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The $\beta-$Change of Special Finsler Spaces

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Abstract: We have considered the $\beta-$change of Finsler metric L given by $L^* = f(L,\beta)$, where f is any positively homogeneous function of degree one in L and β. We have obtained the $\beta-$change of $C-$reducible Finsler spaces, $S3-$like Finsler spaces and $T-$tensor. Particular case when b_i in β is concurrent vector field has been studied.

Key Words: $\beta-$change, Finsler metric, $T-$tensor, $C-$reducible, $S3-$like Finsler spaces.

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§1. Introduction

Let $F^n = (M^n, L)$ be an $n-$dimensional Finsler space on the differentiable manifold M^n, equipped with the fundamental function $L(x,y)$. B. N. Prasad and Bindu Kumari [1] and C. Shibata [2] considered the $\beta-$change of Finsler metric given by

$$L^*(x,y) = f(L,\beta), \tag{1.1}$$

where f is positively homogeneous function of degree one in L and β and β given by $\beta(x,y) = b_i(x)\,y^i$ is a one-form on M^n. The Finsler space (M^n, L^*) obtained from F^n by the $\beta-$change (1.1) will be denoted by F^{*n}. The Homogeneity of f in (1.1) gives

$$Lf_1 + \beta f_2 = f, \tag{1.2}$$

where the subscripts '1' and '2' denote the partial derivatives with respect to L and β respectively.

Differentiating (1.2) with respect to L and β respectively, we get

$$Lf_{11} + \beta f_{12} = 0 \quad \text{and} \quad Lf_{12} + \beta f_{22} = 0.$$

Hence, we have

$$\frac{f_{11}}{\beta^2} = -\frac{f_{12}}{L\beta} = \frac{f_{22}}{L^2},$$

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which gives

$$f_{11} = \beta^2\omega, \quad f_{22} = L^2\omega, \quad f_{12} = -\beta L\omega,$$

where the Weierstrass function ω is positively homogeneous function of degree -3 in L and β. Therefore

$$L\omega_1 + \beta\omega_2 + 3\omega = 0. \tag{1.3}$$

Again ω_2 is positively homogeneous of degree -4 in L and β, so

$$L\omega_{21} + \beta\omega_{22} + 4\omega_2 = 0. \tag{1.4}$$

Throughout the paper we frequently use above equations (1.2) to (1.4) without quoting them. The concept of concurrent vector field has been given by Matsumoto and K. Eguchi [6] and S. Tachibana [7], which is defined as follows:

The vector field b_i is said to be a concurrent vector field if

$$(i) \qquad b_{i|j} = -\,g_{ij}, \qquad (ii) \qquad b_i|_j = 0, \tag{1.5}$$

where small and long solidus denote the $h-$ and $v-$covariant derivatives respectively.

It has been proved by by Matsumoto that b_i and its contravariant components b^i are functions of coordinates alone. Therefore from (1.5)(ii), we have

$$C_{ijk}\, b^i = 0.$$

§2. **Fundamental Quantities of** F^{*n}

To find the relation between fundamental quantities of F^n and F^{*n}, we use the following results

$$\dot\partial_i\beta = b_i, \qquad \dot\partial_i L = l_i, \qquad \dot\partial_j l_i = L^{-1}h_{ij}, \tag{2.1}$$

where $\dot\partial_i$ stands for $\dfrac{\partial}{\partial y^i}$ and h_{ij} are components of angular metric tensor of F^n given by $h_{ij} = g_{ij} - l_i\, l_j = L\dot\partial_j\dot\partial_j L$.

The successive differentiation of (1.1) with respect to y^i and y^j gives:

$$l_i^* = f_1 l_i + f_2 b_i, \tag{2.2}$$

$$h_{ij}^* = \frac{ff_1}{L}h_{ij} + fL^2\omega m_i m_j, \tag{2.3}$$

where $m_i = b_i - \frac{\beta}{L}\, l_i$. The quantities corresponding to F^{*n} will be denoted by putting star on the top of those quantities.

