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Pure Edge-Neighbor-Integrity of Graphs

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Abstract: In a communication network, several vulnerability measures are used to determine the resistance of the network to disruption of operation after the failure of certain stations or communication links. This study introduces a new vulnerability parameter, pure edge-neighbor-integrity of graphs. The pure edge-neighbor-integrity of a graph G is defined to be $PENI(G) = \min_{\Re \subseteq E(G)}\{|\Re| + \varpi_e(G/\Re)\}$, where $\Re$ is any edge subversion strategy of G and $\varpi_e(G/\Re)$ is the number of edges in the largest component of $G/\Re$. A set $\Re \subseteq E(G)$, is said to be a $PENI$-set of G if $PENI(G) = |\Re| + \varpi_e(G/\Re)$. In this paper, several properties and bounds on the $PENI$ are presented over here and the relation between $PENI$ with other parameters is investigated. The $PENI$ of some classes of graphs is also computed.

Key Words: Vulnerability, integrity, neighbor-integrity, edge-neighbor-integrity.

AMS(2010): 05C40, 05C99, 05C76.

§1. Introduction

Networks appear in many different applications and settings. The most common networks are telecommunication networks, computer networks, the internet, road and rail networks and other logistic networks. In all applications, vulnerability and reliability are crucial and important features. Network designers often build a network configuration around specific processing, performance and cost requirements. But there is little consideration given to the stability of the networks communication structure when under the pressure of link or node loses. This lack of consideration makes the networks have low survivability. Therefore, network design process must identify the critical points of failure and be able to modify the design to eliminate them [18].

A network can be modeled by a graph whose vertices represent the stations and whose edges represent the communication lines. Vulnerability measures the resistivity of the network to the disruption of its operation due to the failure of certain stations or communication links. Losing links or nodes eventually lead to a loss of the effectiveness of the network. Communication networks must be constructed so as to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. Many graph

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theoretical parameters have been used in the past to describe the stability of communication networks, including connectivity, integrity, toughness and binding number. However, these parameters do not take into account the effect that the removal of a vertex has on the neighbors of that vertex. If a station is destroyed, the adjacent stations are betrayed and become useless to the network as a whole. The neighbor integrity is a measure of the vulnerability of graphs to the disruption caused by the consecutive removal of a vertex and all of its adjacent vertices [8, 9, 10, 15] a probabilistic basis. However, sometimes it is important to take subjective reliability estimates into consideration. Among the relevant issue of importance, we are particularly interested in one of the vulnerabilities. That is, in an unfriendly external environment, how vulnerable is such a distributed system to certain external destruction and how much computing power can be sustained in the face of destruction.

The concept of network vulnerability is motivated by the design and analysis of networks under a hostile environment. Several graph theoretic models under various assumptions have been proposed for the study and assessment of network vulnerability. Graph integrity, introduced by Barefoot et al. [4, 5], is one of these models that has received wide attention [2, 11].

In 1994, Margaret B. Cozzens and Wu [7] introduced a new graph parameter called the edge-neighbor-integrity. They consider the edge analogue of (vertex)neighbor-integrity a measure of the vulnerability of graphs to disruption caused by the removal of edges, their incident vertices, and all of their incident edges. The integrity of a graph $G = (V, E)$, which was introduced as a useful measure of the vulnerability of the graph, is defined as follows: $I(G) = \min\{|S| + m(G-S) : S \subseteq V(G)\}$, where $m(G-S)$ denotes the order of the largest component. Barefoot, Entringer and Swart defined the edge-integrity of a graph G with edge set $E(G)$ by $I'(G) = \min\{|S| + m(G-S) : S \subseteq E(G)\}$. The weak integrity was introduced by Kirlangic [14] and is defined as $I_w(G) = \min\{|S| + m_e(G-S) : S \subseteq V(G)\}$, where $m_e(G-S)$ denotes the number of edges in a largest component of $G-S$. Let u be a vertex in G. $N(u) = \{v \in V(G)|u \neq v, v \text{ and } u \text{ are adjacent}\}$ is the open neighbourhood of u, and $N[u] = \{u\} \cup N(u)$ denotes the closed neighborhood of u. A vertex u in G is said to be subverted if the closed neighborhood $N(u)$ is deleted from G. A set of vertices $S = \{u_1, u_2, \cdots, u_n\}$ is called a vertex subversion strategy of G if each of the vertices in S has been subverted from G. Let G/S be the survival-subgraph when S has been a vertex subversion strategy of G. The closed neighborhood of a vertex subset $S, N[S]$, is $\cup_{u\in S} N[u]$. Hence $G/S = G - N[S] = G - (\cup_{u\in S} N[u])$. The vertex-neighbor-integrity of a graph $G, VNI(G)$, is defined to be $VNI(G) = min_{S\subseteq V(G)}\{|S| + \omega(G/S)\}$, where S is any vertex subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S. The edge $e = (v, w)$ in G is said to be subverted if the edge e, all of its incident edges, and the two ends of e, namely v and w, are removed from G. (For simplicity, an edge $e = (v, w)$ is subverted if the two ends of the edge e, namely v and w, are deleted from G.) A set of edges $\Re = \{e_1, e_2, \cdots, e_n\}$ is called an edge subversion strategy of G if each of the edges in $\Re$ has been subverted from G. Let $G/\Re$ be the survival-subgraph when $\Re$ has been an edge subversion strategy of G. The edge-neighbor-integrity of a graph G, is defined to be $ENI(G) = min_{\Re\subseteq E(G)}\{|\Re| + \varpi(G/\Re)\}$, where $\Re$ is any edge subversion strategy of G, and $\varpi(G/\Re)$ is the maximum order of the components of $G/\Re$. We now introduce a new measure of stability of a graph G in this sense and it is called pure edge-neighbor-integrity. Formally, the pure edge-neighbor-integrity $PENI(G)$ of a graph G is defined as $PENI(G) = \min_{\Re \subseteq E(G)}\{|\Re| + \varpi_e(G/\Re)\}$, where $\Re$ is any edge subversion strategy of G and $\varpi_e(G/\Re)$ is the number of edges of a largest component of $G/\Re$. Any set $\Re$ with property that $PENI(G) = |\Re| + \varpi_e(G/\Re)$ is called a $PENI-$ set of G. $\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges, with vertex set $V(G) = \{v_1, v_2, \cdots, v_p\}$. The distance between the vertices v_i and v_j is the length of the shortest path joining v_i and v_j. The shortest v_iv_j path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic, denoted by $diam(G)$. The order and size of G are denoted by p and q, respectively. We use Bondy and Murty [6, 12] for terminology and notations not defined here. In general, the degree of a vertex v in a graph G is the number of edges of G incident with v and it is denoted by $degv$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G), (\delta(G))$. We denote the minimum number of edges in edge cover of G (i.e., edge cover number) by $\alpha_1(G)$ and the minimum number of edges in independent set of edges of G (i.e., edge independence number) by $\beta_1(G)$. A vertex of degree one is called a pendant vertex. The symbols $\alpha(G)$, $\kappa(G)$, $\lambda(G)$, and $\beta(G)$ denote the vertex cover number, the connectivity, the edge-connectivity, and the independence number of G, respectively.

