number or SVM number.

Definition 5.1 *Let f be a an injective function of a (p,q) - graph $G(V,E)$ defined from E to the set $\{1,2,3,\cdots,n\}$ that induces for each vertex v the label defined by the rule $f^v(v) = Round \left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v. Let $f(E) \cup f^v(V) \subseteq \{1,2,3,\cdots,n\}$. If n is the smallest positive integer satisfying these conditions together with the condition that all the vertex labels as well as the edge labels are distinct, then n is called the super vertex mean number (or SVM number) of the graph $G(V,E)$, and is denoted by $SV_m(G)$.*

Observation 5.2 It is observed that $SV_m(G) = p + q$, for all SVM graphs G whose order is p and size is q. And for other graphs G, $SV_m(G) \geq p + q + 1$. For graphs containing an isolated vertex or a leaf, the super vertex mean number is not defined.

Therefore, for any (p,q) − graph G, $p + q \leq SV_m(G) \leq \infty$. For example, the SVM number of C_4, $SV_m(C_4) = 9$.

§6. Conclusion

In this paper, we have proved that all the linear cyclic snakes are super vertex mean graphs. In the case of super mean labeling, the vertex analogue of SVM, it was easy to obtain a general formula for linear cyclic snakes as well as other cyclic snakes represented the string $s_1, s_2, s_3, \cdots, s_{k-2}$, where each s_i need not be equal. This is because when we calculate the induced edge label for an edge, by finding the average of the labels of the two vertices which are the end points of that particular edge, we need to consider only those two vertices. Therefore the average remains the same as in the case of cycles.

But for super vertex mean labeling, when we find the induced vertex labeling of the connecting vertices of a cyclic snake we have to consider four edges that are incident on those vertices to get the average. Thus it becomes pretty difficult to obtain a general formula for cyclic snakes represented the string $s_1, s_2, s_3, \cdots, s_{k-2}$, where each s_i need not be equal. Another possibility emerges is that we try to explore the SVM labeling of KC – snakes, which is defined as a connecting graph in which each of the k many blocks is isomorphic to a cycle C_n for some n and the block - cut point graph is a path. As in the case of kC_n – snakes, a kC –snake too can be represented by a string of integers, $s_1, s_2, \cdots, s - k - 2$. It remains still an open problem to label a kC –snake which has either equal s_i or different s_i.

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Equal Degree Graphs of Simple Graphs

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Abstract: This paper introduces equal degree graphs of simple existed graphs. These graphs exhibited some properties which are co-related with the older one. We characterize graphs for which their equal degree graphs are connected, completed, disconnected but not totally disconnected. We also obtain several properties of equal degree graphs and specify which graphs are isomorphic to equal degree graphs and complement of equal degree graphs. Furthermore, the relation between equal degree graphs and degree Prime graphs is determined.

Key Words: Parameters of the graph, simple graphs, equal degree graphs, degree prime graphs, degree graph isomorphism, Smarandachely k-degree graph, Smarandachely degree graph.

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§1. Introduction

The evaluations of new graphs are involving sets of objects and binary relations among them. So the construction and preparation of graphs varies author to author, and thus it is difficult to pin point its formulation to a single source. Thus the graphs discovered many times, and each discovery being independent of the other. For this reason, there are various types of graphs each with its own definition.

Many authors, starting from 2003, the parameters vertex degree and degree sequence of graphs were used again in Graph theory, and several types of graphs have been introduced. In this sequel we have introduced equal degree graphs of various simple graphs and characterized their properties.

For any finite group G, the definition and notation of degree graph of a simple group was introduced by Lewis and white [5]. This graph is defined as follows: Let G be a finite group and let $cd(G)$ be the set of irreducible character degrees of G. Then the degree graph $\triangle(G)$ is the graph whose set of vertices is set of primes that divide degrees in $cd(G)$, with an edge between p and q if pq divides a for some $a \in cd(G)$.

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In [6], the authors introduced the degree pattern of a finite group G and denoted by $D(G)$, where $D(G) = (deg(p_1), deg(p_2), \cdots, deg(p_k))$, here $p_1 < p_2 < \cdots < p_k$ are distinct primes in prime decomposition of n. Further, the authors S.F.Kapur, Albert and Curtiss [7] introduced the notation $D(G)$ for degree sets of connected graphs, trees, planner and outer planner graphs. According to these authors, the notation $D(G)$ is a degree set of degrees of vertices of G.

The authors Manoussakis, Patil and Sankar [8] was proved that for any finite non empty set S of non negative integers, there exist a disconnected graph G such that $D(G) = S$, and also the minimum order of such a graph is determined.

There are several ways to produce new graphs from the existing graphs in Graph theory. Recently the authors M.Sattanathan and R.Kala [1] introduced a special way to produce the degree prime graph $DP(G)$ for any finite simple undirected graph G. The $PD(G)$ of a graph G having the same vertex set of G and two vertices are adjacent in $PD(G)$ if and only if their unequal degrees are relatively prime in G. By the motivation of these degree prime graphs, we construct and study the equal degree graphs of simple graphs with usual notation $D(G)$. We suspect that these graphs will be used to solve many computational problems in computer engineering and applied sciences.

Throughout this paper, G and $D(G)$ represent finite simple undirected graphs having without loops and without multiple edges of same order. We have introduced degree graph $D(G)$ of G, which is defined as a graph with same vertex set as G and two vertices of G are adjacent in $D(G)$ if and only if their degrees are equal in G. In this paper we studied interrelations between G and $D(G)$, and hence we obtain several properties and their consequences of $D(G)$ with illustrations and examples. Further we characterize G for which $D(G)$ either is connected, disconnected, totally disconnected or complete.

§2. Basic Definitions and Notations

In this section we consider basic definitions and their graph theoretical notations. Throughout the text, we consider G is an abstract graph structure which is a finite undirected graph without loops and multiple edges. We represent G as $G = G(V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$. We are only going to deal with finite graphs, so we define $|V| = n$ to the order of G and $|E| = m$ to be the size of G where n and m are called graph parameters. Further if there is an edge e in G between the vertices u and v, we briefly write $e = uv$ or $e = (u, v)$ and say edge e joins the vertices u and v. A vertex is said to be isolated if it is not adjacent to any other vertex.

The complement of a graph $G(V, E)$ is the graph $G^c(V, E^c)$ having the same vertex set as G, and its edge set E^c is the complement of E, that is, uv is an edge of G^c if and only if uv is not an edge of G. A complete graph of order n is denoted by K_n. A graph of order n with no edges in an empty graph and is denoted by $N_n = K_n^c$ which isomorphic to totally disconnected. A path of length n is denoted by P_n. A cycle of length $n(n \geqslant 3)$ is a cycle of length n and is denoted by C_n.

We now turn to graphs whose vertex sets can be partitioned in special ways. A graph G is a partite graph if $V(G)$ can be partitioned into subsets, called partite sets. A graph G is a $k-$partite graph if $V(G)$ can be partitioned into k subsets $V_1, V_2, \cdots, V_k$ (partite sets) such that uv is an edge of G if u and v belongs to different partite sets. In addition, if every two vertices in different partite sets are joined by an edge, then G is a complete $k-$partite graph. If $|V_i| = n_i$ for $1 \leqslant i \leqslant k$, then we denote this complete $k-$partite graph by $K_{n_1,n_2,\cdots,n_k}$. Thus

(1) $K_{1,1,\cdots,1} \cong K_n$;
(2) $K_{n,n}$ is a complete bipartite;
(3) $K_{1,n}$ is a Star;
(4) K_{n_1,n_2} is a complete bipartite.

In a graph G, the degree of a vertex u is the number of edges of G which are incident to u and denoted by $d(u)$, $deg(u)$ or $deg_G(u)$. A graph is regular if all its vertices are of the same degree and $r-$regular if all of its vertices are of degree r. A $3-$regular graph is a cubic graph Q_8. If $d_i, 1 \leqslant i \leqslant n$, be the degree of the vertices v_i of a graph G then the sequence $d(G) = (d_1, d_2, \cdots, d_n)$ is the degree sequence of G. Usually, we order the vertices so that the degree sequence is monotonically increasing, that is, $\delta(G) = d_1 \leq d_2 \leq \cdots \leq d_n = \triangle(G)$. Also two graphs with the same degree sequence are said to be degree equivalent.

We need the following results for preparing equal degree graphs.

Theorem 2.1([2]) *The sum of the degrees of graphs is even, being twice the number of edges.*

Theorem 2.2([2]) *In any graph there is an even number of vertices of odd degree.*

Theorem 2.3([2]) *If $d_1, d_2, \cdots, d_n$ is the degree sequence of some graph, then, necessarily $\sum_{i=1}^{n} d_i$ is even, and $0 \leq d_i \leq n-1$ for $1 \leq i \leq n$. But converse is not true.*

Theorem 2.5([2]) *Let G be a graph of order $n \geq 2$, then $d(G)$ contains at least two numbers are same.*

Theorem 2.6([2]) *Let G be a r-regular graph of order n. Then $|E(G)| = \dfrac{rn}{2}$.*

Theorem 2.7([3]) *Let G be a r-regular graph of order n. Then G^c is $(n-r-1)$-regular graph of order n.*

Theorem 2.8([3]) *A graph is $1-$regular if and only if it is of even order and is the disjoint union of some $K_2^{'}s$.*

If $n = 1$, then G is called trivial graph, otherwise G is called non-trivial graph. In this paper we consider non-trivial graphs only. For further details and notations we refer [4].

§3. Equal Degree Graphs

We have already mentioned that the best known parameters of a graph are order and size. But another parameter of a graph is degree of a vertex which is a meaningful to have a term for the number of edges meeting at a vertex. By using this parameter we define equal degree graph as follows.

