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Some Properties of a h-Randers Finsler Space

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Abstract: The purpose of the present paper is to obtain the relation between imbedding class numbers of tangent Riemannian spaces to (M^n, L) and (M^n, L^*) where L^*(x, y) is obtained from the transformation of L(x, y) is given by

$$L^*(x, y) \to L(x, y) + b_i(x, y)y^i$$

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§1. Introduction

In 1971 Matsumoto [5] introduced the transformation of Finsler metric

$$\bar{L}(x, y) \to L(x, y) + b_i y^i \tag{1.1}$$

and obtain the relation between the imbedding class numbers of a tangent Riemannian spaces to (M^n, L) and a Finsler space $(M^n, \bar{L})$ which is obtained by the transformation of the Finsler metric L by the relation given by in the equation (1.1). Since a concurrent vector field is a function of (x) i.e., position only, assuming $b_i(x)$ as a concurrent vector field, Matsumoto [6] studied the R3-likeness of Finsler spaces (M^n, L) and $(M^n, \bar{L})$. Singh and Prasad [14,11] generalized the concept of concurrent vector field and introduced the semi-parallel and concircular vector fields which are functions of (x) only. Assuming $b_i(x)$ as a concicular vector field, Prasad, Singh and Singh [11] studied the R3-likeness of (M^n, L) and $(M^n, \bar{L})$.

If $L(x, y)$ is a metric function of Riemannian space then $\bar{L}(x, y)$ reduces to the metric function of Rander's space. Such a Finsler metric was first introduced by G. Randers [13]from the standpoint of general theory of relativity and applied to the theory of the electron microscope by R. S. Ingarden [3] who first named it us Randers space. The geometrical properties of this space have been studied by various workers [2, 7, 9, 12, 15].. In 1970 Numata [10] has studied the properties of $(M^n, \bar{L})$ which is obtained from Minkowski space (M^n, L) by transformation

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(1.1). In all those works the function $b_i(x)$ are assumed to be functions of (x) only.

In 1980, Izumi [4] while studying the conformal transformation of Finsler spaces, introduced the h-vector b_i which is v-covariantly constant with respect to Cartan's connection $C\Gamma$ and satisfies the relation

$$LC^h_{ij}b_h = \rho h_{ij}$$

Thus the h-vector b_i is not only a function of (x) but it is also a function of directional arguments satisfying $L\dot{\partial}_j b_i = \rho h_{ij}$. The purpose of the present paper is to obtain the relation between imbedding class numbers of tangent Riemannian spaces to (M^n, L) and (M^n, L^*) where L^*(x, y) is obtained from the transformation of L(x, y) is given by

$$L^*(x, y) \to L(x, y) + \beta(x, y), \tag{1.2}$$

where $\beta(x, y) = b_i(x, y)y^i$, i.e. $b_i(x, y)$ is the function of position and direction both.

§2. An h-Vector in (M^n, L)

Let b_i be a vector field in the Finsler space (M^n, L). If $b_i(x, y)$ satisfies the conditions

$$b_i|_j = 0, \tag{2.1}$$

$$LC^h_{ij}b_h = \rho h_{ij}, \tag{2.2}$$

then the vector field b_i is called an h-vector [4]. Here $|_i$ denotes the v-covariant derivative with respect to y^i in the case of Cartan's connection $C\Gamma$, C^h_{ij} is the cartan's C-tensor, h_{ij} is the angular metric tensor and ρ is given by

$$\rho = \frac{LC^i b_i}{(n-1)}, \tag{2.3}$$

where C^i is the torsion tensor given by $C^i_{jk}g^{jk}$.