A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G [16].

The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G [12]. In the present work, the basic properties of pure edge-neighbor-integrity and of $PENI-$sets are explored, bounds and relationship between pure edge-neighbor-integrity and other graphical parameters are considered. Finally, the pure edge-neighbor-integrity of binary operations of some graphs are determined. We need the following to prove main results.

Lemma 1.1([13]) *If $D \subseteq E(G)$, then $L(G - D) = L(G) - D$.*

Theorem 1.1([14]) *If a graph G of order n is isomorphic to a cycle graph or a tree, then $I_w(G) = I(G) - 1$.*

Theorem 1.2([12]) *For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Lemma 1.2 *For any graph G, $\beta_1(G) \leq \alpha(G)$.*

Theorem 1.3([1]) *For any connected graph G of even order p, $\gamma' = \frac{p}{2}$ if and only if G is isomorphic to K_p or $K_{\frac{p}{2},\frac{p}{2}}$.*

Theorem 1.4([2]) *The integrity of*

(a) *the complete graph K_p is p;*

(b) *the complete bipartite graph $K_{m,n}$ is $1 + min\{m, n\}$.*

§2. Main results

Proposition 2.1 *(a) For any complete graph K_p, $PENI(K_p) = \lfloor \frac{p}{2} \rfloor$;*

(b) For any path P_p with $p \geq 3$, $PENI(P_p) = \lceil 2\sqrt{p+2} \rceil - 4$;

(c) For any cycle C_p,

$$PENI(C_p) = \begin{cases} 1, & \text{if } p = 3 \ ; \\ 2, & \text{if } p = 4 \ ; \\ \lceil 2\sqrt{p} \rceil - 3, & \text{if } p \geq 5. \end{cases}$$

(d) For the star $K_{1,p-1}$, $PENI(K_{1,p-1}) = 1$;

(e) For the double star $S_{n,m}$, $PENI(S_{n,m}) = 1$;

(f) For the complete bipartite graph $K_{n,m}$, $PENI(K_{n,m}) = \min\{n, m\}$;

(g) For the wheel graph $W_{1,p-1}$, $p \geq 5$, $PENI(W_{1,p-1}) = \lceil 2\sqrt{p} \rceil - 3$.

Remark 2.1 (1) If H is a subgraph of G, then $PENI(H) \leq PENI(G)$;

(2) Pure edge-neighbor integrity of a connected graph for $p \geq 2$, takes its minimum value at $K_{1,p-1}$ and its maximum value at K_p complete graph;

(3) $0 \leq PENI(G) \leq q$.

Lemma 2.1 *If G is a non-trivial graph, then for all $v \in V(G)$, $PENI(G-v) \geq PENI(G)-1$, the bound is sharp for $G = K_4$.*

Proposition 2.2 *(a) If G has enough components close in size to the largest one, then $PENI(G) = \varpi_e(G)$. In particular, if $G = pH$ with $p \geq \varpi_e(H)$, then $PENI(G) = \varpi_e(H)$;*

(b) Suppose that G is disconnected and $m(G) = k$, if G has at least $k-1$ components of order k, then empty set is an $PENI(G)$-set of G.

Lemma 2.2 *If $\Re$ is $PENI$-set of G, then $\varpi_e(G/\Re) = PENI(G/\Re)$ and ϕ is $PENI$-set of $G/\Re$.*

Proof Let $\Re$ is $PENI$-set of G and $\Re^\star$ be $PENI$-set of $G/\Re$. Thus

$$\begin{aligned} |\Re| + \varpi_e(G/\Re) &= PENI(G) \\ &\leq \varpi_e(G/(\Re \cup \Re^\star)) + |\Re \cup \Re^\star| \\ &= |\Re| + \varpi_e[(G/\Re)/\Re^\star] + |\Re^\star| \\ &= |\Re| + PENI(G/\Re). \end{aligned}$$

So, $\varpi_e(G/\Re) \leq PENI(G/\Re)$, but $\varpi_e(G/\Re) \geq PENI(G/\Re)$. This completes the proof. □

Lemma 2.3 *If $D \subseteq E(G)$,$PENI(L(G-D)) = PENI(L(G)-D))$.*

Proof The proof follows by Lemma 1.1. □

Theorem 2.1 *If G is a simple graph such that $\overline{G} \cong L(G)$, then $PENI(G) = PENI(L(G)) =$*

$PENI(\overline{G})$ if and only if $G = C_5$ or G is the graph shown in the Figure 1.