Definition 3.1 *Let G be a simple graph with vertex set $V = \{1, 2, \cdots, n\}, n > 1$, Then the equal degree graph of G is $D(G)$ having the same vertex set as G and two vertices $u, v \in V$ are adjacent in $D(G)$ if and only if $deg_G(u) = deg_G(v)$.*

Generally, we know *Smarandachely k-degree graphs D_kG* of a graph G in which vertices $u, v \in V$ are adjacent if and only if $|deg_G(u) - deg_G(v)| = k$ for integers $k \leq \Delta(G) - \delta(G)$ and *Smarandachely degree graph SG* in which u, v are adjacent if $|deg_G(u) - deg_G(v)| \geq 1$. Clearly, $D_0G = DG$. The definition of equal degree graphs should be noted that the following.

1. $D(G)$ is a non-trivial graph;
2. $D(G)$ has at least one edge;
3. $D(G)$ is also a simple undirected graph having without multiple edges.

The following Fig.1 shows that simple graphs and their equal degree graphs of order 4.

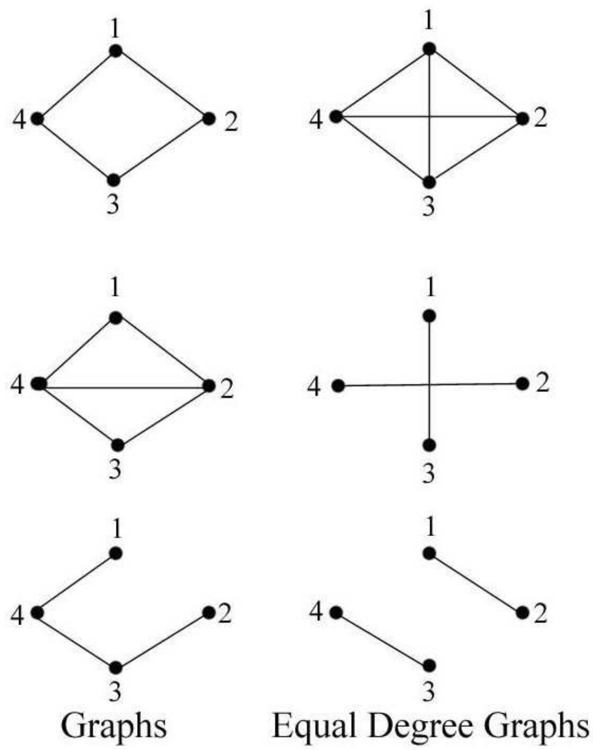

Fig.1 Graphs and their equal degree graphs

We now characterize graphs G for which $D(G)$ is either regular or not. The following propositions are immediate.

Proposition 3.2 *For any graph G, the graph $D(G)$ is never a $0-$regular graph.*

Proof Suppose $D(G)$ is a $0-$regular graph. Then its degree sequence is $d(D(G)) = (0, 0, \cdots, 0)$ for any graph G. This shows that $D(G)$ is isomorphic to empty graph, and thus $D(G)$ is a trivial graph, which is a contradiction to the definition of equal degree graphs. Thus the result follows. $\square$

Remark 3.3 (i) $D(G)$ is 1-regular if and only if $D(G)$ is a graph of order 2. (ii) $D(G)$ is 1-regular if and only if G is a graph of order 4 and $d(G) = (0, 0, 1, 1)$ or $(2, 2, 3, 3)$.

Proposition 3.4 *For any graph G of order $n > 4$, $D(G)$ is never $1-$regular. In particular, $D(G)$ is never $2, 3, \cdots, (n-2)$-regular.*

Proof Suppose G is a graph with size $n > 4$. We show that $D(G)$ is not $1-$regular. Assume that $D(G)$ is $1-$regular. Then the degree sequence of G is $d(D(G)) = (1, 1, \cdots, 1)(n \text{ times})$.

$$(1 + 1 + \cdots + 1)(n \text{ times}) = \begin{cases} \text{even}, & \text{if n is even} \\ \text{odd}, & \text{if n is odd} \end{cases}$$

Case 1. If n is odd, then $1 + 1 + \cdots + 1(n \text{ times})$ is odd, which contradicts to the Theorem 2.2. So in this case the result is not true.

Case 2. If n is even, then $1 + 1 + \cdots + 1(n \text{ times})$ is odd, which is impossible because the graph $D(G)$ does not contain an even number of vertices of same odd degree.

From the above cases our assumption is not true, and hence $D(G)$ is never $1-$regular for any graph G of order $n > 4$. □

Remark 3.5 Similarly we can show that $D(G)$ is never 2-regular, 3-regular,$\cdots$, $(n-2)$-regular but it should be $(n-1)$-regular.

Theorem 3.6(Fundamental Theorem) *For any graph G of order n, the degree graph $D(G)$ is either complete or disconnected but not totally disconnected.*

Proof Suppose $D(G)$ is totally disconnected. Then obviously $D(G)$ is isomorphic to N_n. But by the definition of degree graph, $D(G)$ is never isomorphic to N_n. Hence $D(G)$ is not totally disconnected.

Now we prove that $D(G)$ is either complete or disconnected. Suppose $D(G)$ is disconnected. Then there is nothing to prove. If possible assume that $D(G)$ is connected, then the Proposition 3.4 and Remark 3.5 shows that $D(G)$ is $(n-1)$-regular, and hence $D(G)$ is complete. □

§4. Complete Equal Degree Graphs

In this section we are going to prove that the equal degree graphs are complete. Further we characterize graphs G for which $D(G)$ is complete and also we show that $G \cong D(G)$.

Proposition 4.1 *The degree graph of regular graph is complete.*

Proof Let V be a vertex set of r-regular graph G. Then the degree sequence of G is $d(G) = (r, r, \cdots, r)$, that is, $d(i) = r$ for each $1 \leq i \leq n$. We show that $D(G)$ is complete. For this let $i, j \in V$, $i \neq j$, then $deg(i) = r$ and $deg(j) = r$. Therefore, $deg(i) = deg(i)$, for all $i \neq j$ in $V(G)$. Thus i and j are adjacent in $D(G)$. This shows that every two distinct pair of vertices is joined by an edge in $D(G)$. Hence $D(G)$ is complete. □

This proposition has a number of useful consequences.

Corollary 4.2 *Let G be a connected graph of order $n > 4$. Then $D(G)$ is either complete or disconnected.*

Proof The proof is divided into cases following.

Case 1. Suppose G is a connected regular graph of order $n > 4$. Then, by the Proposition 4.1, $D(G)$ is complete.

Case 2. Suppose G is a connected but not regular. Then the degree sequence of G contains at least two distinct positive integers, say s and t. That is, if $u, v \in G$, then $deg(u) \neq deg(u)$ implies u is not adjacent to v in $D(G)$. Hence $D(G)$ is disconnected. □

Corollary 4.3 *For each $n > 1$, we have*

(*i*) $D(K_n) = K_n$;
(*ii*) $D(C_n) = K_n$;
(*iii*) $D(N_n) = K_n$;
(*iv*) $D(K_{n,n}) = K_{2n}$;
(*v*) $D(Q_8) = K_8$.

Proof (*i*) The complete graph K_n is $(n-1)$-regular, and thus $D(K_n) = K_n$.

(*ii*) For each $n \geq 3$, the cycle C_n is 2-regular. Hence $D(C_n) = K_n$.

(*iii*) For each, the empty graph N_n is 0-regular graph. Hence $D(N_n) = K_n$.

(*iv*) Since the completed bipartite graph $K_{n,n}$ is n-regular and the order of $K_{n,n}$ is $2n$. Thus $D(K_{n,n}) = K_{2n}$.

(*v*) Q_8 is 3-regular of order 8, and thus $D(Q_8) = K_8$. □

Corollary 4.4 *Let G^c be the complement of r-regular graph G of order n, then $D(G) = D(G^c) \cong K_n$.*

Proof We deduces this consequence from the Theorem 2.6 and Proposition 4.1 as follows.

We know that G is r-regular graph of order n if and only if G^c is $(n-r-1)$-regular graph of same order n. Thus the Proposition 4.1 shows that $D(G) \cong K_n$ if and only if $D(G^c) \cong K_n$. Hence $D(G) = D(G^c) \cong K_n$. □

Corollary 4.5 *Let G be any graph of order n. For fixed $m \in Z$, if there exists an integer n such that $deg(v) = mn$ for each vertex v of G. Then, $D(G) = K_n$.*

Proof Obviously follows from Proposition4.1, since G is mn-regular graph of order n.

We now characterize the graphs G which attain bounds for $|E(D(G))|$. We know that

$$0 \leq |E(G)| \leq \frac{n(n-1)}{2}$$

for any simple graph G of order $n \geq 1$. But the following result specifies that the bounds for $|E(D(G))|$. $\square$

Theorem 4.6 *If G is any graph of order $n > 1$, then* $0 < |E(G)| \leq \frac{n(n-1)}{2}$.

Proof From the definition of equal degree graphs, $|V(G)| = |V(D(G))| = n$, and $n > 1$. For any non-trivial graph G, we have $deg_G u \leq (n-1)$ for each $u \in V(G)$. This is also true in $D(G)$, that is, $deg_{D(G)} u \leq (n-1)$ for each $u \in V(D(G))$. From Theorem2.1 we have

$$2|E(D(G))| = \sum_{d \in V(D(G))} deg(u) \Rightarrow 2|E(D(G))| \leqslant n(n-1) \Rightarrow |E(D(G))| \leqslant \frac{n(n-1)}{2}.$$

It is one extreme of the required inequality. At the other extreme, a degree graph $D(G)$ may possess at least one edge at all. That is, $|E(D(G))| \neq 0$. Hence

$$0 < |E(G)| \leq \frac{n(n-1)}{2}. \quad \square$$

Remark 4.7 The above inequality says that the following two specifications for $D(G)$:

(1) $D(G)$ or $D(G^c)$ has at least one edge or at most $\binom{n}{2}$ edges;

(2) $D(G)$ is never totally disconnected. In particular, $D(G) \ncong N_n$ for each G of order $n > 1$.

