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Spectra and Energy of Signed Graphs

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Abstract: The energy of a signed graph Σ is defined as $\varepsilon(\Sigma) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of Σ. In this paper, we study the spectra and energy of a class of signed graphs which satisfy pairing property. We show that it is possible to compare the energies of a pair of bipartite and non-bipartite signed graphs on n vertices by defining quasi-order relation in such a way that the energy is increasing. Further, we extend the notion of extended double cover of graphs to signed graphs to find the spectra of unbalanced signed bipartite graphs and also we construct non-cospectral equienergetic signed bipartite graphs.

Key Words: Signed graph, Smarandachely signed graph, signed energy, extended double cover(EDC) of signed graphs, equienergetic signed bipartite graphs.

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§1. Introduction

A signed graph is an ordered pair $\Sigma = (G, \sigma)$, where G is the underlying graph of Σ and $\sigma : E \rightarrow \{+1, -1\}$, called signing (or a signature), is a function from the edge set $E(G)$ of G into the set $\{+1, -1\}$. It is said to be homogeneous if its edges are all positive or negative otherwise heterogeneous, and a Smarandachely signed if $|e_+ - e_-| \geq 1$, where e_+, e_- are numbers of edges signed by $+1$ or -1 in $E(G)$, respectively. Negation of a signed graph is the same graph with all signs reversed. In figure, we denote positive edges with solid lines and negative edges with dotted lines.

The adjacency matrix of a signed graph is the square matrix $A(\Sigma) = (a_{ij})$ where (i, j) entry is $+1$ if $\sigma(v_i v_j) = +1$ and -1 if $\sigma(v_i v_j) = -1$, 0 otherwise. The characteristic polynomial of the signed graph Σ is defined as $\Phi(\Sigma : \lambda) = \det(\lambda I - A(\Sigma))$, where I is an identity matrix of order n. The roots of the characteristic equation $\Phi(\Sigma : \lambda) = 0$, denoted by $\lambda_1, \lambda_2, \cdots, \lambda_n$ are called the eigenvalues of signed graph Σ. If the distinct eigenvalues of $A(\Sigma)$ are $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ and their multiplicities are $m_1, m_2, \cdots, m_n$ then the spectrum of Σ is

$$Spec(\Sigma) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & ... & \lambda_n \\ m_1 & m_2 & ... & ... & m_n \end{pmatrix}.$$

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Two signed graphs are cospectral if they have the same spectrum. The spectral criterion for balance in signed graph is given by B. D. Acharya as follows:

Theorem 1.1([1]) *A signed graph is balanced if and only if it is cospectral with the underlying graph. i.e.* $Spec(\Sigma) = Spec(G)$.

The sign of a cycle in a signed graph is the product of the signs of its edges. Thus a cycle is positive if and only if it contains an even number of negative edges. A signed graph is said to be balanced if all of its cycles are positive otherwise unbalanced.

In a signed graph Σ, the degree of a vertex v is defined as $sdeg(v) = d(v) = d^+_\Sigma(v) + d^-_\Sigma(v)$, where $d^+_\Sigma(v)(d^-_\Sigma(v))$ is the number of positive(negative) edges incident with v. It is said to be regular if all its vertices have same degree. The net degree of a vertex v of a signed graph Σ is $d^\pm_\Sigma(v) = d^+_\Sigma(v) - d^-_\Sigma(v)$. It is said to be net-regular of degree k if all its vertices have same net-degree equal to k.

Spectra of graphs is well documented in [5] and signed graphs is discussed in [7, 8, 9, 11]. For standard terminology and notations in graph theory we follow D. B. West [15] and for signed graphs we follow T. Zaslavsky [16].

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of Σ, then $\varepsilon(\Sigma) = \sum_{i=1}^n |\lambda_i|$. Two signed graphs Σ_1 and Σ_2 are said to be equienergetic if $\varepsilon(\Sigma_1) = \varepsilon(\Sigma_2)$. Naturally, cospectral signed graphs are equienergetic. Equienergetic signed graphs are constructed in [3, 13].