Lemma 2.1([4]) *If b_i is an h-vector then the function ρ and are independent of y.*

Since The v-covariant derivation of $b^2 = g^{ij}b_ib_j$ and the fact that g^{ij} is v-covariantly constant yield

$$b\dot{\partial}_k b = g^{ij}b_ib_j|_k.$$

In the view of (2.1) we have

$$\dot{\partial}_k b = 0.$$

Thus we have

Lemma 2.2 *The magnitude b of an h-vector is independent of y.*

From (2.1), Ricci identity [8] and the fact that $S_{ihjk} = g_{hr}S^r_{ijk}$ is skew-symmetric in h and

i we have

$$b_i|_j|_k - b_i|_k|_j = -S^h_{ijk} b_h = 0.$$

Thus we have

Lemma 2.3 *For an h-vector b_i we have $S_{hijk} b^h = 0$, where S_{hijk} are components of v-curvature tensor of Cartan's connection $C\Gamma$.*

The concept of concurrent vector field in (M^n, L) has been introduced by Tachibana [16] and its properties have been studied by Matsumoto [6]. A vector field b_i in (M^n, L) is said to be concurrent if it satisfies the condition (2.1) and

$$b_{i|j} = -g_{ij}, \tag{2.4}$$

where $_{|j}$ denotes h-covariant differentiation with respect to x^i in the sense of Cartan's connection $C\Gamma$.

Applying Ricci Identity [8]

$$b_{i|j}|_k - b_i|_{k|j} = -b_h P^h_{ijk} - b_{i|h} C^h_{jk} - b_i|_h P^h_{jk}$$

and using (2.1) and (2.4) we have

$$P^h_{ijk} b_h + C_{ijk} = 0.$$

Since $P_{imjk} = g_{mh} P^h_{ijk}$ is skew-symmetric in i and m, contraction of above equation with $b^i = g^{ij} b_j$ gives $C_{ijk} b^i = 0$. Hence we have the following

Lemma 2.4 *An h-vector b_i with $\rho \neq 0$ is not a concurrent vector field.*

§3. Properties of the h-Randers Finsler Space

Let b_i be an h-vector in the Finsler space (M^n, L) and (M^n, L^*) be another Finsler space whose fundamental function $L^*(x, y)$ is given by (1.2).

Since b_i is an h-vector, from (2.1) and (2.2), we get

$$\dot\partial_j b_i = L^{-1} \rho h_{ij}, \tag{3.1}$$

which after using the indicatory property of h_{ij} yields $\dot\partial_j \beta = b_j$.

Definition 3.1 *Let M^n be an n-dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x, y), (y^i = \dot x^i)$ of M^n. A change in the fundamental function L by the equation (1.2) on the same manifold M^n is called h-Randers change. A space equipped with fundamental metric L^* is called h-Randers changed Finsler space F^{*n}.*

Now differentiating (1.2) with respect to y^i we have

$$l_i^* = l_i + b_i, \tag{3.2}$$

where $l_i = \dot{\partial}_i L$ is the normalized supporting element in (M^n, L) and $l_i^* = \dot{\partial}_i L^*$ is the normalized element of support in (M^n, L^*). The quantities of (M^n, L^*) will be denoted by starred letter. Now differentiating (3.2) with respect to y^j then the angular metric tensor $h_{ij}^* = \dot{\partial}_j l_i^*$ is given by

$$h_{ij}^* = \sigma h_{ij}, \tag{3.3}$$

where $\sigma = LL^{-1}(1+\rho)$. Hence we have

$$g_{ij}^* = \sigma g_{ij} + (1-\sigma) l_i l_j + (l_i b_j + l_j b_i) + b_i b_j. \tag{3.4}$$

From (3.4) the relation between the contravariant components of the fundamental tensors can be derived as follows

$$g^{*ij} = \sigma^{-1} g^{ij} - (1+\rho^2)\sigma^{-3}(1-b^2-\sigma) l^i l^j - (1+\rho)\sigma^{-2}(l^i b^j + l^j b^i), \tag{3.5}$$

where b is the magnitude of the vector b_i.