Corollary 4.8 *Let G be a r-regular graph of order $n > 1$. Then*

$$|E(G)| = \frac{rn}{2} \quad \textit{and} \quad |E(D(G))| = \binom{n}{2}.$$

Proposition 4.9 *Let G be a graph of order n. Then $|E(G)| = |E(D(G))|$ if and only if G and $D(G)$ are $(r-1)$-regular graphs.*

Proof Let G be a r-regular graph of order $n > 1$. By the Corollary 4.7,

$$|E(G)| = \frac{rn}{2} \quad \text{and} \quad |E(D(G))| = \binom{n}{2}.$$

Therefore,

$$|E(G)| = |E(D(G))| \Leftrightarrow \frac{rn}{2} = \binom{n}{2} \Leftrightarrow \frac{rn}{2} = \frac{n(n-2)}{2} \Leftrightarrow = n-1 \Leftrightarrow G$$

is $(n-1)$-regular, and hence $D(G)$ is also $(n-1)$-regular. $\square$

Proposition 4.10 *If G^c is a complement of G, then $D(G^c) = D(G)$.*

Proof Let G^c be the complement of a graph G of order $n > 1$. Then the following cases arise on the regularity of G.

Case 1. Suppose G is a regular graph. Then the result obviously follows from Proposition 4.1.

Case 2. Suppose G is not a regular graph. We show that $D(G^c) = D(G)$. If possible assume that $D(G^c) \neq D(G)$, then the following three subcases arise.

Subcase 2.1 If $V(D(G^c)) \neq V(D(G))$ then obviously $V(G^c) \neq V(G)$, which is a contradiction to the fact that $V(G^c) = V(G)$.

Subcase 2.2 If $E(D(G^c)) \neq E(D(G))$, then $V(D(G^c)) + V(D(G)) = \frac{n(n-1)}{2}$. This shows that either $D(G)$ or $D(G^c)$ has at most $\frac{n(n-1)}{4}$ edges, which is a contradiction to the fact that $D(G)$ or $D(G^c)$ has at most $\frac{n(n-1)}{2}$ edges.

Subcase 2.3 If $V(D(G^c)) \neq V(D(G))$, $E(D(G^c)) \neq E(D(G))$, then trivially it is not true from cases 2.1 and 2.2.

From the above three subcases 2.1, 2.2 and 2.3, we conclude that $D(G^c) \neq D(G)$ is not true. Hence $D(G^c) = D(G)$. □

Remark 4.11 The converse of the above result is not true. For example, $D(N_n) = D(N_n^c)$ but $N_n \neq N_n^c$.

Theorem 4.12 *Let G_1 and G_2 be same regular graphs of order n. Then $D(G_1) = D(G_2)$. But converse is not true.*

Proof Suppose G_1 and G_2 be regular graphs of same order $n > 1$. By the Proposition 4.1, $D(G_1) \cong K_n$ and $D(G_2) \cong K_n$, and thus $D(G_1) = D(G_2)$. But converse of this result is not true. This illustrates the Figure 2. Consider the graphs G_1 and G_2 on four vertices and their degree graphs. □

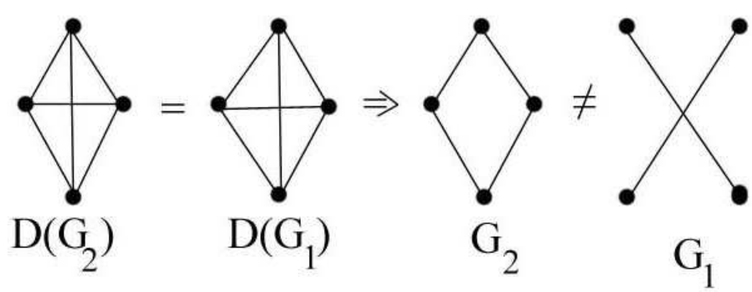

Fig.2 Graphs G_1, G_2, $D(G_1)$ and $D(G_2)$.

Theorem 4.13 *Let G_1 and G_2 be graphs of same order $n > 1$ such that $d(G_1) = d(G_2)$. Then $D(G_1) = D(G_2)$. But converse is not true.*

Proof Suppose G_1 and G_2 be non-regular degree equivalent graphs of same order $n > 1$. Then their degree sequences are equal. That is, $d(G_1) = d(G_1) = (d_1, d_2, \cdots, d_n)$ where $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, $0 \leqslant d_i \leqslant n-1$ for each $1 \leqslant i \leqslant n$. By the definition of degree graphs, G_1 and G_2 ($G_1 \neq G_2$) are both realize same degree graph, that is, $D(G_1) = D(G_2)$.

Converse of the Theorem 4.13 is not true, in general. For the degree sequences $d(G_1) = (2, 2, 2, 2)$ and $d(G_2) = (3, 2, 3, 2)$, we have $D(G_1) = D(G_2)$ implies that $d(G_1) \neq d(G_2)$. □

Theorem 4.14 *If G_1 and G_2 are two graphs such that $G_1 \cong G_2$, then $D(G_1) \cong D(G_2)$. But the converse is not true.*

Proof Suppose $G_1 \cong G_2$. Then there exists an isomorphism φ from G_1 onto G_2. We show that $D(G_1) \cong D(G_2)$. For this let $(u, v) \in E(D(G_1))$, then by the definition of degree graphs, $deg_{G_1}u = deg_{G_1}v \Rightarrow deg_{G_2}\varphi(u) = deg_{G_2}\varphi(v) \Rightarrow (\varphi(u), \varphi(v)) \in E(D(G_2)) \Rightarrow D(G_1) \cong D(G_2)$.

But converse of this result is not true. For example, $D(N_4) \cong D(C_4)$ but $N_4 \ncong C_4$. □

§5. Disconnected Equal Degree Graphs

In this section we characterize the graphs G for which $D(G)$ is disconnected.

Theorem 5.1 *Let G be a graph of order $n + k$. Then $D(G) \cong K_n \cup N_k$ where n is the number of vertices of same degree and k is the number vertices of unequal degree.*

Proof Let $|V(G)| = n + k$.Then V can be partitioned into two subsets S_1 and S_2 such that $S_1 = \{u_1, u_2, \cdots, u_n\}$ and $S_2 = \{v_1, v_2, \cdots, v_k\}$ where $deg_G u_i = deg_G u_j$ for all $1 \leqslant i \neq j \leqslant n$ and $deg_G v_i \neq deg_G v_j$ for all $1 \leqslant i \neq j \leqslant k$. By the definition of Degree graphs, $< S_1 >\cong K_n$ and $< S_2 >\cong N_k$. Hence $D(G) \cong K_n \cup N_k$. □

This theorem gives the following consequences.

Corollary 5.2 $D(P_n) =< S_1 > \cup < S_2 >$ *where* $< S_1 >\cong K_2$ *and* $< S_2 >\cong K_{n-2}$.

Proof Let v_1 and v_{n+1} be the internal and external vertices of the path $P_n : v_1e_1v_2e_2v_3 \cdots v_ne_nv_{n+1}$. Then the vertex set $V = V(P_n)$ can be partitioned two disjoint sets S_1 and S_2 such that $S_1 = \{v_1, v_{n+1}\}$ and $S_2 = \{v_1, v_2, \cdots, v_n\}$.

Case 1. In this case $deg_G v_1 = deg_G v_{n+1} \neq deg_G v_j$ for $j = 2, 3, .., n$. This shows that there exists only one edge between v_1 and v_{n+1} in S_1, and which are not adjacent to the vertices in S_2. Thus the degree graph of S_1 is an induced sub graph $< S_1 >$ which is isomorphic to K_2.

Case 2. Suppose $v_i, v_j \in S_2$ for every $i \neq j$. Then $deg_G v_i = deg_G v_j = 2 \neq 1 = deg_G v_1 = deg_G v_{n+1}$ for each $i \neq j$ such that $2 \leqslant i, j \leqslant n$. Thus the degree graph of S_2 is also an induced subgraph $< S_2 >$ which is isomorphic to K_{n-2}.

Case 3. Suppose $u \in S_1$ and $v \in S_2$. Then there is no edge between u and v in the equal degree graph whose vertex set is $V = S_1 \cup S_2$ since $deg_G u = 1 \neq 2 = deg_G v$.

From the cases 1, 2 and 3 we conclude that $D(P_n)$ is disconnected with two disjoint components $< S_1 >$ and $< S_2 >$. □

The proofs of the following consequences are obvious.

Corollary 5.3 *The following results on $D(G)$ are true:*

(1) $D(K_{1,n}) = K_1 \cup K_{n-1}$;
(2) $D(K_{n_1,n_2,\cdots,n_k}) = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$;
(3) $D(W_n) = K_1 \cup K_{n-1}$.

Corollary 5.4 *Let G be a graph of order $p+q$. Then $D(G) = K_p \cup K_q$ where p is the number of vertices of same degree and q is the number of vertices of another same degree.*

Corollary 5.5 *Let T be a tree of order $n_1+n_2+n_3$. Then $D(T) = K_{n_1} \cup K_{n_2} \cup K_{n_3}$ where n_1 is the number of pendent vertices, n_2 is the number of non-pendent vertices of same degree and n_3 is another non-pendent vertex of another same degree.*

Theorem 5.6 *Let D be a connected Euler graph. Then $D(G)$ is either complete or disjoint union of complete components.*

Proof Consider the two cases on the regularity and non-regularity of a connected Euler graph G.