The cartesian product $\Sigma_1 \times \Sigma_2$ of two signed graphs $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$ is defined as the signed graph $(V_1 \times V_2, E, \sigma)$ where the edge set E is that of the Cartesian product of the underlying unsigned graphs and the signature function σ for the labeling of the edges is defined by

$$\sigma((u_i, v_j)(u_k, v_l)) = \begin{cases} \sigma_1(u_i, u_k), & \text{if j = l} \\ \sigma_2(v_j, v_l), & \text{if i = k} \end{cases}$$

The Kronecker product of $\Sigma_1 \bigotimes \Sigma_2$ of two signed graphs $\Sigma_1 = (V_1, E_1, \sigma_1)$ and $\Sigma_2 = (V_2, E_2, \sigma_2)$ is the signed graph $(V_1 \times V_2, E, \sigma)$ where the edge set E is that of the Kronecker product of the underlying unsigned graphs and the signature function σ for the labeling of the edges is defined by $\sigma((u_i, v_j)(u_k, v_l)) = \sigma_1(u_i, u_k)\sigma_2(v_j, v_l)$.

Generally, quasi-order relation is used to compare the energies of bipartite graphs. In this paper, we use quasi-order method to compare the energies of two signed graphs of order n which are bipartite and unbalanced non-bipartite signed graphs. Fundamental question in the energy theory is to find the maximal and minimal energy graphs over a significant class of graphs. It is natural to find for signed graphs also. Here we give maximum energy signed graphs which belong to the class of pairing property. Further, we study the spectra and energy of extended double cover (EDC) of signed graphs and also construct non-cospectral equienergetic signed bipartite graphs.

§2. Energy of Signed Graphs in Δ_n

A graph G is a bipartite graph if and only if $\lambda_i = -\lambda_{n+1-i}$, for $1 \leq i \leq \frac{1}{2}(n-1)$. This result is known as *pairing theorem* by Coulson and Rushbrooke [6]. But non-bipartite signed graphs also satisfy pairing property and examples are given in [3]. The class of signed graphs satisfying pairing property we denote it as Δ_n.

The following result is given by Bhat and Pirzada in [3] which gives the spectral criterion of signed graphs on Δ_n.

Theorem 2.1 *Let Σ be a signed graph of order n which satisfies the pairing property. Then the following statements are equivalent:*

(1) *spectrum of Σ is symmetric about the origin;*

(2) $\Phi_\Sigma(\lambda) = \lambda^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k} \lambda^{n-2k}$, *where b_{2k} are non-negative integers for all $k = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$;*

(3) *Σ and $-\Sigma$ are cospectral, where $-\Sigma$ is the signed graph obtained by negating sign of each edge of Σ.*

Now it is possible to define a quasi-order relation over Δ_n in such a way that the energy is increasing. Note that Δ_n consists of signed bipartite as well as unbalanced non-bipartite signed graphs which satisfy pairing property.

Definition 2.2 *Let Σ_1 and Σ_2 be two signed graphs of order n in Δ_n. From Theorem* 2.1 *we can express*

$$\Phi_\Sigma(\lambda) = \lambda^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k} \lambda^{n-2k}$$

where b_{2k} are non-negative integers for all $k = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$. If $b_{2k}(\Sigma_1) \le b_{2k}(\Sigma_2)$ for all k where $1 \le k \le \lfloor \frac{n}{2} \rfloor$ then we can write $\Sigma_1 \le \Sigma_2$. Further, if $b_{2k}(\Sigma_1) < b_{2k}(\Sigma_2)$ for all k where $1 \le k \le \lfloor \frac{n}{2} \rfloor$ then we write $\Sigma_1 < \Sigma_2$. Hence

$$\Sigma_1 \preceq \Sigma_2 \Rightarrow \varepsilon(\Sigma_1) \le \varepsilon(\Sigma_2),$$

$$\Sigma_1 \prec \Sigma_2 \Rightarrow \varepsilon(\Sigma_1) < \varepsilon(\Sigma_2),$$

which implies that the energy is increasing in a quasi order relation over Δ_n.

In [13], it is shown that Coulson's Integral formula remains valid for signed graphs also.

Theorem 2.3([13]) *If Σ is a signed graph then the energy of signed graph Σ is*

$$\varepsilon(\Sigma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[n - \frac{i\lambda\phi'(i\lambda)}{\phi(i\lambda)} \right] d\lambda.$$

Following result is the consequence of Coulson's Integral formula for signed graphs.