From the lemma (2.1) and (3.2) we have

$$\dot{\partial}_i \sigma = \frac{(1+\rho)}{L} m_i, \tag{3.6}$$

$$m_i = b_i - \frac{\beta}{L} l_i. \tag{3.7}$$

Now differentiating (3.3) with respect to y^k (3.2), (3.6), (3.3) and the fact

$$\dot{\partial}_k h_{ij} = 2C_{ijk} - L^{-1}(h_{ik} l_j + h_{jk} l_i),$$

we have

$$C_{ijk}^* = \sigma C_{ijk} + (1+\rho)\frac{h_{ij} m_k + h_{jk} m_i + h_{ki} m_j}{2L}. \tag{3.8}$$

From the definition of m_i, it is evident that

$$\begin{aligned} &(a)\ m_i l^i, && (b)\ m_i b^i = b^2 - \tfrac{\beta^2}{L^2} = m^i m_i, \\ &(c)\ h_{ij} m^i = h_{ij} b^i = m_j, && (d)\ C_{ij}^h m_h = L^{-1}\rho h_{ij}. \end{aligned} \tag{3.9}$$

From (2.1), (3.5), (3.8) and (3.9) we have

$$\begin{aligned} C_{ij}^{*r} \;=\; & C_{ij}^r + \frac{(h_{ij} m^r + h_j^r m_i + h_i^r m_j)}{2L^*} - \frac{1}{L^*}[\{\rho \\ & + \frac{L}{2L^*}(b^2 - \frac{\beta^2}{L^2})\} h_{ij} + \frac{L}{L^*} m_i m_j] l^r. \end{aligned} \tag{3.10}$$

Proposition 3.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then the normalized supporting element l_i^*, angular metric tensor h_{ij}^*, fundamental metric tensor g_{ij}^* and (h)hv-torsion tensor C_{ijk}^* of F^{*n} are given by (3.2), (3.3), (3.4) and (3.8) respectively.*

Proposition 3.2 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then the reciprocal of the fundamental metric tensor g_{ij}^* is given by (3.5).*

The curvature tensor S_{hijk} of (M^n, L^*) is given by

$$S_{hijk}^* = C_{hkm}^* C_{ij}^{*m} - C_{hjm}^* C_{ik}^{*m}. \tag{3.11}$$

From the equation (3.8) and (3.10), we have

$$\begin{aligned} C_{hkm}^* C_{ij}^{*m} &= \sigma C_{hkm} C_{ij}^m + \alpha h_{ij} h_{hk} + \frac{(1+\rho)}{2L}\{C_{ijk} m_h + C_{hjk} m_i \\ &\quad + C_{hik} m_j + C_{hij} m_k\} + \frac{(1+\rho)}{4LL^*}\{2h_{ij} m_k m_h \\ &\quad + 2h_{hk} m_i m_j + h_{ik} m_j m_h + h_{ih} m_j m_k + h_{jk} m_i m_h \\ &\quad + h_{jh} m_i m_k\}, \end{aligned} \tag{3.12}$$

where $\alpha = \frac{(1+\rho)\rho}{4L^2} + \frac{1+\rho}{4LL^*}(b^2 - \frac{\beta^2}{L^2})$. Thus from (3.11) we have

$$S_{hijk}^* = \sigma S_{hijk} + h_{ij} d_{hk} + h_{hk} d_{ij} - h_{ik} d_{jh} - h_{hj} d_{ik}, \tag{3.13}$$

where $d_{ij} = \frac{\sigma}{2} h_{ij} + \frac{1+\rho}{4LL^*} m_i m_j$.

If we define the tensor A_{ij} and B_{ij} as

$$A_{ij} = \frac{h_{ij} + d_{ij}}{\sqrt{2}}, \qquad B_{ij} = \frac{h_{ij} - d_{ij}}{\sqrt{2}}, \tag{3.14}$$

then S_{hijk}^* is written as

$$S_{hijk}^* = \sigma S_{hijk} - (A_{hj} A_{ik} - A_{hk} A_{ij}) + (B_{hj} B_{ik} - B_{hk} B_{ij}). \tag{3.15}$$

Thus we have

Proposition 3.3 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then the curvature tensor S_{hijk}^* is given by (3.15).*