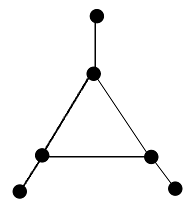

Figure 1 G

Proposition 2.3 *If a connected graph G is isomorphic to its line graph, then $PENI(G) = PENI(L(G))$. But the converse is not true, for example the graph G is given in the following Figure 2.*

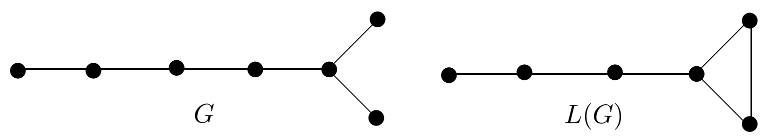

Figure 2 G and $L(G)$

Notice that $PENI(G) = 2 = PENI(L(G))$, but G and $L(G)$ are not isomorphic.

Lemma 2.4 *Let G be a connected graph of order at least 3. If $PENI(G) = 1$, then the diameter of G is ≤ 3.*

Proof The diameter of G is ≥ 4 is Supposed, then G contains a path P_5. Hence for any edge e in $G, \varpi_e(G/e) \geq 1$, and for any two edges e_1 and e_2 in $G, \varpi_e(G/\{e_1, e_2\}) \geq 0$. Thus $EENI(G) \geq 2$, a contradiction. Hence, the diameter of G is ≤ 3. □

Lemma 2.5 *For any a graph G, $PENI(G) = VNI(L(G))$.*

Proof Since every edge dominating set in G is a dominating set in the line graph of G, the set of edges S that satisfies $PENI(G)$ equal to the set of vertices S that satisfies $VNI(L(G))$, this completes the proof. □

Lemma 2.6 *For any (p, q) graph G, $\lceil \frac{q}{\Delta'+1} \rceil \leq PENI(G) \leq q - \beta_1$, where Δ' denotes the maximum degree of an edge in G.*

Observation 2.1 For any connected graph G, $PENI(G) = q - \beta_1$ if and only if $G \cong P_p, 3 \leq p \leq 6$, $G \cong p_8$, $G \cong C_4$ or G in the Figure 3.

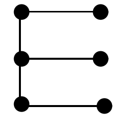

Figure 3 G

Corollary 2.1 *For any connected (p, q) graph, $PENI(G) = p - q$ if and only if G is isomorphic to $K_{1,p-1}$ or $S_{n,m}$.*

Observation 2.2 Let G be a graph, and let $\Re$ be $PENI$-set of G such that $|\Re| = 1$, then the following hold

(a) $PENI(G) = 1$;
(b) $|E - \Re| = \sum_{e \in \Re} deg(e)$;
(c) $\Delta^{'}(G) = q - 1$.

Corollary 2.2 *For any connected graph G of even order p, $PENI(G) = \frac{p}{2}$ if and only if G is isomorphic to K_p or $K_{\frac{p}{2},\frac{p}{2}}$.*

Theorem 2.2 *For any integer $n \geq 1$, there does not exist any graph G satisfy $PENI(G) = I(G) = \gamma'(G) = n$.*

Proof Let G be a graph of order p. By Theorem 1.3 and Corollary 2.2, $PENI(G) = \frac{p}{2} = \gamma'(G)$ if p is even and $G \cong K_p$ or $G \cong K_{\frac{p}{2},\frac{p}{2}}$, but from Theorem 1.4, $I(K_p) = p$, and $I(K_{\frac{p}{2},\frac{p}{2}}) = \frac{p}{2} + 1$. Hence the result. □

Theorem 2.3 *For any integer $k \geq 1$, there exists a graph G of size $q \geq k$ with $PENI(G) = \gamma(G) = k$, where $\gamma(G)$ is domination number.*

Proof The result is true for $k = 1, 2$, since $G_1 = K_2, G_2 = K_3$ have the desired property. For $k \geq 3$, consider the graph G_k which is obtained from k disjoint copies of the complete graph K_3 and joining the vertex v_i in the i^{th} copy with the vertex v_{i+1} in the $(i+1)^{th}$ copy, and joining the vertex u_i in the i^{th} copy with the vertex w_i in the $(i+1)^{th}$ copy. The graph G_3 shown in Figure 4.

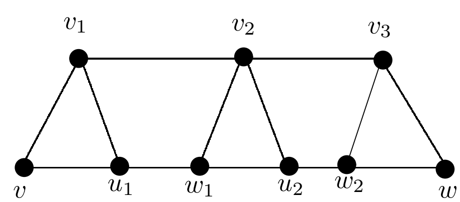

Figure 4 G_3

Consider $D = \{v_1, v_2, v_3, \cdots, v_k\}$ be a dominating set for G_k, and $|D| = k$. Let's claim that set D is a minimum dominating set. Since each $v_i, 2 \leq i \leq k-1$, is adjacent to w_{i-1} and u_i. If v_i is removed from set D, then w_{i-1} and u_i will not be dominated by any vertex. Hence D is a minimum domination set. Therefore, $\gamma(G_k) = k$. Consider $\Re = \{(v_1, v), (v_2, w_1), (v_3, w_2), \cdots, (v_i, w_{i-1}), 1 \leq i \leq k\}$. Then $|\Re| = k$, and $\varpi_e(G_k/\Re) = 0$. Therefore, $PENI(G_k) \leq |\Re| + \varpi_e(G_k/\Re) = k$. Consider $\Re_1 = \{(v_1, v), (v_2, w_1), (v_3, w_2), ..., (v_{i-1}, w_{i-2}), 1 \leq i \leq k\}$. Then $|\Re_1| = k - 1$, and $\varpi_e(G_k/\Re_1) = 4$, this implies that $|\Re_1| + \varpi_e(G_k/\Re_1) > |\Re| + \varpi_e(G_k/\Re)$. If $\varpi_e(G_k/\Re) = 1$, then $|\Re| \geq k$. Thus, $PENI(G_k) \geq k + 1$. Therefore, $PENI(G_k) = k$. □