Corollary 2.4 *Let Σ be a signed graph. Then*

$$\varepsilon(\Sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[\lambda^k \phi_\Sigma(\frac{i}{\lambda})\right] d\lambda.$$

Theorem 2.5 *Let $\Sigma \in \Delta_n$. Then the energy of a signed graph can be expressed as*

$$\varepsilon(\Sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k}(\Sigma)\lambda^{2k}\right] d\lambda.$$

and if $\Sigma_1, \Sigma_2 \in \Delta_n$ and $\Sigma_1 < \Sigma_2$ then $\varepsilon(\Sigma_1) < \varepsilon(\Sigma_2)$.

Proof Coulson's Integral formula can be expressed as

$$\varepsilon(\Sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[\lambda^k \phi_\Sigma(\frac{i}{\lambda})\right] d\lambda.$$

Since $\Sigma \in \Delta_n$, from Theorem 2.1 we can deduce

$$\phi_\Sigma\left(\frac{i}{\lambda}\right) = \left(\frac{i^n}{\lambda^n}\right)\left[1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k}(\Sigma)\lambda^{2k}\right]$$

and substituting in the above expression, we get

$$\begin{aligned}\varepsilon(\Sigma) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[i^n(1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k}(\Sigma)\lambda^{2k})\right] d\lambda \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln \left[1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k}(\Sigma)\lambda^{2k}\right] d\lambda.\end{aligned}$$

But $\frac{1}{\pi}$ p.v. $\int_{-\infty}^{+\infty} \frac{1}{\lambda^2} ln[i^n] d\lambda = 0$ where p.v. is the principal value of Cauchy's integral. Hence $\varepsilon(\Sigma)$ is a monotonically increasing function on the coefficients of $b_{2k}(\Sigma)$. □

Now the question is which signed graphs are having maximum signed energy in Δ_n.

Theorem 2.6([14]) *Let Σ be a signed graph with n vertices and m edges, then*

$$\sqrt{2m + n(n-1)|\det(A(\Sigma))|^{2/n}} \le \varepsilon(\Sigma) \le \sqrt{2mn}.$$

Corollary 2.7 $\varepsilon(\Sigma) = \sqrt{2mn} = n\sqrt{r}$ *if and only if* $\Sigma^T\Sigma = (\Sigma)^2 = rI_n$, *where r is the maximum degree of Σ and I_n is the identity matrix of order n.*

Proof Notice that $\varepsilon(\Sigma) = n\sqrt{r}$ if and only if there exists a constant t such that $|\lambda_i|^2 = t$ for all i and Σ is an r-regular signed graph. Hence equality holds if and only if $\Sigma^T\Sigma = (\Sigma)^2 = tI$ and $t = r$. □

The following two examples are given by the present author in [12, 14].

Example 2.8 Following unbalanced signed cycle, we denote it as (C_4^-).

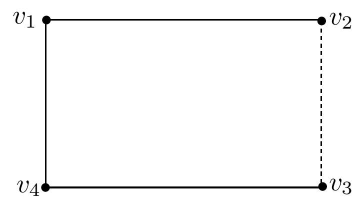

Fig.1 Signed cycle with maximum signed energy

$$A(C_4^-) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial is $\phi(C_4^-) = \lambda^4 - 4\lambda^2 + 4$ and Spec (C_4^-)=$\{(\sqrt{2})^2, (-\sqrt{2})^2\} \in \Delta_n$. Hence $\varepsilon(C_4^-) = 4\sqrt{2} = n\sqrt{r}$.

Example 2.9 Following unbalanced signed complete graph, we denote it as (K_6^-).

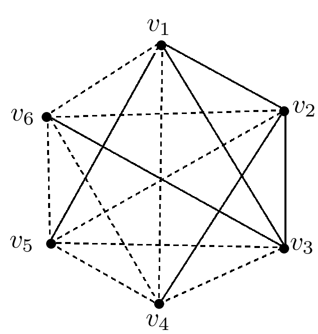

Fig.2 Signed Complete graph with maximum signed energy

$$A(K_6^-) = \begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -1 & -1 \\ 1 & -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & 1 & -1 & -1 & 0 \end{pmatrix}$$

which is a symmetric conference matrix having the characteristic polynomial $\phi(K_6^-) = \lambda^5 - 15\lambda^3 + 75\lambda - 125$ and Spec $A(K_6^-)$=$\{(\sqrt{5})^3, (-\sqrt{5})^3\}\in \Delta_n$. The signed energy of $\varepsilon(K_6^-) = 6\sqrt{5} = n\sqrt{r}$.