If $|_j$ denotes v-covariant differentiation with respect to y^j in (M^n, L^*) then we have

$$h_{ij}|_k - h_{ik}|_j = \frac{(h_{ij} l_k - h_{ik} l_j)}{L}, \tag{3.16}$$

$$m_i|_j - m_j|_i = \frac{(m_i l_j - m_j l_i)}{L}, \tag{3.17}$$

$$d_{ij}|_k - d_{ik}|_j = \frac{(d_{ij} l_k - d_{ik} l_j)}{L}. \tag{3.18}$$

Hence from (3.14), (3.16) and (3.18), we get

$$A_{ij}|_k - A_{ik}|_j = \frac{(B_{ij} l_k - B_{ik} l_j)}{L}, \tag{3.19}$$

$$B_{ij}|_k - B_{ik}|_j = \frac{(A_{ij} l_k - A_{ik} l_j)}{L}. \tag{3.20}$$

§4. Imbedding Class Numbers of Tangent Riemannian Space to (M^n, L) and (M^n, L^*)

The tangent vector space M_x^n to M^n at every point x is regarded as n-dimensional Riemannian space (M_x^n, g_x) with Riemannian metric $g_x = g_{ij}(x,y)dy^i dy^j$. Thus the component C_{jk}^i of Cartan's C-tensor are the Christoffel symbols associated with g_x, i.e.

$$C_{jk}^i = \frac{1}{2} g^{ih} (\cdot\partial_k g_{jh} + \dot{\partial}_j g_{hk} + \dot{\partial}_h g_{jk}).$$

Hence C_{jk}^i defines the Riemannian connection on M_x^n. It is observed from the definition if S_{hijk} that the curvature tensor of the Riemannian space (M_x^n, g_x) at a point x. The space (M_x^n, g_x) equipped with such a Riemannian connection will be called the tangent Riemannian space.

In the theory of Riemannian space, we know that any n-dimensional Riemannian space V^n, can be imbedded isometrically in a Euclidean space of dimension $\frac{n(n-1)}{2}$. If $n+r$ is the lowest dimension of the Euclidean space in which V^n is imbedded isometrically then the integer r is called imbedding class number of V^n. The fundamental theorem of isometric imbedding [1] states that the tangent Riemannian n-space (M_x^n, g_x) is locally imbedded isometrically in an Euclidean $n+r$ space if and only if there exist r numbers, and $\lambda = \pm 1$, r symmetric tensor $H_{(P)ij}$ and $\frac{r(r-1)}{2}$ covariant vector fields $H_{(PQ)i} = H_{(QP)i}$, $\quad Q = 1, 2, 3, \cdots, r$ satisfying the Gauss equations,

$$S_{hijk} = Sigma \lambda_{(P)} \{H_{(P)hj} H_{(P)ik} - H_{(P)hk} H_{(P)ij}\},$$

where summation is given over P.

The Codazzi equations

$$H_{(P)ij}|_k - H_{(P)ik}|_j = \Sigma \lambda_{(Q)} \{H_{(Q)ij} H_{(QP)k} - H_{(Q)ik} H_{(QP)j}\},$$

where summation is given over Q and Ricci-Kuhne equations

$$\begin{aligned} & H_{(PQ)i}|_j - H_{(PQ)j}|_i + \Sigma \lambda_{(R)} \{H_{(RP)i} H_{(RQ)j} \\ & - H_{(RP)j} H_{(PQ)i}\} + g^{hk} \{H_{(P)hi} H_{(Q)kj} - H_{(P)hj} H_{(Q)ki}\} = 0. \end{aligned}$$

For a special case when (M_x^n, g_x) is of imbedding class 1, the above equations reduce to

$$S_{hijk} = \lambda(H_{hj}H_{ik} - H_{hk}H_{ij}), \tag{4.1}$$

$$H_{ij}|_k - H_{ik}|_j = 0. \tag{4.2}$$

Since $S_{hijk}y^k = 0$, from (3.21), we have

$$H_{hj}H_{i0} - H_{h0}H_{ij} = 0$$

contracting above equation by y^i, we have

$$H_{hj}H_{00} - H_{h0}H_{0j} = 0,$$

which implies that $H_{0j} = 0$ or $H_{ij} = H_{00}^{-1}H_{h0}H_{0j}$. In the latter case we get $S_{hijk} = 0$. In the theory of spaces of imbedding class 1, [17] introduced the concept of type number t, which is the rank of matrix $\parallel H_{ij} \parallel$ provided to the rank is more than 1. If the rank is 0 or 1, then S vanishes. Therefore if (M_x^n, g_x) is of imbedding class 1, the second fundamental tensor H_{ij} satisfies $H_{ij}y^j = 0$ and thus the type number t is less than n.