Corollary 2.3 *For every integer $n \geq 1$, there exists graph G with $PENI(G) = n$.*

Lemma 2.7 *Let G be a graph of order p, $PENI(G) = 0$ if and only if $G \cong \overline{K}_p$.*

Theorem 2.4 *For any graph G of order p, $PENI(G) \leq I_w(G) \leq I'(G)$.*

Proof Clearly, $I_w(G) \leq I'(G)$. If G is complete, then $PENI(G) = \lfloor \frac{p}{2} \rfloor \leq |p-1| = I_w(G)$. G is non-complete is supposed and $S' = \{u_1, u_2, \cdots, u_p\}$ be an I_w-set of G. Then S' is a vertex cut-set of G, and u_i, where $1 \leq i \leq p$, is not an isolated vertex of G. Let $\Re = \{(u_i, v_i) \in E(G)/$ for some vertex $v_i \in V, u_i \in S',\ where\ i = 1, 2, \cdots, p\}$ thus $|\Re| = |S'| = p$. Therefore,

$$G/\Re = G - \{u_1, u_2, \cdots, u_p, v_1, v_2, \cdots, v_p\}$$
$$= G - (S' \cup \{v_i \in V(G)/(u_i, v_i) \in \Re, u_i \in S'\}) \subseteq G - S',$$

it follows that $\varpi_e(G/\Re) \leq m_e(G - S')$, then

$$PENI(G) \leq |\Re| + \varpi_e(G/\Re)$$
$$\leq |S'| + m_e(G - S') = I_w(G).$$ □

Observation 2.3 If $PENI(G) = I_w(G)$, then the induced subgraph of G, $< S >$ must be a null graph, where S is an I_w-set of G. But the converse is not true, for example in the graph in Figure 5. $S = \{u_1, u_2, u_3\}$ is an I_w-set of G is noted. Therefore, $I_w(G) = 4$ and $\Re = \{e_1, e_2\}$ is a $PENI$- set of G. Thus $PENI(G) = 2$. $< S >$ is a null graph, but $I_w(G) \neq PENI(G)$.

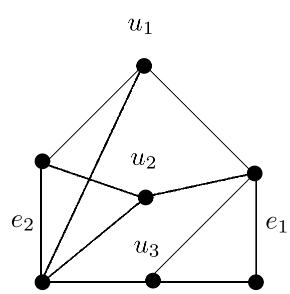

Figure 5

Lemma 2.8 *If diam$(L(G)) = 1$, then $PENI(G) = 1$.*

Proof Since diam$(L(G)) = 1$, then G is either K_3 or $K_{1,p-1}$. Hence the result. □

Remark 2.2 If G is a graph with $\alpha(G) = 1$, $PENI(L(G)) = \lfloor \frac{p}{2} \rfloor$.

Theorem 2.5 *For any graph G, $VNI(G) \leq PENI(G) + 1$.*

Proof Let $\Re = \{(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)\}$ be a $PENI$-set of G, let S be a set of one end vertex of each edge in $\Re$. Thus $|S| \leq |\Re|$ and $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\} \subseteq N[S]$. Therefore $G/S = G - N[S] \subseteq G - \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\} = G/\Re$, and $|S| + \omega(G/S) \leq |\Re| + \omega(G/\Re) \leq |\Re| + \varpi_e(G/\Re) + 1 = PENI(G) + 1$. So $VNI(G) = min_{S \subseteq V(G)}\{|S| + \omega(G/S)\} \leq |S'| + \omega(G/S') \leq PENI(G) + 1$. □

Theorem 2.6 *For any graph G, $PENI(G) \geq ENI(G) - 1$.*

Proof Let $\Re$ be a $PENI$-set of G. Since $ENI(G) \leq |\Re| + \varpi(G/\Re)$ and $\varpi(G/\Re) \leq \varpi_e(G/\Re) + 1$, for every $\Re \subseteq E(G)$, hence the result. □

Theorem 2.7 *For any graph G and $e \in E(G)$, $PENI(G-e) \geq PENI(G) - 1$.*

Proof Let $\Re^\star$ be a $PENI$-set of $G - e$, and $PENI(G-e) = |\Re^\star| + \varpi_e((G-e)/\Re^\star)$, let $\Re^{\star\star} = \Re^\star \cup \{e\}$. Then $|\Re^{\star\star|} = |\Re^\star| + 1$. Then $\Re^{\star\star}$ is $PENI$-set of G and $\varpi_e(G/\Re^{\star\star}) = \varpi_e(G/e)/\Re^\star)$ Therefore,

$$\begin{aligned} PENI(G) &\leq |\Re^{\star\star}| + \varpi_e(G/\Re^{\star\star}) \\ &\leq |\Re^\star| + \varpi_e[(G/e)/\Re^\star] + 1 \\ &= PENI(G-e) + 1. \end{aligned}$$

Then $PENI(G-e) \geq PENI(G) - 1$. □

Theorem 2.8 *For any graph G, $PENI(G) \leq \alpha_1(G)$.*

Proof Let $\Re$ be $PENI$-set of G such that $PENI(G) = |\Re| + \varpi_e(G/\Re)$ and let C be a minimum edge covering of G. Since each vertex of G is an end vertex of some edge in C, we have $G/C = \phi$ and $\varpi_e(G/C) = 0$.