From (2.2) and (2.3) we get the following relations between metric tensors of F^n and F^{*n}

$$g_{ij}^* = \frac{ff_1}{L}g_{ij} - \frac{p\beta}{L}l_i l_j + (fL^2\omega + f_2^2)b_i b_j + p(l_i b_j + l_j b_i), \tag{2.4}$$

where $p = (f_1 f_2 - f\beta L\omega)$.

The contravariant components of the metric tensor of F^{*n} will be dertived from (2.4) as follows:

$$g^{*ij} = \frac{L}{ff_1} g^{ij} + \frac{pL^3}{f^3 f_1 t}\left(\frac{f\beta}{L^2} - \triangle f_2\right) l^i l^j - \frac{L^4\omega}{ff_1 t} b^i b^j - \frac{pL^2}{f^2 f_1 t}(l^i b^j + l^j b^i), \tag{2.5}$$

where we put $b^i = g^{ij} b_j, \quad l^i = g^{ij} l_j, \quad b^2 = g^{ij} b_i b_j$ and

$$t = f_1 + L^3\omega\triangle, \qquad \triangle = b^2 - \frac{\beta^2}{L^2}. \tag{2.6}$$

Putting $q = 3f_2\omega + f_2\omega$, we find that

$$\begin{aligned}
&(a)\ \ \dot\partial_i f = \frac{f}{L} l_i + f_2 m_i,\\
&(b)\ \dot\partial_i f_1 = -\beta L\omega m_i,\\
&(c)\ \dot\partial_i f_2 = L^2\omega m_i,\\
&(d)\ \dot\partial_i \omega = -\frac{3\omega}{L} l_i + \omega_2 m_i,\\
&(e)\ \dot\partial_i b^2 = -2C_{..i},\\
&(f)\ \dot\partial_i \triangle = -2C_{..i} - \frac{2\beta}{L^2} m_i
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&(a)\ \dot\partial_i p = -\beta L q m_i,\\
&(b)\ \dot\partial_i t = -2L^3\omega C_{..i} + (L^3\triangle\omega_2 - 3\beta L\omega) m_i,\\
&(c)\ \dot\partial_i q = -\frac{3q}{L} l_i + (4f_2\omega_2 + 3\omega^2 L^2 + f\omega_{22}) m_i,
\end{aligned} \tag{2.8}$$

where '.' denotes the contraction with b^i, viz. $C_{..i} = C_{jki} b^j b^k$.

Differentiating (2.4) with respect to y^k, using (2.1) and (2.7), we get the following relation between the Cartan's $C-$tensors ($C^*_{ijk} = \frac{1}{2}\dot\partial_k\, g^*_{ij}$ and $C_{ijk} = \frac{1}{2}\dot\partial_k g_{ij}$):

$$C^*_{ijk} = \frac{ff_1}{L} C_{ijk} + \frac{p}{2L}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{qL^2}{2} m_i m_j m_k. \tag{2.9}$$

It is to be noted that

$$m_i l^i = 0, \quad m_i m^i = \triangle = m_i b^i, \quad h_{ij} l^j = 0, \quad h_{ij} m^j = h_{ij} b^j = m_i, \tag{2.10}$$

where $m^i = g^{ij} m_j = b^i - \frac{\beta}{L} l^i$.

To find $C^{*i}_{jk} = g^{*ih} C^*_{jhk}$ we use (2.5), (2.9), (2.10), we get

$$\begin{aligned}
C^{*i}_{jk} = C^i_{jk} &+ \frac{p}{2ff_1}(h_{jk} m^i + h^i_j m_k + h^i_k m_j) + \frac{qL^3}{2ff_1} m_j m_k m^i\\
&- \frac{L}{ft} C_{.jk} n^i - \frac{pL\triangle}{2f^2 f_1 t} h_{jk} n^i - \frac{2pL + L^4\triangle q}{2f^2 f_1 t} m_j m_k n^i,
\end{aligned} \tag{2.11}$$

where $n^i = fL^2\omega b^i + pl^i$.

We have the following relations corresponding to the vectors with components n^i and m^i:

$$C_{ijk}m^i = C_{.jk}, \quad C_{ijk}n^i = fL^2\omega C_{.jk}, \quad m_i m^i = fL^2\omega\triangle. \tag{2.12}$$

§3. The $\beta-$Change of $C-$reducible Finsler Space

Let F^n be a $C-$reducible Finsler space. Then [5]

$$C_{hjk} = \frac{1}{n+1}(h_{hj}C_k + h_{hk}C_j + h_{jk}C_h), \tag{3.1}$$

where $C_k = C_{hjk}g^{hj}$.