Again by virtue of Lemma 2.3 and equation (4.1), we get

$$H_{hj}H_{ik} - H_{hk}H_{ij}b^h = 0.$$

From this equation we have

$$H_{hj}b^hb^jH_{ik} - H_{hk}b^hH_{ij}b^j = 0.$$

This gives

$$H_{hk}b^h = 0, \quad or \quad H_{ik} = \frac{H_{hk}b^hH_{ij}b^j}{H_{hj}b^hb^j}.$$

In the latter case $S_{hijk} = 0$. Thus for an imbedding class 1, $H_{hk}b^k = 0$. Now we shall put

$$H^*_{(1)ij} = \sqrt{\sigma}H_{ij}, \qquad \varepsilon^*_1 = \varepsilon, \tag{4.3}$$

$$H^*_{(2)ij} = A_{ij}, \qquad \varepsilon^*_2 = -1, \qquad 4.4$$

$$H^*_{(3)ij} = B_{ij}, \qquad \varepsilon^*_3 = 1, \qquad 4.5$$

then from (3.15) and (4.1), we get

$$S^*_{hijk} = \Sigma\lambda^*_P\{H^*_{(P)hj}H^*_{(P)ik} - H^*_{(P)hh}H^*_{(P)ij}\},$$

where summation is varies from $P = 1, 2, 3$. Thus the above equation is noting but Gauss equation of (M_x^n, g_x^*).

Now we put

$$H^*_{(21)i} = -H^*_{(12)i} = 0, \tag{4.6}$$

$$H^*_{(31)i} = -H^*_{(13)i} = 0, \tag{4.7}$$

$$H^*_{(32)i} = -H^*_{(23)i} = \frac{1}{L} l_i \tag{4.8}$$

and using (4.2), (4.3), (3.3), Lemma 2.1 and the fat that $H_{i0} = 0$, we get

$$H^*_{(1)ij}|_k - H^*_{(1)ik}|_j = 0. \tag{4.9}$$

Again in view of (4.4), (4.5), (4.6), (4.7) and (4.8), equations (3.19) and (3.20) reduce to

$$H^*_{(2)ij}|_k - H^*_{(2)ik}|_j = \Sigma\lambda^*_Q\{H^*_{(Q)ij}H^*_{(Q2)k} - H^*_{(Q)ik}H^*_{(Q2)j}\}, \quad 4.10$$

$$H^*_{(3)ij}|_k - H^*_{(3)ik}|_j = \Sigma\lambda^*_Q\{H^*_{(Q)ij}H^*_{(Q3)k} - H^*_{(Q)ik}H^*_{(Q3)j}\}, \quad 4.11$$

where summation is varies from $Q = 1, 2, 3$.

The equations (4.9), (4.10) and (4.11) are the Codazzi equations of (M^n_x, g^*_x). Now we have to verify Ricci-Kuhne equations, we have from (3.10),

$$l_i|_j = L^{-1}h_{ij+L^{*-1}}[\{\rho + (2L^*)^{-1}(v^2 - \frac{\beta^2}{L^2})\}h_{ij} + L^{*-1}m_i m_j]$$

from which we get $l_i|_j - l_j|_i = 0$. Hence from (4.10), we get

$$H^*_{(32)i}|_j - H^*_{(23)j}|_i = 0,$$

which are the Ricci-Kuhne equations of (M^n_x, g^*_x) as

$$M^*_{(12)} - M^*_{(21)} = 0, \quad and \quad M^*_{(13)} - M^*_{(31)} = 0.$$

Thus from above we have

Theorem 4.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then if the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L) is of imbedding class 1, then the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L^*) is at most of imbedding class 3.*

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