Case 1. Let G be a regular graph. The Proposition 4.1 shows that $D(G)$ is complete.

Case 2. Let G be a non- regular graph of order $n_1+n_2+\cdots+n_k$. Then n_1 is the number of vertices of degree 2, n_2 is the number of vertices of degree 4, and so on n_k is the number of vertices of degree $2k$. The Theorem 5.1 shows that $D(G)$ is isomorphic to the disjoint union of complete components $K_{n_1}, K_{n_2}, \cdots, K_{n_k}$. Hence $D(G) = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$. □

Example 5.7 For each $n \geqslant 3$, the cycle C_n is a regular Euler graph, and thus $D(C_n) = K_n$.

Example 5.8([3]) The graphs D_{12} and M_{16} are the Davids and Mohammeds graphs of order 12 and 16 respectively, which are non-regular Euler graphs, and their Degree graphs $D(D_6) = K_6 \cup K_6'$ and $D(M_{11}) = K_4 \cup K_7'$, which are disconnected graphs.

Theorem 5.9 *Let G be a simple graph of order $n > 4$. Then $D(D(G)) \cong D(G) \Leftrightarrow G$ is r-regular graph.*

Proof For any simple graph G of order $n > 4$, we have G is r-regular $\Leftrightarrow D(G) \cong K_n$, by the Proposition 4.1 $\Leftrightarrow D(D(G)) \cong D(K_n) \Leftrightarrow D(D(G)) \cong K_n$(since $D(K_n) = K_n$) $\Leftrightarrow D(D(G)) \cong D(G)$. □

§6. Relation Between $D(G)$ and $DP(G)$

In [1], the authors M. Sattanathan and R. Kala introduced Degree prime graphs and studied their characterizations. According to these authors $DP(G)$ is a graph whose vertex set is same as $V(G)$ and $u, v \in V(G)$ are adjacent in $DP(G)$ if and only if

$$deg_G u \neq deg_G v, \quad gcd(deg_G u,\ deg_G v) = 1.$$

In this paper, we are going to study the relation between $D(G)$ and $DP(G)$. Fundamentally we observe that the following:

For any graph G, we have

(1) $D(G) \neq DP(G)$;

(2) $D(DP(G)) \neq DP(D(G))$.

But the following result specifies that the relation between $D(G)$ and $DP(G)$ for some graphs G.

Theorem 6.1 *If G is either totally disconnected, regular or complete, then $D(G) \cong (DP(G))^c$.*

Proofv For any totally disconnected, regular or complete graph G, we know that $D(G) \cong K_n$ and $DP(G) \cong N_n$. But $K_n = N_n^c$. This shows that the required result is obviously true. *Box*

Here, we present an open problem following:

Problem 6.2 *Let G be a graph. Then*

(1) *Find the cardinality of the set $S = \{G : D(G) \text{ is complete }\}$;*
(2) *Find the cardinality of the set $S = \{G : D(D(G)) = D(G)\}$;*
(3) *For the finite family of graphs $\{G_i\}$, show that*

$$\bigcup_{i=1}^{n} D(G_i) = D\left(\bigcup_{i=1}^{n} G_i\right) \quad \text{and} \quad \bigcap_{i=1}^{n} D(G_i) = D\left(\bigcap_{i=1}^{n} G_i\right);$$

(4) *Find the graph G such that $D(D(\cdots(D(G))\cdots))$(n times)$= G$.*

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4−Remainder Cordial Labeling of Some Graphs

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Abstract: Let G be a (p, q) graph. Let f be a function from $V(G)$ to the set $\{1, 2, \cdots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \cdots, k\}$, where $v_f(x)$ denote the number of vertices labeled with x and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labeled with even integers and number of edges labelled with odd integers. A graph with admits a k-remainder cordial labeling is called a k-remainder cordial graph. In this paper we investigate the 4- remainder cordial behavior of grid, subdivision of crown, Subdivision of bistar, book, Jelly fish, subdivision of Jelly fish, Mongolian tent graphs.

Key Words: k-Remainder cordial labeling, Smarandache k-remainder cordial labeling, grid, subdivision of crown, subdivision of bistar, book, Jelly fish, subdivision of Jelly fish, Mongolian tent.

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§1. Introduction

We considered only finite and simple graphs.The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uwv. The product graph G_1XG_2 is defined as follows:

Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_2$ and u_1 adj $v_1]$. The graph $P_m \times P_n$ is called the planar grid. Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. A mongolian tent $M_{m,n}$ is a graph obtained from $P_m \times P_n$ by adding one extra vertex above the grid and joining every other of the top row of $P_m \times P_n$ to the new vertex. Cahit [1], introduced the concept of cordial labeling of graphs. Ponraj et al. [4, 6], introduced remainder cordial labeling of graphs and investigate the remainder cordial labeling behavior of path, cycle,

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star, bistar, complete graph, $S(K_{1,n})$, $S(B_{n,n})$, $S(W_n)$, P_n^2, $P_n^2 \bigcup K_{1,n}$, $P_n^2 \bigcup B_{n,n}$, $P_n \cup B_{n,n}$, $P_n \bigcup K_{1,n}$, $K_{1,n} \bigcup S(K_{1,n})$, $K_{1,n} \bigcup S(B_{n,n})$, $S(K_{1,n}) \bigcup S(B_{n,n})$, etc., and also the concept of k-remainder cordial labeling introduced in [5]. In this paper we investigate the 4-remainder cordial labeling behavior of Grid, Subdivision of crown, Subdivision of bistar, Book, Jelly fish, Subdivision of Jelly fish, Mongolian tent, etc,. Terms are not defined here follows from Harary [3] and Gallian [2].

§2. k-Remainder Cordial Labeling

Definition 2.1 *Let G be a (p,q) graph. Let f be a function from $V(G)$ to the set $\{1,2,\cdots,k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The labeling f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, otherwise, Smarandachely if $|v_f(i) - v_f(j)| \geq 1$ or $|e_f(0) - e_f(1)| \geq 1$ for integers $i,j \in \{1,\cdots,k\}$, where $v_f(x)$ and $e_f(0)$, $e_f(1)$ respectively denote the number of vertices labeled with x, the number of edges labeled with even integers and the number of edges labelled with odd integers. Such a graph with a k-remainder cordial labeling is called a k-remainder cordial graph.*

First we investigate the 4−remainder cordial labeling behavior of the planar grid.

Theorem 2.2 *The planar grid $P_m \times P_n$ is 4−remainder cordial.*

Proof Clearly this grid has $m-$rows and $n-$columns. We assign the labels to the vertices by row wise.

Case 1. $m \equiv 0 \pmod 4$

Let $m = 4t$. Then assign the label 1 to the vertices of $1^{st}, 2^{nd}, \cdots, t^{th}$ rows. Next we move to the $(t+1)^{th}$ row. Assign the label 4 to the vertices of $(t+1)^{th}, (t+2)^{th}, \ldots, (2t)^{th}$ rows. Next assign the label to the vertices $(2t+1)^{th}$ row. Assign the labels 2 and 3 alternatively to the vertices of $(2t+1)^{th}$ row. Next move to $(2t+2)^{th}$ row. Assign the labels 3 and 2 alternatively to the vertices of $(2t+2)^{th}$ row. In general i^{th} row is called as in the $(i-2)^{th}$ row, where $2t+1 \leq i \leq 3t$. This procedure continued until we reach the $(4t)^{th}$ row.

Case 2. $m \equiv 1 \pmod 4$

As in Case 1, assign the labels to the vertices of the first, second, $\cdots, (m-1)^{th}$ row. We give the label to the m^{th} row as in given below.

Subcase 2.1 $n \equiv 0 \pmod 4$

Rotate the row and column and result follows from Case 1.

Subcase 2.2 $n \equiv 1 \pmod 4$

Assign the labels $4,3,4,3,\cdots,4,3$ to the vertices of the first, second, $\cdots$, $\left(\frac{n-1}{2}\right)^{th}$ columns. Next assign the label 2 to the vertices of $(\frac{n+1}{2})^{th}$ column. Then next assign the labels $2,1,2,1,\cdots,$

$2,1$ to the vertices of $\frac{n+3}{2}, \frac{n+5}{2}, \cdots, (\frac{2n}{2}-2)^{th}$ columns. Assign the remaining vertices.

Subcase 2.3 $n \equiv 2 \pmod 4$

Assign the labels $4,3,4,3,\cdots,4,3$ to the vertices of $1^{st}, 2^{nd}, \cdots, (\frac{n-2}{2})^{th}$ columns. Next assign the label 2 to the vertices of $\left(\frac{n}{2}\right)^{th}$ column. Then next assign the labels $2,1,2,1,\cdots,2,1$ to the vertices of $\frac{n}{2}+1, \frac{n}{2}+2, \cdots, \left(\frac{2n}{2}-1\right)^{th}$ columns. Finally assign the label 1 to the remaining vertices of n^{th}column.

Subcase 2.4 $n \equiv 3 \pmod 4$

Assign the labels $4,3,4,3,\cdots,4,3$ alternatively to the vertices of $1^{st}, 2^{nd}, \cdots, \left(\frac{n+1}{2}\right)^{th}$ columns. Then next assign the labels $1,2,1,2,\cdots$ to the vertices of $\frac{n+3}{2}, \frac{n+5}{2}, \cdots, \left(\frac{2n}{2}-1\right)^{th}$ columns. Finally assign the label 1 to the remaining vertices of n^{th} column. Hence f is a $4-$remainder cordial labeling of $P_m \times P_n$.