Using equation (3.1) in equation (2.9), we get

$$C^*_{hjk} = (p_k h_{hj} + p_j h_{hk} + p_h h_{jk}) + \frac{qL^2}{2}\, m_h m_j m_k, \tag{3.2}$$

where

$$p_k = \frac{ff_1}{L(n+1)}C_k + \frac{p}{2L}m_k. \tag{3.3}$$

Using equation (2.3) in equation (3.2), we get

$$C^*_{hjk} = \frac{L}{ff_1}(p_k h^*_{hj} + p_j h^*_{hk} + p_h h^*_{jk}) + q_h\, m_j m_k + q_j\, m_h m_k + q_k\, m_j m_h, \tag{3.4}$$

where

$$q_h = \frac{qL^2}{6}m_h - \frac{L^3\omega}{f_1}p_h. \tag{3.5}$$

Now suppose that the transformation (1.1) is such that $(n+1)(f_1\omega_2+3\beta L\omega^2)m_h = 6f_1\omega C_h$, then $q_h = 0$. So equation (3.4) reduces to

$$C^*_{hjk} = \frac{L}{ff_1}(p_k h^*_{hj} + p_j h^*_{hk} + p_h h^*_{jk}) \tag{3.6}$$

which will give $\dfrac{C^*_k}{n+1} = \dfrac{L}{ff_1}\, p_k$, so that

$$C^*_{hjk} = \frac{1}{n+1}(C^*_k h^*_{hj} + C^*_j h^*_{hk} + C^*_h h^*_{jk}) \tag{3.7}$$

Hence F^{*n} is also a $C-$reducible. Therefore we have the following result.

Theorem 3.1 *Under the $\beta-$change of Finsler metric with the condition $(n+1)(f_1\omega_2 + 3\beta L\omega^2)m_h = 6f_1\omega C_h$, the $C-$reducible Finsler space is transformed to a $C-$reducible Finsler space.*

In the theorem (3.1) we have assumed that $(n+1)(f_1\omega_2+3\beta L\omega^2)m_h = 6f_1\omega C_h$. However if this condition is not satisfied then a $C-$reducible Finsler space may not be transformed to a $C-$reducible Finsler space. In the following we discuss under what condition a $C-$reducible Finsler space is transformed to a $C-$reducible Finsler space by $\beta-$change of Finsler metric.

In both the spaces F^n and F^{*n} are $C-$reducible then from (3.1) and its corresponding equation for F^{*n} we find, on using (2.9), that

$$\begin{aligned}&\frac{fL^2\omega}{t}[(Q_h m_j m_k + Q_j m_h m_k + Q_k m_j m_h) - f_1(C_{..h}h_{jk} + C_{..j}h_{hk}\\&+ C_{..k}h_{jh})] = \left(\frac{p}{2L} - \frac{ff_1 r}{L(n+1)}\right)(h_{jk}m_h + h_{hj}m_k + h_{hk}m_j)\\&+\left(\frac{qL^2}{2} - 3fL^2\omega r\right)m_h m_j m_k,\end{aligned} \tag{3.8}$$

where $Q_h = tC_h - L^3\omega C_{..h}$ and $r = (n-2)pt + f_1(3p + L^3 q\triangle)$. Thus, we have the following result.

Theorem 3.2 *A $C-$reducible Finsler space is transformed to a $C-$reducible Finsler space by a $\beta-$change of Finsler metric if and only if (3.8) holds.*

The condition (3.8) of theorem (3.2) is too complicated to study any geometrical concept of Finsler space. So we consider that our β in $\beta-$change of Finsler metric is such that b_i is a concurrent vector field [6] so that $C_{.i} = 0$, $C_{..i} = 0$. Hence equation (3.8) reduces to

$$\begin{aligned}&fL^2\omega(C_h m_j m_k + C_j m_h m_k + C_k m_j m_h) = \left(\frac{p}{2L} - \frac{ff_1 r}{2L}\right)(h_{jk}m_h + h_{hj}m_k\\&+ h_{hk}m_j) + \left(\frac{qL^2}{2} - 3f^2\omega r\right)m_h m_j m_k.\end{aligned}$$

Contracting this equation with g^{jk}, we find

$$2fL^3\omega\triangle C_h = \{(n+1)(p - ff_1 r) + (qL^3 - 6f^2 L\omega r)\triangle\}\, m_h.$$

Hence we have the following result.