Lemma 2.10([3, 8]) *Let Σ_1 and Σ_2 be two signed graphs with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{n_1}$ and $\mu_1, \mu_2, \cdots, \mu_{n_2}$. Then*

(1) *the eigenvalues of $\Sigma_1 \times \Sigma_2 = \lambda_i + \mu_j$, for all $i = 1, 2, \cdots, n_1, j = 1, 2, \cdots, n_2$;*

(2) *the eigenvalues of $\Sigma_1 \bigotimes \Sigma_2 = \lambda_i\mu_j$, for all $i = 1, 2, \cdots, n_1, j = 1, 2, \cdots, n_2$.*

Theorem 2.11 *There exists an infinite family of signed graphs having maximum signed energy in Δ_n.*

Proof Let Σ_1, Σ_2 be two signed graphs in Δ_n with orders n_1 and n_2 having maximum energies $n_1\sqrt{r_1}, n_2\sqrt{r_2}$ respectively. The Kronecker product of $\Sigma_1 \bigotimes \Sigma_2$ is a symmetric matrix of order n_1n_2. From Lemma 2.10, $\Sigma_1 \bigotimes \Sigma_2$ has maximum energy $n_1n_2\sqrt{r_1r_2}$. □

Here we note that maximum energy signed graphs belong to the class of Δ_n.

§3. Spectra of Signed Biparitite Graphs in Δ_n.

In [2], N. Alon introduced the concept of extended double cover of a graph. Here we extend this notion to signed graphs in order to establish the spectrum of various signed bipartite graphs. The ordinary spectrum of EDC of graph is given by Z. Chen in [4].

Lemma 3.1([4]) *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eignenvalues of the graph G. Then the eigenvalues of extended double cover of graph are $\pm(\lambda_1 + 1), \pm(\lambda_2 + 1), \cdots, \pm(\lambda_n + 1)$.*

Now we define extended double cover of signed graph Σ as follows:

Definition 3.2 *Let Σ be a signed graph with vertex set $\{v_1, v_2, \cdots, v_n\}$. Let Σ^* be a signed bipartite graph with $V(\Sigma^*) = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ where,*

1) *v_i is adjacent to u_i and either $\sigma(v_iu_i) = +1$ or $\sigma(v_iu_i) = -1$;*

2) *v_i is adjacent to u_j if v_i is adjacent to v_j in Σ;*

3) *$\sigma(v_iu_j) = +1$ if $\sigma(v_iv_j) = +1$ and $\sigma(v_iu_j) = -1$ if $\sigma(v_iv_j) = -1$.*

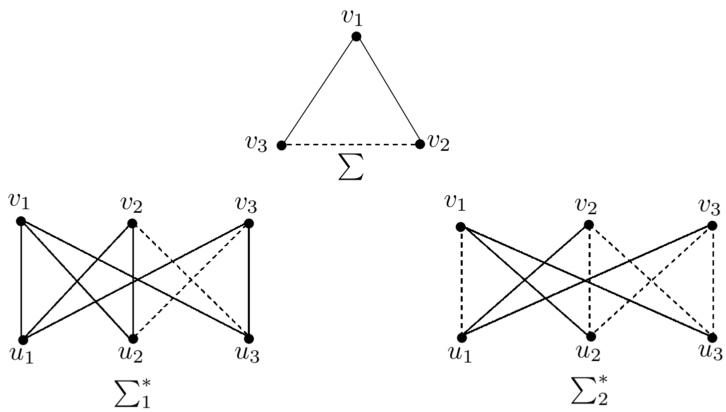

Fig.3 Extended double covers of signed graph Σ.

Then Σ^* is known as extended double cover of signed graph of signed graph Σ and in short we write it as EDC of Σ. Since we get two EDCs of signed graph, we denote it as Σ_1^* if $\sigma(v_iu_i) = +1$ and Σ_2^* if $\sigma(v_iu_i) = -1$.

We need the following Lemma from [10] for further investigation.

Lemma 3.3([10]) *Let* $A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ *be a symmetric* 2×2 *block matrix. Then the spectrum of* A *is the union of the spectra of* $A_0 + A_1$ *and* $A_0 - A_1$.