All other cases follow by symmetry. $\square$

Next is the graph $K_2 + mK_1$.

Theorem 2.3 *If* $m \equiv 0,1,3 \pmod 4$ *then* $K_2 + mK_1$ *is* $4-$*remainder cordial.*

Proof It is easy to verify that $K_2 + mK_1$ has $m+2$ vertices and $2m$ edges. Let $V(K_2 + mK_1) = \{u, u_i, v : 1 \leq i \leq m\}$ and $E(K_2 + mK_1) = \{uv, uu_i, vu_i : 1 \leq i \leq m\}$.

Case 1. $m \equiv 0 \pmod 4$

Let $m = 4t$. Then assign the label $3,3$ respectively to the vertices u, v. Next assign the label 1 to the vertices $u_1, u_2, \cdots, u_{t+1}$. Then next assign the label 2 to the vertices $u_{t+2}, u_{t+3}, \cdots, u_{2t+1}$. Then followed by assign the label 3 to the vertices $u_{2t+2}, u_{2t+3}, \ldots, u_{3t}$. Finally assign the label 4 to the remaining non-labelled vertices $u_{3t+1}, u_{3t+2}, \cdots, u_{4t}$.

Case 2. $m \equiv 1 \pmod 4$

As in Case 1, assign the labels to the vertices $u, v, u_i, (1 \leq i \leq m-1)$. Next assign the label 2 to the vertex u_m.

Case 3. $m \equiv 3 \pmod 4$

Assign the labels to the vertices $u, v, u_i, (1 \leq i \leq m-2)$ as in case(ii). Finally assign the labels $3,4$ respectively to the vertices u_{m-1}, u_m. The table given below establish that this labeling f is a $4-$remainder cordial labeling.

Nature of m	$e_f(0)$	$e_f(1)$
$m \equiv 0 \pmod 4$	$m+1$	m
$m \equiv 1 \pmod 4$	m	$m+1$
$m \equiv 3 \pmod 4$	m	$m+1$

Table 1

This completes the proof. $\square$

The next graph is the book graph B_n.

Theorem 2.4 *The book B_n is $4-$remainder cordial for all n.*

Proof Let $V(B_n) = \{u, v, u_i, v_i : 1 \le i \le n\}$ and $E(B_n) = \{uv, uu_i, vv_i, u_iv_i : 1 \le i \le n\}$.

Case 1. n is even

Assign the labels $3, 4$ to the vertices u and v respectively. Assign the label 1 to the vertices $u_1, u_2, \cdots, u_{\frac{m}{2}}$ and assign 4 to the vertices $u_{\frac{m}{2}+1}, u_{\frac{m}{2}+2}, \cdots, u_n$. Next we consider the vertices v_i. Assign the label 2 to the vertices $v_1, v_2, \cdots, v_{\frac{n}{2}}$. Next assign the label 3 to the remaining vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \cdots, u_n$, respectively.

Case 2. n is odd

Assign the labels $3, 4$ to the vertices u and v respectively. Fix the labels $4, 2, 1$ to the vertices $u_1, u_2, \cdots, u_{\frac{n}{2}+1}$ and also fix the labels $3, 1, 2$ respectively to the vertices $v_1, v_2, \cdots, v_{\frac{n}{2}+1}$. Assign the labels to the vertices $u_4, u_5, \cdots, u_n$ as in the sequence $2, 1, 2, 1 \ldots, 2, 1$. In similar fashion, assign the labels to the vertices $v_4, v_5, \cdots, v_n$ as in the sequence $3, 4, 3, 4 \ldots, 3, 4$. The table 2 shows that this vertex labeling f is a $4-$ remainder cordial labeling.

Nature of n	$e_f(0)$	$e_f(1)$
n is even	$m+1$	m
n is odd	m	$m+1$

Table 2

This completes the proof. □

Now we consider the subdivision of $B_{n,n}$.

Theorem 2.5 *The subdivision of $B_{n,n}$ is $4-$remainder cordial.*

Proof Let $V(S(B_{n,n})) = \{u, v, u_i, v_i, w_i, x, x_i : 1 \le i \le n\}$ and $E(S(B_{n,n})) = \{uu_i, vv_i, u_iw_i, v_ix_i, ux, xv : 1 \le i \le n\}$.It is clearly to verify that $S(B_{n,n})$ has $4n+3$ vertices and $4n+2$ edges.

Assign the labels $1, 4, 3$ to the vertices u, x and v respectively. Assign the labels $1, 3$ alternatively to the vertices $u_1, u_2, \cdots, u_n$. Next assign the labels $2, 4$ alternatively to the vertices $w_1, w_2, \cdots, w_n$. We now consider the vertices v_i and x_i. Assign the labels $2, 4$ alternatively to the vertices $v_1, v_2, \cdots, v_n$. Then finally assign the labels $3, 1$ alternatively to the vertices $x_1, x_2, \cdots, x_n$. Obviously this vertex labeling is a $4-$remainder cordial labeling. □

Next, we consider the subdivision of crown graph.

Theorem 2.5 *The subdivision of $C_n \odot K_1$ is $4-$remainder cordial.*

Proof Let $u_1u_2 \ldots u_n$ be a cycle C_n. Let $V(C_n \odot K_1) = V(C_n \bigcup \{v_i : 1 \le i \le n\})$ and $E(C_n \odot K_1) = E(C_n \bigcup \{u_i, v_i : 1 \le i \le n\})$.The subdivide edges u_iu_{i+1} and u_iv_i by x_i and y_i respectively. Assign the label 2 to the vertices $u_i, (1 \le i \le n)$ and 3 to the vertices

$x_i, (1 \le i \le n)$. Next assign the label 1 to the vertices $y_i, (1 \le i \le n)$. Finally assign the label 4 to the vertices $v_i, (1 \le i \le n)$. Clearly, this labeling f is a $4-$remainder cordial labeling. □

Now we consider the Jelly fish $J(m, n)$.

Theorem 2.6 *The Jelly fish $J(m, n)$ is $4-$remainder cordial.*

Proof Let $V(J(m, n)) = \{u, v, x, y, u_i, v_j : 1 \le i \le m$ and $1 \le j \le n\}$ and $E(J(m, n)) = \{uu_i, vv_j, ux, uy, vx, vy : 1 \le i \le m$ and $1 \le j \le n\}$. Clearly $J(m, n)$ has $m + n + 4$ vertices and $m + n + 5$ edges.

Case 1. $m = n$ and m is even.

Assign the label 2 to the vertices $u_1, u_2, \cdots, u_{\frac{n}{2}}$ and assign the label 4 to the vertices $u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}, \cdots, u_n$. Next assign the label 1 to the vertices $v_1, v_2, \cdots, v_{\frac{n}{2}}$ and assign 3 to the vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_n$. Finally assign the labels $3, 4, 2, 1$ respectively to the vertices u, x, y, v.

Case 2. $m = n$ and m is odd.

In this case assign the labels to the vertices $u_i, v_i (1 \le i \le m - 1)$ and u, v, x, y as in Case 1. Next assign the labels $2, 1$ respectively to the vertices u_n and v_n.

Case 3. $m \ne n$ and assume $m > n$.

Assign the labels $3, 4, 1, 2$ to the vertices u, x, y, v respectively. As in Case 1 and 2, assign the labels to the vertices $u_i, v_i (1 \le i \le n)$.

Subcase 3.1 $m - n$ is even. Assign the labels to the vertices $u_{n+1}, u_{n+2}, \cdots, u_m$ as in the sequence $3, 4, 2, 1; 3, 4, 2, 1; \cdots$. It is easy to verify that u_n is received the label 1 if $m - n \equiv 0 \pmod 4$.

Subcase 3.2 $m - n$ is odd. Assign the labels to the vertices $u_i (n \le i \le m)$ as in the sequence $4, 3, 2, 1; 4, 3, 2, 1; \cdots$. Clearly, u_n is received the label 1 if $m - n \equiv 0 \pmod 4$. □

For illustration, a $4-$remainder cordial labeling of Jelly fish $J(m, n)$ is shown in Figure 1.

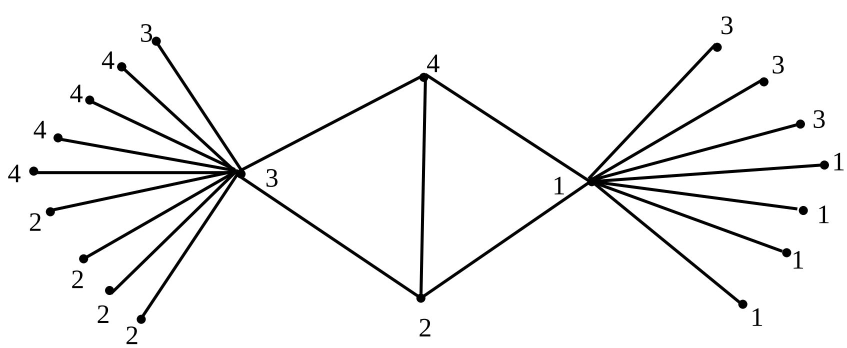

Figure 1

Theorem 2.8 *The subdivision of the Jelly fish $J(m, n)$ is $4-$remainder cordial.*

Proof Let $V(S(J(m,n))) = \{u, u_i, x_i, v, v_j, y_j : 1 \leq i \leq m, 1 \leq j \leq n\} \bigcup \{w_i : 1 \leq i \leq 7\}$ and $E(S(J(m,n))) = \{uu_i, u_ix_i, vv_j, v_jy_j : 1 \leq i \leq m, 1 \leq j \leq n\} \bigcup \{uw_1, uw_2, w_1w_5, w_5w_6, w_6w_7, w_5w_3, w_3v, vw_4, w_4w_7, w_2w_7\}$.