Theorem 3.3 *If a $C-$reducible Finsler space is transformed to a $C-$reducible Finsler space by a concurrent $\beta-$change of Finsler metric, then the vector C_h is along the direction of the vector m_h.*

§4. The $\beta-$Change of $v-$Curvature Tensor

To find the $v-$curvature tensor of F^{*n} with respect to Cartan's connection, we use the following:

$$C^h_{ij}h_{hk} = C_{ijk}, \quad h^k_j h^i_k = h^i_j, \quad h_{ij}n^i = fL^2\omega m_j. \tag{4.1}$$

The $v-$curvature tensors S^*_{hijk} of F^{*n} [4] is defined as

$$S^*_{hijk} = C^{*r}_{hk} C^*_{rij} - C^{*r}_{hj} C^*_{ikr}. \tag{4.2}$$

From (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14) we get the following relation between $v-$curvature tensors of F^n and F^{*n} [1]:

$$S^*_{hijk} = \frac{ff_1}{L} S_{hijk} + d_{hj}d_{ik} - d_{hk}d_{ij} + E_{hk}E_{ij} - E_{hj}E_{ik}, \tag{4.3}$$

where

$$d_{ij} = L\sqrt{\frac{s}{t}} C_{.ij} - \frac{pf_1}{2L^2\sqrt{ts}} h_{ij} + \frac{2\omega p - qf_1}{2\sqrt{ts}} Lm_i m_j, \tag{4.4}$$

$$E_{ij} = \frac{p}{2L^2\sqrt{f\omega}} h_{ij} - \frac{p\omega - qf_1}{2f_1\sqrt{f\omega}} Lm_i m_j \tag{4.5}$$

and $s = ff_1\omega$.

Now suppose that b_i is a concurrent vector field and F^n is an $S3-$like Finsler space [4], then $C_{.ij} = 0$,

$$S_{hijk} = \frac{S}{L^2} (h_{hk}h_{ij} - h_{hj}h_{ik}),$$

where S is any scalar function of x and y.

In view of these equations, we have from (4.3)

$$\begin{aligned} S^*_{hijk} &= \left(\frac{ff_1 S}{L^3} + \frac{p^2 f_1^2}{4L^4 ts} - \frac{p^2}{4L^4 f\omega}\right)(h_{hk}h_{ij} - h_{hj}h_{ik}) \\ &+ \left\{\frac{p(p\omega - qf_1)}{4L^2 ff_1\omega} - \frac{pf_1(2\omega p - qf_1)}{4Lts}\right\}(h_{hj}m_i m_k + h_{ik}m_h m_j \\ &- h_{hk}m_i m_j - h_{ij}m_h m_k). \end{aligned} \tag{4.6}$$

Now suppose that the transformed Finsler space F^{*n} is also $S3-$like. Then

$$S^*_{hijk} = \frac{S^*}{L^{*2}} (h^*_{hk}h^*_{ij} - h^*_{hj}h^*_{ik}). \tag{4.7}$$

Now from (2.3), it follows that

$$\begin{aligned} (h^*_{hk}h^*_{ij} - h^*_{hj}h^*_{ik}) &= \left(\frac{ff_1}{L}\right)^2 (h_{hk}h_{ij} - h_{hj}h_{ik}) \\ &+ f^2 f_1 L\omega(h_{hk}m_i m_j + h_{ij}m_h m_k - h_{hj}m_k m_i - h_{ik}m_h m_j). \end{aligned} \tag{4.8}$$