The following Lemma gives the relation between the spectrum of a signed graph and its EDC of signed graph.

Lemma 3.4 *Let* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *be the eigenvalues of a signed graph then the spectrum of EDCs of signed graph is*

(1) $Spec(\Sigma_1^*) = \left\{ \pm(\lambda_1 + 1), \quad \pm(\lambda_2 + 1), \quad \cdots \quad, \pm(\lambda_n + 1) \right\}$

(2) $Spec(\Sigma_2^*) = \left\{ \pm(\lambda_1 - 1), \quad \pm(\lambda_2 - 1), \quad \cdots \quad, \pm(\lambda_n - 1) \right\}$

Proof Let the adjacency matrix of the signed graph Σ be A. Then the adjacency matrix of EDC of signed graph of Σ is $\begin{pmatrix} 0 & A+I \\ A+I & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & A-I \\ A-I & 0 \end{pmatrix}$, where I is an identity matrix.

From Lemma 3.3, it is clear that the eigenvalues of Σ^* are $\pm(\lambda_i + 1)$ if $\sigma(v_iu_i) = +1$ and $\pm(\lambda_i - 1)$ if $\sigma(v_iu_i) = -1$ for each eigenvalue λ of Σ. □

Theorem 3.5 *Let* Σ *be a connected signed graph. Then* Σ_1^*, Σ_2^* *and* $(\Sigma \times K_2)$ *are co-spectral if and only if* Σ *belongs to the class of* Δ_n.

Proof Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of a signed graph Σ then

(*i*) $Spec(\Sigma_1^*) = \left\{ \pm(\lambda_1 + 1), \quad \pm(\lambda_2 + 1), \quad \cdots \quad, \pm(\lambda_n + 1) \right\}$;

(*ii*) $Spec(\Sigma_2^*) = \left\{ \pm(\lambda_1 - 1), \quad \pm(\lambda_2 - 1), \quad \cdots \quad, \pm(\lambda_n - 1) \right\}$;

(iii) $Spec(\Sigma\times K_2) = \left\{ (\lambda_1 \pm 1), \quad (\lambda_2 \pm 1), \quad ..... \quad ,(\lambda_n \pm 1) \right\}.$

So, $Spec(\Sigma_1^*) = Spec(\Sigma_2^*) = Spec(\Sigma\times K_2)$ if and only if $\lambda_i = -\lambda_{n+1-i}$, for $i = 1, 2, \cdots, n$. Hence the proof. □

Now we give spectra of various signed bipartite graphs.

Proposition 3.6 *Let $(P_n)_1^*$ and $(P_n)_2^*$ be the extended double covers of signed path P_n. Then the spectrum is*

$$(1)\quad Spec(P_n)_1^* = \begin{pmatrix} \pm(2cos\frac{\pi i}{n+1}+1) \\ n \end{pmatrix},\ i=1,\cdots,n.$$

$$(2)\quad Spec(P_n)_2^* = \begin{pmatrix} \pm(2cos\frac{\pi i}{n+1}-1) \\ n \end{pmatrix},\ i=1,\cdots,n.$$

Remark 3.7 If Σ is a signed path then EDCs of signed paths are balanced. Hence EDCs of signed paths are having same energy as underlying graph.

Proposition 3.8 *Let C_n^+ (C_n^-) be the positive(negative) signed cycles on C_n. Then the spectrum of EDCs are respectively*

(1) *If n is odd, then $Spec(C_n^+)_1^* = \left[\pm(2cos\frac{2\pi i}{n}+1), i=1,2,\cdots,n\right]$ and $Spec(C_n^+)_2^* = \left[\pm(2cos\frac{2\pi i}{n}-1), i=1,2,\cdots,n\right]$;*

(2) *If n is even, then $(i)Spec(C_n^-)_1^* = \left[\pm(2cos\frac{(2i+1)\pi}{n}+1), i=1,2,\cdots,n\right]$ and $Spec(C_n^-)_2^* = \left[\pm(2cos\frac{(2i+1)\pi}{n}-1), i=1,2,\cdots,n\right]$.*

If the signed graph is $+K_n$ then EDCs of $+K_n$ are $(K_n)_1^* = +K_{n,n}$ and $(K_n)_2^*$. Spec$(K_{n,n})$ $= \{\pm n, 0^{2n-2}\}$. Following result gives the spectrum of $(K_n)_2^*$ which is an unbalanced net-regular signed complete bipartite graph.