Thus

$$\begin{aligned} PENI(G) &= |\Re| + \varpi_e(G/\Re) \\ &\leq |C| + \varpi_e(G/C) = |C| = \alpha_1(G). \end{aligned}$$ □

Theorem 2.9 *For any graph G, $PENI(G) \leq \beta_1(G)$.*

Proof Let $\Re$ be $PENI$-set of G such that $PENI(G) = |\Re| + \varpi_e(G/\Re)$ and let M be a maximum matching in G. It is clear $G/M = \phi$ or a set of isolated vertices, hence $\varpi_e(G/M) = 0$. Then

$$\begin{aligned} PENI(G) &= |\Re| + \varpi_e(G/\Re) \\ &\leq |M| + \varpi_e(G/M) = |M| = \beta_1(G). \end{aligned}$$ □

Theorem 2.10 *For any graph G, $PENI(G) \leq \alpha(G)$.*

Proof The proof follows from Lemma 1.2 and Theorem 2.9. □

Theorem 2.11 *For any tree T, $PENI(T) \geq \delta(T)$.*

Proof Let $\Re$ be a $PENI$-set of T such that $PENI(T) = |\Re| + \varpi_e(T/\Re)$. Then $\varpi_e(T/\Re) \geq \delta(T/\Re) \geq \delta(T) - |\Re|$, So, $PENI(T) = |\Re| + \varpi_e(T/\Re) \geq |\Re| + \delta(T) - |\Re| = \delta(T)$. □

Lemma 2.9 *For any tree T, $PENI(T) \geq \lambda(T)$.*

Proof The proof follows from Theorems 2.11 and 1.2. □

Lemma 2.10 *For any tree T, $PENI(T) \geq \kappa(T)$.*

Proof The proof follows from Lemma 2.9 and Theorem 1.2. □

Notice that $\alpha_1(G), \beta_1(G)$ and $\alpha(G)$ are upper bounds of $PENI(G)$, while $\delta(G), \lambda(G)$ and $\kappa(G)$ are lower bounds of $PENI(G)$.
However, the independence number β, has no such relationship with $PENI(G)$. For example,

(1) $PENI(K_{1,n}) < \beta(K_{1,n})$;

(2) $PENI(K_p) > \beta(K_p)$;

(3) $PENI(K_{n,m}) = \begin{cases} n = m = \beta(K_{n,m}), \text{ if } n = m\ ; \\ min\{n, m\} < \beta(K_{n,m}), \text{ if } n \neq m. \end{cases}$

Corollary 2.4 *For any graph G, $PENI(G) \leq \lfloor \frac{p}{2} \rfloor$.*

Proof Let M be a maximum matching of G. Then $|M| = \beta_1(G) \leq \lfloor \frac{p}{2} \rfloor$. Two cases are discussed.

Case 1. If $\beta_1(G) = \lfloor \frac{p}{2} \rfloor$, then $G/M = \phi$ (if p is even) or a single vertex (if p is odd), hence $PENI(G) \leq |M| + \varpi_e(G/M) = \lfloor \frac{p}{2} \rfloor$.

Case 2. If $\beta_1(G) < \lfloor \frac{p}{2} \rfloor$, then by Theorem 2.9, we have $PENI(G) \leq \beta_1(G) < \lfloor \frac{p}{2} \rfloor$. □

Theorem 2.12 *For any graph G, $PENI(G) \geq \lceil \frac{I(G)}{2} \rceil - 1$.*

Proof Let $\Re = \{(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)\}$ be a $PENI$-set of G. So $PENI(G) = |\Re| + \varpi_e(G/\Re) = n + \varpi_e(G/\Re)$.

Let $S^\star = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. Since $\Re$ may not be edge independent in G, $|S^\star| \leq 2n$. Then

$$\begin{aligned} I(G) &= min_{S \subseteq V(G)}\{|S| + m(G - S)\} \\ &\leq |S^\star| + m(G - S^\star) \leq 2n + \varpi_e(G/\Re) + 1 \\ &\leq 2(n + \varpi_e(G/\Re)) + 1 = 2PENI(G) + 1. \end{aligned}$$

Therefore, $PENI(G) \geq \lceil \frac{I(G)}{2} \rceil - 1$. □

Corollary 2.5 *For any graph G, $PENI(G) \geq \lceil \frac{I_w(G)}{2} \rceil$.*

§3. Pure-Edge Neighbor Integrity of Some Graph Operators

Definition 3.1([12]) *The (Cartesian)product $G \times H$ of graphs G and H has $V(G) \times V(H)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.*

Theorem 3.1 *For a graph* $K_2 \times P_p$,

$$PENI(K_2 \times P_p) = \begin{cases} \frac{p}{2} + 1, & p \text{ is even;} \\ \frac{p-1}{2} + 1, & p \text{ is odd.} \end{cases}$$

Proof The number of vertices of graph $K_2 \times P_p$ is $2p$ and the number of edges is $3p-2$. The graph $K_2 \times P_p$ is shown in Figure 6, we have two cases.