Case 1. $m = n$.

Assign the label 2 to the vertices $u_1, u_2, \cdots, u_m$ and 3 to the vertices $x_1, x_2, \cdots, x_m$. Next assign the label 1 to the vertices $v_1, v_2, \cdots, v_m$ and assign the label 4 to the vertices $y_1, y_2, \cdots, y_m$. Finally assign the labels $3, 2, 3, 2, 3, 1, 4, 4$ and 1 respectively to the vertices $u, w_1, w_5, w_6, w_7, w_2, w_3, v$ and w_4.

Case 2. $m > n$.

Assign the labels to the vertices $u, u_i, v, v_i, x_i, y_i, w_1, w_2, w_3, w_4, w_5, w_6, w_7, (1 \leq i \leq n)$ as in case(i). Next assign the labels $1, 4$ to the next two vertices x_{n+1}, x_{n+2} respectively. Then next assign the labels $1, 4$ respectively to the vertices x_{n+3}, x_{n+4}. Proceeding like this until we reach the vertex x_n.That is the vertices $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, \cdots$ are labelled in the pattern $1, 4; 1, 4; 1, 4; 1, 4; \cdots$. Similarly the vertices $u_{n+1}, u_{n+2}, \cdots$ are labelled as $2, 3; 2, 3; 2, 3; \cdots, 2, 3$. The Table 3, establish that this vertex labeling f is a 4−remainder cordial labeling of $S(J(m,n))$.

Nature of m and n	$e_f(0)$	$e_f(1)$
$m = n$	$2n + 5$	$2n + 5$
$m > n$	$m + n + 5$	$m + n + 5$

This completes the proof. □

Theorem 2.9 *The graph $C_3^{(t)}$ is 4−remainder cordial.*

Proof Let $V(C_3^{(t)}) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(C_3^{(t)}) = \{uu_i, vv_i, u_iv_i : 1 \leq i \leq n\}$. Assume $t \geq 3$. Fix the label 3 to the central vertex u and fix the labels $1, 2, 2, 4, 3, 4$ respectively to the vertices u_1, u_2, u_3, v_1, v_2 and v_3. Next assign the labels $1, 2$ to the vertices u_4, u_5. Then assign the labels $1, 2$ respectively to the next two vertices u_6, u_7 and so on. That is the vertices u_4, u_5, u_6, u_7 are labelled as in the pattern $1, 2, 1, 2 \cdots, 1, 2$. Note that the vertex u_n received the label 1 or 2 according as n is even or odd. In a similar way assign the labels to the vertices v_4, v_5, v_6, v_7 as in the sequence $4, 3, 4, 3, 4, 3, \cdots$. Clearly 4 is the label of u_n according as n is even or odd. The Table 4 establish that this vertex labeling is a 4−remainder cordial labeling of $C_3^{(t)}$, $t \geq 3$.

Nature of n	$e_f(0)$	$e_f(1)$
n is even	$\frac{3n}{2}$	$\frac{3n}{2}$
n is odd	$n + 1$	$n + 2$

Table 4

For $t = 1, 2$ the remainder cordial labeling of graphs $C_3^{(1)}$ and $C_3^{(2)}$ are given below in Figure 2.

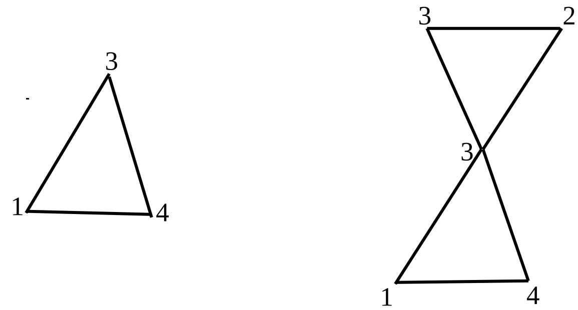

Figure 2

This completes the proof. □

Theorem 2.10 *The Mongolian tent* $M_{m,n}$ *is* 4−*remainder cordial.*

Proof Assign the label 3 to the new vertex.

Case 1. $m \equiv 0 \pmod 4$ and $n \equiv 0, 2 \pmod 4$.

Consider the first row of M_n. Assign the labels $2, 3, 2, 3, \cdots 2, 3$ to the vertices in the first row. Next assign the labels $3, 2, 3, 2 \cdots 3, 2$ to the vertices in the second row. This procedure is continue until reach the $\frac{n}{2}^{th}$ row. Next assign the labels $1, 4, 1, 4, \cdots, 1, 4$ to the vertices in the $\frac{n}{2} + 1^{th}$ row. Then next assign the labels $4, 1, 4, 1, \cdots, 4, 1$ to the vertices in the $\frac{n}{2} + 2^{th}$ row. This proceedings like this assign the labels continue until reach the last row.

Case 2. $m \equiv 2 \pmod 4$ and $n \equiv 0, 2 \pmod 4$.

In this case assign the labels to the vertices as in Case 1.

Case 3. $m \equiv 1 \pmod 4$ and $n \equiv 0, 2 \pmod 4$.

Here assign the labels by column wise to the vertices of M_n. Assign the labels $2, 3, 2, 3, \cdots 2, 3$ to the vertices in the first column. Next assign the labels $3, 2, 3, 2 \cdots 3, 2$ to the vertices in the second column. This method is continue until reach the $\frac{n}{2}^{th}$ column. Next assign the labels $1, 4, 1, 4, \cdots, 1, 4$ to the vertices in the $\frac{n}{2} + 1^{th}$ column. Then next assign the labels $4, 1, 4, 1, \cdots, 4, 1$ to the vertices in the $\frac{n}{2} + 2^{th}$ column. This procedure is continue until reach the last column.

Case 4. $m \equiv 3 \pmod 4$ and $n \equiv 0, 2 \pmod 4$.

As in Case 3 assign the labels to the vertices in this case. The remainder cordial labeling of graphs $M_{7,4}$ is given below in Figure 3..

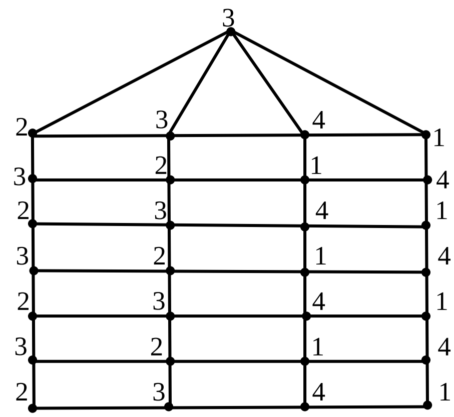

Figure 3

This completes the proof. □

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Motion Planning in Certain Lexicographic Product Graphs

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Abstract: Let G be an undirected graph with n vertices in which a robot is placed at a vertex say v, and a hole at vertex u and in all other $(n-2)$ vertices are obstacles. We refer to this assignment of robot and obstacles as a configuration C_u^v of G. Suppose we have a one player game in which the robot or obstacle can be slide to an adjacent vertex if it is empty i.e. if it has a hole. The goal is to take the robot to a particular destination vertex using minimum number of moves. In this article, we give the minimum number of moves required for the motion planning problem in Lexicographic products of some graphs. In addition, we proved the necessary and sufficient condition for the connectivity of the lexicographic product of two graphs.

Key Words: Motion in a graph, lexicographic product graphs, k-factor of graphs.

AMS(2010): 05C85, 05C75, 68R10, 91A43.

§1. Introduction

Given a graph G, with a robot placed at one of it's vertices and movable obstacles at some other vertices. Assuming that we are allowed to slide the robot and obstacles to an adjacent vertex if it is empty. Let $u, v \in V(G)$, and suppose that the robot is at v and the hole at u and obstacles at other vertices we refer to this as a configuration C_u^v. Now we use $u \xleftarrow{r} v$ and $u \xleftarrow{o} v$ to denote respectively, the robot move and the obstacle move from vertex v to an adjacent vertex u where $u, v \in E(G)$. A simple move is referred to as moving an obstacle or the robot to an adjacent empty vertex while a graph G is k-reachable if there exists a k-configuration such that the robot can reach any vertex of the graph in a finite number of simple moves. The objective is to find a minimum sequence of moves that takes the robot from (source) vertex p to a (destination) vertex t.

For two vertices $u, v \in V(G)$, let $d_G(u, v)$ denotes the distance between u and v in G. Most of the distances used in this article are in G so we use $d(u, v)$ instead of $d_G(u, v)$ to represent the distance between the vertices u and v in G. We denote the complete, complement of a complete, cycle, path graph and complete graph minus one factor on n vertices by K_n, $\overline{K}_n$, C_n, P_n and $K_n - I$ respectively.

The motion planning problem in graph was proposed by Papadimitriou et al [9] where it was shown that with arbitrary number of holes, the decision version of such problem is NP-

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complete and that the problem is complex even when it is restricted to planar graphs. They also gave time algorithm for trees. The result in [9] was improve in [2]. Robot motion planning on graphs (RMPG) is a graph with a robot placed at one of its vertices and movable obstacle at some of the other vertices while generalization of RMPG problem is the Multiple robot motion planning in graph (MRMPG) whereby we have k different robots with respective destinations. Ellips and Azadeh [5] studied MRMPG on trees and introduced the concept of minimal solvable trees. Auletta et al [1] also studied the feasibility of MRMPG problem on trees and gave an algorithm that, on input of two arrangements of k robots on a tree of order n, decides in time $O(n)$ whether the two arrangements are reachable from one another. Parberry [8] worked on grid of order n^2 with multiple robots while Deb and Kapoor [4,3] generalized and apply the technique used in [8] to calculate the minimum number of moves for the motion planning problem for the cartesian product of two given graphs.