In view of (4.6), (4.7) and (4.8), we have

$$\begin{aligned} &\left(\frac{ff_1 S}{L^3} + \frac{p^2 f_1^2}{4L^4 ts} - \frac{p^2}{4L^4 f\omega} - \frac{S^* f_1^2}{L^2}\right)(h_{hk}h_{ij} - h_{hj}h_{ik}) \\ &+ \left\{\frac{p(p\omega - qf_1)}{4L^2 ff_1\omega} - \frac{pf_1(2\omega p - qf_1)}{4Lts} - S^* f_1 L\omega\right\}(h_{hk}m_i m_j \\ &+ h_{ij}m_h m_k - h_{hj}m_i m_k - h_{ik}m_h m_j) = 0. \end{aligned} \tag{4.9}$$

Contracting (4.9) by $g^{ij}g^{hk}$, we get

$$\begin{aligned}&\left(\frac{ff_1S}{L^3}+\frac{p^2f_1^2}{4L^4ts}-\frac{p^2}{4L^4f\omega}-\frac{S^*f_1^2}{L^2}\right)(n-1)(n-2)\\&+\ 2\left\{\frac{p(p\omega-qf_1)}{4L^2ff_1\omega}-\frac{pf_1(2\omega p-qf_1)}{4Lts}-S^*f_1L\omega\right\}\triangle=0.\end{aligned}\tag{4.10}$$

Hence, we have the following result.

Theorem 4.1 *If a $S3-$like Finsler space is transformed to a $S3-$like Finsler space under the concurrent $\beta-$change, then equation (4.10) holds.*

§5. The $\beta-$Change of $T-$Tensor

The $T-$tensor of Finsler space F^n is defined by [3]:

$$T_{hijk}=LC_{hij}|_k+l_hC_{ijk}+l_iC_{hjk}+l_jC_{hik}+l_kC_{hij},\tag{5.1}$$

where

$$C_{hijk}|_k=\frac{\partial C_{hij}}{\partial y^k}-C_{rij}C^r_{hk}-C_{hrj}C^r_{ik}-C_{hir}C^r_{jk}.\tag{5.2}$$

To find the $T-$tensor of F^{*n}, first of all we find

$$C^*_{hij}||_k=\frac{\partial C^*_{hij}}{\partial y^k}-C^*_{rij}C^{*r}_{hk}-C^*_{hrj}C^{*r}_{ik}-C^*_{hir}C^{*r}_{jk},$$

where $||$ denotes $v-$covariant derivative in F^{*n}. The derivatives of m_i and h_{ij} with respect to y^k are given by

$$\begin{aligned}\dot\partial_km_i&=-\frac{\beta}{L^2}h_{ik}-\frac{1}{L}l_im_k,\\\dot\partial_k(h_{ij})&=2C_{ijk}-\frac{1}{L}(l_ih_{jk}+l_jh_{ki}).\end{aligned}\tag{5.3}$$

From (2.7), (2.8), (2.9) and (5.3), we get

$$\begin{aligned}\frac{\partial C^*_{hij}}{\partial y^k}&=\frac{ff_1}{L}\frac{\partial C_{hij}}{\partial y^k}+\frac{p}{L}(C_{hij}m_k+C_{ijk}m_h+C_{jhk}m_i+C_{ihk}m_j)\\&-\frac{p\beta}{2L^3}(h_{ij}h_{hk}+h_{hj}h_{ik}+h_{ih}h_{jk})+\frac{p}{2L^2}(h_{jk}l_hm_i+h_{hk}l_jm_i\\&+h_{hk}l_im_j+h_{ik}l_hm_j+h_{jk}l_im_h+h_{ik}l_jm_h+h_{ij}l_hm_k+h_{hj}l_im_k\\&+h_{ih}l_jm_k+h_{ij}l_km_h+h_{jh}l_km_i+h_{hi}l_km_j)-\frac{\beta q}{2}(h_{ij}m_hm_k\\&+h_{jh}m_im_k+h_{hi}m_jm_k+h_{ik}m_jm_h+h_{jk}m_im_h+h_{hk}m_im_j)\\&-\frac{qL}{2}(l_im_jm_hm_k+l_jm_hm_im_k+l_hm_im_jm_k+l_km_im_jm_h)\\&+\frac{L^2}{2}(4f_2\omega_2+3L^2\omega^2+f\omega_{22})\,m_im_jm_hm_k.\end{aligned}\tag{5.4}$$