Proposition 3.9 *Let $(K_n)_2^*$ be the EDC of $+K_n$. Then the spectrum of $(K_n)_2^*$ is*

$$Spec(K_n)_2^* = \begin{pmatrix} -2 & -k & k & 2 \\ n-1 & 1 & 1 & n-1 \end{pmatrix},$$

where $k = d^{\pm}(K_n)_2^* = n-2$.

Remark 3.10 From above Proposition 3.9, $\varepsilon(K_n)_2^* = 2(3n-8)$.

Theorem 3.11([13]) *The spectrum of heterogeneous unbalanced signed complete graph (K_n^{net}) is*

$$Spec(K_n^{net}) = \begin{pmatrix} 5-n & 1+4\cos(\frac{2\pi i}{n}) \\ 1 & 1 \end{pmatrix},\ i=1,\cdots,n-1.$$

where (K_n^{net}) is a net regular signed complete graph defined on $+K_n$.

Proposition 3.12 *If $(K_n^{net})_1^*$ and $(K_n^{net})_2^*$ are the net-regular signed complete bipartite graphs of EDCs of K_n^{net}. Then the spectrum is*

(1)

$$Spec(K_n^{net})_1^* = \begin{pmatrix} \pm k & \pm(2+4\cos(\frac{2\pi i}{n})) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, (n-1),$$

where $k = (6-n)$ gives net regularity of $(K_n^{net})_1^*$.

(2)

$$Spec(K_n^{net})_2^* = \begin{pmatrix} \pm k & \pm(1+4\cos(\frac{2\pi i}{n})) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1,$$

where $k = (4-n)$ gives net regularity of $(K_n^{net})_2^*$.

From the above Propositions, we are having the following result.

Theorem 3.13 *EDCs of signed graphs are net-regular if and only if signed graph Σ is net-regular.*

§4. Equienergetic Signed Graphs in Δ_n

Here we construct equienergetic signed bipartite graphs on $4n$ vertices which are non-cospectral and equienergetic.

Theorem 4.1 *There exists a pair of non-cospectral equienergetic signed bipartite graphs on $4n$ vertices where n is odd and $n \geq 3$.*

Proof Let Σ be a signed cycle of order n and of odd length with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and let the extended double covers of signed graph Σ be Σ_1^* and Σ_2^*.

Case 1. If Σ is balanced then

$$Spec(\Sigma) = \begin{pmatrix} 2 & \lambda_i \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

By Lemma 3.4,

$$Spec(\Sigma_1^*) = \begin{pmatrix} \pm 3 & \pm(\lambda_i + 1) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

and

$$Spec(\Sigma_2^*) = \begin{pmatrix} \pm 1 & \pm(\lambda_i - 1) \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

Hence Σ_1^* and Σ_2^* are non-cospectral bipartite signed graphs on $2n$ vertices where n is odd and Σ_1^* is balanced and Σ_2^* is unbalanced.

Further, let H_1, H_2 and K_1, K_2 be second iterated extended double cover signed graphs of Σ_1^* and Σ_2^* respectively. By Theorem 3.5, Spec H_1 = Spec H_2 and Spec K_1= Spec K_2. Let Spec S = Spec H_1 = Spec H_2 and Spec T= Spec K_1 = Spec K_2.

$$Spec(S) = \begin{pmatrix} \pm(4) & \pm(2) & \pm(\pm(\lambda_i + 1) + 1) \\ 1 & 1 & 2(n-1) \end{pmatrix}, \ i = 1, \cdots, n-1.$$

and

$$Spec(T) = \begin{pmatrix} \pm(2) & \pm(0) & \pm(\pm(\lambda_i - 1) + 1) \\ 1 & 1 & 2(n-1) \end{pmatrix}, \ i = 1, \cdots, n-1.$$

Hence $S = (\Sigma_1^*)^*$ and $T = (\Sigma_2^*)^*$ are non-cospectral bipartite signed graphs on $4n$ vertices where n is odd.