The MRMPG problem of grid graph of order n^2 with n^2-1 robots is known as (n^2-1)-puzzle. The objective of (n^2-1)-puzzle is to verify whether two given configuration of the grid graph of order n^2 are reachable from each other and if they are reachable then to provide a sequence of minimum number of moves that takes one configuration to the other. The (n^2-1)-puzzle have been studied extensively in [7, 8, 10, 11].

Our work was motivated by Deb and Kapoor [3] whereby they gave minimum sequence of moves required for the motion planning problem in Cartesian product of two graphs having girth 6 or more. They also proved that the path traced by the robot coincides with a shortest path in case of Cartesian product graphs of graphs. In this paper, we consider the case of lexicographic product graphs. Here we give the minimum number of moves required for the motion planning problem in the Lexicographic product of two graphs say G and H, where G and H are specified in each of our cases.

1.1 Lexicographic Product of Graphs

Definition 1.1 *The lexicographic product $G \circ H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ in which (u_i, v_j) and (u_p, v_q) are adjacent if one of the following condition holds:*

(i) $\{u_i, u_p\} \in E(G)$;

(ii) $u_i = u_p$ *and* $\{v_j, v_q\} \in E(H)$.

The graphs G and H are known as the factors of $G \circ H$. Now onwards G and H are simple graphs with $V(G) = \{1, 2, 3, \cdots, m\}$ unless otherwise stated.

Suppose we are dealing with p-copies of a graph G and we are denoting these p-copies of G by G^i, where $i = \{1, 2, 3, \cdots, p\}$. Then for each vertex $u \in V(G)$ we denote the corresponding vertex in the i^{th} copy G^i by u^i. The girth of a graph G, denoted by $g(G)$ is the length of the shortest cycle contained in graph G.

Example 1.2 Let $G = P_2$ and $H = C_4$. The graph $G \circ H$ is shown in Figure 1 below.

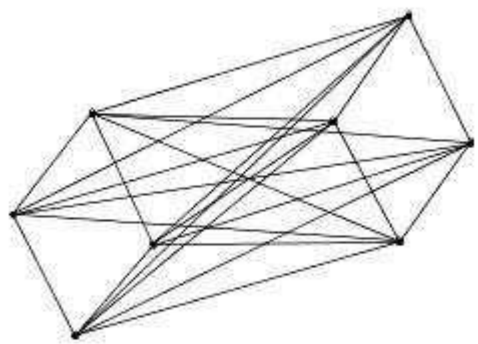

Figure 1 $P_2 \circ C_4$

Remark 1.3 The Lexicographic product $G \circ H$, of graphs G and H, is the graph obtained by replacing each vertex of G by a copy of H and every edge of G by the complete bipartite graph $K_{[H],[H]}$.

1.2 Connectivity of Lexicographic Product of Two Graphs

Here we aim at proving a corollary which Deb and Kapoor [3] mentioned as concerning the condition for which the lexicographic product $G \circ H$ is connected.

Proposition 1.4 (See [6]) *Suppose that u^i and v^j are two vertices in $G \circ H$. Then*

$$d_{(G\circ H)}(u^i, v^j) = \begin{cases} d_H(u,v), & if\ i=j\ and\ d_G(i)=0 \\ \min\{d_H(u,v),2\}, & if\ i=j\ and\ d_G(i)\neq 0 \\ d_G(i,j), & if\ i\neq j \end{cases}$$

Theorem 1.5 *Let G and H be two non-trivial graphs. Then $G \circ H$ is connected if and only if G is connected.*

Proof Assume that $G \circ H$ is connected. We only need to show that graph G is connected. Given that $u, v \in V(H)$ and $i, j \in V(G)$. Let u^i be an arbitrary vertex in $G \circ H$. Since $G \circ H$ is connected it means that the vertex u^i has an edge with at least a vertex in $G \circ H$ (especially in one of the factors in the product graph), let this vertex be v^j. Next, by definition of the lexicographic product graph, for u^i and v^j to have an edge in $G \circ H$ then $\{i, j\} \in E(G)$. Which implies that there is a path between i and j in G. But u^i is an arbitrary vertex in $G \circ H$ we conclude that there is a path between each pair of vertices in graph G. Therefore G is connected. Conversely, suppose that graph G is connected. It suffices us to show that $G \circ H$ is connected. Since G is connected it implies that for all $i, j \in V(G)$ where i and j are distinct $d(i, j) \neq 0$. We shall prove this by contradiction. Assume that the graph $G \circ H$ is disconnected. If $G \circ H$ is disconnected it means that there exist an arbitrary pair of vertices

(u^i, v^j) in $G \circ H$ such that $d_{G\circ H}(u^i, v^j) = 0$ for all $i, j \in V(G)$ and $u, v \in V(H)$. Since i and j are distinct then by Proposition 1.4 we have that $d_{G\circ H}(u^i, v^j) = d_G(i, j)$. But $d_G(i, j) \neq 0$ therefore $d_{G\circ H}(u^i, v^j) \neq 0$ a contradiction. Since the pair (u^i, v^j) is arbitrary we conclude that the product graph $G \circ H$ is connected. This completes the proof. $\square$

§2. Robot Moves in Lexicographic Product of a Graph and Complement of Complete Graph

Definition 2.1 *An edge u^p, v^q in $G \circ H$ is said to be a G-edge if $u = v$ and $\{p, q\} \in E(G)$.*

Definition 2.2 *Given two graphs G and H. For any $u^p, v^q \in V(G \circ H)$,we call the distance between u and v in H to be the H-distance between u^p and v^q in $G \circ H$. We use $d_G(u^p, v^q)$ and $d_H(u^p, v^q)$ to denote the G-distance and H-distance between u^p, v^q in $G \circ H$, respectively.*

Now when there is no confusion about the graph in question G, we use $d(u, v)$ instead of $d_G(u, v)$ to represent the distance between u and v in G.

In view of the above definition, we now have this proposition.

Proposition 2.3 *Given two graphs G and H. Let $\{i, j\}, \{j, k\} \in E(G)$ and $u, v \in V(H)$. Then $d_{G\circ H - u^i}(v^i, u^j) = d_{G\circ H}(v^i, u^j)$.*

Proof To prove this, notice that $d_{G\circ H-u^i}(v^i, u^j) = 1$ which is same as $d_{G\circ H}(v^i, u^j)$.

Each vertex set of copy H^i is adjacent to all other vertices in copy H^j for all $\{i, j\} \in E(G)$ in Lexicographic product graphs. $\square$

The following two results was given in [4].

Lemma 2.4([4]) *Given two graphs G and H. Let $\{i, j\}, \{j, k\} \in E(G)$ such that $\{i, k\} \notin E(G)$ and $u, v, w \in V(H)$. Consider the configuration $C^{v^j}_{u^i}$ of $G \circ H$. Then we require at least three moves to move the robot from v^j to w^k.*

Lemma 2.5([4]) *Given two graphs G and H. Let $u, v, w \in V(H)$ such that $\{u, v\}, \{v, w\} \in E(H)$ and $\{u, w\} \notin E(H)$. For some $i \in V(G)$, consider the configuration $C^{v^i}_{u^i}$ of $G \circ H$. Then minimum of three moves are required to move the robot from v^i to w^i.*

Proposition 2.6 *Let G be a graph and H a complement of a complete graph on n vertices. Let $\{i, j\}, \{j, k\} \in E(G)$ and $u, v \in V(H)$. Then starting from the configuration $C^{u^i}_{v^i}$ we need at least two moves to move the robot to u^j.*

Proof To move the robot from u^i to u^j before it, the hole is required to move from v^i to u^j. This takes just a move since $d_{G\circ H-u^i}(v^i, u^j) = 1$. Then the move $u^j \overset{r}{\leftarrow} u^i$ takes the robot to u^j. Hence the result follows. $\square$

Corollary 2.7 *Let G be any graph and H a complement of a complete graph on n vertices. Let $\{i, j\}, \{j, k\} \in E(G)$ and $u, v \in V(H)$, where u and v are distinct. Then starting from the configuration $C^{v^j}_{v^i}$ we need at least three moves to move the robot to v^k.*

Proof Observe that $\{v^i, v^j\}, \{v^j, v^k\} \in E(G \circ H)$. For the robot to move to v^k before it, the hole must move from v^i to v^k. This takes $d_{G\circ H}(v^i, v^k) = 2$. Then the move $v^k \xleftarrow{r} v^j$ moves the robot from v^j to v^k. Hence the result follows. $\square$

Definition 2.8 *A robot move in $G \circ H$ is called a G-move if the edge along which the move take place is a G-edge.*

Definition 2.9 *Let T be a sequence of moves that take the robot from u^p to v^q in $G \circ H$. An H-move (respectively G-move) in T of the robot is said to be a secondary H-move (respectively G-move) if it is preceded by an H-move (respectively G-move). An H-move (respectively G-move) in T of the robot is said to be a primary H-move (respectively G-move) if it is preceded by a G-move (respectively H-move). Also the edge corresponding to a primary G-move (respectively H-move) in T is said to be a primary G-edge (respectively H-edge). In view of the above definitions we have the following remark.*

Remark 2.10 Given graph G and H a complement of a complete graph on n vertices.

(i) In view of Proposition 2.6, to perform the first move of the robot we require at least 2 moves;

(ii) In view of Corollary 2.7, to perform each secondary move of the robot we require at least 3 moves.