Case 1. p is even. Consider $\Re = \{e_{2+2j}, 0 \le j < \frac{p}{2}\}$, $|\Re| = \frac{p}{2}$, and $\varpi_e((K_2 \times P_p)/\Re) = 1$. Therefore,

$$PENI(K_2 \times P_p) \le |\Re| + \varpi_e((K_2 \times P_p)/\Re) = \frac{p}{2} + 1. \tag{1}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times P_p)/\Re) = 0$, then $|\Re| \ge p-1$. So

$$PENI(K_2 \times P_p) \ge p-1. \tag{2}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times P_p)/\Re) \ge 2$, then $|\Re| \ge \frac{p}{2}, p > 2$. Thus

$$PENI(K_2 \times P_p) \ge \frac{p}{2} + 2. \tag{3}$$

Therefore, the inequalities (1), (2) and (3) lead to $PENI(K_2 \times P_p) = \frac{p}{2} + 1$.

Case 2. p is odd. Consider $\Re = \{e_{2+2j}, 0 \le j < \frac{p-1}{2}\}$, $|\Re| = \frac{p-1}{2}$, and $\varpi_e((K_2 \times P_p)/\Re) = 1$. Therefore,

$$PENI(K_2 \times P_p) \le |\Re| + \varpi_e((K_2 \times P_p)/\Re) = \frac{p-1}{2} + 1. \tag{4}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times P_p)/\Re) = 0$, then $|\Re| \ge p-1$. So

$$PENI(K_2 \times P_p) \ge p-1. \tag{5}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times P_p)/\Re) \ge 2$, then $|\Re| \ge \frac{p-1}{2} + 1$. Thus

$$PENI(K_2 \times P_p) > \frac{p-1}{2} + 1. \tag{6}$$

Therefore, these inequalities (4), (5) and (6) lead to $PENI(K_2 \times P_p) = \frac{p-1}{2} + 1$. □

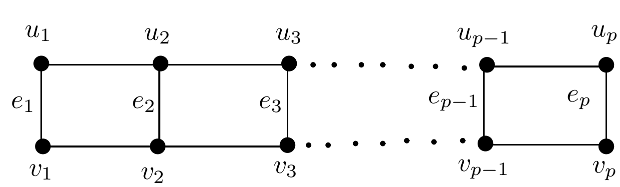

Figure 6 $K_2 \times P_p$

Theorem 3.2 *For a graph $K_2 \times C_p$,*

$$PENI(K_2 \times C_p) = \begin{cases} \frac{p}{2} + 1, & p \text{ is even and } p > 2; \\ \frac{p+1}{2} + 1, & p \text{ is odd and } p \geq 3. \end{cases}$$

Proof The number of vertices of graph $K_2 \times C_p$ is $2p$ and the number of edges is $3p$. The graph $K_2 \times C_p$ is shown in Figure 7, two cases are considered.

Case 1. p is even. Consider $\Re = \{e_{1+2j}, 0 \leq j < \frac{p}{2}\}$, $|\Re| = \frac{p}{2}$, and $\varpi_e((K_2 \times C_p)/\Re) = 1$. Therefore,

$$PENI(K_2 \times C_p) \leq |\Re| + \varpi_e((K_2 \times C_p)/\Re) = \frac{p}{2} + 1. \tag{7}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times C_p)/\Re) = 0$, then $|\Re| \geq p - 1$. So

$$PENI(K_2 \times C_p) \geq p - 1. \tag{8}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times C_p)/\Re) \geq 2$, then $|\Re| \geq \frac{p}{2} + 1$. Thus

$$PENI(K_2 \times C_p) \geq \frac{p}{2} + 3. \tag{9}$$

Therefore, these inequalities (7), (8) and (9) lead to

$$PENI(K_2 \times C_p) = \frac{p}{2} + 1.$$

Case 2. (i) p is odd, $p = 3$. Consider $S = \{e_1, e_2\}, |S| = 2$, and $\varpi_e((K_2 \times C_p)/\Re)) = 1$. Thus,

$$PENI(K_2 \times C_p) \leq |\Re| + \varpi_e((K_2 \times C_p)/\Re) = 3.$$

(ii) $p > 3$, Consider $\Re = \{e_{1+2j}, 0 \leq j < \frac{p+1}{2}\}$, $|\Re| = \frac{p+1}{2}$ and $\varpi_e((K_2 \times P_p)/\Re) = 1$. Therefore,

$$PENI(K_2 \times C_p) \leq |\Re| + \varpi_e((K_2 \times C_p)/\Re) = \frac{p+1}{2} + 1. \tag{10}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times C_p)/\Re) = 0$, then $|\Re| \geq p - 1$. So

$$PENI(K_2 \times C_p) \geq p - 1. \tag{11}$$

If $\Re$ is set of any edges such that $\varpi_e((K_2 \times C_p)/\Re) \geq 2$, then $|\Re| \geq \frac{p+1}{2}$. Thus

$$PENI(K_2 \times C_p) \geq \frac{p+1}{2} + 2. \tag{12}$$

Therefore, these inequalities (10), (11) and (12) lead to

$$PENI(K_2 \times C_p) = \frac{p+1}{2} + 1.$$

□

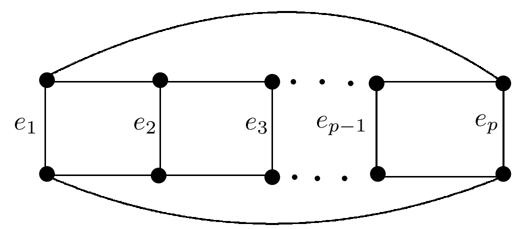

Figure 7 $K_2 \times C_p$

Theorem 3.3 *$PENI(K_2 \times K_{1,p-1}) = 2$.*

Proof The number of vertices of graph $K_2 \times K_{1,p-1}$ is $2p$. The set $\Re = \{e\}$ as shown in Figure 8 is chosen. If we remove the edge e, $p-1$ components such that $\varpi_e((K_2 \times K_{1,p-1})/\Re) = 1$, thus $|\Re| + \varpi_e((K_2 \times K_{1,p-1})/\Re) = 2$. Therefore, $PENI(K_2 \times K_{1,p-1}) = 2$. If $\Re$ is set of any edges such that $\varpi_e((K_2 \times K_{1,p-1})/\Re) = 0$, then $|\Re| \geq p-1$. So $PENI(K_2 \times K_{1,p-1}) \geq p-1$.