$$\varepsilon(S) = 2[4 + 2 + \sum_{i=1}^{n-1} | \pm (\lambda_i + 1) + 1|],$$

$$\varepsilon(T) = 2[2 + 0 + \sum_{i=1}^{n-1} | \pm (\lambda_i - 1) + 1|].$$

If $\varepsilon(S) = \varepsilon(T)$ then $4 = \sum_{i=1}^{n-1}(| \pm(\lambda_i - 1) + 1 | - | \pm(\lambda_i + 1) + 1 |)$, then we know that

$$4 = \sum_{i=1}^{n-1}(|2 - \lambda_i| + |\lambda_i| - |\lambda_i + 2| - |\lambda_i|),$$

$$4 = \sum_{i=1}^{n-1}(|\lambda_i - 2| - |\lambda_i + 2|).$$

Since Σ is a balanced signed cycle $\lambda_i = 2cos\frac{2\pi i}{n}, \ i = 1, \cdots, n-1$,

$$4 = \sum_{i=1}^{n-1}(|2cos\theta_i - 2| - |2cos\theta_i + 2|),$$

$$1 = \sum_{i=1}^{n-1}(sin^2(\frac{\theta_i}{2}) - cos^2(\frac{\theta_i}{2})),$$

$$-1 = \frac{1}{2}\sum_{i=1}^{n-1} 2cos\theta_i.$$

Since $\sum_{i=1}^{n-1} \lambda_i = -2$, so $\varepsilon(S) = \varepsilon(T)$.

Case 2. If Σ is unbalanced then

$$Spec(\Sigma) = \begin{pmatrix} -2 & \lambda_i \\ 1 & n-1 \end{pmatrix}, \ i = 1, \cdots, n-1.$$

By Lemma 3.4,

$$Spec(\Sigma_1^*) = \begin{pmatrix} \pm 1 & \pm(\lambda_i + 1) \\ 1 & n-1 \end{pmatrix},\ i = 1, \cdots, n-1.$$

and

$$Spec(\Sigma_2^*) = \begin{pmatrix} \pm 3 & \pm(\lambda_i - 1) \\ 1 & n-1 \end{pmatrix},\ i = 1, \cdots, n-1.$$

By a similar argument as in Case 1, we get $\varepsilon(S) = \varepsilon(T)$. Hence the proof. □

Example 4.2 Consider the signed graphs Σ_1^* and Σ_2^* as shown in Fig.3. By Lemma 3.4, the characteristic polynomials of Σ_1^* and Σ_2^* are

$$\phi(\Sigma_1^*) = (\lambda + 2)^2(\lambda - 2)^2(\lambda + 1)(\lambda - 1)$$

$$\phi(\Sigma_2^*) = \lambda^4(\lambda + 3)(\lambda - 3)$$

The characteristic polynomials of $(\Sigma_1^*)^*$and $(\Sigma_2^*)^*$ are

$$\phi(\Sigma_1^*)^* = \lambda^2(\lambda + 1)^2(\lambda - 1)^2(\lambda + 3)^2(\lambda - 3)^2(\lambda + 2)(\lambda - 2),$$

$$\phi(\Sigma_2^*)^* = (\lambda + 1)^4(\lambda - 1)^4(\lambda + 4)(\lambda - 4)(\lambda + 2)(\lambda - 2).$$

Hence $Spec(\Sigma_1^*)^* \neq Spec(\Sigma_2^*)^*$ but $\varepsilon(\Sigma_1^*)^* = \varepsilon(\Sigma_2^*)^* = 20$.

Another example of equienergetic signed bipartite graphs on $4n$ vertices is given below.

Example 4.3 Consider the signed graphs Σ_1^* and Σ_2^* as shown in Fig.3. By Lemma 2.10, the characteristic polynomials of $(\Sigma_1^* \times K_2)$ and $(\Sigma_2^* \times K_2)$ are

$$\phi(\Sigma_1^* \times K_2) = \lambda^2(\lambda + 1)^2(\lambda - 1)^2(\lambda + 3)^2(\lambda - 3)^2(\lambda + 2)(\lambda - 2),$$

$$\phi(\Sigma_2^* \times K_2) = (\lambda + 1)^4(\lambda - 1)^4(\lambda + 4)(\lambda - 4)(\lambda + 2)(\lambda - 2).$$

Hence $Spec(\Sigma_1^* \times K_2) \neq Spec(\Sigma_2^* \times K_2)$ but $\varepsilon(\Sigma_1^* \times K_2) = \varepsilon(\Sigma_2^* \times K_2) = 20$.

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