From equation (2.9), (2.10), (2.11) and (2.12), we have

$$\begin{aligned}
C^*_{rij}\,C^{*r}_{hk} &= \frac{ff_1}{L}C_{rij}C^r_{hk} + \frac{p}{2L}(C_{hjk}m_i + C_{hik}m_j + C_{hij}m_k \\
&+ C_{ijk}m_h) + \frac{f_1 p}{2Lt}(C_{.ij}h_{hk} + C_{.hk}h_{ij}) - \frac{ff_1L^2\omega}{t}C_{.ij}C_{.hk} \\
&+ \frac{p^2\triangle}{4fLt}h_{ij}h_{hk} + \frac{L^2(qf_1 - 2p\omega)}{2t}(C_{.ij}m_hm_k + C_{.hk}m_im_j) \\
&+ \frac{p(p + L^3q\triangle)}{4Lft}(h_{ij}m_km_h + h_{hk}m_im_j) + \frac{p^2}{4Lff_1}(h_{ij}m_km_h \\
&+ h_{hk}m_im_j + h_{jk}m_im_k + h_{jk}m_im_h + h_{ih}m_jm_k + h_{ik}m_jm_h) \\
&+ \frac{L^2\{2pqt + (qf_1 - 2p\omega)(2p + L^3q\triangle)\}}{4ff_1t}m_im_jm_hm_k.
\end{aligned} \tag{5.5}$$

From equation (5.4) and (5.5), we get

$$\begin{aligned}
C^*_{hij}||_k &= \frac{ff_1}{L}C_{hij}|_k - \frac{p}{2L}(C_{hij}m_k + C_{ijk}m_h + C_{hjk}m_i + C_{ihk}m_j) \\
&- \frac{p(2f\beta t + L^2p\triangle)}{4fL^3t}(h_{ij}h_{hk} + h_{hj}h_{ik} + h_{ih}h_{jk}) - \bigg(\frac{\beta q}{2} \\
&+ \frac{f_1p^2 + f_1L^3pq\triangle + 3p^2}{4Lff_1t}\bigg)(h_{ij}m_km_h + h_{hk}m_im_j + h_{jh}m_im_k \\
&+ h_{ik}m_jm_h + h_{hi}m_jm_k + h_{jk}m_im_h) - \frac{p}{2L^2}\{l_h(h_{jk}m_i \\
&+ h_{ij}m_k + h_{ik}m_j) + l_j(h_{hk}m_i + h_{ik}m_h + h_{ih}m_k) + l_i(h_{hk}m_j \\
&+ h_{jk}m_h + h_{hj}m_k) + l_k(h_{ij}m_h + h_{jh}m_i + h_{hi}m_j)\} - \frac{qL}{2}(l_im_jm_hm_k \\
&+ l_jm_hm_im_k + l_hm_im_jm_k + l_km_im_jm_h) - \frac{f_1p}{2Lt}(C_{.ij}h_{hk} + C_{.hj}h_{ik} \\
&+ C_{.hk}h_{ij} + C_{.ik}h_{hj} + C_{.hi}h_{jk} + C_{.jk}h_{hi}) + \frac{ff_1L^2\omega}{t}(C_{.ij}C_{.hk} \\
&+ C_{.hj}C_{.ik} + C_{.hi}C_{.jk}) - \frac{L^2(qf_1 - 2p\omega)}{2t}(C_{.ij}m_km_h + C_{.hk}m_im_j \\
&+ C_{.hj}m_im_k + C_{.ik}m_jm_h + C_{.hi}m_jm_k + C_{.jk}m_hm_i) \\
&+ \bigg[\frac{L^2}{2}(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22}) \\
&- \frac{3L^2\{2pqt + (qf_1 - 2p\omega)(2p + L^3q\triangle)\}}{4ff_1t}\bigg]m_im_jm_hm_k.
\end{aligned} \tag{5.6}$$

Using equations (2.2), (2.9) and (5.6), we get the following relation between $T-$tensors of