Theorem 2.11 *Let G be a graph and H a complement of a complete graph on n vertices. Let $i, j, k \in V(G)$ and $u, v, w \in V(H)$. Then each robot and obstacle moves in a minimum sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{u^j}^{w^j}$ in $G \circ H$ is a G-move. Also such a minimum sequence involves exactly a number of G-moves of the robot and $3a$ moves in total, where $a = d(i, j) \geq 1$.*

Proof Since $\{u, v\}, \{v, w\} \notin E(H)$ it means $d_H(u^i, v^i) = d_H(v^i, w^i) = 0$. The first part of this lemma follows. Now, let T be a sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{u^j}^{w^j}$ in $G \circ H$. First, let z be the number of robot moves in T. By Proposition 2.6, we need at least two moves to accomplish the first move of the robot. Observe that each remaining $z-1$ robot moves in T is a secondary G-move. So by Remark 2.10, we need minimum of $3(z-1)$ moves to accomplish the $z-1$ secondary G-moves (since the first robot move places the hole at a preceding copy of the robot). Now, if $w^j \xleftarrow{r} w^a$ is the z^{th} robot move in T, it will leave the graph $G \circ H$ with the configuration $C_{w^a}^{w^j}$. Since $d_{G\circ H}(w^a, u^j) = 1$, so we need minimum of one more move to take the hole from w^a to u^j. Hence T involves minimum $3z$ moves. Notice that, the expression $3z$ takes the minimum value when z is minimum.

Next, let $d(i, j) = a$ and $[i = i_0, i_1, i_2, ..., i_a = j]$ be a path of length a connecting i and j in G. Then $[v^i = v^{i_0}, v^{i_1}, v^{i_2}, \cdots, v^{i_a} = w^j]$ is a path of length a in $G \circ H$ joining v^i to w^j. So the sequence of moves $u^i \xleftarrow{o^*} v^{i_1} \xleftarrow{r} v^{i_0} \xleftarrow{o^*} v^{i_2} \xleftarrow{r} v^{i_1} \xleftarrow{o^*} v^{i_3} \xleftarrow{r} v^{i_2} \xleftarrow{o^*} v^{i_4} \cdots v^i_{a-2} \xleftarrow{o^*} v^{i_a} \xleftarrow{r} v^i_{a-1} \xleftarrow{o^*} u^j$ takes the robot from v^i to w^j along this path. Also it involves exactly a number of G-moves of the robot. Furthermore, from the given sequence above obstacle moves involves exactly $2 + 2(a-1)$ moves. Therefore a minimum sequence of moves T that takes the configuration $C_{u^i}^{v^i}$ to $C_{u^j}^{w^j}$ involves exactly $3a$ moves. $\square$

Theorem 2.12 *Let G be a graph and H a complement of a complete graph on n vertices. Let $\{i,j\} \in E(G)$ and $u,v,w \in V(H)$. Then each robot move in a minimum sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{w^j}^{w^i}$ in $G \circ H$ involves exactly 5 moves.*

Proof Combining Proposition 2.6 and Corollary 2.7 gives the result. □

The above Lemma gives the minimum number of moves required to take the robot from a given factor to itself and to another factor $G \circ H$. The proof of this lemma is immediate from Theorems 2.11 and 2.12.

Lemma 2.13 *Consider the graph $G \circ H$. Let $u,v \in V(H)$ with the initial configuration $C_{u^i}^{v^i}$, where G is any graph and H a complement of a complete graph on n vertices. Then*

(i) to move the robot from H^i to H^i we require at least 5 G-moves;

(ii) to move the robot from H^i to H^j we require at least $1+2(a-1)+a$ G-moves. Where $a = d(i,j) \geq 1$.

Corollary 2.14 *Let G be a path and H a complement of a complete graph on n vertices. Let $i,j,k \in V(G)$ and $u,v,w \in V(H)$. Then each robot and obstacle moves in a minimum sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{w^j}^{u^j}$ in $G \circ H$ is a G-move. Also such a minimum sequence involves exactly $3a$ moves in total, where $a = d(i,j) \geq 1$.*

Proof The proof of this corollary is immediate from Lemma 2.13. □

Corollary 2.15 *Let G be a path and H a complement of a complete graph on n vertices. Let $i,j \in V(G)$ and $u,v,w \in V(H)$. Then each robot move in a minimum sequence of moves that takes $C_{u^i}^{v^i}$ to $C_{w^j}^{w^i}$ in $G \circ H$ involves exactly 5 moves.*

proof The proof of this corollary is immediate from Lemma 2.13. □

Note that for such a product (Corollaries 2.14 and 2.15) there is no shortest path. Next, we consider the case when graph G is a complete graph. We would do this by stating the lemma below without proof.

Lemma 2.16 *Let G be complete and H a complement of a complete graph on n vertices. Let $i,j \in V(G)$ and $u,v,w \in V(H)$. Then starting from configuration $C_{u^i}^{v^i}$,*

(i) we require at least 5 moves to move the robot to w^i;

(ii) we require a minimum of 2 moves to move the robot to w^k.

§3. Robot Moves in $K_n - I \circ H$

In this section we investigate a case where graph G is a complete graph minus a 1-factor and H a complete graph or it's complement.

Lemma 3.1 *Let G be a complete graph minus a 1-factor and H a complete graph (or it's complement). Let $i,j \in V(G)$ but $\{i,j\} \notin E(G)$ and $u,v \in V(H)$. Then each robot move in a* *minimum sequence of moves that takes $C^{u^i}_{v^i}$ to $C^{u^j}_{v^j}$ is a G-move. Also such a minimum sequence involves exactly 6 moves.*

Proof let T be a sequence of moves that takes $C^{u^i}_{v^i}$ to $C^{u^j}_{v^j}$. By Proposition 2.6, two moves is required to move the robot to an adjacent vertex. Next by Corollary 2.7, four additional moves is required to take the robot and hole to their required destination, while the remaining move is the last move of the hole. So, the sequence T of moves $v^i \xleftarrow{o^*} u^k \xleftarrow{r} u^i \xleftarrow{o^*} v^k \xleftarrow{o^*} u^j \xleftarrow{r} u^k \xleftarrow{o^*} v^j$ takes the robot and the hole to the required destination, and each move in this sequence is a G-move. Also T involves exactly six moves. □

Lemma 3.2 *Let G be a complete graph minus a 1-factor and H a complete graph (or it's complement). Let $i, j \in V(G)$ but $\{i, j\} \notin E(G)$ and $u, v \in V(H)$. Then each robot move in a minimum sequence of moves that takes $C^{u^i}_{v^j}$ to $C^{u^j}_{v^i}$ is a G-move. Also such a minimum sequence involves exactly 6 moves.*

Proof The proof can be drawn in the same line as that of Lemma 3.1. □

Lemma 3.3 *Let G be a complete graph minus a 1-factor and H a complement of a complete graph with n vertices. Let $i, j \in V(G)$ but $\{i, j\} \notin E(G)$ and $u, v \in V(H)$. Then each robot move in a minimum sequence of moves that takes $C^{u^i}_{v^j}$ to $C^{v^i}_{v^j}$ is a G-move. Also such a minimum sequence involves exactly 6 moves.*

Proof Let T be a sequence of moves that takes $C^{u^i}_{v^j}$ to $C^{v^i}_{v^j}$. By Proposition 2.6, we need at least 2 moves to accomplish the first G-move of the robot. Notice that the last move of the robot is also a G-move. Now if $v^i \xleftarrow{r} v^m$ is the last move of the robot, it will leave the graph $G \circ H$ with the configuration $C^{v^i}_{v^m}$ and this would require 3 moves. Since $d_{G\circ H}(v^m, v^j) = 1$ so we need minimum of one more move to take the hole from v^m to v^j. Therefore T involves exactly 6 moves. □

Lemma 3.4 *Let G be a complete graph minus a 1-factor and H a complete graph. Let $i, j \in V(G)$ but $\{i, j\} \notin E(G)$ and $u, v \in V(H)$. Then starting from the configuration $C^{u^i}_{u^j}$ of $G \circ H$, we require at least 3 moves to move the robot to v^i.*

Proof For the robot to move to v^i before it, the hole must be moved from u^j to v^i. This takes 2 moves since $d_{G\circ H}(u^j, v^i) = 2$. Then the move $v^i \xleftarrow{r} u^i$ takes the robot from u^i to v^i. Hence the result follows □

In view of Lemmas 3.1, 3.2 and 3.3 we have the following theorem.

Theorem 3.5 *Consider the graph $G \circ H$. Let G be a complete graph minus a 1-factor and H a graph. Let $i, j \in V(G)$ where $\{i, j\} \notin E(G)$ and $u, v \in V(H)$. According as the hole is either at the i^{th} or j^{th} copy of $G \circ H$. Then to move the robot from*

(i) H^i *to* H^j *we require* 5 *moves if* H *is either* K_n *or* $\overline{K_n}$*;*

(ii) H^i *to* H^i *we require* 5 *moves if* H *is* $\overline{K_n}$*.*

Finally, we have the corollary below which is as a result of Lemma 3.4.

Corollary 3.6 *Consider the graph $G \circ H$. Let G be a complete graph minus a 1-factor and H a complete graph. Let $i, j \in V(G)$ but $\{i, j\} \notin E(G)$ and $u, v \in V(H)$. Then to move the robot from H^i to H^i we require 3 moves according as the hole is either at the j^{th} or i^{th} copy of $G \circ H$ respectively.*

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To know what people really think, pay regard to what they do rather than what they say.

By Rene Descartes, a French philosopher.

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