If $\varpi((K_2 \times K_{1,p-1})/\Re) \geq 2$, then trivially $|\Re| + \varpi_e((K_2 \times K_{1,p-1})/\Re) > 2$. Thus $HI(K_2 \times K_{1,p-1}) = 2$. □

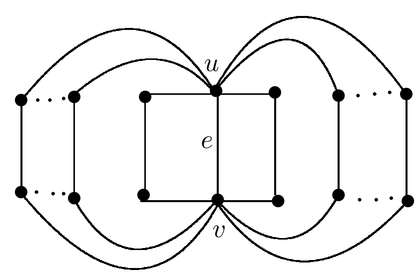

Figure 8 $K_2 \times K_{1,p-1}$

Definition 3.2([12]) *For a simple connected graph G the square of G denoted by G^2, is defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 in G.*

Theorem 3.4 *For a graph P_p^2,*

$$PENI(P_p^2) = \begin{cases} \frac{p}{3}, & \text{if } p \equiv 0(mod\ 3), \\ \frac{p-1}{3}, & \text{if } p \equiv 1(mod\ 3), \\ \frac{p-2}{3} + 1, & \text{if } p \equiv 2(mod\ 3). \end{cases}$$

Proof Let $V(P_p) = \{v_1, v_2, \cdots, v_p\}$. Then, $|V(P_p^2)| = p$ and $|E(P_p^2)| = 2p-3$. The graph P_5^2 is shown in Figure 9.

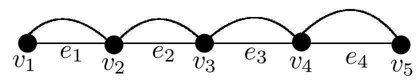

Figure 9 P_5^2

An edge set $\Re$ of P_p^2 as below is considered.

(1) If $p \equiv 0(mod\ 3)$, then $p = 3k$ for some integer $k \geq 1$. Consider

$$\Re = \{e_{2+3i}/0 \leq i \leq k-1\} \ and \ |\Re| = k.$$

We have, $|\Re| = \frac{p}{3}$, and $\varpi_e(P_p^2/\Re) = 0$;

(2) If $p \equiv 1(mod\ 3)$, then $p = 3k+1$ for some integer $k \geq 1$. Consider

$$\Re = \{e_{2+3i}/0 \leq i \leq k-1\} \ and \ |\Re| = k.$$

We have, $|\Re| = \frac{p-1}{3}$, and $\varpi_e(P_p^2/\Re) = 0$;

(3) If $p \equiv 2(mod\ 3)$ then, $p = 3k-1$ for some integer $k \geq 1$. Consider

$$\Re = \{e_{1+3i}/0 \leq i \leq k-1\} \ and \ |\Re| = k.$$

We have, $|\Re| = \frac{p-2}{3} + 1$ and $\varpi_e(P_p^2/\Re) = 0$.

To discuss the minimality of $|\Re| + \varpi_e(P_p^2/\Re)$. Consider any edge set $\Re_1$ of P_p^2 such that, $|\Re_1| \leq |\Re|$, then due to the construction of P_p^2 (i.e., to convert $P_p^2/\Re_1$ into disconnected graph, include at least one edge in $\Re_1$) must be included. It generates a large value of $\varpi_e(P_p^2/\Re_1)$ such that,

$$|\Re| + \varpi_e(P_p^2/\Re) \leq |\Re_1| + \varpi_e(P_p^2/\Re_1) \tag{13}$$

Let $\Re_2$ be any edge set of P_p^2 such that $\varpi_e(P_p^2/\Re_2) \geq 1$. Then

$$|\Re| + \varpi_e(P_p^2/\Re) \leq |\Re_2| + \varpi_e(P_p^2/\Re_2). \tag{14}$$

Therefore, these inequalities (13) and (14) lead to

$$|\Re| + \varpi_e(P_p^2/\Re) = min\{|X| + \varpi_e(G/X) : X \subseteq E(G)\} = PENI(P_p^2). \qquad \square$$

Definition 3.3([17]) *The lollipop graph $L_{p,d}$ is obtained from a complete graph K_{p-d} and a path P_d, by joining one of the end vertices of P_d to all the vertices of K_{p-d}.*

Theorem 3.5 *For a lollipop graph $L_{p,d}$,*

$$PENI(L_{p,d}) = \lfloor \frac{p-d+1}{2} \rfloor + \lceil 2\sqrt{d+1} \rceil - 4.$$

Proof The number of the vertices of $L_{p,d}$ is p and the number of edges is $d-1+\frac{(p-d+1)(p-d)}{2}$.