Finsler spaces F^n and F^{*n}:

$$\begin{aligned}
T^*_{hijk} &= \frac{f^2 f_1}{L^2} T_{hijk} + \frac{f(f_1 f_2 + f\beta L\omega)}{2L}(C_{hij}m_k + C_{ijk}m_h + C_{hjk}m_i \\
&+ C_{ihk}m_j) + \frac{f^2 L^2 f_1 \omega}{t}(C_{.ij}C_{.hk} + C_{.hj}C_{.ik} + C_{.hi}C_{.jk}) - \frac{f f_1 p}{2Lt} \\
&(C_{.ij}h_{hk} + C_{.hk}h_{ij} + C_{.hj}h_{ik} + C_{.ik}h_{hj} + C_{.hi}h_{jk} + C_{.jk}h_{hi}) \\
&- \frac{fL^2(qf_1 - 2p\omega)}{2t}(C_{.ij}m_k m_h + C_{.hk}m_i m_j + C_{.hj}m_i m_k \\
&+ C_{.ik}m_j m_h + C_{.hi}m_j m_k + C_{.jk}m_h m_i) - \frac{p(2f\beta t + L^2 p\triangle)}{4L^3 t}(h_{ij}h_{hk} \\
&+ h_{hj}h_{ik} + h_{ih}h_{jk}) - \left(\frac{f\beta q}{2} + \frac{f_1 p^2 + f_1 L^3 pq\triangle + 3p^2}{4Lf_1 t} - \frac{pf_2}{L}\right) \\
&(h_{ij}m_k m_h + h_{hk}m_i m_j + h_{jh}m_i m_k + h_{ik}m_j m_h + h_{hi}m_j m_k \\
&+ h_{jk}m_i m_h) + \left[\frac{fL^2}{2}(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22}) + \frac{4L^2 f_2 q}{2}\right. \\
&\left.- \frac{3L^2\{2pqt + (qf_1 - 2p\omega)(2p + L^3 q\triangle)\}}{4f_1 t}\right] m_i m_j m_h m_k.
\end{aligned} \tag{5.7}$$

If b_i is a concurrent vector field in F^n, then $C_{.ij} = 0$. Therefore from (5.7), we have

$$\begin{aligned}
T^*_{hijk} &= \frac{f^2 f_1}{L^2} T_{hijk} - \frac{p(2f\beta t + L^2 p\triangle)}{4L^3 t}(h_{ij}h_{hk} + h_{hj}h_{ik} + h_{ih}h_{jk}) \\
&- \left(\frac{f\beta q}{2} + \frac{f_1 p^2 + f_1 L^3 pq\triangle + 3p^2}{4Lf_1 t} - \frac{pf_2}{L}\right) \\
&(h_{ij}m_k m_h + h_{hk}m_i m_j + h_{jh}m_i m_k + h_{ik}m_j m_h + h_{hi}m_j m_k \\
&+ h_{jk}m_i m_h) + \left[\frac{fL^2}{2}(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22}) + \frac{4L^2 f_2 q}{2}\right. \\
&\left.- \frac{3L^2\{2pqt + (qf_1 - 2p\omega)(2p + L^3 q\triangle)\}}{4f_1 t}\right] m_i m_j m_h m_k.
\end{aligned} \tag{5.8}$$

If b_i is a concurrent vector field in F^n, with vanishing $T-$tensor then $T-$tensor of F^{*n} is given by

$$\begin{aligned}
T^*_{hijk} &= -\frac{p(2f\beta t + L^2 p\triangle)}{4L^3 t}(h_{ij}h_{hk} + h_{hj}h_{ik} + h_{ih}h_{jk}) \\
&- \left(\frac{f\beta q}{2} + \frac{f_1 p^2 + f_1 L^3 pq\triangle + 3p^2}{4Lf_1 t} - \frac{pf_2}{L}\right) \\
&(h_{ij}m_k m_h + h_{hk}m_i m_j + h_{jh}m_i m_k + h_{ik}m_j m_h + h_{hi}m_j m_k \\
&+ h_{jk}m_i m_h) + \left[\frac{fL^2}{2}(4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22}) + \frac{4L^2 f_2 q}{2}\right. \\
&\left.- \frac{3L^2\{2pqt + (qf_1 - 2p\omega)(2p + L^3 q\triangle)\}}{4f_1 t}\right] m_i m_j m_h m_k.
\end{aligned} \tag{5.9}$$

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