The graph $L_{p,d}$ consists of a complete graph of order $p-d+1$ and a path of order $d-1$. By Proposition 2.1, it follows that

$$PENI(L_{p,d}) = PENI(P_{d-1}) + PENI(K_{p-d+1}) = \lfloor \frac{p-d+1}{2} \rfloor + \lceil 2\sqrt{d+1} \rceil - 4. \qquad \Box$$

Definition 3.4([17]) *A broom graph $B_{p,d}$ consists of a path P_d, together with $(p-d)$ end vertices all adjacent to the same end vertex of P_d.*

Theorem 3.6 *For a broom graph $B_{p,d}$,*

$$PENI(B_{p,d}) = \lceil 2\sqrt{d} \rceil - 3.$$

Proof Let $V(B_{p,d}) = \{u_1, u_2, \cdots, u_d, v_1, v_2, \cdots, v_{p-d}\}$ such that $u_1, u_2, \cdots, u_d$ is a path on d vertices and $v_1, v_2, \cdots, v_{p-d}$ are end vertices that are adjacent to u_d. An edge e as shown in Figure 10 is chosen,

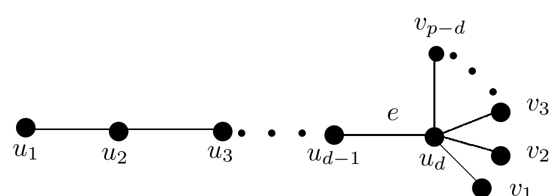

Figure 10 $B_{p,d}$

and e is deleted, we get $p-d+1$ components, namely $(p-d)$ isolated vertices and a path of order $(d-2)$. By Proposition 2.1, it follows that

$$PENI(B_{p,d}) = 1 + PENI(P_{d-2}) = 1 + \lceil 2\sqrt{d} \rceil - 4.$$

Thus

$$PENI(B_{p,d}) = \lceil 2\sqrt{d} \rceil - 3. \qquad \Box$$

Corollary 3.1 *For any broom graph, if $p-d=2$, then*

$$PENI(B_{p,d}) = PENI(L(B_{p,d})).$$

Definition 3.5([12]) *The join of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1 + G_2$ consists of vertex set $V = V_1 \cup V_2$, and edge set $E = E_1 \cup E_2$ and all edges joining V_1 with V_2.*

Theorem 3.7 *For a joint graph $K_2 + P_p$,*

$$PENI(K_2 + P_p) = \lceil 2\sqrt{p+2} \rceil - 3.$$

Proof Let K_2 be a complete graph with vertices u_1, u_2 and P_p, a path with vertices $v_1, v_2, \cdots, v_p$. Let G be the graph $K_2 + P_p$. Then, $V(G) = \{u_1, u_2, v_1, \cdots, v_p\}, |V(G)| = p+2$, and $|E(G)| = 3p$.

The graph $K_2 + P_p$ is shown in Figure 11.

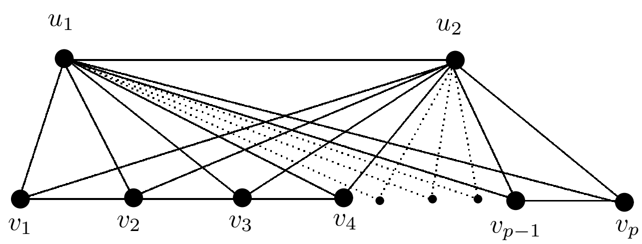

Figure 11 $K_2 + P_p$

Consider $\Re_1 = \{(u_1, u_2)\}$, $|\Re_1| = 1$. Then, $G/\Re_1 = P_p$, so that $\varpi_e(G/\Re_1) = p-1$. Let $\Re_2 = \{e_k = (v_k, v_{k+1}), 1 \le k \le p-1 / e_k \in PENI-set\ of\ P_p\}$. Take $E_1 = \{e_k / e_k \in PENI-set\ of\ P_p\}$ so that $|\Re_2| = |E_1|$. Consider $\Re = \Re_1 \cup \Re_2$. Thus, $|\Re| = |\Re_1| + |\Re_2| = |\Re_1| + |E_1|$ and $G/\Re = P_p/E_1$. So $\varpi_e(G/\Re) = \varpi_e(P_p/E_1)$. By Proposition 2.1, we have

$$\begin{aligned} |\Re| + \varpi_e(G/\Re) &= |\Re_1| + |E_1| + \varpi_e(P_p/E_1) \\ &= |\Re_1| + PENI(P_p) = 1 + \lceil 2\sqrt{p+2} \rceil - 4 \\ &= \lceil 2\sqrt{p+2} \rceil - 3. \end{aligned} \tag{15}$$

To claim that $|\Re| + \varpi_e(G/\Re)$ is minimum. Suppose $\Re_3$ is any edge set of G such that $\Re_3 = \Re_1 \cup \{e\}$ and $|\Re_3| = 2$. Then $|\Re_3| + \varpi_e(G/\Re_3) \ge |\Re| + \varpi_e(G/\Re)$. Let $\Re_5$ be edge set of G such that $\Re_5 = \Re_2$. Then, $\varpi_e(G/\Re_5) \ge p$. Hence, $|\Re_5| + \varpi_e(G/\Re_5) \ge |\Re_2| + p > |\Re| + \varpi_e(G/\Re)$. Therefore, from the above discussion, it follows that $|\Re| + \varpi_e(G/\Re)$ is minimum. Hence, from equation (15) and the minimality of $|\Re| + \varpi_e(G/\Re)$, we have

$$PENI(K_2 + P_p) = \lceil 2\sqrt{p+2} \rceil - 3.$$ □

Theorem 3.8 *For a joint graph $K_2 + C_p$,*

$$PENI(K_2 + C_p) = \begin{cases} p-1, & p=3,\ 4; \\ \lceil 2\sqrt{p} \rceil - 2, & p \ge 5. \end{cases}$$

Proof The proof is similar to that of the Theorem 3.7. □

Theorem 3.9 *For a joint graph $K_2 + K_p$,*

$$PENI(K_2 + K_p) = \lfloor \frac{p+2}{2} \rfloor.$$

Proof Since $K_2 + K_p = K_{p+2}$ is a complete graph of order $p + 2$, by Proposition 2.1,

$$PENI(K_2 + K_p) = PENI(K_{(p+2)}) = \lfloor \frac{p+2}{2} \rfloor. \qquad \square$$

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